JOURNAL OF MATHEMATICAL PSYCHOLOGY 38, 86–101 (1994)

Additive Conjoint Measurement for Multiattribute Utility

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This paper shows that the role of risky alternatives can be greatly reduced in the elicitation procedures of multiattribute utility. This reduction can be achieved by invoking methods from additive conjoint measurement; it is desirable because risky choices involve more cognitive problems, thus more biases and unreliability, than riskless ones. Existing results of multiattribute utility are generalized to obtain a complete axiomatization of the new elicitation procedure. The approach has been developed in a medical decision analysis project to advise on the choice between surgery and radiotherapy for laryngeal cancer. © 1994 Academic Press, Inc.

1. INTRODUCTION

This paper argues for a more extensive use of additive conjoint measurement in *multiattribute utility*. We show that, under some customary assumptions, risk attitudes can be elicited from riskless decisions up to one single "global" parameter of risk aversion. Thus, the use of risky choices in the elicitation procedure can be reduced to a minimum. This is desirable because risky choices are subject to many biases and are usually found to be unreliable.

The traditional approach to multiattribute utility reflects the history of the topic. That history started with the foundation of expected utility maximization for risky decisions by von Neumann and Morgenstern (1944) and Savage (1954). The relevant parameters for risk aversion were subsequently found and characterized by Pratt (1964) and Arrow (1965), and many attractive subclasses of utility were characterized, with utility determined up to one or a few parameters. One of those parameters usually measures risk aversion. Methods were developed for eliciting the utility functions of individuals. Keeney and Raiffa (1976) give an accessible account of this traditional approach. All the stages of the traditional approach were developed within the realm of expected utility and were based on observed choices between *risky* alternatives.

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0022-2496/94 $6.00

Nowadays, however, it is generally believed that expected utility, while normatively appealing, simply is not empirically valid. Observed decisions violate expected utility. This restricts the usefulness of expected utility for descriptive applications. But even for prescriptive applications it poses problems. In prescriptive applications of multiattribute utility, advice is given to individuals concerning optimal choices between n-dimensional alternatives under risk. Expected utility is usually taken as the normative model and some subjective parameters concerning beliefs and/or utilities of the decision maker remain to be elicited. A problem then is that the parameters to be elicited are distorted because the decision maker is deviating from expected utility. This also explains why different methods for eliciting utility, being subject to different distortion effects, give systematically different results; see Hershey *et al.* (1982, Sect. 8).

Many biases are involved in risky choices; see Slovic *et al.* (1988) and numerous other references. Consequently, the reliability of risky choices is very low. Camerer (1989, Sect. 3.1) found that 31.6% of the subjects reversed preference in repeated risky choices and called it "distressingly close" to the 50% that would result from random choices; he pointed out, however, that this was a typical finding. Surveys of nonexpected utility are given in Machina (1987), Fishburn (1988), and Camerer and Weber (1992).

An additional problem regarding risky choices, in particular when subjects are not students but rather individuals selected randomly from the society, is that most nonacademic people have little awareness of the concept of probability. Many of them are not used to deliberate risky decisions, in particular if choices must be made under stress and within a short span of time; for further comments, see the application described in Section 4. Torrance (1987) reports problems in conveying probabilities to subjects and describes visual aids.

Because of the cognitive problems involved in risky choices, this paper proposes a new approach to multiattribute utility theory. The approach aims at minimizing these cognitive problems by minimizing the role of risky choices in the elicitation procedure; the desirability of this was already mentioned in Bell and Raiffa (1982, (a) in the Introduction). Whereas the risk structure is used from the very beginning in the traditional approach, in our procedure it is only used at the very end, when it is really needed. As much as possible, we attempt to infer the structure of the individual's preferences from riskless judgments. We axiomatize and scale representing functions in terms of riskless preferences and only require risky preference judgments where they are essential for the full description of the individual's utilities.

In accordance with the conditions customary in multiattribute utility, additive representability on the riskless alternatives is guaranteed by means of the axioms from additive conjoint measurement; see for instance Krantz *et al.* (1971), hereafter abbreviated KLST. Note that, at this stage, with only riskless decisions considered, there is not yet a meaningful difference between additive representation and multiplicative representation with positive factors. Henceforth, additive representability for riskless decisions is always assumed.

In Section 2 we discuss a meaningfulness pitfall, related to the old debate about difference/identity of risky and riskless cardinal utility. Section 3 shows that the axiomatization of additive or multiplicative risky utility can now be simplified. The characterizing condition, utility independence, only has to be imposed on one attribute. It then is automatically implied for all the other attributes. This was shown for three or more attributes, and under a continuity assumption for risky utility, in Fishburn and Keeney (1974), Keeney and Raiffa (1976, Theorems 6.2 and 6.11), and von Stengel (1993). In the application described in Section 4 we shall, however, deal with two attributes. Hence Section 3 extends the existing results to two attributes, and provides fully rigorous proofs. The extension to two attributes had already been suggested informally in Bell and Raiffa (1982, end of Sect. 3.1).

Further, we do not presuppose continuity for risky utility. Continuity of riskless utility (implied mainly by restricted solvability for riskless preferences) is shown to suffice. It is obvious that risky utility represents the same preferences over outcomes as riskless utility, so that risky utility is a strictly increasing transform of riskless utility. A priori, without continuity of riskless utility or other conditions, risky utility can very well be a noncontinuous transform of riskless utility. We shall see, however, that the other conditions do imply continuity of the transformation, thus of risky utility. Again, this generalization of previous results is motivated by the aim to reduce the role of risky choices as much as possible.

By utility independence and standard uniqueness results, the relation between risky and riskless utility is narrowed down to a one-parameter class. The parameter, an exponent, is an index of risk aversion in the sense of Dyer and Sarin (1982). Actually, the analysis of this paper provides arguments that such a parameter, a measure of concavity of utility, should not be identified with risk aversion (see Sect. 5). We nevertheless follow the traditional risk aversion terminology.

Section 4 describes an application to a medical decision problem. This initiated the research leading to this paper. By means of the approach of this paper a refinement can be obtained of existing methods of determining QALY's (quality adjusted life years). Not only is an exponent (the risk parameter) inferred from data, as is customary in the literature, but also the value function for length of life (and quality of voice), to which the exponent is applied. In the literature a fairly ad hoc value function for length of life is usually chosen. Examples are given in Section 4. This paper elicits the entire utility function from preferences, while minimizing the role of risky preferences.

Section 5 reports an empirical study. We describe a concrete method of elicitation and show that the risk parameter, for the elicitation of which risky choices and probabilities must be invoked, exhibits a great variance. Counterintuitive negative values are sometimes found for the risk parameter, which can be explained by the usual deviations from expected utility, for instance through distortions of probabilities. These findings illustrate once more the problematic nature of elicitation through risky alternatives and add to the motivation of this paper.

Finally, Section 6 concludes.

2. Risky Cardinal Utility, Riskless Cardinal Utility, and a Meaningfulness Pitfall

$X_1, ..., X_n$ are non-empty sets, with $n \geqslant 2$. For simplicity of presentation we assume that all X_j are non-degenerate intervals. *Alternatives* are elements of the product space $X := \times_{j=1}^{n} X_j$. Its coordinates are also called *attributes*. For an alternative $x = (x_1, ..., x_n)$ and a $y_i \in X_i$, $y_i x$ denotes (x with x_i replaced by y_i). By $P = \{P, Q, ...\}$ we denote a set of probability distributions over alternatives. Elements of P are called *risky alternatives*. For simplicity of presentation, we assume that P is the set of all "simple" probability distributions over X. P is *simple* if it assigns probability one to a finite set. The usual notation for a simple probability distribution is $(p^1, x^1; ...; p^m, x^m)$ where, for each j, alternative x^j results with probability p^j and the length can be any natural number m.

A preference relation $\succcurlyeq$ is given on P. We assume throughout, without further mention, that $\succcurlyeq$ satisfies the usual conditions of completeness (every pair of risky alternatives is comparable) and transitivity; note that completeness implies reflexivity. These conditions are necessary for the existence of a *representing function* W; i.e., $W: P \to \mathbb{R}$ satisfies $P \succcurlyeq Q \Leftrightarrow W(P) \geqslant W(Q)$. Usual notations are $\succ$ for strict preference (i.e., the asymmetric part), $\sim$ for indifference (i.e., the symmetric part), and $\preccurlyeq$ and $\prec$ for reversed preferences.

Risky alternatives can be "mixed", i.e., for $0 \leqslant \lambda \leqslant 1$, $\lambda P + (1 - \lambda) Q$ assigns probability $\lambda P(A) + (1 - \lambda) Q(A)$ to each set A of alternatives and is again an element of P. We assume that the preference relation $\succcurlyeq$ on P satisfies the usual von Neumann–Morgenstern axioms, that we take in the version most popular nowadays, i.e., the one by Jensen (1967). So $\succcurlyeq$ satisfies *mixture independence*, meaning $P \succcurlyeq Q \Leftrightarrow \lambda P + (1 - \lambda) R \succcurlyeq \lambda Q + (1 - \lambda) R$ for all $P, Q, R \in P$, $0 < \lambda < 1$. Further $\succcurlyeq$ satisfies *mixture continuity*, i.e., for each $P \succ Q$ there exist $0 < \lambda, \mu < 1$ such that $P \succ \lambda P + (1 - \lambda) Q$ and $\mu P + (1 - \mu) Q \succ Q$. It is well-known that these conditions are necessary and sufficient for the existence of a risky utility $u: X \to \mathbb{R}$, the expectation of which represents $\succcurlyeq$ on P. See for instance Fishburn (1970, 1982). It is also well-known that this function u is *cardinal* (or an interval scale).

The approach of this paper can also be studied for non-expected utility models that have cardinal utility functions for riskless alternatives. See for instance Miyamoto (1988) or Miyamoto and Eraker (1988). We shall, however, restrict attention to expected utility. The question remains of course how to choose the function u. In decision analysis further independence conditions are usually imposed on $\succcurlyeq$, invoking all the probabilistic structure. This paper uses an alternative approach that avoids the use of risky choices as much as possible.

We identify any alternative with the degenerate probability distribution assigning probability one to the alternative. This induces a weak order $\overline{\succcurlyeq}$ over the alternatives. Note that the risky utility u represents $\overline{\succcurlyeq}$. For simplicity, *monotonicity* is assumed throughout, i.e., $[\forall i: x_i \geqslant y_i] \Rightarrow x \overline{\succcurlyeq} y$ and $[\forall i: x_i \geqslant y_i \,\&\, \exists i: x_i > y_i] \Rightarrow x \overline{\succ} y$. We do not derive additive representability of $\overline{\succcurlyeq}$ from conditions on $\succcurlyeq$, but instead we apply the additive conjoint measurement techniques from KLST directly to $\succcurlyeq$. Below, quantifiers "for all" are omitted.

(1a) The relation $\succcurlyeq$ satisfies *restricted solvability* ($x_i v \succcurlyeq w \succcurlyeq z_i v \Rightarrow \exists y_i : y_i v \sim w$).

(1b) The relation $\succcurlyeq$ satisfies the *Archimedean axiom*. The definition can for instance be found in KLST. For understanding of our analysis the definition is not needed, hence it is not repeated here.

(2a) The relation $\succcurlyeq$ satisfies *independence*: $[v_i x \succcurlyeq v_i y \Leftrightarrow w_i x \succcurlyeq w_i y]$ for all i, v_i, w_i, x, y.

(2b) If $n=2$, then $\succcurlyeq$ also satisfies the *Thomsen condition*, i.e., $[(y_1, a_2) \sim (x_1, b_2) \;\&\; (x_1, c_2) \sim (v_1, a_2) \Rightarrow (y_1, c_2) \sim (v_1, b_2)]$.

We call the combination of the above conditions (1a)–(2b) the *additivity axioms* and assume throughout that they are all satisfied. We first discuss conditions (2a) and (2b); the technical conditions (1a) and (1b) are discussed below. We could have used alternative intuitive conditions instead of (2a) and (2b), e.g., "generalized triple cancellation"; see Wakker (1989). It is well-known that the four conditions above imply *additive representability*; i.e., there exist functions $V_1 : X_1 \to \mathbb{R}, ..., V_n : X_n \to \mathbb{R}$, such that $(x_1, ..., x_n) \mapsto V_1(x_1) + \cdots + V_n(x_n) = V(x)$ represents $\succcurlyeq$. See for instance KLST. Note here that axiom 2a implies joint factor independence in Chapter 6 of KLST: if one can replace one common coordinate by another, without affecting preference, then by repetition one can replace any finite number of common coordinates without affecting preference. The functions $V_1, ..., V_n$ are called *additive value functions* (for $\succcurlyeq$). It is well-know that the function V is cardinal.

The representation can as well be taken multiplicatively, by applying the exponential function to V and $V_1, ..., V_n$. The only additivity axiom that is not necessary for additive representability is restricted solvability. Monotonicity, as we assume throughout this paper, is equivalent to strict increasingness of all additive value functions. It can straightforwardly be verified that restricted solvability excludes discontinuities of the strictly increasing additive value functions, so that they are also continuous. Conversely, restricted solvability is implied by continuity. So the axioms imposed above are necessary and sufficient for the existence of continuous strictly increasing additive value functions for $\succcurlyeq$. So we could have used a continuity condition for $\succcurlyeq$ instead of the technical conditions (1a) and (1b).

Obviously, since both the additive function V obtained above and the risky utility function u, to be maximized in expectation, represent $\succcurlyeq$, they are related by a strictly increasing transformation ϕ. That is, $u = \varphi \circ V$ for a strictly increasing function φ. We discuss now what kind of function φ can be. A "meaningfulness" pitfall must be avoided here. That is, u and V both being cardinal, one may be tempted to identify these functions, i.e., assume that one is a linear transform of the other. Such an identification of a riskless and a risky cardinal utility function has often been taken for granted in the past; classical examples are Bernoulli (1738) and Ramsey (1931). Recently, the issue has been discussed extensively. We refer the reader to Tversky (1967), Fishburn (1989), and Wakker (1992, Sect. 2), and the many references in these papers. The model of this paper provides a very clear case where a risky and riskless cardinal utility should not be identified.

A point that has received relatively little attention in the literature is that there can be many different kinds of riskless cardinal utility functions and that between these there may be differences as large as between the risky utility function and any riskless one. Riskless utility is mostly taken as an index of strength of preference. We deal, however, with another case. In our setup, riskless cardinal utility derives from additive decomposability for multiattribute alternatives and need not be an index of strength of preference.

Example 4.1 shows that in our setup the identification of risky and riskless utility is clearly unreasonable. Thus, observations of riskless preferences ($\overline{\succcurlyeq}$) do not suffice to elicit the entire preference structure of the risk–preference relation $\succcurlyeq$ and choices between risky alternatives must be observed. We shall adopt an axiom, *utility independence*, that reduces the family of transformations to a one-parameter family. In principle, if observations were deterministic and error-free, then one observed indifference between risky alternatives would suffice to calculate the parameter, thus the entire preference structure $\succcurlyeq$. Of course, in practice several observations of risky choices must be made to obtain a better estimate of the risk parameter.

Keeney and Raiffa (1976, e.g. Sects. 5.2, 5.6.6) give many arguments in favor of utility independence. Also Miyamoto and Eraker (1988) found that most subjects satisfy the condition. In the application sketched in Section 4 it was necessary to minimize the number of risky decision questions. This motivated our adoption of utility independence; it is, admittedly, based on practical considerations rather than on normative considerations or introspection.

3. A Weakening of Utility Independence

At this stage, where the riskless V is only required to represent $\overline{\succcurlyeq}$, the difference between an additive form for V and a multiplicative form (where by monotonicity all values must be positive) is not yet meaningful. The axiom that has been used in the literature to characterize an additive or multiplicative risky utility function u is utility independence. As soon as u is additive, it is an additive representation for $\overline{\succcurlyeq}$, so, by cardinality, a linear transform of V; V can then be taken identical to u. If u is multiplicative (where by monotonicity all of its factors can be taken positive) then its logarithm is an additive representation for $\overline{\succcurlyeq}$, so is a linear transform of V; then V can be taken identical to that logarithm.

Definition 3.1. Let $J \subset \{1, ..., n\}$, $z \in \times_{j=1}^{n} X_j$ and let $\succcurlyeq_J^z$ be the preference relation generated on the probability distributions over $\times_{j \in J} X_j$ by fixing the coordinates of $\{1, ..., n\} \backslash J$ at levels identical to those of z. J is *utility independent* if

$\succcurlyeq_j^z$ is independent of z. We say that *utility independence* holds if all subsets of $\{1, \ldots, n\}$ are utility independent.

Utility independence is necessary and sufficient for additive or multiplicative decomposability of u, see for instance Keeney and Raiffa (1976, Sect. 5.4.3 and Remark below Theorem 6.1). In the present setup, with additive representability of $\overline{\succcurlyeq}$ presupposed, the result can be strengthened. It suffices to require utility independence for one attribute. This was also shown in Fishburn and Keeney (1974), Keeney and Raiffa (1976, Theorems 6.2 and 6.11), and von Stengel (1993), for at least three attributes and under continuity of the risky u, instead of only continuity of the riskless V as we have here. In view of the application in Section 4 we need the extension to two attributes. Below we call a function φ *linear* if $\varphi: \mu \mapsto \sigma\mu + \tau$ for real σ, τ, and *exponential* if $\varphi: \mu \mapsto \sigma e^{\lambda\mu} + \tau$. Increasingness implies $\sigma > 0$ in the linear case and $\sigma\lambda > 0$ in the exponential case. We say that u is a linear/exponential transform of V, if $u = \phi \circ V$ for a linear/exponential function ϕ. A comparison of the proof below with that of the most closely related result, Theorem 6.11 in Keeney and Raiffa (1976), is given at the end of the proof.

THEOREM 3.2. *Under additive representability of riskless preferences, utility independence of one attribute implies that u is a linear or exponential transform of the additive representation V of* $\overline{\succcurlyeq}$. *Thus it implies utility independence in full strength, and continuity of u.*

Proof. Let coordinate 1 be utility independent. We first show, for the range $V(X)$ of V:

For each $\mu \in V(X)$ there exists an open neighborhood S within $V(X)$ on which φ is linear or exponential. (3.1)

Let $\mu = V(x)$. By continuity, $V(X)$ is an interval. First we define S, next we derive (3.1) for μ, S. The definition of S is first given for $\mu \in \operatorname{int}(V(X))$; here int denotes topological interior, obtained by deleting boundary elements. By continuity we can take $\mu = V(x)$ with x such that no x_j is maximal or minimal in X_j. Take an open interval $]v_1, w_1[\subset X_1$ around x_1 so small that for each $y_1 \in]v_1, w_1[$, each value in $S :=]V(v_1x), V(w_1x)[$ can be obtained as $V(y_1z)$ for an alternative z. Next we define S for $\mu = \max(V(X))$. Then $\mu = V(x)$ where each x_j is maximal within X_j. Take an interval $]v_1, x_1] \subset X_1 (v_1 < x_1)$ so small that for each $y_1 \in]v_1, x_1]$, each value in $S :=]V(v_1x), V(x)]$ can be obtained as $V(y_1z)$ for an alternative z. Finally, we define S for $\mu = \min(V(X))$. Then $\mu = V(x)$ where each x_j is minimal within X_j. Now take an interval $[x_1, v_1[\subset X_1$ $(v_1 > x_1)$ so small that for each $y_1 \in [x_1, v_1[$, each value in $S := [V(x), V(v_1x)[$ can be obtained as $V(y_1z)$ for an alternative z.

Let, for any $\varepsilon \neq 0$, α, $\alpha+\varepsilon$, and $\alpha+2\varepsilon$ in S, $\lambda := (\varphi(\alpha+\varepsilon) - \varphi(\alpha))/(\varphi(\alpha+2\varepsilon) - \varphi(\alpha))$. Let a_1, b_1, c_1 be such that $V(a_1x) = \alpha$, $V(b_1x) = \alpha+\varepsilon$, $V(c_1x) = \alpha+2\varepsilon$. So $V_1(c_1) - V_1(b_1) = V_1(b_1) - V_1(a_1) = \varepsilon$. Substitutions shows that $((1-\lambda), a_1x; \lambda, c_1x) \sim$

b_1x. By utility independence of coordinate 1 the same holds if we replace $x_2, ..., x_n$ by any other $y_2, ..., y_n$. This means that $(\varphi(\alpha'+\varepsilon)-\varphi(\alpha'))/(\varphi(\alpha'+2\varepsilon)-\varphi(\alpha'))=\lambda$ for all α', $\alpha'+\varepsilon$, and $\alpha'+2\varepsilon$ in S (set $V(a_1 y)=\alpha'$). This shows that for any $\alpha \in S$ and any $\varepsilon \neq 0$, on the set of points in S of the form $\alpha + k\varepsilon$, for any integer k, φ is an exponential or linear function. Applying this result to $\varepsilon/2$ instead of ε, ..., $\varepsilon \times 2^{-m}$ instead of ε, etc., we see that φ is an exponential or linear function on all points in S of the form $\alpha + k\varepsilon \times 2^{-m}$ for integers k and natural numbers m. This uniquely determines the strictly increasing function φ on the interval S as a linear or exponential function.

Any compact subinterval T of $V(X)$ can be covered by finitely many intervals S of the form as described in (3.1), such that each consecutive pair overlaps. From this it follows that the linear or exponential functions on these intervals all fit together into one linear or exponential function φ on T. By expanding T we see that this holds true on the entire $V(X)$. We can conclude only now:

$$\varphi, \text{ and thus } u, \text{ are continuous.} \tag{3.2}$$

Let us mention differences between the above proof and the proof of Theorem 6.11 in Keeney and Raiffa (1976). First, our proof does not impose boundedness of outcomes or continuity of u (and allows for $n=2$). Second, the "constant risk aversion" as derived for, in our notation, φ is established only for special values in the proof of Keeney and Raiffa (1976); this also holds for the argument in Section 3.1 of Bell and Raiffa (1982). We preferred to give more elaboration on this point, with the step from local to global made explicit; in a rigorous proof that step should be elaborated. Third, no results of the economic theory of constant absolute risk aversion have been invoked in our proof. ∎

Note as a corollary of the above theorem that as soon as u is additively or multiplicatively decomposable, it satisfies utility independence, so is a linear or exponential transform of V. This could also have been derived from the cardinality of the additive representation that u or its logarithm generates for $\succcurlyeq$. Henceforth we assume utility independence. Then u is of one of the following forms:

$$u: (x_1, ..., x_n) \mapsto e^{\lambda V_1(x_1)} \times \cdots \times e^{\lambda V_n(x_n)} \text{ for } \lambda > 0 \tag{3.3a}$$

$$u: (x_1, ..., x_n) \mapsto V_1(x_1) + \cdots + V_n(x_n) \text{ (parametrized by "}\lambda = 0\text{")} \tag{3.3b}$$

$$u: (x_1, ..., x_n) \mapsto -e^{\lambda V_1(x_1)} \times \cdots \times e^{\lambda V_n(x_n)} \text{ for } \lambda < 0. \tag{3.3c}$$

For the representation of risky preferences, each of these forms can be multiplied by any positive real number and any real number can be added up. We take λ as the parameter identifying the above cases, where case (3.3b) is parametrized by $\lambda = 0$; indeed it can be seen as a limiting case for $\lambda \to 0$.

If V is taken as a value function in the sense of Dyer and Sarin (1982), then it can be seen that the above forms have a constant (Arrow–Pratt-like) degree of risk aversion, identical to the parameter λ. For one-dimensional outcomes, constant risk aversion was empirically confirmed in Krzysztofowicz (1983), but disconfirmed in Keller (1985). Note that the "risk aversion parameter" λ in our prescriptive model is the same for all attributes in the following sense: if only the jth attribute is varied and this is used to find the parameter λ to relate the riskless value function V_j to the risky utility function u, then the same λ is found for all attributes. Krzysztofowicz (1983) and Keller (1985) tested empirically whether the risk parameter is invariant across different attributes, and found that it is not. Of course, a problem for all these empirical investigations is that the risky decisions of subjects may violate more basic assumptions, such as transitivity or expected utility maximization, which distorts the outcomes of the investigations.

In the application in Section 4 we give normative arguments that λ should be positive there. These arguments have nothing to do with risk attitude and are based solely on considerations referring to utility per se. This is another instance where the interpretation of concavity/convexity of utility as an expression of risk aversion/proneness is not convincing; criticisms on this interpretation have often been expressed.

Now the riskless choices reveal the preference structure up to the parameter λ. In principle, one observed non-trivial indifference between risky alternatives, in addition to the riskless preferences $\bar{\succcurlyeq}$, will now suffice to reveal the entire preference structure $\succcurlyeq$.

4. An Application to Medical Decision Making

The research leading to this paper was initiated by a medical decision problem (see Maas and Stalpers, 1992). For patients suffering from laryngeal cancer (to be precise, with tumors of the T_2- or T_3-category), a choice must be made between two treatments, namely radiotherapy and surgery (laryngectomy). Under surgery the vocal cords are removed, so that the patient must learn to speak artificially. For instance, (s)he must belch up air from the stomach (so-called esophageal speech) or use an electronic device. This device transforms a soundless articulation of the oral cavity into a somewhat metallically sounding speech (so-called electro-laryngeal speech). Patients will sometimes remain mute. The probability of tumor recurrence after surgery is lower than after radiotherapy, so that a longer length of life can be expected. Radiotherapy has no or minor effect on the quality of voice.

So a tradeoff must be made between quality of voice and length of life; for a detailed description of quality of life after loss of voice, see Maas (1991). The evaluation of quality of voice and its tradeoff against length of life is highly dependent on the personal circumstances of the patient, e.g., whether the patient is a singer or a tailor. Hence individual elicitation of utilities is essential in this medical decision problem. The Department of Radiotherapy of the Radboud hospital in Nijmegen, the Netherlands, therefore initiated joint research with the Department of Mathematical Psychology of the Nijmegen Institute for Cognition and Information.

The choice between radiotherapy and surgery is essentially a two-attribute utility problem. A choice must be made between risky alternatives, i.e., probability distributions over two-attribute alternatives. One attribute refers to length of life, the other to quality of voice. It is well-known that people have problems in understanding the notion of probability. It was also found in this project that patients have great difficulty in understanding risky alternatives. This is even more so because, contrary to most experiments in the literature, the subjects in this project are not students, but patients who constitute a sample drawn from the general public. In our application, patients are under great emotional stress; they are reluctant to deliberate risky decisions and their judgements are often unreliable. This motivated our development of an alternative method, minimizing the number of risky questions.

There is extensive literature on the biases induced by risk. This further motivated our approach to rely as much as possible on information inferred from riskless choices. Therefore, patients are first asked to choose between many pairs of riskless alternatives (none of these involving 0 years of life). From these choices an additive (or, equivalently, multiplicative) value function is derived, representing choices under certainty. Next, choices between risky alternatives are elicited from patients. The number of these choices is reduced to the minimum. Finally, the elicited utilities of the patients are used to formulate an advice.

The most basic problem encountered in prescriptive decision making is that in practice people violate transitivity. A procedure was developed to resolve intransitivities, see Maas (1990) and Maas *et al.* (1992). It was decided that no solution is constructed if patients are "too" intransitive (see Maas, 1993); then a decision by usual medical methods (e.g., a medical protocol) is recommended. For such patients we feel that the normative model deviates too much from the observed preferences and that its implications would therefore be unreliable. The possibility of detecting intransitivities motivated the use of pair comparisons; the direct ranking of alternatives was found to give unreliable results. For the nine patients considered thus far, one was too intransitive. For the other eight patients a perfectly additive representation was found.

Additive representability of riskless choices is implied by utility independence. Utility independence of survival duration from health quality was tested and confirmed in Miyamoto and Eraker (1988); they also argued for a multiplicative representation, thus for utility independence in full force. The condition was questioned in Loomes and McKenzie (1989, p. 301). Next the obtained additive representation under certainty had to be transformed into a risky utility function. The following example, similar to an example in Pliskin *et al.* (1980), shows that the two functions cannot be identified.

EXAMPLE 4.1. Suppose preference must be determined between two risky alternatives:

$$P = (.50, (10 \text{ years, good voice}); .50, (1 \text{ year, bad voice}))$$

$$Q = (.50, (10 \text{ years, bad voice}); .50, (1 \text{ year, good voice})).$$

Choice of P indicates an all-or-nothing decision, choice of Q could be based on an equity principle (being sure that one of the levels is optimal; this is similar to risk aversion). The majority of people prefer alternative P to Q. There are reasons to consider this a normative choice. The argument for P against Q is based on the desire to combine the good voice with the longest period of life, because then one has most time to enjoy it.

The two risky alternatives generate the same marginal probability distributions, (.50, 10 years; .50, 1 year) and (.50, good voice; .50, bad voice), over the attributes. Fishburn (1965) showed that it is necessary and sufficient for additive decomposability of the risky utility u (so $u = V$) that risky alternatives are indifferent as soon as they induce the same marginal distributions. Since P is not equivalent to Q, the additive model is rejected here and necessarily $u \neq V$. There is interaction between the attributes: for the majority of people a good voice is of more value when combined with a longer life. This implies that λ will be positive. Preference of Q over P would imply that λ is negative. In this application, our model seems only reasonable when λ is positive. General comments along these lines are provided in Keeney and Raiffa (1976, Subsect. 5.4.5).

So, indeed, risky and riskless utility cannot be identified and the choices under certainty do not provide all the information needed to obtain the patients' (risky) utility functions. Hence preferences between risky alternatives must be observed. For these preferences utility independence was assumed. Then the risky utility function u can be obtained from the riskless utility function V as in Formula (3.3.a); i.e., it is e^V to the power λ ($\lambda > 0$ by Example 4.1).[1] Here only one parameter, the exponent λ, remains to be determined. This exponent λ is a measure of risk proneness; i.e., a larger λ indicates lower risk aversion. For determining λ, quality of voice can be kept constant by Theorem 3.2, and only length of life is varied. Let V_1 denote the additive value function for length of life.

In the literature on QALY's, parametric families of utility functions have often been adopted where, as in our case, only an exponent λ remained to be estimated from risky choices. See for instance Pliskin *et al.* (1980) and Miyamoto and Eraker (1985, 1988, 1989). These papers adopted so-called log/power or linear/exponential families to value length of life. This can be seen, for positive λ's as appropriate in the application of this section, to be the special case where V_1 is either the identity function or the logarithmic function. Note here that V_1 appears as exponent in (3.3a). Axiomatic characterizations have been given in Pliskin *et al.* (1980) and in Miyamoto and Eraker (1989); the latter also tested their axioms and rejected them for a substantial proportion of subjects. Also Loomes and McKenzie (1989, p. 300) and Mehrez and Gafni (1989) criticize the mentioned models. This added to the motivation of this paper, where the V_1 function is not chosen arbitrarily, but is estimated from data, thus refining existing methods.

[1] Here we face a problem that is generally encountered in applications of representation theory: Data are discrete and finite; the theories, however (e.g., Theorem 3.2), require infinite data, even continua. These continua must be taken as thought experiments, meant to guide intuition.

5. An Experiment

An experiment was carried out among 45 students. Fourteen students were discarded for being too intransitive. Also we discarded eight students who violated monotonicity; the latter was only observed in risky decisions. Finally, five students were discarded who exhibited lexicographic preferences among the presented riskless alternatives. For the latter students no non-trivial tradeoffs could be determined and too little information could be obtained about the riskless additive V. This is no problem in the sense that decision advice is trivial when preferences are lexicographic. There remained 18 transitive monotonic non-lexicographic students; their data were analyzed. By discarding the other students, we assured that our analysis would focus on the innovative aspects of our approach and would not be disturbed by more basic violations of rationality. The problem of such more basic disturbances of rationality, and the desirability to exclude them, was mentioned in Miyamoto and Eraker (1988, p. 8, column two).

Binary choices were elicited for riskless alternatives, i.e., all combinations of the voice qualities mute, artificial, and normal and the lengths of life 2, 4, 6, 9, 12, and 15 years. For each subject an additive representation V was obtained from his/her choices by means of the program Unicon (see Roskam, 1974); the additive representation was perfect for each subject, i.e., Kruskal's stress was 0 in each case. This seems not to have been the case in McNeil *et al.* (1981); see Fig. 2 there. Miyamoto and Eraker (1988) found that many subjects who display the pattern of preferences of that Fig. 2 were lexicographic at short durations; therefore, our exclusion of lexicographic preferences may contribute to our different finding. Of course, our finding of an additive representation can partly be explained by the restricted number of choices that were used to test additive representability. But it is also based on the exclusion of intransitive and nonmonotonic preferences; possibly, violations of sophisticated conditions found in the literature can partly be explained by more basic inconsistencies, such as violations of transitivity and monotonicity.

By Theorem 3.2 (again, taking for granted that actual data are always finite), it suffices to restrict attention to risky alternatives for which only length of life varies. Arguments in favor of this are provided in Miyamoto and Eraker (1988, p. 7). Hence normal quality of voice is assumed and suppressed from the notation.

For riskless choices the parameter λ is meaningless; it can be chosen arbitrarily. We use the term meaningless in the measurement–theoretical sense: a property can only be *meaningful* if it is invariant under permissible transformations. For a multiplicative representation, all power functions are permissible transformations. They can, however, change a concave function into a non-concave function and vice versa. The value function measures all the personal characteristics of subjects *other than* (global, average) risk aversion. For instance, *variability* of local risk aversion is captured by the value function. That is, a person with a small variation in local risk aversion will exhibit approximately the same local risk aversion for small and large lengths of life, a person with a high variation may exhibit a high local risk aversion for small lengths of life and a small local risk aversion for large lengths of life.

A choice of a parametric family of value functions, with parameters subsequently determined to give best fit with respect to the observed values for 2, 4, 6, 9, 12, and 15 years, may easily introduce unwarranted restrictions. Hence we decided to use interpolation, which in a sense stays as close as possible to the data, in a maximally objective way.

In the experiment, students had to indicate a length of life y such that they were indifferent between a pair of risky choices (remember that quality of voice is supposed normal). Five indifferences were presented; they are illustrated in Fig. 1.

I_1 to I_4 involve certainty equivalents (CE) and are called CE-indifferences. From each indifference λ was calculated, giving five values of λ for each student. The mean of these values, M, was taken as estimation. Table 1 presents the 18 obtained values M, as well as the standard deviations SD, per student. Note that the SD is rather high. This confirms the often observed unreliability of risky choices and contributes to the motivation of this paper. Analysis of variance showed a significant effect ($p = 0.002$) of person on parameter λ; this is of course a minimal requirement to exclude total randomness.

It is well-known that the CE-indifferences $I_1, ..., I_4$ encounter more systematic biases than I_5. As found in Cohen and Jaffray (1988) and numerous other studies, greater risk aversion is exhibited in choice making if certain outcomes are involved. This was called the "certainty effect" in Kahneman and Tversky (1979). However, our experiment used matching procedures. There attention is focused on the outcomes and then the CE-equivalence method, as used in $I_1, ..., I_4$, exhibits additional risk seeking (see for instance Hershey *et al.*, 1982). We did not find a significant difference between values of λ as calculated for CE-indifferences and those calculated for I_5 ($t = 1.11$, $df = 17$, $p = 0.284$).

The focusing on outcomes also implies that subjects distinguish insufficiently between I_1 and I_4. This can be seen to lead to greater risk aversion, i.e., lower λ, for I_4. Indeed we found a significant difference ($t = 4.66$, $df = 17$, $p = 0.000$) in the expected direction. As many as 9 out of 18 estimations of λ were negative for I_4. In Example 4.1 it was argued that in this application negative λ's are counterintuitive. Apart from the 9 negative λ's in I_4, 14 negative λ's were found among the remaining 72 estimations. For 3 out of 18 students M, the mean of λ, was negative. The finding of negative λ's can be explained, first, by the unreliability of observed risky decisions and, second, by violations of the model of this paper. For instance,

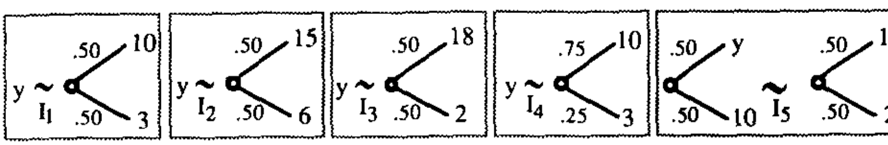

FIG. 1. The five indifferences $I_1, ..., I_5$. Students were asked to indicate the value of y for which they were indifferent.

TABLE 1

Means and Standard Deviations for Risk Parameter λ

M	0.728	1.02	4.70	−0.801	1.16	1.85	−0.575	2.07	2.91
SD	0.832	1.83	1.52	3.59	0.839	0.914	1.58	2.90	2.45
M	0.634	−0.057	0.678	2.93	1.17	0.006	0.446	0.909	0.720
SD	1.56	0.942	2.35	2.65	0.893	0.171	1.39	0.922	1.24

the distortion of probabilities, as suggested above by the comparison of I_1 and I_4, is such a violation.

6. Conclusion

The performance of multiattribute utility can be improved by invoking results from additive conjoint measurement for riskless choices. Given the additive representability of riskless choices as usually assumed in multiattribute utility, axiomatizations can be simplified and generalized. Risky preferences can be determined in a more reliable way by observing, as much as possible, riskless decisions; this also simplifies the cognitive task of subjects. The analysis also contributes to the discussions on risky versus riskless utility, the discussions on the interpretation of concavity of utility as risk aversion, and the measurement of QALY's.

Acknowledgments

The use of additive conjoint measurement in decision analysis was initiated by Thom Bezembinder. This paper received helpful comments from Thom Bezembinder, John Miyamoto, and two anonymous referees.

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RECEIVED: May 20, 1992