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NOTES AND COMMENTS

A GENERAL RESULT FOR QUANTIFYING BELIEFS

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THIS NOTE GENERALIZES THE RESULTS by Sarin and Wakker (1992a) and Machina and Schmeidler (1992) to derive a "cumulative capacity functional." Thus we obtain a general decision theoretic foundation for the representation of beliefs by capacities ("nonadditive probabilities"). Choquet expected utility and probabilistically sophisticated preferences are two special cases of our cumulative capacity model.

Notations and definitions are as in Sarin and Wakker (1992a), and are summarized as follows. $\mathscr{C}$ is the set of consequences; S is the state space; $\mathscr{A}$ is the sigma-algebra on S of events; $\mathscr{A}^{ua}$ is the sub-sigma-algebra of unambiguous events; $\mathscr{F}$ is the set of acts (maps from S to $\mathscr{C}$, assumed finite-valued and $\mathscr{A}$-measurable in this note); $\mathscr{F}^{ua}$ is the set of $\mathscr{A}^{ua}$-measurable acts; $f_A h$ coincides with act f on A, with act h on A^c; α denotes both a consequence and the related constant act; $\succcurlyeq$ is the preference relation over acts, that also denotes the induced ordering of consequences and an induced ordering of events defined by $A \succcurlyeq B$ if $\alpha_A \beta \succcurlyeq \alpha_B \beta$ for outcomes $\alpha \succ \beta$. The latter "more-likely-than" relation will be used in P4 below. For further discussions and details the reader is referred to Sarin and Wakker (1992a). Next we list the conditions used in the main result. The statement of P4 below (as well as P4D discussed after) has been simplified as compared to P4 in Sarin and Wakker (1992a); the simplification is possible because this note only considers acts with a finite range.

POSTULATE P1: *The preference relation $\succcurlyeq$ over the acts is a weak ordering.*

POSTULATE P2* (Sure-Thing Principle for Unambiguous Two-Consequence Acts): *For all consequences $\alpha \succ \beta$ and unambiguous events A, B, H with $A \cap H = B \cap H = \varnothing$:*

$$\alpha_A \beta \succcurlyeq \alpha_B \beta \Leftrightarrow \alpha_{A \cup H} \beta \succcurlyeq \alpha_{B \cup H} \beta.$$

POSTULATE P3: *For all events $A \in \mathscr{A}$, acts $f \in \mathscr{F}$, and consequences α, β:*

$$\alpha \succcurlyeq \beta \Rightarrow \alpha_A f \succcurlyeq \beta_A f.$$

The reversed implication holds as well if $A \in \mathscr{A}^{ua}$, A is nonnull, and $f \in \mathscr{F}^{ua}$.

POSTULATE P4 (Cumulative Dominance): *For all acts f, g we have:*

$$f \succcurlyeq g \text{ whenever } \{s \in S: f(s) \succcurlyeq \alpha\} \succcurlyeq \{s \in S: g(s) \succcurlyeq \alpha\} \text{ for all consequences } \alpha.$$

POSTULATE P5 (Nontriviality): *There exist consequences α, β such that $\alpha \succ \beta$.*

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POSTULATE P6 (Fineness of the Unambiguous Events): *If $\alpha \in \mathscr{C}$ and, for $f \in \mathscr{F}^{ua}$, $g \in \mathscr{F}$, we have $f \succ g$, then there exists a partition $(A_1, \ldots, A_m)$ of S, with all elements in $\mathscr{A}^{ua}$, such that $\alpha_{A_j} f \succ g$ for all j; the same holds with $\prec$ instead of $\succ$.*

P4 considers cumulative events (receiving consequence α or anything better), and the more-likely-than relation has been defined accordingly. An alternative formulation can be given in terms of decumulative events (receiving consequence α or anything worse). Then one defines $A \succcurlyeq^* B$ if $\beta_A \alpha \preccurlyeq \beta_B \alpha$ for some $\alpha \succ \beta$, and requires $f \succcurlyeq g$ whenever $\{s: g(s) \preccurlyeq \alpha\} \succcurlyeq^* \{s: f(s) \preccurlyeq \alpha\}$ for all consequences α. This "dual" condition is equivalent to P4, as one sees by complement taking. Nehring (1993) considers a condition P4D, requiring (together with his definition L1) $f \succcurlyeq g$ whenever $\{s: g(s) \preccurlyeq \alpha\} \succcurlyeq \{s: f(s) \preccurlyeq \alpha\}$ for all consequences α. That is, the condition considers decumulative events where, however, the more likely than relation $\succcurlyeq$ has been derived from cumulative events. Hence condition P4D is not truly dual to P4; it imposes the restriction $v(A) + v(A^c) = 1$ and excludes Ellsberg-type preferences.

Postulates P1–P6 are used to derive a capacity measure v over events in a general decision model that will now be described. For an act f and a capacity v, the *cumulative distribution function* $F_{f,v}$: $\mathscr{C} \to [0,1]$ is defined by $F_{f,v}$: $\alpha \mapsto v(\{s \in S: f(s) \succcurlyeq \alpha\})$. If $\alpha^1 \succcurlyeq \cdots \succcurlyeq \alpha^m$ and $\{\alpha^1, \ldots, \alpha^m\} \supset \text{range}(f)$, then we may denote $F_{f,v}$ by $(\alpha^1, v_1; \ldots; \alpha^m, v_m)$, where $v_j := v(\{s \in S: f(s) \succcurlyeq \alpha^j\})$ for all j.

A function $\hat{V}$ is a *cumulative distribution functional* if its range is $\mathbb{R}$ and its domain consists of all the cumulative distribution functions generated by simple probability distribution functions over $\mathscr{C}$. Further, a cumulative distribution functional $\hat{V}$ is required to satisfy (strict first-order) *stochastic dominance* (i.e., $\hat{V}(F_{f,v}) > \hat{V}(F_{g,v})$ whenever $F_{f,v} \neq F_{g,v}$ and $F_{f,v} \geq F_{g,v}$ on its entire domain) and *mixture continuity* (i.e., continuity of $\lambda \mapsto \hat{V}(\lambda F_f + (1-\lambda) F_g)$ on $[0,1]$). A function V: $\mathscr{F} \to \mathbb{R}$ is a *cumulative capacity functional* if it *agrees* with a cumulative distribution functional, i.e., there exist a capacity v (the capacity *related to* V) and a cumulative distribution functional $\hat{V}$ such that $V(f) = \hat{V}(F_{f,v})$ for all acts f. Under the conditions of the theorem below, the capacity related to $\hat{V}$ will be determined uniquely. We call V *mixture continous* if the associated cumulative distribution functional is mixture continuous. Finally, a capacity v is *convex-ranged* on $\mathscr{A}^{ua}$ if for every $A \supset C$ in $\mathscr{A}^{ua}$ and every μ between $v(A)$ and $v(C)$ there exists an event B such that $A \supset B \supset C$ and $v(B) = \mu$.

THEOREM 1: *Suppose P5 (nontriviality) holds. Then the following two statements are equivalent:*

(i) There exists a cumulative capacity functional V that represents $\succcurlyeq$. On $\mathscr{A}^{ua}$ the capacity v, related to V, is additive and convex-ranged; the functional V is mixture continuous.

(ii) Postulates P1, P2, P3, P4, and P6 are satisfied.*

Further, the function V is ordinal and the capacity v in (i) is unique.

Here we sketch the proof of the implication (ii) $\Rightarrow$ (i). As in Savage (1954), an additive convex-ranged probability P is obtained on the unambiguous events $\mathscr{A}^{ua}$. Then for any event A, with $\alpha \succ \beta$ consequences as in P5, and U an appropriate unambiguous event obtained through P6, $\alpha_U \beta \sim \alpha_A \beta$ gives the capacity $v(A) = P(U)$. For all acts f that satisfy $\alpha \succcurlyeq f(s) \succcurlyeq \beta$ for all states, and U' an appropriate unambiguous event obtained through P6, $\alpha_{U'} \beta \sim f$ gives a cumulative capacity functional $V(f) = P(U')$. Finally, V is extended to all acts, and shown to satisfy all requirements. For an elaborated proof, see Sarin and Wakker (1992b).

A special case of a cumulative capacity functional $V(f) = \hat{V}(\alpha^1, v_1; \ldots; \alpha^m, v_m)$ is Choquet expected utility, $\sum_{i=1}^m (v_i - v_{i-1}) u(\alpha^i)$, which is obtained by strengthening P2 to include all unambigous acts. As an example of a special case of $V(f)$ we give a weighted utility-like form:

$$V(f)=\frac{\sum_{i=1}^{m} w(\alpha_i)(v_i-v_{i-1})u(\alpha_i)}{\sum_{i=1}^{m} w(\alpha_i)(v_i-v_{i-1})},$$

where u is the utility function, w is the "weighting function," and we set $v_0=0$.

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