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Theory and Methodology

A note on 'Stability of the constant cost dynamic lot size model' by K. Richter *

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Abstract: In a paper by K. Richter the stability regions of the dynamic lot size model with constant cost parameters are analyzed. In particular, an algorithm is suggested to compute the stability region of a so-called generalized solution. In general this region is only a subregion of the stability region of the optimal solution. In this note we show that in a computational effort that is of the same order as the running time of Richter's algorithm, it is possible to partition the parameter space in stability regions such that every region corresponds to another optimal solution.

Keywords: Parametric programming, combinatorial analysis, inventory

1. Introduction

We consider the constant cost dynamic lot size problem with set-up cost $c > 0$, unit holding cost $h > 0$ and a planning horizon consisting of T periods. Richter (1987) has analyzed the stability region of this model, i.e., the following question was studied: given an optimal solution for the cost parameters c and h, for which other pairs of parameters (c', h') is the solution still optimal? To answer this question, the notion of a generalized solutuion was introduced. A generalized solution can be viewed as a complete description of the output of the well-known dynamic programming algorithm of Wagner and Whitin (1958). The generalized solutions of two different pairs of cost parameters are equal if and only if the optimal solutions for the planning horizons consisting of periods 1 to t are equal for all t, $1 \leq t \leq T$. It follows that two pairs (c, h) and (c', h') can have different generalized solutions, although they correspond to the same optimal solution. In other words, the stability region of an optimal solution is in general the union of stability regions of several generalized solutions. Richter has shown how one can obtain bounds, such that the generalized solution for (c, h) is a generalized solution for (c', h') if and only if c'/h' lies within these bounds. This result implies that the stability regions of the generalized solutions are convex cones in $\mathbb{R}^2_+$. Although it was not proven or stated explicitly, Richter assumed that the convexity property also holds for the stability regions of the optimal solutions, because to find these he only explores neighbouring stability regions of generalized solutions. The correctness of this approach

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follows from Richter and Vörös (1989a), where the 'convex cone property' is proven for an even more general case.

Richter did not give much attention to the computational aspects of his method to compute the stability regions of the generalized solutions. His exposition suggests an algorithm that requires $O(T^2)$ elementary operations. In fact, this algorithm was implemented by him in an interactive BASIC program. The running time of his method to determine the stability region of an optimal solution is not directly clear, because this involves the computation of the stability regions of several generalized solutions.

2. Computing stability regions of optimal solutions

In a recent paper by Wagelmans, Van Hoesel and Kolen (1991) it is shown that the dynamic lot size problem can be solved in $O(T \log T)$ time and that some special cases can even be solved in linear time. Among these special cases is the problem originally considered in Wagner and Whitin (1958), and therefore also the constant cost model. We shall first use this result to show that it is possible to compute the stability region of an optimal solution in $O(T^2)$ time. Subsequently we shall show that even a stronger result holds.

Because of the 'convex cone property' we can normalize the cost coefficients by taking $h = 1$. This means that we are in fact considering a parametric version of the constant cost dynamic lot-sizing with one parameter that appears linearly in the objective function. This problem has also been discussed in Richter and Vörös (1989b) and a related problem is analyzed in Richter (1986). However, in both cases no explicit normalization of the cost coefficients is made and the analysis is again based on generalized solutions.

The question we want to solve now is: given an optimal solution for set-up cost c, compute the values c_{low} and c_{up} such that the solution is optimal if and only if the set-up cost belongs to the interval $[c_{\text{low}}, c_{\text{up}}]$. It has been shown in Gusfield (1983) that if one has the disposal of a 'suitable' algorithm to solve the non-parametric problem that runs in $O(A)$ time, then this question can be solved in $O(A^2)$ time. The term 'suitable' refers to the fact that all operations of the algorithm must preserve linearity in the cost coefficients that are parameterized. Many well-known combinatorial algorithms possess this property. In particular, the linear time algorithm to solve the constant cost dynamic lot size problem is 'suitable'. Therefore, it is possible to compute the stability region of an opimal solution in $O(T^2)$ time.

The result above also holds for the more general case where both the set-up costs and the unit holding costs are linear functions of a parameter λ, i.e.,

$$c_t = a_t + b_t\lambda \quad \text{and} \quad h_t = f_t + g_t\lambda \quad \text{for } t = 1, \dots, T,$$

and λ is restricted to the region for which all cost coefficients are non-negative. Given a value for λ and a corresponding optimal solution, one can compute the interval $[\lambda_{\text{low}}, \lambda_{\text{up}}]$ that contains exactly all values of λ for which the solution is optimal in $O(T^2)$ time. In this general case it is not easy to give a good estimate of the computational effort needed to compute all stability regions of optimal solutions, because it is not directly clear how many different solutions can become optimal if λ varies over its feasible region. However, in the case of the constant cost model this question is easy to answer.

It is well-known that if one considers the optimal value of the problem as a function of the changing parameter, i.e. c, then this function is piecewise linear and concave. If the slope of the function changes then another solution becomes optimal. In our case the slope of a linear part of the function is exactly equal to the number of set-ups in the corresponding solution. Because of the concavity of the function this number is non-increasing for increasing c. It follows that the function consists of at most T linear parts. There exists a simple method that is often attributed to Eisner and Severance (1976) to compute this function and the optimal solutions. If the non-parametric problem can be solved in $O(A)$ time and the function has $O(B)$ pieces, then the computational effort of this method can be bounded by $O(AB)$ elementary operations (see Gusfield, 1980, for a detailed description of an implementation). This means that in our case we can determine the function in $O(T^2)$ time.

3. Conclusion

We have shown that it is possible to compute the stability regions for all possible optimal solutions in $O(T^2)$ time, i.e., in the same computational effort that Richter used for the determination of only a part of the stability region of one optimal solution.

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