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WARP Does Not Imply SARP for More Than Two Commodities

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The only examples available in the literature to show that the Weak Axiom of Revealed Preference does not imply the Strong Axiom of Revealed Preference, the examples of Gale and Shafer, apply only to the case of three commodities. This paper constructs examples for four or more commodities. *Journal of Economic Literature* Classification Numbers: D11, C60.

1. INTRODUCTION

One of the most famous open questions in the history of economics was the question of whether the Weak Axiom of Revealed Preference (WARP) for consumer demand functions, as introduced by Samuelson [10], was sufficient to guarantee maximization of a utility function. Here the usual regularity conditions (continuity, efficiency) are assumed to be satisfied; WARP is the condition that excludes cycles of length 2 in the revealed preference relation. Ville [12] and Houthakker [4] independently proved that an apparently stronger condition, the Strong Axiom of Revealed Preference (SARP; excluding cycles of any length) was necessary and sufficient to construct a utility function, as if maximized by the consumer. As pointed out for instance by Arrow [1], this did not completely settle the question. It was still unclear whether the weaker WARP condition would, in the presence of the usual regularity conditions, nevertheless suffice to imply SARP. Some hope for equivalence of WARP and SARP came from the result of Rose [9], who showed that for two commodities indeed WARP is strong enough to imply SARP. For three or more commodities the question remained open. Finally, Gale [3] constructed an actual coun-

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terexample, i.e., an example where WARP was satisfied, but SARP was violated; see Example 1 below. The question seemed to be settled, WARP does not imply SARP, and the subject was laid to rest. What has often been overlooked is that Gale's construction (as well as Shafer's, [11]) applies only to the case of three commodities and leaves the question open for more commodities. Indeed, for more commodities no counterexamples have yet been constructed in the literature. This paper provides such.

A theoretical argument to show that also for more than three commodities WARP does not imply SARP has already been provided by Kihlstrom, Mas-Colell, and Sonnenschein [5]. They showed that, for a demand function with a negative definite matrix of substitution terms, there exists an open neighborhood such that all demand functions in the open neighborhood satisfy WARP. For most of these the matrix of substitution terms is not symmetric; that symmetry is known to be a necessary condition for SARP. This does not show how to actually construct a counterexample of a (smooth!) demand function with a nonsymmetric matrix of substitution terms within that open neighborhood. It does formalize the intuition, ascribed to Samuelson by Gale, that it should be possible to perturb a demand function maximizing a utility function in such a way that it continues to satisfy WARP, but no longer satisfies SARP.

This paper shows in a constructive way that WARP cannot imply SARP, by giving actual counterexamples. These examples are obtained by a nontrivial embedding of Gale's counterexample in higher dimensions. As an intermediate tool, demand functions defined on convex compact nonlinear budget sets are used. These have recently been studied in Peters and Wakker [7].

2. Preparations

Let $X = \mathbb{R}^n_+$ be the set of commodity bundles, with Σ^n the collection of budget sets B of the form

$$B = \{x \in \mathbb{R}^n_+ : p \cdot x \leqslant \alpha\}$$

for some price vector $p \in \mathbb{R}^n_{++}$ and income $\alpha \geqslant 0$, and D the demand function assigning to each budget set the commodity bundle chosen from the budget set. It is implicitly assumed that for each commodity j there exists a budget set B with $D_j(B) > 0$. Otherwise commodity j could be suppressed, and D would essentially be of lower dimension. R is the (directly) revealed preference relation, i.e., xRy if there exists a budget set from which x is chosen whereas also y is contained in the budget set. Conversely, if $\succcurlyeq$ is a binary relation on X, then $\succcurlyeq$ *rationalizes* D if for every budget set B we have

$$\{D(B)\} = \{x \in B : x \succcurlyeq y \text{ for every } y \text{ in } B\}, \tag{1}$$

i.e., D uniquely maximizes $\succcurlyeq$ on B. Obviously, not every binary relation rationalizes a demand function, and not every demand function can be rationalized by a binary relation. Revealed preference theory has studied the question which demand functions can be rationalized by a weak order. A *weak order* is a binary relation that is transitive and complete (every pair of commodity bundles is comparable), hence reflexive. Under the usual continuity conditions a weak order is represented by a utility function, thus the corresponding demand function maximizes the same utility function. We write xPy if xRy and $x \neq y$. P is called the directly revealed strict preference relation. For $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, we write $x \geqslant y$ if $x_i \geqslant y_i$ for $i = 1, ..., n$; $x \leqslant y$ is similar.

$\text{EFF}(B) := \{x \in B$: there is no $y \in B$ with $y \geqslant x$, $y \neq x\}$ denotes the *efficient subset* of B. Throughout, D is assumed to satisfy *efficiency*, i.e., $D(B) \in \text{EFF}(B)$ for every budget set B. D satisfies the SARP if there does not exist a cycle $x = x^0Px^1P...Px^k = x$, where $k > 0$ is the length of the cycle. D satisfies the WARP if there exist no cycles of length two. WARP has recently been characterized by [6, Section 7] and [2, Section 3]. Ville [12] and Houthakker [4] showed that D can be rationalized by a weak order if and only if it satisfies SARP; Richter [8] extended this to multi-demand functions. We say that $x \in X$ is *indirectly revealed preferred to* $y \in X$, notation $x\bar{R}y$, if there exists a sequence $x = x^0Rx^1R...Rx^k = y$. We write $x\bar{P}y$ if $x\bar{R}y$ and $x \neq y$. Obviously, SARP holds if and only if $\bar{P}$ is irreflexive, i.e., $x\bar{P}x$ for no $x \in X$.

We first present Gale's example, slightly modified to allow restriction to $\mathbb{R}^3_{++}$.

EXAMPLE 1 (The example of Gale for linear budget sets). Let A be the matrix

$$\begin{pmatrix} -3 & 4 & 0 \\ 0 & -3 & 4 \\ 4 & 0 & -3 \end{pmatrix}.$$

For each price vector p for which Ap is nonnegative, $(\alpha/pAp)\,Ap$ is the demand vector of the three-dimensional budget set associated with price p and income α. Gale [3, Section 4] shows how to extend this demand function, which we denote by D^3, to the remaining budget sets: The preferences as revealed by the price vectors with Ap nonnegative together with WARP turn out to completely determine D^3. Since the details of this extension of D^3 play no role in our analysis, they are omitted. Gale shows that WARP is satisfied by D^3. The following cycle shows that SARP is violated. It is a small modification of Gale's cycle and has been constructed in [7] to show that zero coordinates can be avoided.[1] Let $x^1=(1, 0.001, 0.001)$, $x^2=(0.6, 0.001, 0.3)$, $x^3=(0.3, 0.001, 0.6)$, $x^4=(0.001, 0.001, 1)$, and let $p^1=(9.028, 16.021, 12.025)$, $p^2=(10.212, 13.209, 9.916)$, $p^3=(12.312, 12.009, 9.016)$, and $p^4=(16.021, 12.025, 9.028)$. Define $B^i := \{x: p^i x \leqslant p^i x^i\}$. Because, by construction, each x^i is a multiple of Ap^i and is efficient in B^i, $D^3(B^i)=x^i$ for all i. This and $x^{i+1} \in B^i$ implies $x^i P x^{i+1}$ for $i=1, 2, 3$. Consequently, $x^1 \bar{P} x^4$, i.e., $(1, 0.001, 0.001)\ \bar{P}(0.001, 0.001, 1)$. By interchanging the appropriate numbers one similarly shows $(0.001, 0.001, 1)\ \bar{P}(0.001, 1, 0.001)$ and $(0.001, 1, 0.001)\ \bar{P}(1, 0.001, 0.001)$. A cycle has been revealed: D^3 violates SARP. ∎

The following example, extending Gale's example to nonlinear budget sets, will be needed in Section 3.

EXAMPLE 2 (The example of Gale extended to convex compact generalized budget sets). This example considers the collection $\Sigma^{3'}$ of convex compact subsets of X, for $n=3$. In [7] it was shown that the demand function D^3 as constructed in Example 1 can be extended to a generalized demand function F^3 on $\Sigma^{3'}$, such that F^3 still satisfies efficiency and WARP. First it is shown that for each $B' \in \Sigma^{3'}$ there is an element x' that is the D^3 solution of a budget set tangential to B' at x', thus containing B' as a subset. Then WARP implies $F^3(B')=x'$. The existence of x' follows from an application of Kakutani's fixed point theorem. Further details are omitted here. We only mention that continuity and surjectivity of D^3 are essential in this construction. Obviously, F^3 also violates SARP, as the cycle constructed in Example 1 is also a cycle for F^3. ∎

3. EMBEDDING GALE'S EXAMPLE IN HIGHER DIMENSIONS

In this section and Section 4 (see Proposition 1) the following theorem is proved.

THEOREM. *For every dimension $n \geqslant 3$ there exists a continuous demand function that satisfies WARP but not SARP.*

[1] An alternative way to transform a demand function D on Σ^3 generating a cycle with nonnegative coordinates into a demand function D' generating a cycle with strictly positive coordinates is as follows. Choose any $\alpha > 0$. If a budget set B has a nonempty intersection B' with $[\alpha, \infty[^3$, then $D'(B) = D((B' - (\alpha, \alpha, \alpha)) \cap \mathbb{R}^3_+) + (\alpha, \alpha, \alpha)$; otherwise, $D'(B)$ is the intersection of B with the line segment connecting the origin and (α, α, α). Then D' contains all cycles that D does, and D' does not contain other cycles.

To prove this theorem, first Gale's example is embedded in higher dimensions. In the next section we derive continuity.

Suppose $n > 3$. F^3 is the demand function of Example 2, defined on the generalized three-dimensional budget sets in $\Sigma^{3'}$. D^n is the n-dimensional demand function yet to be constructed, with domain Σ^n. With every n-dimensional budget set B we will associate a three-dimensional convex nonlinear set $f(B) \subset \mathbb{R}^3_+$; $D^n(B)$ will be the (unique, as will be demonstrated) element of B related to $F^3(f(B))$.

Let $h: \mathbb{R}_+ \to \mathbb{R}_+$ be any surjective strictly increasing (hence continuous) and strictly concave function; obviously, $h(0) = 0$. For example, $h(\mu) = \ln(1 + \mu)$ can be taken. Let $f: \mathbb{R}^n_+ \to \mathbb{R}^3_+$, $f: (x_1, ..., x_n) \mapsto (x_1, x_2, x_3 + h(x_4) + \cdots + h(x_n))$. For every $B \in \Sigma^n$, the set $f(B)$ is *comprehensive*, i.e., for each $x \in f(B)$ and $y \in \mathbb{R}^3_+$, if $y \leqslant x$, then $y \in f(B)$. Further, $f(B)$ is convex, as follows from comprehensiveness and the fact that the f value of a convex combination dominates the convex combination of f values.

So $f(B)$ is in the domain of F^3, and there exists an element of B, denoted $D^n(B)$, with f value $F^3(f(B))$. Lemma 1 below implies uniqueness:

$$D^n(B) \text{ is the unique element of } B \text{ with } f(D^n(B)) = F^3(f(B)). \qquad (2)$$

So indeed D^n defines a map from Σ^n to X, with $D^n(B) \in B$.

LEMMA 1. *Let $\bar{z}_1, \bar{z}_2 \in \mathbb{R}_+$, and suppose that the set $\{z \in B: z_1 = \bar{z}_1, z_2 = \bar{z}_2\}$ is nonempty. Then f_3 attains its maximum over that set at a unique point.*

Proof. Because $\{z \in B: z_1 = \bar{z}_1, z_2 = \bar{z}_2\}$ is nonempty and compact, and f_3 is continuous, the maximum exists. Suppose $z = (\bar{z}_1, \bar{z}_2, z_3, ..., z_n)$ and $z' = (\bar{z}_1, \bar{z}_2, z'_3, ..., z'_n)$ are points with $f_3(z) = f_3(z')$. If $z_j \neq z'_j$ for some $j \geqslant 4$, then, by the strict concavity of h, $f_3(z/2 + z'/2) = z_3/2 + z'_3/2 + h(z_4/2 + z'_4/2) + \cdots + h(z_n/2 + z'_n/2) > z_3/2 + z'_3/2 + h(z_4)/2 + h(z'_4)/2 + \cdots + h(z_n)/2 + h(z'_n)/2 = f_3(z)/2 + f_3(z')/2 = f_3(z)$. Hence the fourth up to nth coordinates of a maximum location of f_3 are uniquely determined. Given these, so is the third: a maximum location of f_3 must be unique. ∎

$D^n(B)$ can be obtained by choosing the first two coordinates equal to those of $F^3(f(B))$ and next finding the unique maximum of f_3 given those first two coordinates. D^n inherits efficiency and WARP from F^3:

LEMMA 2. *D^n satisfies efficiency and WARP.*

Proof. To derive efficiency, suppose $x \in B$ and $x \geqslant D^n(B)$. Then, because $f(x) \in f(B)$, because $f(x) \geqslant f(D^n(B)) = F^3(f(B))$, and because F^3 is efficient, we must have $f(x) = f(D^n(B))$. If one coordinate of x were strictly greater than that of $D^n(B)$ then, since $x \geqslant D^n(B)$ and h is strictly increasing, one coordinate of $f(x)$ would be strictly greater than that of $f(D^n(B))$; this is impossible. So $x = D^n(B)$.

To derive WARP, suppose for contradiction there were two budget sets B and B' containing $x \neq y$ with $D^n(B) = x$, $D^n(B') = y$. Then, by (2), $F^3(f(B)) = f(x) \neq f(y) = F^3(f(B'))$, contradicting WARP of F^3. ∎

We construct a cycle "isomorphic" to the one of F^3, thus obtaining a violation of SARP. Our construction is such that it can also be applied if the domain is restricted to $\mathbb{R}^n_{++}$.

LEMMA 3. *D^n violates SARP.*

Proof. Fix $z_4 > 0, ..., z_n > 0$ so small that $\mu := h(z_4) + \cdots + h(z_n) < 0.001$. Let f_z be the restriction of f to $\mathbb{R}^3_+ \times \{(z_4, ..., z_n)\}$. Once coordinates 4 to n have been fixed at $z_4, ..., z_n$, the restriction f_z becomes an affine bijection from $\mathbb{R}^3_+ \times \{(z_4, ..., z_n)\}$ to $\mathbb{R}^3_+ + (0, 0, \mu)$. Observe that the points $z_4, ..., z_n$ have been chosen sufficiently small to ensure that $\mathbb{R}^3_+ + (0, 0, \mu)$ contains the entire cycle for F^3 as constructed in Example 1. To make this possible, the cycle of Gale was modified into one with strictly positive coordinates. This (adapted) cycle will be embedded in dimension n by means of the inverse mapping f_z^{-1}. Let, for $j = 4, ..., n$, $h_j: \mathbb{R} \to \mathbb{R}$ be affine and tangential to h at z_j, i.e., $h_j(z_j) = h(z_j)$ for all j, $h_j > h$ elsewhere. So $h_j(x_j) = h'_j x_j + h_j(0)$, where $h'_j > 0$ is the derivative of h at z_j, i.e., the slope of h_j.[2] Let $\bar{f}: (x_1, ..., x_n) \mapsto (x_1, x_2, x_3 + h_4(x_4) + \cdots + h_n(x_n))$; this function is affine. Consider the first two points, x^1, x^2, of the cycle in Example 1, with $x^1 P x^2$ revealed by a linear budget set $B^3 \subset \mathbb{R}^3_+$.

Let (p_1, p_2, p_3) and α be the price vector and income corresponding to B^3. Then $(\bar{f})^{-1}(B^3) = \{(x_1, ..., x_n) \in \mathbb{R}^n_+ : p_1 x_1 + p_2 x_2 + p_3 x_3 + p_3 h_4(x_4) + \cdots + p_3 h_n(x_n) \leqslant \alpha\} = \{(x_1, ..., x_n) \in \mathbb{R}^n_+ : p_1 x_1 + p_2 x_2 + p_3 x_3 + p_3 h'_4 x_4 + \cdots + p_3 h'_n x_n \leqslant \alpha - h_4(0) - \cdots - h_n(0)\}$. On $\mathbb{R}^3_+ \times \{(z_4, ..., z_n)\}$, $\bar{f} = f = f_z$, so $f_z^{-1}(x^1), f_z^{-1}(x^2) \in (\bar{f})^{-1}(B^3)$ (note that $f_z^{-1}(x^1), f_z^{-1}(x^2)$ exist because μ is sufficiently small). Elsewhere $\bar{f} \geqslant f$, hence, for each $y \in (\bar{f})^{-1}(B^3)$, $\bar{f}(y) \geqslant f(y)$. Since $\bar{f}(y)$ is an element of B^3, so is $f(y)$; $f((\bar{f})^{-1}(B^3)) \subset B^3$ follows. Because $f((\bar{f})^{-1}(B^3))$ contains $f(f_z^{-1}(x^1)) = x^1$, it contains $F^3(B^3) = x^1$. Therefore, by WARP, $x^1 = F^3(f((\bar{f})^{-1}(B^3)))$. It follows by (2) that $D^n((\bar{f})^{-1}(B^3)) = f_z^{-1}(x^1)$. So the linear n-dimensional budget set $(\bar{f})^{-1}(B^3)$, which also contains $f_z^{-1}(x^2)$, has revealed $f_z^{-1}(x^1)\, P f_z^{-1}(x^2)$. Similarly, the entire cycle of Gale's example can be embedded by f_z^{-1} in the n-dimensional space. ∎

[2] If h is not differentiable at z_j, take the right derivative. This always exists because h is concave.

Note that the above proof in fact showed that a cycle of length k for F^3 generates a cycle of length k for D^n. Preservation of WARP means that the absence of cycles of length 2 for F^3 implies the same for D^n. Similarly it can be shown that absence of cycles of length k for F^3 implies the same for D^n. Shafer [11] provides examples to show that absence of cycles of length k does not ensure absence of longer cycles, for any k and dimension 3. Unfortunately, the technique of this paper does not apply to the examples of Shafer because surjectivity of the demand function is essential for our construction technique. Shafer's demand functions are not surjective, so for the extension of these demand functions to higher dimensions a different construction must be invoked.

Note that, by fixing coordinates 4, ..., n at $z_4, ..., z_n$, F^3 is "embedded" in D^n. A simpler embedding could be obtained by setting $z_4, ..., z_n$ equal to 0 and $h \equiv 0$. Then, however, commodities 4, ..., n would never be bought, and the example would be essentially three-dimensional.

4. Continuity and Variations in Domain

This section concludes the proof of the theorem and gives some additional results.

Proposition 1. *The demand function D^n as constructed in Section 3 is continuous.*

Proof. We first show that for a sequence $B^j \in \Sigma^n$, converging to $B' \in \Sigma^n$, for which also $D^n(B^j)$ converges to some $\hat{z}$, we have $D^n(B') = \hat{z}$. Obviously, $\hat{z} \in B'$. For nonlinear compact convex sets, we use the Hausdorff metric; for the linear budget sets this complies with the usual notions of convergence and continuity. Now every step in $B \mapsto f(B) \mapsto F^3(f(B))$ is continuous: Continuity of the first step follows mainly because one can restrict attention to a restriction of f to a compact set, which is uniformly continuous. The second step, continuity of F^3, was established in [7]. So $B \mapsto F^3(f(B))$ is continuous. This and (2) imply $f(D^n(B')) = F^3 f((B')) = \lim_{j \to \infty} F^3(f(B^j))$. The latter is identical to $\lim_{j \to \infty} f(D^n(B^j))$ which, by continuity of f, equals $f(\hat{z})$. $f(D^n(B')) = f(\hat{z})$ and $\hat{z} \in B'$ imply by (2) that $D^n(B') = \hat{z}$.

Now continuity of D^n follows by a standard argument: Any converging sequence B^j is contained within a compact set, hence for every subsequence there is a sub-subsequence for which the D^n solutions converge; as established before, that convergence must be to the D^n value of the limiting budget set. ∎

Kim and Richter [6] introduce the C-axiom, which can be considered a continuous extension of WARP. It is defined as follows. If $x^k P y^k$ for all $k \in \mathbb{N}$, $x^k \to x$, $y^k \to y$, and $x \neq y$, then not yPx. We show in the proof below that the function D^n as constructed above satisfies the C-axiom.

PROPOSITION 2. *For every dimension $n \geqslant 3$ there exists a continuous demand function that satisfies the C-axiom but not SARP.*

Proof. It suffices to show that the demand function D^n as constructed in the previous section satisfies the C-axiom. We first consider the demand function constructed by Gale, i.e., D^3. Let x^k, y^k, x, y be as in the definition of the C-axiom. As demonstrated in Gale [3, Section 4], $x^k P y^k$ is revealed by the price vector $A^{-1}x^k$, where the matrix A is as in Example 1. These price vectors converge to $A^{-1}x$. Continuity of D^3 implies xPy, in accordance with the C-axiom. So D^3 satisfies the C-axiom. The C-axiom now also follows for F^3, the extension of D^3 to nonlinear sets, because all revealed preferences can be revealed by linear budget sets; see the end of Example 2. Finally, for $n \geqslant 4$, a violation of the C-axiom would, through the mapping f, generate a violation of the C-axiom for F^3. ∎

The following variations in domain are possible. The function D^n as constructed in Section 3 can be extended to all compact convex subsets of $\mathbb{R}^n_+$ without any problem. The construction for linear budget sets immediately extends to arbitrary convex compact comprehensive subsets, thus, by efficiency and WARP, to any convex compact subset. Also the derivation of continuity and the C-axiom remains unaffected. These observations also hold if the domain is restricted to the compact convex subsets within $\mathbb{R}^n_{++}$ (see also Footnote 1 in Example 1).

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