

COMPARISONS OF RISK AVERSION, WITH AN APPLICATION
TO BARGAINING

Peter WAKKER, Nijmegen

Hans PETERS, Maastricht

Tom van RIEL, Nijmegen

ABSTRACT: Key results of Pratt (1964), Arrow (1971), Yaari (1969), and Kihlstrom and Mirman (1974) on the comparability of the risk aversion of different decision makers, are extended to completely general consequence sets, without any restriction on the utility functions, for decision making under risk, and to topologically connected consequence spaces with continuous utility functions, for decision making under uncertainty. An application to bargaining game theory is given.

1. INTRODUCTION. Around 1965 Pratt and Arrow, independently, found a key tool to compare the risk aversion of different decision makers who maximize expected utility. Yaari (1969) gave further results for the context of decision making under uncertainty. All this was done for real numbers (amounts of money) as consequences. Kihlstrom and Mirman (1974) gave results for multidimensional consequences. In all these papers differentiability assumptions for the utility functions were made, so the Euclidean space structure on the consequence set was used. In Kihlstrom, Roth and Schmeidler (1981) these results on risk aversion were introduced into bargaining game theory, to define and study "risk sensitivity" of bargaining solutions. In Peters and Tijs (1981) a slightly different definition of "more risk averse than" has been introduced to define and study risk sensitivity.

The present paper extends results for decision making under risk (DMUR) to completely general consequence sets and utility functions. For decision making under uncertainty (DMUU) matters are more complicated. Here different decision makers may, in principle, have different subjective probabilities. Hence we shall assume here that the utility functions are continuous w.r.t. a connected topology on the consequence set. This still is less restrictive than the assumptions, usually made in literature. Remark 4.2, under expected utility maximization an ex-

tension of Remark 1 in Yaari (1969), now shows that it is only possible to compare the risk aversion of decision makers who (can be thought to) have the same subjective probabilities.

Our work enables the extension of comparisons of risk aversion to cases where consequences (contrary to money or commodity bundles) have no (known) physical quantification. Although for such cases risk aversion as such is undefined, "more risk averse than" can be defined; as we shall see in subsection 2.4; and use in section 6.

2. ELEMENTARY DEFINITIONS

2.1. GENERAL DEFINITIONS. Let X be a nonempty set. Elements of X are called alternatives, and denoted by x, y, v, w , etc. By \succ , a binary relation on X , we denote the preference relation of a person T on X . We write $x < y$ if $y \succ x$, $x > y$ or $y < x$ if $x \succ y$ and not $y \succ x$, and $x \approx y$ if $x \succ y$ and $y \succ x$. \succ is a weak order if it is complete ($x \succ y$ or $y \succ x$ for all $x, y \in X$) and transitive. A weak order induces an equivalence relation \approx . \succ is trivial if $x \succ y$ for all x, y .

C is another nonempty set. Its elements are consequences, and denoted by $\alpha, \beta, \gamma, \delta$. Intuitively, an alternative x will yield a consequence; but T has not sufficient information to know for sure which one. It is custom to distinguish between two ways to model the insufficiency of information: decision making under risk (DMUR) and decision making under uncertainty (DMUU).

2.2. DEFINITIONS FOR DMUR. Here X is $L^S(C)$, the set of simple lotteries on C , i.e. probability measures on $(C, 2^C)$ that assign probability 1 to a finite subset of C . Simple lotteries (= alternatives) are also denoted as $(p_j; x_j)_{j=1}^n$, where n can be any natural number, $x_j \in C$ and $p_j \geq 0$ for all j , $\sum_{j=1}^n p_j = 1$. It assigns probability p_j to every consequence x_j . For any $\alpha \in C$, we write $\bar{\alpha}$ for $(1; \alpha)$.

DEFINITION 2.1. A function $U: C \rightarrow \mathbb{R}$ is a von Neumann-Morgenstern (vNM) utility function (for \succ) if:

$$(p_j; x_j)_{j=1}^n \succ (q_i; y_i)_{i=1}^m \Leftrightarrow \sum_{j=1}^n p_j U(x_j) \geq \sum_{i=1}^m q_i U(y_i)$$

for all $(p_j; x_j)_{j=1}^n$ and $(q_i; y_i)_{i=1}^m$ in X . Here $\sum_{j=1}^n p_j U(x_j)$, notation $EU((p_j; x_j)_{j=1}^n)$, is the expected utility of $(p_j; x_j)_{j=1}^n$.

2.3. DEFINITIONS FOR DMUU. Here we deal with a finite state space, i.e. a set $\{s_1, \dots, s_n\}$ of (possible) states (of nature). Exactly one state is the true state, the others are untrue. T does not know for sure which state is true, and cannot influence the truth of the s_j 's. And now X is the cartesian product C^n . An act (= alternative) (x_1, \dots, x_n) is interpreted to yield consequence x_j , with j such that s_j is true. For $\alpha \in C$, we write $\bar{\alpha}$ for (α, \dots, α) . As in DMUR, $\bar{\alpha}$ yields consequence α with certainty. For DMUU, we shall throughout assume:

ASSUMPTION 2.2. C is a connected topological space; C^n is endowed with the product topology.

For instance C may be \mathbb{R} , or \mathbb{R}_+^m .

DEFINITION 2.3. $((p_j)_{j=1}^n, U)$ is a subjective expected utility (SEU) model (for \succ) if $U: C \rightarrow \mathbb{R}$ (U is the (subjective) utility function), $p_j \geq 0$ for all j , $\sum p_j = 1$ (p_j is the subjective probability of T for state s_j), and:

$$x = (x_1, \dots, x_n) \succ (y_1, \dots, y_n) = y \Leftrightarrow \sum_{j=1}^n p_j U(x_j) > \sum_{j=1}^n p_j U(y_j)$$

for all $x, y \in X$. Here $\sum_{j=1}^n p_j U(x_j)$, notation $SEU(x)$, is the subjective expected utility of x .

2.4. FURTHER GENERAL DEFINITIONS. In this subsection we introduce two ways to compare the risk aversion of two decision makers T^1 and T^2 , with preference relations \succ^1, \succ^2 . The first definition has been introduced by Yaari (1969, p. 316) (his formulation in terms of "acceptance sets" is logically equivalent to ours), as an alternative for the formulations used in Pratt (1964) and Arrow (1971). The second, slightly different property (that we have renamed) has been introduced in Peters

and Tijs (1981).

DEFINITION 2.4. \succ^1 (or T^1) is more risk averse than (MRA) \succ^2 (or T^2) if $x \succ^1 \bar{\alpha} \Rightarrow x \succ^2 \bar{\alpha}$ for all $x \in X$, $\alpha \in C$.

DEFINITION 2.5. \succ^1 (or T^1) is less risk prone than (LRP) \succ^2 (or T^2) if $x \succ^1 \bar{\alpha} \Rightarrow x \succ^2 \bar{\alpha}$ for all $x \in X$, $\alpha \in C$.

Note that the trivial \succ is both the least risk averse, and the least risk prone, preference relation.

3. THE MAIN RESULT FOR DMUR

THEOREM 3.1. Let, for DMUR, U^k be a vNM utility function for \succ^k , $k = 1, 2$. Then:

(3.1.a) \succ^1 MRA $\succ^2 \Leftrightarrow U^2 = \psi \cdot U^1$ with $\psi: U^1(C) \rightarrow U^2(C)$ nondecreasing and convex.

(3.1.b) \succ^1 LRP $\succ^2 \Leftrightarrow U^1 = \phi \cdot U^2$ with $\phi: U^2(C) \rightarrow U^1(C)$ nondecreasing and concave.

PROOF: To derive " \Rightarrow " in (3.1.a), suppose \succ^1 MRA \succ^2 . Then in particular $\bar{\beta} \succ^1 \bar{\alpha} \Rightarrow \bar{\beta} \succ^2 \bar{\alpha}$, i.e. $U^1(\bar{\beta}) \geq U^1(\bar{\alpha}) \Rightarrow U^2(\bar{\beta}) \geq U^2(\bar{\alpha})$, for all α, β . By Lemma A7.7 $U^2 = \psi \cdot U^1$ for a nondecreasing $\psi: U^1(C) \rightarrow U^2(C)$. To show convexity of ψ , suppose $U^1(\mu)$ is a weighted mean $\sum_{j=1}^n p_j U^1(\mu_j)$. Then $(p_j; \mu_j)_{j=1}^n \succ^1 \bar{\mu}$, by

MRA $(p_j; \mu_j)_{j=1}^n \succ^2 \bar{\mu}$. Hence $\sum_{j=1}^n p_j U^2(\mu_j) \geq U^2(\mu)$, i.e.

$$\sum_{j=1}^n p_j \psi(U^1(\mu_j)) \geq \psi(U^1(\mu)) = \psi\left(\sum_{j=1}^n p_j U^1(\mu_j)\right): \psi \text{ is convex.}$$

To derive " \Leftarrow " in (3.1.a), suppose $U^2 = \psi \cdot U^1$ for a convex nondecreasing ψ . And suppose $(p_j; x_j)_{j=1}^n \succ^1 \bar{\alpha}$, i.e.

$$\sum_{j=1}^n p_j U^1(x_j) \geq U^1(\bar{\alpha}). \text{ To derive is } (p_j; x_j)_{j=1}^n \succ^2 \bar{\alpha}, \text{ i.e.}$$

$$\sum_{j=1}^n p_j U^2(x_j) \geq U^2(\bar{\alpha}), \text{ or } \sum_{j=1}^n p_j \psi(U^1(x_j)) \geq \psi(U^1(\bar{\alpha})). \text{ This is}$$

direct if $U^1(x_j) \geq U^1(\bar{\alpha})$ for all j , so suppose $U^1(x_j) < U^1(\bar{\alpha})$.

Let $0 < p \leq 1$ be s.t. $(1-p)U^1(x_1) + p \sum_{j=1}^n p_j U^1(x_j) = U^1(\alpha)$.

Since $(1-p) + p \sum_{j=1}^n p_j = 1$, convexity of ψ implies

$(1-p)\psi(U^1(x_1)) + p \sum_{j=1}^n p_j \psi(U^1(x_j)) \geq \psi(U^1(\alpha))$. Since

$\psi(U^1(x_1)) \leq \psi(U^1(\alpha))$ and $p > 0$, we have $\sum_{j=1}^n p_j \psi(U^1(x_j)) \geq \psi(U^1(\alpha))$,

or $\sum_{j=1}^n p_j U^2(x_j) \geq U^2(\alpha)$.

For (3.1.b), note that $\succ^1 \text{LRP} \succ^2$ if and only if $[x \prec^2 \bar{\alpha} \Rightarrow x \prec^1 \bar{\alpha}]$ for all x, α . Now the proof of (3.1.b) is completely analogous to that of (3.1.a), mainly by interchanging \prec and \succ , 1 and 2, MRA and LRP, \geq and \leq , ψ and ϕ , concave and convex, $>$ and $<$. □

4. THE MAIN RESULT FOR DMUU. Recall that we assume C to be a connected topological space.

THEOREM 4.1. Let $((p_j^k)_{j=1}^n, U^k)$ be a SEU-model for \succ^k , with U^k continuous, $k = 1, 2$. Equivalent are:

(4.1.i) $\succ^1 \text{MRA} \succ^2$ [respectively $\succ^1 \text{LRP} \succ^2$].

(4.1.ii) (a), (b) or (c) below applies:

(a) (The nondegenerate case) $U^2 = \psi \cdot U^1$ for a convex nondecreasing continuous ψ [respectively $U^1 = \phi \cdot U^2$ for a concave nondecreasing continuous ϕ] and $(p_j^1)_{j=1}^n = (p_j^2)_{j=1}^n$.

(b) (The degenerate case of certainty) $U^2 = \psi \cdot U^1$ for a nondecreasing continuous, possibly nonconvex ψ [respectively $U^1 = \phi \cdot U^2$ for a nondecreasing continuous ϕ , possibly nonconcave] and $p_j^1 = 1 = p_j^2$ for some j .

(c) (The degenerate case of triviality) \succ^2 is trivial [respectively \succ^1 is trivial].

PROOF: (iic) \Rightarrow (i) and (iib) \Rightarrow (i) are straightforward. (iia) \Rightarrow (i) is exactly as the " \Leftarrow " 's in Theorem 3.1.

So now we assume (i), and derive (ii). First the case where $\succ^1 \text{MRA} \succ^2$, so the parts not between brackets [...]. If \succ^2 is trivial, i.e. U^2 is constant, nothing remains to be proved. So let U^2 not be constant. As in the proof of Theorem 3.1., $U^2 = \psi \cdot U^1$

with ψ nondecreasing. U^1 and ψ are nonconstant too. By Lemma A.7.7 ψ is continuous.

For all $(\lambda)U^1(\alpha), (v)U^1(\gamma)$ in the connected $U^1(C)$ we can find $U^1(\beta)$ with $U^1(\beta) = p_1^1 U^1(\alpha) + (1-p_1^1)U^1(\gamma)$. So

$(\alpha, \gamma, \dots, \gamma) \approx^1 \bar{\beta}$. By (i) $(\alpha, \gamma, \dots, \gamma) \geq^2 \bar{\beta}$, i.e.

$p_1^2 U^2(\alpha) + (1-p_1^2)U^2(\gamma) \geq U^2(\beta)$. So:

$$(4.2) \text{ (Key inequality) } p_1^2 \psi(\lambda) + (1-p_1^2)\psi(v) \geq \psi(p_1^1 \lambda + (1-p_1^1)v)$$

for all $\lambda, v \in U^1(C)$.

Since ψ is a continuous nondecreasing nonconstant function on a connected domain, $p_1^1 = 1 \Leftrightarrow p_1^2 = 1$ and $p_1^1 = 0 \Leftrightarrow p_1^2 = 0$ straightforwardly follow from (4.2). Analogously $p_j^1 = 1 \Leftrightarrow p_j^2 = 1$ and $p_j^1 = 0 \Leftrightarrow p_j^2 = 0$ follow for all $j \neq 1$. For the case where some $p_j^k = 1$, everything has been proved, we are in case (b) then.

Remains the case where not only U^2 is nonconstant, $U^2 = \psi \cdot U^1$ with ψ nondecreasing and continuous, ψ and U^1 nonconstant, but where also $0 < p_j^1 < 1$ for some j , say $j=1$. Then also $0 < p_j^2 < 1$. First convexity of ψ is derived. Let $\sigma > \tau \in U^1(C)$. If $p_1^1 \geq p_1^2$, then $p_1^1 \psi(\sigma) + (1-p_1^1)\psi(\tau) \geq p_1^2 \psi(\sigma) + (1-p_1^2)\psi(\tau) \geq (4.2)$ $\psi(p_1^1 \sigma + (1-p_1^1)\tau)$. If $p_1^1 \leq p_1^2$, then $(1-p_1^1)\psi(\sigma) + p_1^1 \psi(\tau) \geq (1-p_1^2)\psi(\sigma) + p_1^2 \psi(\tau) \geq (4.2) \psi((1-p_1^1)\sigma + p_1^1 \tau)$. By Theorem A.7.6 ψ is convex.

Finally in the above mentioned remaining case we must show that $p_j^1 = p_j^2$ for all j . We do it only for $0 < p_1^1 < 1$ and $0 < p_1^2 < 1$, other cases are analogous or have been handled above. Since ψ is convex and nonconstant on a connected domain, there must exist $\mu \in \text{int } U^1(C)$ where ψ is differentiable with $\psi'(\mu) > 0$. Let $\epsilon \neq 0$ with $|\epsilon|$ so small that $\lambda := \mu + (1-p_1^1)\epsilon$ and $v := \mu - p_1^1 \epsilon$ in $U^1(C)$. Now $\mu = p_1^1 \lambda + (1-p_1^1)v$. By (4.2) $p_1^2 \psi(\mu + (1-p_1^1)\epsilon) + (1-p_1^2)\psi(\mu - p_1^1 \epsilon) \geq \psi(\mu)$. Letting ϵ approach zero, this can only be if $(p_1^2(1-p_1^1)\epsilon)\psi'(\mu) - (p_1^1(1-p_1^2)\epsilon)\psi'(\mu) \geq 0$

for all ε close to zero, both positive and negative.

$\psi'(\mu)$ being > 0 , we conclude that $p_1^2 = p_1^1$.

For the parts not between brackets, (i) \Rightarrow (ii) has been derived. So we turn to the parts between brackets.

[Let $\succ^1 \text{LRP} \succ^2$. Then $x \prec^2 \bar{\alpha} \Rightarrow x \prec^1 \bar{\alpha}$. As in Theorem 3.1, the proof is completely analogous to that where $\succ^1 \text{MRA} \succ^2$.] \square

REMARK 4.2. Note that, also in (4.1.ii.c) equality of subjective probabilities can be arranged, because the probabilities for the trivial preference relation can be chosen completely arbitrary.

5. SOME GENERAL OBSERVATIONS. In literature usually risk aversion of preference relations \succ^1 and \succ^2 is compared under the presupposition that \succ^1 and \succ^2 induce the same ordering on the certain alternatives, i.e. have $U^1 = \phi \cdot U^2$ for strictly increasing ϕ . Our theorems, in the present general context, show as an implication of comparability of risk aversion, that $U^1 = \phi \cdot U^2$ or $U^2 = \psi \cdot U^1$, with ϕ , respectively ψ , only non-decreasing. Note that a nondecreasing concave (convex) $\phi(\psi)$ can be not-strictly-increasing only where it is maximal (minimal). In the case of strict increasingness, MRA and LRP by Lemma A 7.5 are equivalent; MRA and LRP hold simultaneously, only in the case of strict increasingness.

By Theorem A 7.4 and Lemma A 7.7 a characterization of comparability of risk aversion can be obtained by means of quotients of differences of utilities, in the spirit of (e) in Theorem 1 in Pratt (1964), also in our present general context.

Finally we refer to observation 85 in section 3.4 of Hardy, Littlewood and Polya (1959). This mathematical result on comparability of "means" (= "certainty equivalents") in fact is very close to the results of Pratt (1964), and to our Theorem 3.1 for $C \subset \mathbb{R}$.

6. RISK SENSITIVITY OF BARGAINING SOLUTIONS FOR GENERAL

CONSEQUENCE SETS. A (two-person) bargaining situation is a

triple $\langle C, U^1, U^2 \rangle$ where C is a nonempty set of consequences and $U^1, U^2: C \rightarrow \mathbb{R}_+$ are bounded functions, with

$(U^1(\alpha), U^2(\alpha)) > 0$ for some $\alpha \in C$. Such a bargaining situation $\langle C, U^1, U^2 \rangle$ is interpreted to involve two bargainers who either agree on some $x \in L^S(C)$ giving $EU^i(x)$ to bargainer i , or disagree, in which case each bargainer ends up with 0 utility.

A (two-person) bargaining game is a compact convex subset S of \mathbb{R}_+^2 , with $x > 0$ for some $x \in S$, and with $y \in S$ if $y \in \mathbb{R}_+^2$ with $y \leq x$ for some $x \in S$. B denotes the family of all bargaining games. A bargaining solution is a map $f: B \rightarrow \mathbb{R}^2$ with $f(S) \in S$ for every $S \in B$. Note that, if $\Gamma = \langle C, U^1, U^2 \rangle$ is a bargaining situation, then

$$S_\Gamma := \{y \in \mathbb{R}_+^2: y \leq x \text{ for some } x \in \text{conv cl} \{(U^1(\alpha), U^2(\alpha)): \alpha \in C\}\}$$

(where "cl" means "closure") is a bargaining game since S is compact (cf. Rockafellar (1970, Th. 17.2)). The following property for bargaining solutions was introduced in Kihlstrom, Roth, and Schmeidler (1981). For its interpretation, see (3.1.a), (3.1.b); and also section 5, first paragraph.

DEFINITION 6.1. The bargaining solution f is risk sensitive on a family A of bargaining situations if, for all $i, j \in \{1, 2\}$ with $i \neq j$, we have $f_i(S_{\Gamma'}) \geq f_i(S_\Gamma)$ for every $\Gamma = \langle C, U^1, U^2 \rangle \in A$ where Γ' is the bargaining situation obtained from Γ by replacing U^j by $\phi \cdot U^j$ with $\phi: U^j(C) \rightarrow \mathbb{R}_+$ strictly increasing, concave, and continuous.

(Strict increasingness and continuity are assumed for convenience.) As far as the authors know, all results in the literature on risk sensitivity are derived for bargaining situations $\langle C, U^1, U^2 \rangle$ with C a compact subset of a Euclidean space and with the U^i continuous. Here, in particular in view of Theorem 3.1, we omit these restrictions. Also, most results in the literature are derived for bargaining situations $\Gamma = \langle C, U^1, U^2 \rangle$ satisfying:

$$(6.1) \quad P(S_T) \subset \text{cl}\{(U^1(\alpha), U^2(\alpha)) : \alpha \in C\}.$$

$(P(T) := \{x \in T : y=x \text{ if } y \in T \text{ and } y \geq x\})$ is the Pareto set of $T \subset \mathbb{R}^2$.) Assumption (6.1) would, in Definition 6.1, imply that ϕ is defined on the second coordinate of every point of $P(S_T)$; which simplifies the analysis. Here, we relax assumption (6.1) and, instead, denote by BS a family of bargaining situations such that:

$$(6.2) \quad \text{For every } S \in B, \text{ there is a } \Gamma = \langle C, U^1, U^2 \rangle \in \text{BS with} \\ S = S_\Gamma \text{ and } P(S_\Gamma) = \text{cl}\{(U^1(\alpha), U^2(\alpha)) : \alpha \in C\}.$$

Further, by BSC ("certain") we denote the subfamily of bargaining situations in BS satisfying (6.1). For such families BS with BS \neq BSC, only few results on risk sensitivity of bargaining solutions have been derived (see Roth and Rothblum (1982), and Peters and Tijs (1985)). The following theorem enables us to extend risk sensitivity results, obtained on BSC, to BS.

THEOREM 6.2. Let the bargaining solution f be risk sensitive on BSC. Then f is risk sensitive on BS.

In order to prove this theorem, we give some notations and a lemma. For a nonempty compact convex subset Y of \mathbb{R}^2 , we denote by $\pi^Y : [\bar{p}_1^Y, \bar{p}_1^Y] \rightarrow \mathbb{R}$ the function whose graph is $P(Y)$ (here, \bar{p}^Y and \underline{p}^Y are the left and right endpoints of $P(Y)$, respectively).

LEMMA 6.3. Let $X \subset \mathbb{R}^2$ be nonempty and compact, with $X = P(X)$, and $Y := \text{conv}(X)$. Let $\phi : \{x_2 : (x_1, x_2) \in X \text{ for some } x_1\} \rightarrow \mathbb{R}$ be a continuous strictly increasing concave function such that $X' \subset P(Y')$ where $X' := \{(x_1, \phi(x_2)) : x \in X\}$ and $Y' := \text{conv}(X')$. Then $\pi^{Y'} = \psi \cdot \pi^Y$ with $\psi : [\bar{p}_2^Y, \bar{p}_2^Y] \rightarrow \mathbb{R}$ continuous, strictly increasing, and concave.

PROOF: The proof is in a few steps, some of which will be only outlined, for briefness' sake. For ever $x \in X$, denote by \bar{x} the point in $P(Y)$ with first coordinate x_1 . Define the

function $\bar{\phi}: W := \{\bar{x}_2: x \in X\} \rightarrow \mathbb{R}$ by $\bar{\phi}(\bar{x}_2) := \phi(x_2)$ for every $\bar{x}_2 \in W$. Since $X = P(X)$, $\bar{\phi}$ is well-defined, and since, moreover, ϕ is strictly increasing, so is $\bar{\phi}$. We proceed with showing that $\bar{\phi}$ is concave. In view of Lemma A7.4, it is sufficient to show, for $\bar{x}_2, \bar{y}_2, \bar{z}_2 \in W$ with $\bar{x}_2 > \bar{z}_2$ and $\bar{y}_2 = \alpha\bar{x}_2 + (1-\alpha)\bar{z}_2$ for some $0 < \alpha < 1$, that

$\bar{\phi}(\bar{y}_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2)$. There are eight cases, depending on whether $x, y, z \in P(Y)$ or not, or equivalently, depending on whether $x=\bar{x}, y=\bar{y}, z=\bar{z}$, or not. The case $x=\bar{x}, y=\bar{y}, z=\bar{z}$, follows from the concavity of ϕ . The other cases are dealt with in (i) - (iii) below.

(i) Suppose $x=\bar{x}, y=\bar{y}, z \neq \bar{z}$. We write $\bar{y}_2 = \beta\bar{x}_2 + (1-\beta)z_2$ with $1 > \beta > \alpha$. Then $\bar{\phi}(\bar{y}_2) =$

$$\phi(y_2) \geq \beta\phi(x_2) + (1-\beta)\phi(z_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2).$$

(ii) Suppose $x=\bar{x}, z=\bar{z}, y \neq \bar{y}$. There are $s, t \in X \cap P(Y)$ with $x_2 \geq s_2 > \bar{y}_2 > t_2 \geq z_2$ such that $\bar{y} = \delta s + (1-\delta)t$ for some $0 < \delta < 1$. Concavity of ϕ implies

$$\delta\bar{\phi}(\bar{s}_2) + (1-\delta)\bar{\phi}(\bar{t}_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2). \text{ Hence}$$

$$\bar{\phi}(\bar{y}_2) \geq \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2) \text{ since otherwise}$$

$(\bar{y}_1, \bar{\phi}(\bar{y}_2)) = (y_1, \phi(y_2)) \in X' \subset P(Y')$ would be contradicted.

(iii) The case $x \neq \bar{x}, y=\bar{y}, z=\bar{z}$, is analogous to (i). In the remaining cases at least two out of $\{\bar{x}, \bar{y}, \bar{z}\}$ are not in X . Suppose, e.g., $x \neq \bar{x}, y \neq \bar{y}, z=\bar{z}$. First prove the lemma for the case $\{x\} = X \setminus P(Y)$; then replace x by \bar{x} and ϕ by the so obtained $\bar{\phi}$. This construction brings us back in case (ii). Similarly for the other cases.

(iv) We now describe ψ . Let $\lambda \in [p_2^Y, \bar{p}_2^Y]$. If $\lambda = \bar{x}_2$ for some $\bar{x}_2 \in W$, then $\psi(\lambda) = \bar{\phi}(\bar{x}_2)$. Otherwise, $\lambda = \alpha\bar{x}_2 + (1-\alpha)\bar{z}_2$ where $0 < \alpha < 1$, and $\bar{x}_2 := \min\{\bar{y}_2 \in W: \bar{y}_2 > \lambda\}$ and $\bar{z}_2 := \max\{\bar{y}_2 \in W: \bar{y}_2 < \lambda\}$ exist in view of the compactness of X : then $\psi(\lambda) = \alpha\bar{\phi}(\bar{x}_2) + (1-\alpha)\bar{\phi}(\bar{z}_2)$. ψ can be seen to be strictly increasing and concave since $\bar{\phi}$ is; and hence continuous (also for $\lambda = \bar{p}_2^Y$). \square

PROOF OF THEOREM 6.2. Let $\Gamma = \langle C, U^1, U^2 \rangle \in BS$, and let $\Gamma' = \langle C, U^1, \phi \cdot U^2 \rangle$ where $\phi: U^2(C) \rightarrow \mathbb{R}_+$ is strictly increasing, continuous, and concave. We show:

$$(6.3) \quad f_1(S_{\Gamma'}) \geq f_1(S_{\Gamma}).$$

On account of ϕ 's properties, we can continuously extend it to a function on $cl(U^2(C))$; we call this extension also ϕ . Next, we apply Lemma 6.3 with $X := \{z \in Z: (z_1, \phi(z_2)) \in P(S_{\Gamma'})\}$ where $Z := cl\{(U^1(\alpha), U^2(\alpha)): \alpha \in C\}$.

Then, with notations as in Lemma 6.3, $S_{\Gamma} = \{x \in \mathbb{R}_+^2: x \leq y \text{ for some } y \in P(Y)\}$ and $S_{\Gamma'} = \{x \in \mathbb{R}_+^2: x \leq y \text{ for some } y \in P(Y')\}$.

In view of (6.2), $\Delta = \langle D, V^1, V^2 \rangle \in BSC$ exists with $S_{\Gamma} = S_{\Delta}$. Then $S_{\Gamma'} = S_{\Delta'}$, where $\Delta' = \langle D, V^1, \psi \cdot V^2 \rangle$, with ψ as in Lemma 6.3, is well-defined in view of (6.2). Since f is risk sensitive on BSC, we have $f_1(S_{\Delta'}) \geq f_1(S_{\Delta})$, from which (6.3) follows. \square

7. (APPENDIX) CONVEX FUNCTIONS ON NON-CONVEX DOMAINS. In this appendix we adapt some more or less elementary results on convex functions to the case of nonconvex domains. No literature on this case is known to us, apart from the following definition, which is given in Peters and Tijs (1981, (2.2)).

DEFINITION A 7.1. Let V be a subset of a real vector space, $\phi: V \rightarrow \mathbb{R}$. Then ϕ is convex [respectively concave] if:

$$\phi\left(\sum_{j=1}^n p_j \mu_j\right) \leq [\text{respectively } \geq] \sum_{j=1}^n p_j \phi(\mu_j) \text{ whenever } 0 \leq p_j \leq 1$$

for all j , $\sum_{j=1}^n p_j = 1$, and μ_1, \dots, μ_n and $\sum_{j=1}^n p_j \mu_j$ are in V .

REMARK A 7.2. ϕ is concave if and only if $-\phi$ is convex. All results below for convex ϕ 's can be reformulated for concave ϕ 's by the substitution $\phi \mapsto -\phi$.

We shall only consider domains in \mathbb{R} . Our main tool:

DEFINITION A 7.3. Let $V \subset \mathbb{R}$, $\phi: V \rightarrow \mathbb{R}$, $\mu, \nu \in V$, $\mu < \nu$. Then $[\phi(\nu) - \phi(\mu)] / (\nu - \mu)$ is the average increase of ϕ over $[\mu, \nu]$, denoted $AI_{\phi}[\mu, \nu]$.

THEOREM A 7.4. Let $V \subset \mathbb{R}$, $\phi: V \rightarrow \mathbb{R}$. Equivalent are:

(A 7.4. i) ϕ is convex.

(A 7.4.ii) Let $0 \leq p \leq 1, \mu, \tau$ and $p\mu + (1-p)\tau \in V$. Then

$$\phi(p\mu + (1-p)\tau) \leq p\phi(\mu) + (1-p)\phi(\tau).$$

(A 7.4.iii) If $\mu < \nu < \tau$ for $\mu, \nu, \tau \in V$, then

$$AI_\phi(\mu, \nu) \leq AI_\phi(\nu, \tau).$$

(A 7.4.iv) If $\mu < \nu, \sigma < \tau, \mu \leq \sigma, \nu \leq \tau$, for $\mu, \nu, \sigma, \tau \in V$,
 then $AI_\phi(\mu, \nu) \leq AI_\phi(\sigma, \tau)$.

PROOF: We write AI instead of AI_ϕ . (i) \Rightarrow (ii) is immediate.

For (ii) \Rightarrow (iii), let $p = (\tau - \nu) / (\tau - \mu)$, so $\nu = p\mu + (1-p)\tau$.

Now $\phi(\nu) \leq p\phi(\mu) + (1-p)\phi(\tau) \Leftrightarrow AI(\mu, \nu) \leq AI(\nu, \tau)$.

For (iii) \Rightarrow (iv), suppose (iii). If $\nu = 0$, (iv) equals (iii).

If $\nu < \sigma$, then by (iii): $AI(\mu, \nu) \leq AI(\nu, \sigma) \leq AI(\sigma, \tau)$. Finally,

if $\sigma < \nu$, then, with the convention $AI(\mu, \sigma) := AI(\sigma, \nu)$ in case

$\mu = \sigma$, and $AI(\nu, \tau) := AI(\sigma, \nu)$ in case $\nu = \tau$, (iii) gives:

$AI(\mu, \sigma) \leq AI(\sigma, \nu) \leq AI(\nu, \tau)$. Now $AI(\mu, \nu) \leq AI(\sigma, \tau)$ follows,

since $AI(\mu, \nu)$ is a weighted mean of $AI(\mu, \sigma)$ and $AI(\sigma, \nu)$, and

$AI(\sigma, \tau)$ is a weighted mean of $AI(\sigma, \nu)$ and $AI(\nu, \tau)$.

For (iv) \Rightarrow (i), assume (iv). Let $\mu \in V$ be a weighted mean

$\sum_{k=1}^n p_k \mu_k$ of elements of V . Let $I = \{i: \mu_i < \mu\}$, $J = \{j: \mu_j > \mu\}$.

Then $\sum_I p_i [\mu - \mu_i] = \sum_J p_j [\mu_j - \mu]$. To prove is $\phi(\mu) \leq \sum_{k=1}^n p_k \phi(\mu_k)$,

i.e. $\sum_I p_i [\phi(\mu) - \phi(\mu_i)] \leq \sum_J p_j [\phi(\mu_j) - \phi(\mu)]$. Let $I \neq \emptyset \neq J$ (other

cases are trivial). Let $\mu_k = \max\{\mu_i: i \in I\}$. Now

$$\sum_I p_i [\phi(\mu) - \phi(\mu_i)] = \sum_I p_i [\mu - \mu_i] AI(\mu_i, \mu) \leq (\text{by (iv)}) \sum_I p_i [\mu - \mu_i] AI(\mu_k, \mu)$$

$$= \sum_J p_j [\mu_j - \mu] AI(\mu_k, \mu) \leq (\text{by (iv)}) \sum_J p_j [\mu_j - \mu] AI(\mu, \mu_j) =$$

$$\sum_J p_j [\phi(\mu_j) - \phi(\mu)]. \quad \square$$

LEMMA A 7.5. Let $V \subset \mathbb{R}$, $\phi: V \rightarrow \mathbb{R}$ strictly increasing,

$\psi: \phi(V) \rightarrow V$ the inverse of ϕ . Then ϕ is concave if and only if ψ is convex.

PROOF: ϕ is concave $\Leftrightarrow [\phi(\nu) - \phi(\mu)] / (\nu - \mu) \geq [\phi(\tau) - \phi(\nu)] / (\tau - \nu)$

for all $\mu < \nu < \tau$ in $V \Leftrightarrow (\nu - \mu) / [\phi(\nu) - \phi(\mu)] \leq (\tau - \nu) / [\phi(\tau) - \phi(\nu)]$

for all $\mu < \nu < \tau$ in $V \Leftrightarrow [\psi(\tilde{\nu}) - \psi(\tilde{\mu})] / (\tilde{\nu} - \tilde{\mu}) \leq [\psi(\tilde{\tau}) - \psi(\tilde{\nu})] / (\tilde{\tau} - \tilde{\nu})$

for all $\tilde{\mu} < \tilde{\nu} < \tilde{\tau}$ in $\phi(V) \Leftrightarrow \psi$ is convex. We applied Remark

A 7.2, and two times (A 7.4.iii) \Leftrightarrow (A 7.4.i). □

THEOREM A7.6. Let $V \subset \mathbb{R}$ convex. Let $\phi: V \rightarrow \mathbb{R}$ continuous. Let for every $\mu > \nu \in V$ there exists $0 < p < 1$ such that $\phi(p\mu + (1-p)\nu) \leq p\phi(\mu) + (1-p)\phi(\nu)$. Then ϕ is convex.

PROOF: See observation 88 in section 3.7 of Hardy, Littlewood and Polya (1959). \square

LEMMA A7.7. Let C be a nonempty set. Let $U^1: C \rightarrow \mathbb{R}$ and $U^2: C \rightarrow \mathbb{R}$. Equivalent are:

(A 7.7.i): $U^2 = \phi \cdot U^1$ for a nondecreasing $\phi: U^1(C) \rightarrow \mathbb{R}$.
 (A 7.7.ii): $U^1(\alpha) \geq U^1(\beta) \Rightarrow U^2(\alpha) \geq U^2(\beta)$ for all $\alpha, \beta \in C$.
 Furthermore, if C is a connected topological space, and U^1 and U^2 are continuous, then so is ϕ in (i).

PROOF: (i) \Rightarrow (ii) is direct. For (ii) \Rightarrow (i), note that by (ii): $U^1(\alpha) = U^1(\beta) \Leftrightarrow [U^1(\alpha) \geq U^1(\beta) \text{ and } U^1(\beta) \geq U^1(\alpha)] \Rightarrow U^2(\alpha) = U^2(\beta)$. So $U^2 = \phi \cdot U^1$ for some ϕ . Nondecreasingness is straightforward. For the furthermore statement note that a nondecreasing function from a connected $U^1(C)$ onto a connected $U^2(C)$ must be continuous (it cannot make "jumps"). \square

REFERENCES

- (1) K.J. ARROW (1971): Essays in the Theory of Risk-Bearing. North-Holland, Amsterdam.
- (2) G.H. HARDY, J.E. LITTLEWOOD, G. POLYA (1959): Inequalities. University Press, Cambridge.
- (3) R.E. KIHLSSTROM, L.J. MIRMAN (1974): Risk Aversion with Many Commodities. *Journal of Economic Theory* 8, 361-388.
- (4) R.E. KIHLSSTROM, A.E. ROTH, D. SCHMEIDLER (1981): Risk Aversion and Nash's Solution to the Bargaining Problem. In: Game Theory and Mathematical Economics (O. Moeschlin, D. Pallaschke, eds.), North-Holland, Amsterdam, 65-71.
- (5) H. PETERS, S. TIJS (1981): Risk Sensitivity of Bargaining Solutions. *Methods of Operations Research* 44, 409-420.
- (6) H. PETERS, S. TIJS (1985): Risk Aversion in n-Person Bargaining. *Theory and Decision* 18, 47-72.
- (7) J. PRATT (1964): Risk Aversion in the Small and in the Large. *Econometrica* 32, 122-136.
- (8) R.T. ROCKAFELLAR (1970): Convex Analysis. Princeton University Press, Princeton, New Jersey.
- (9) A.E. ROTH, U.G. ROTHBLUM (1982): Risk Aversion and Nash's Solution for Bargaining Games with Risky Outcomes. *Econometrica* 50, 639-647.

- (10) M.E. YAARI (1969): Some Remarks on Measures of Risk Aversion and on their Uses. *Journal of Economic Theory* 1, 315-329.

Peter Wakker, Tom van Riel
Department of Mathematics
University of Nijmegen
Toernooiveld
6525 ED NIJMEGEN
The Netherlands

Hans Peters
Faculty of Economics
University of Limburg
P.O. Box 616
6200 MD MAASTRICHT
The Netherlands