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AGREEING PROBABILITY MEASURES FOR COMPARATIVE PROBABILITY STRUCTURES[1]

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It is proved that fine and tight comparative probability structures (where the set of events is assumed to be an algebra, not necessarily a σ-algebra) have agreeing probability measures. Although this was often claimed in the literature, all proofs the author encountered are not valid for the general case, but only for σ-algebras. Here the proof of Niiniluoto (1972) is supplemented. Furthermore an example is presented that reveals many misunderstandings in the literature. At the end a necessary and sufficient condition is given for comparative probability structures to have an almost agreeing probability measure.

1. Introduction. The axioms, definitions and basic results can be found in most of the references. For the basic results we shall refer to Niiniluoto (1972) (see his paper to Lemma 3). We shall use his notations and formulations, except that we prefer "comparative" to "qualitative", and "agreeing with" to "realizing". So we write CP (= Comparative Probability) instead of QP (= Qualitative Probability).

One minor correction:

We say that two events B and C are *almost equivalent* (notation $B \sim * C$) if the following two conditions are satisfied:

(i) $B \cup E \geq C$ for all $E > \phi$ such that $B \cap E = \phi$,

(ii) $C \cup F \geq B$ for all $F > \phi$ such that $C \cap F = \phi$.

Our present formulation avoids the unintended consequence that $A \sim * X$ (X is the sure event) for every event A.

It seems that the first clear and precise statement of the axioms of CP was given by de Finetti (1931).

We emphasize the fact that the set of events, I, is an algebra, and not necessarily a σ-algebra, and that our probability measures are only assumed to be finitely additive, not necessarily countably additive.

Elements of a partition are always assumed to be events. Let us repeat one definition of Savage (1954): An *(n-fold) almost uniform partition* (abbreviation: AUP(n)) of an event B is an (n-*fold*) partition of B such that $C \leq D$ whenever C and D are unions respectively of r and $r + 1$ elements of the partition and $1 \leq r < n$.

We shall ascribe properties of the CP-structure to the CP-relation, so we shall say $\geq$ has an agreeing probability measure, $\geq$ is fine, $\geq$ contains an atom, etc. We say that $\geq$ is AUP(∞) if there exists an AUP(n) of X for every $n \in \mathbb{N}$.

One important new definition: We say that two events A and B *differ by no more than n times C* ($n \in \mathbb{N}$, C an event) if there are events $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ such that $A_j \leq C$ and $B_j \leq C$ for all $j \leq n$ and, furthermore, $A - \cup_{j=1}^{n} A_j \leq B$ and $B - \cup_{j=1}^{n} B_j \leq A$.

If A and B differ by no more than n times C, they evidently also differ by no more than m times D for any $m \geq n$ and any event $D \geq C$. Instead of "1 times C" we also say "C". So $\geq$ is fine iff for every $B > \phi$ there is an $n \in \mathbb{N}$ such that ϕ and X differ by no more than n

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times B. Finally let us emphasize the importance of Niiniluoto (1972) Lemma 1.b: for all A, B, C, D in I, if $A \cap B = \phi$, $A \geq C$, $B \geq D$, then $A \cup B \geq C \cup D$. It will be used many times without further mention.

2. The Theorems.

THEOREM 1. *If $\geq$ is* AUP(∞) *then there is a unique probability measure P that almost agrees with $\geq$.*

PROOF. See Savage (1954), Section III.3, first part of Theorem 2 and its proof. □

REMARK. Although Savage assumes $I = 2^X$, his proof up to 8.a is also valid for the general case of I being an algebra. In the literature often the mistake is made of also applying other theorems of Section III.3 of Savage (1954) to the general case, although in Section III.4 at the top of page 43, Savage points out that this is not correct if I is not a σ-algebra.

LEMMA 1. *Let $\geq$ be atomless. Then for every event $G > \phi$ there is a sequence of events $(G_n)_{n=1}^{\infty}$ in I such that $G_1 = G$, $G_{n+1} \subset G_n$, $G_n > \phi$, and $G_{n+1} \leq G_n - G_{n+1}$ for all n.*

PROOF. Having constructed $G_m (m \in N)$, we construct G_{m+1} as follows: G_m is not an atom so there is $A \in I$, $A \subset G_m$ such that $\phi < A < G_m$. If $A \leq G_m - A$ we take $G_{m+1} = A$, otherwise we take $G_{m+1} = G_m - A$. □

LEMMA 2. *Let $\geq$ be fine, $n \in N$, $G > \phi$ an event. Then for each event A there is a partition $\{A_1, \cdots, A_n\}$ of A such that A_i and A_j differ by no more than G for all $i, j \leq n$.*

PROOF. As $\geq$ is fine there is a partition $\{B_1', \cdots, B_l'\}$ of X such that every $B_j' \leq G$. Taking $B_j = B_j' \cap A$ for all j gives a partition $\{B_1, \cdots, B_l\}$ of A with all elements $\leq G$, for every $A \in I$. Induction hypothesis (m): "If there is an m-fold partition of A with all elements $\leq G$, then there is an n-fold partition of A with all elements differing mutually by no more than G, for every event A." If $m = 1$ we can simply take $A_1 = A$, $A_2 = \cdots = A_n = \phi$. Next we prove that, for any $m_0 \in N$, the induction hypothesis ($m_0 + 1$) follows from the induction hypothesis (m_0). So let $m = m_0 + 1$. Let A be an event, let $\{B_1, \cdots, B_m\}$ be an m-fold partition of A with all elements $\leq G$. $\{B_1, \cdots, B_{m-1}\}$ is an $(m-1)$-fold partition of $\cup_{j=1}^{m-1} B_j$ with all elements $\leq G$. By the induction hypothesis (m_0) there is a partition $\{D_1, \cdots, D_n\}$ of $\cup_{j=1}^{m-1} B_j (m - 1 = m_0)$ such that for all $i, j \leq n$ D_i and D_j differ by no more than G. Let $k \leq n$ be such that $D_k \leq D_j$ for all $j \leq n$. Now define $A_k = D_k \cup B_m$, $A_j = D_j$ for all $j \neq k$. Realizing that $B_m \leq G$ one easily verifies that $\{A_1, \cdots, A_n\}$ is a partition of $\cup_{j=1}^{m} B_j = A$ such that for all $i, j \leq n$ A_i and A_j differ by no more than G. So the induction hypothesis ($m_0 + 1$) is proved. □

THEOREM 2. *If $\geq$ is fine and atomless then it is* AUP(∞).

PROOF. Let $n \in N$. We will construct an AUP(n) of X. First we make an $n(n+1)$-fold partition $\{X_1, \cdots, X_{n\cdot(n+1)}\}$ of X with $\phi < X_1 \leq X_2 \leq \cdots \leq X_{n\cdot(n+1)}$. This can be done by means of Lemma 1 with $G = X$, setting

$$X_{n\cdot(n+1)-k} = G_{k+1} - G_{k+2} \qquad \text{for } 0 \leq k \leq n\cdot(n+1) - 2 \qquad \text{and} \quad X_1 = G_{n\cdot(n+1)}.$$

By lemma 2 we can also take a partition $\{B_1, \cdots, B_n\}$ of X with the B_i mutually differing by no more than X_1, and $B_1 \leq B_2 \leq \cdots \leq B_n$. We shall show that this partition is an AUP(n) of X. To this end we first note that $B_1 \geq \cup_{j=1}^{n} X_j$. For if this were not the case, then the fact that any B_i differs by less than $X_1 \leq X_{n+1}$ from B_1 would imply that $B_i < \cup_{j=1}^{n+1} X_j \leq \cup_{j=0}^{n} x_{jn+i}$, $1 \leq i \leq n$, and hence $X = \cup_{i=1}^{n} B_i < \cup_{i=1}^{n} \cup_{j=0}^{n} X_{jn+i} = X$, which manifestly is not true.

Since for any $r \leq n - 1$ the events $\cup_{j=1}^{r} B_j$ and $\cup_{j=n-r+1}^{n} B_j$ differ by no more than r times X_1, hence by no more than n times X_1, and thus by no more than $\cup_{j=1}^{n} X_j$, it follows that they differ by no more than B_1, and hence by no more than B_{r+1}. Consequently $\cup_{j=1}^{r+1} B_j \geq \cup_{j=n-r+1}^{n} B_j$ for all $r \leq n - 1$. For every $r \leq n - 1$, $\cup_{j=1}^{r+1} B_j$ is the smallest union of $r + 1$ elements of the partition, and $\cup_{j=n-r+1}^{n} B_j$ is the greatest union of r such elements, so the partition $\{B_1, \cdots, B_n\}$ is almost uniform. □

LEMMA 3. *If $\geq$ is fine, then there is a unique probability measure that almost agrees with $\geq$.*

PROOF. Follows from Niiniluoto (1972) Lemmas 4 and 5 and our Theorems 1 and 2. □

LEMMA 4. *Let $\geq$ be fine and let P almost agree with $\geq$. Then for any event B we have $P(B) = 0$ iff $B \sim \phi$.*

PROOF. Suppose $B > \phi$. Then there is a partition $\{X_1, \cdots, X_n\}$ of X with $X_j \leq B$ for all j. So $P(X_j) \leq P(B)$ for all j, as P almost agrees with $\geq$. Thus $P(B) = 0$ implies $P(X_j) = 0$ for all j, contradicting $1 = P(X) = \sum_{j=1}^{n} P(X_j)$. Consequently, $B > \phi$ implies $P(B) > 0$.

The fact that P almost agrees with $\geq$ gives the converse implication. □

LEMMA 5. *If $\geq$ is fine and atomless and almost agrees with P, then $P(A) = P(B)$ iff $A \sim * B$.*

PROOF. Suppose $P(A) - P(B) = \varepsilon > 0$. Then $P(A - B) \geq \varepsilon$. Using lemma 1 we construct a sequence $(G_n)_{n=1}^{\infty}$ with $G_1 = A - B$ (so $G_1 > \phi$). Then $P(G_n) \leq 2^{-n+1} \cdot P(A - B)$ for all n, so there is an m such that $P(G_m) < \varepsilon$. (Since $G_m > \phi$, Lemma 4 implies $P(G_m) > 0$.) Now $G_m \cap B = \phi$, as $G_m \subset A - B$. So $P(B \cup G_m) = P(B) + P(G_m) < P(B) + \varepsilon = P(A)$. Since P almost agrees with $\geq$, this implies $B \cup G_m < A$, so not $B \sim * A$.

Conversely, if A and B are not almost equivalent and $B < A$, then there is an event $G > \phi$ such that $B \cap G = \phi$ and $B \cup G < A$. Then $P(B) + P(G) = P(B \cup G) \leq P(A)$. Since, by Lemma 4, $P(G) > 0$ it follows that $P(B) < P(A)$. □

THEOREM 3. *Let $\geq$ be fine. Then there is a unique probability measure P that almost agrees with $\geq$. Furthermore, $\geq$ has an agreeing probability measure* iff *$\geq$ is tight or $\geq$ contains an atom.*

PROOF. The first assertion is Lemma 3. The second assertion follows from Niiniluoto (1972) Lemmas 4 and 5, and the following argument.

Suppose $\geq$ is atomless. We compare the three relations:

(i) $A \sim B$, (ii) $A \sim * B$, (iii) $P(A) = P(B)$.

Now first suppose $\geq$ has an agreeing probability measure. Since P is the unique almost agreeing probability measure, P must agree with $\geq$. So (iii) implies (i). By lemma 5, (ii) implies (iii). So (ii) implies (i), $\geq$ is tight.

Next suppose $\geq$ is tight. Then (ii) implies (i). By lemma 5, (iii) implies (ii). So (iii) implies (i). Since P was already almost agreeing, this implies that P agrees with $\geq$. □

COROLLARY. *If $\geq$ is fine and tight then there is an agreeing probability measure.*

REMARK: If $\geq$ has an agreeing probability measure and is atomless, then $\geq$ must be tight, but it does not have to be fine, as follows from Nunke and Savage (1952).

3. An example. Let $X = [0, 1)$ and let I be the algebra consisting of all finite unions of intervals $[a_j, b_j)$ of X with $a_j, b_j \in (Q \cup \{\sqrt{2}/4\}) \cap X$, $a_j \leq b_j$ (Q is the set of all rationals). Let P_1 be Lebesgue measure. Define P_2 by $P_2(A) = 2\, P_1\{A \cap [0, \frac{1}{2})\}$ for all $A \in I$. Let $\geq$ be the relation induced by P_1 "refined" by P_2, i.e.:

$$
\begin{array}{llll}
\text{if} & P_1(A) > P_1(B), & & \text{then } A > B; \\
\text{if} & P_1(A) = P_1(B) & \text{and } P_2(A) > P_2(B), & \text{then } A > B; \\
\text{if} & P_1(A) = P_1(B) & \text{and } P_2(A) = P_2(B), & \text{then } A \sim B.
\end{array}
$$

One easily verifies that $\geq$ is a CP-relation and that $(X, I, \geq)$ is a CP-structure. Furthermore, $\geq$ is fine, atomless and AUP(∞). However, $\geq$ is not tight since $[0, \frac{1}{2}) \sim * [\frac{1}{2}, 1)$ while $[0, \frac{1}{2}) > [\frac{1}{2}, 1)$. Now P_1 almost agrees with $\geq$, but does not agree with $\geq$, since $[0, \frac{1}{2}) > [\frac{1}{2}, 1)$ while $P_1[0, \frac{1}{2}) = P_1[\frac{1}{2}, 1)$. By Theorem 3 $\geq$ cannot agree with a probability measure. It is important to note that I is not a σ-algebra.

I is countable, so Condition (C5) on page 19 in Fine (1973) is satisfied, and as remarked before $\geq$ is AUP(∞), but does not agree with a probability measure. This contradicts the assertion in lines 15–17 on page 25 of Fine (1973).

Furthermore, $[0, \sqrt{2}/4)$ cannot be partitioned into two almost equivalent events, i.e., (by Lemma 5) into two events with equal P_1-probability. So the theorem (***) on page 25/26 in Fine (1973) is false, as well as Lemma 7 (iii) and (iv) and Theorem 4 (iii) in Niiniluoto (1972).

Our example is also a counterexample to Theorems 5.4 and 5.5 in Narens (1974). In line 9 of the proof of his Theorem 5.4, the assertion "$y \cup z = x$" is false. Furthermore Conditions (C3) and (C4) of Niiniluoto (1972) must be added to Theorems 3.4, 3.5 and Section 5 in Narens (1974).

Now let us define $\geq_1$ as the CP-relation agreeing with P_1, so $A \geq_1 B$ iff $P_1(A) \geq P_1(B)$. Then $(X, I, \geq_1)$ is a CP-structure, $\geq_1$ is fine, atomless, AUP(∞), and also tight. By the definition of $\geq_1$ P_1 agrees with $\geq_1$. Considering the events $A = [0, \frac{1}{2})$, $B = [\frac{1}{2}, 1)$, $C = D = [0, \frac{1}{4}\sqrt{2})$ one sees that Condition (4) in Definition 2 on page 781 of Luce (1967), which is Axiom 5 on page 207 in Krantz et al. (1971), and "L" on page 25 of Fine (1973), is not satisfied. So Theorem 6 on page 26 in Fine (1973) is not correct, neither is on this point the survey on page 207, 208 in Krantz et al. (1971).

Naren's theorems neither have to be valid if I is a σ-algebra, which can be seen from our example $(X, I, \geq)$, by extending I to the Borel-σ-algebra on X. The other results above are valid for σ-algebra's, and they all result from erroneous application of the terminal conclusion of Theorem 2 on page 34 of Savage (1954).

Finally, for every $n \in \mathbb{N}$, the partition $\{[j/2n, (j+1)/2n) \cup [\frac{1}{2} + j/2n, \frac{1}{2} + (j+1)/2n) \mid 0 \leq j < n\}$ is a partition of X into n equivalent events, and also our structure is "Archimedean" as defined on page 205 of Krantz et al. (1971) (a consequence of the fact that for all events A with $P_1(A) = 0$ also $P_2(A) = 0$). Yet $(X, I, \geq)$ has no agreeing probability measure! So lines 10–12 on page 1582 of Niiniluoto (1972) are not correct, neither is on this point the survey on page 206 etc. in Krantz et al. (1971): De Finetti (1931) and Koopmans (1940) constructed only *almost* agreeing probability measures.

4. More results. The notion of $\geq$ being trisplittable as defined in Narens (1974) is equivalent to the existence of an AUP(3) of every $B \in I$. This implies, as one can see with not much more work than the proofs of Lemma 2 and Theorem 2 that $\geq$ is AUP(∞). By our Theorem 1, due to Savage (1954), this already implies the existence of a unique almost agreeing probability measure. So the conditions in Section 5 of Narens (1974) are unnecessarily strong.

The technique used to prove Theorems 5.1 and 5.2 in Narens (1974) is valuable. It gives rise to the next important theorem. Apparently this result is not new, but the author does not know of any reference to it.

THEOREM 4. *Let* $(X, I, \geq)$ *be a* CP-*structure* (*so the Conditions* (C1) *through* (C5) *of Niiniluto* (1972) *must be satisfied!*). *Then there is an almost agreeing probability measure* iff *there are no two finite sequences of events* $(A_j)_{j=1}^{n}$ *and* $(B_j)_{j=1}^{n}$ *such that* $A_j \geq B_j$ *for all* $j \leq n$ *and* $\sum_{j=1}^{n} (1_{A_j} - 1_{B_j})(x) < 0$ *for all* $x \in X$.

This can be proved in two steps.

(1) The theorem is valid in case I is finite. For the proof of this see Kraft et al. (1959), Theorem 3.

(2) $\geq$ has an almost agreeing probability measure iff every finite substructure of $(X, I, \geq)$ has an almost agreeing probability measure. (See also Kaplan, 1973.) The "if" part is proved completely analogous to Theorems 5.1 and 5.2 in Narens (1974). The "only if" part is trivial.

COROLLARY. *If* $\geq$ *satisfies the finite cancellation axiom as given in Definition* 3.4 *of Narens* (1974) *or in Krantz* et al. (1971) *or, equivalently, satisfies the condition of Scott, given as* (**) *on page* 23 *in Fine* (1973), *or as* 4B *on page* 246 *of Scott* (1964), *then* $\geq$ *has an almost agreeing probability measure.*

The example in Section 3 shows there does not have to be an agreeing probability measure, even though the structure there is "Archimedean".

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