A simple axiomatization of the median procedure on median graphs

Henry Martyn Mulder

Econometrisch Instituut, Erasmus Universiteit

P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

e-mail: hmmulder@few.eur.nl

Beth Novick

Department of Mathematical Sciences, Clemson University

Clemson, SC 29634, USA

e-mail: nbeth@clemson.edu

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Abstract

A profile $\pi = (x_1, ..., x_k)$, of length k, in a finite connected graph G is a sequence of vertices of G, with repetitions allowed. A median x of π is a vertex for which the sum of the distances from x to the vertices in the profile is minimum. The median function finds the set of all medians of a profile. Medians are important in location theory and consensus theory. A median graph is a graph for which every profile of length 3 has a unique median. Median graphs are well studied. They arise in many arenas, and have many applications.

We establish a succinct axiomatic characterization of the median procedure on median graphs. This is a simplification of the characterization given by McMorris, Mulder and Roberts [17] in 1998. We show that the median procedure can be characterized on the class of all median graphs with only three simple and intuitively appealing axioms: anonymity, betweenness and consistency. We also extend a key result of the same paper, characterizing the median function for profiles of even length on median graphs.

Keywords: median graph, median, median function, location function, consensus function, consensus axiom

1 Introduction

Facility location problems involve a set of 'clients' at various locations. One seeks a set of locations acceptable for the provision of a given service. Graphs are a natural model for the locations and interconnections. Hundreds of papers have been written about location problems on graphs using the geodesic metric, see for example the reference lists in [8, 20, 30]. Let $G = (V, E)$ be a graph. Each client is represented by its preferred location in the graph, so by a vertex. Thus the set of clients may be represented by a sequence, or *profile* $\pi = (x_1, x_2, \ldots, x_k)$. Note that π being a sequence, repetitions of vertices are allowed, by which clients having the same preferred location can be represented. Let V^* be the set of all finite sequences of vertices. The location problem is then modelled by a *location function* $L : V^* \rightarrow 2^V - \emptyset$, where 2^V is the power set of V. An appropriate objective function depends on the specific application. To locate a site for an emergency service, one might seek to minimize the greatest distance to any client: hence the center is a good choice. For a facility designed for the delivery of goods, one might want to minimize the average distant to the clients. Here the median set is appropriate. Many versions of 'central' subgraphs have been considered on various classes of graphs, see [13, 37, 38, 32, 33, 31].

In consensus theory, a finite set, or profile, of voters (users, clients) provides a list of preferences for the outcomes of a decision procedure. One seeks 'consensus', namely a set of outcomes which best satisfy the voters. See the list of references in [36, 3, 4] for surveys of such social choice functions. The theory of consensus is widely used in e.g. social choice theory, voting theory, economic theory and biomathematics.

In both settings, that of consensus and that of location, numerous researchers have addressed the issue of identifying an objective function via a succinct 'wish list' of desired properties. The goal here is to identify functions for which this list, or something close, gives a characterization. This method allows one to argue in favor of a particular set of locations (or particular consensus) as being precisely that satisfying certain desirable properties. Another perspective is that one requires that consensus be achieved in a rational way, that is, the objective function should satisfy certain rational rules or 'consensus axioms'. In 1951 Arrow [2] initiated this axiomatic approach for consensus functions by showing that certain sets of axioms could not be satisfied. For a recent survey of this axiomatic approach with an extensive list of references see [6].

Three location functions have been studied axiomatically: the center function, the median function and the mean function. The latter two functions are special instances of the ℓ_p-function, viz. for $p = 1$ and $p = 2$, respectively. This function was introduced in [16]: here $\|\pi\|_p = \sqrt[p]{\sum_{i=1}^{k} [d(x, x_i)]^p}$ is minimized, where $\pi = (x_1, x_2, \ldots, x_k)$. For the center function [19, 27] and the mean function [11, 35, 15, 16], characterizations have been obtained only on trees and tree networks. Characterizations beyond trees seem to be very difficult for these functions.

The median function is more promising. This function satisfies three simple and basic axioms, viz. (A) Anonymity: the clients are anonymous, (B) Betweenness: any location strictly between two clients minimizes the sum of the distances to these two clients, and (C) Consistency: if two sets of clients both prefer location x, then the union of all these clients also prefers location x. It is an easy and well-known result that (A), (B) and (C) are satisfied by the median function for all graphs (in fact for all metric spaces). On most graphs these axioms are not sufficient to characterize the median function. Hence the question arises: *On which graphs is the median function characterized by these three basic axioms*?

Recently it was proved that the median function is characterized by the three basic axioms on hypercubes, see [26]. From results in [17] it follows that it is also characterized by the three basic axioms on trees. In 1990 [24] the first author proposed a 'meta-conjecture', which reads as follows:

Meta-conjecture [Mulder, 1990] *Any 'reasonable' property shared by trees and hypercubes is shared by all median graphs.*

So, in view of the characterization of the median function on trees and hypercubes, this meta-conjecture suggests that the median function should be characterized by the three basic axioms on the class of median graphs. But the proof techniques in [26] were quite specific for hypercubes and could not be generalized.

A median graph is a graph in which any profile of three vertices has a unique median vertex. Median graphs were independently introduced by Avann [1], who called them 'unique ternary distance graphs', by Nebeský [29] and by Mulder and Schrijver [28]. Median graphs are now well studied: see [22, 14, 25, 9] for survey articles. They are important because of the role they play in ternary algebras, ordered sets, discrete lattices, Helly hypergraphs, product graphs and so forth. They have been used in applications in such diverse fields as dynamic search, location theory, social choice theory, biomathematics, mathematical chemistry, computer science, mathematical economics and literary history. Classical examples are trees, hypercubes, and grid graphs.

There is a rich structure theory for median graphs. Notably, in 1978, Mulder [21] showed that every median graph can be obtained from K_1 by a series of 'convex expansions'. Because we make extensive use of the ideas underlying this operation, we will describe it in detail in the sequel. Trees and hypercubes arise as extreme cases of this procedure.

McMorris et al. [17] characterized the median function, by means of (A), (B) and (C), on 'cube-free median' graphs, which include trees. The obstacle in [17] to extend this result to all median graphs were formed by the profiles of even length. By proving some nice and surprising results on median sets of such profiles in median graphs, the same authors [17] could extend the result to all median graphs: A fourth, less intuitively appealing, 'convexity' axiom was needed to deal with the even profiles. This characterization was not shown to be tight, however. It remained an open question whether or not the basic axioms of anonymity, betweenness and consistency in fact imply the more complicated axiom of convexity. In this paper we will settle this open question: surprisingly, the three basic axioms suffice to characterize the median function on all median graphs. Our first approach was to prove some more nice and surprising results for median sets of even profiles. We include these results in Section 4. But in the end we found a direct proof that (A), (B) and (C) characterize the median function by making extensive use of the structure theory developed in [22, 17, 25].

The paper is organized as follows: In Section 2 we present the background on consensus functions on graphs. Section 3 focuses on median graphs, including necessary notation and results. In Section 4 we present our results on median sets of even profiles, which extend those of McMorris et al. [17]. In Section 5 we prove our main result.

2 Preliminaries

Throughout this paper $G = (V, E)$ is a connected graph. All subgraphs considered are induced. Therefore, we may use the same symbol to denote a subgraph as well as its vertex set, equating subgraph H with its vertex set. For any $u, v \in V$, we denote the distance between u and v by $d(u, v)$. The interval between u and v in G is the set

$$I(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\},$$

in other words the set of all vertices 'between' u and v.

Let W be a subset of V. Then W is *convex* in G if it contains all shortest paths between pairs of vertices, that is, $I(u, v)$ is contained in W, for any two vertices u, v in W. Trivially, the intersection of two convex subsets is again convex. If W is a subset of V then the *convex closure* $Con(W)$ is the smallest convex set containing W. A subgraph of G is convex if it is induced by a convex set in G.

Let v be a vertex of G. If there is a unique vertex x in W such that x lies in $I(v, w)$ for all w in W, then x is a *gate* for v in W. The concept of gate was introduced by Dress [7], see also [8, 34]. Note that, if v has a gate in W, then it is the unique vertex in W closest to v. The converse need not be true. Clearly, if v lies in W, then v is its own gate. A subset W of V is called *gated* if each vertex v of G has a gate in W. A subgraph is gated if it is induced by a gated set. A gated set is necessarily convex. For arbitrary graphs the converse is not true. The following property for gated sets probably belongs to folklore, and it is an easy exercise to prove: If W is a gated set, and v is a vertex outside W, then x in W is the gate for v if and only if v is closer to x than to all neighbors of x in W. We refer to this property as the *neighbor-gate property*.

A profile π on G of length k is a nonempty sequence $\pi = (x_1, x_2, \ldots, x_k)$ of vertices of V with repetitions allowed. We denote its length by $k = |\pi|$. When $|\pi|$ is odd, we call π an *odd profile*, otherwise an *even profile*. Let V^* be the set of all profiles of finite length. The concatenation of profiles π_1 and π_2 is denoted by $\pi_1\pi_2$. We write the concatenation of a profile π and a single element profile (v) as πv rather than $\pi(v)$. We refer to a profile π whose elements are contained in a subgraph H as a profile *contained in* H, and, abusing notation slightly, we write $\pi \subseteq H$.

A *consensus function* on G is a function $L : V^* \to 2^V - \emptyset$. For convenience, we write $L(x_1, x_2, \ldots, x_k)$ instead of $L((x_1, x_2, \ldots, x_k))$ for any function L defined on profiles, but will keep the brackets where needed.

A *median* of a profile $\pi = (x_1, x_2, \ldots, x_k)$ is a vertex x in V minimizing the distance sum $\sum_{i=1}^{k} d(x, x_i)$. The *median set* $M(\pi)$ of π is the set of all medians of π. Note that, since G is connected, this defines a consensus function, namely the *median function* $M : V^* \to 2^V - \emptyset$. Trivially, we have $M(x) = \{x\}$, and $M(x, y) = I(x, y)$. Moreover, if $I(u, v) \cap I(v, w) \cap I(w, u) \neq \emptyset$, then $M(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u)$.

The three basic axioms for our consensus functions are

(A) Anonymity: For any profile $\pi = (x_1, x_2, \ \ldots \ , x_k)$ on V and any permutation σ of $\{1, 2, \ \ldots \ , k\}$, we have $L(\pi) = L(\pi^\sigma)$, where $\pi^\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ \ldots \ , x_{\sigma(k)})$.

(B) Betweenness: $L(u, v) = I(u, v)$, for $u, v \in V$.

(C) Consistency: If $L(\pi) \cap L(\rho) \neq \emptyset$ for profiles π and ρ, then

$$L(\pi\rho) \ = \ L(\pi) \ \cap \ L(\rho).$$

It is a simple exercise to prove that the median function satisfies these three basic axioms, see e.g. [17, 25].

Let L be a consensus function satisfying (B) and (C). Take any vertex v in V. Since $L(v) \neq \emptyset$, we have $L(v) \cap L(v) \neq \emptyset$. Hence, by consistency and betweenness, we have $L(v) = L(v) \cap L(v) = L(v, v) = I(v, v)$. So $L(v) = \{v\}$. This property is known as Faithfulness. Then consistency and faithfulness imply that $L(u, u, \ldots, u) = \{u\}$, where $(u, u, \ldots, u)$ is the constant profile containing only u's. This property is called Unanimity.

It belongs to folklore that (B) and (C) are independent. Here are two examples to show independence. The consensus function L on V defined by $L(\pi) = V$, for all π, clearly is anonymous and consistent, but does not satisfy betweenness unless G is K_1 or K_2. The consensus function L on V defined by $L(x, y) = I(x, y)$ for any x and y, and $L(\pi) = V$ for all other π, satisfies anonymity and betweenness, but not consistency, again, unless G is K_1 or K_2.

3 Median graphs

A median graph is a graph G for which $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$, for any three vertices u, v, w in G. Clearly, median graphs are connected. It is easily seen that they are bipartite. In median graphs convex sets are gated. This follows easily from the definition of median graph, using results from [22]. Median graphs possess a beautiful structure and elegant characterizations abound, see e.g. the surveys in [14, 25]. One such characterization is that they are precisely the graphs in which every profile of length 3 has a unique median. The most useful and insightful characterization of median graphs might be the Expansion Theorem in [21]: A graph G is a median graph if and only if

G can be obtained from the one-vertex graph K_1 by successive 'convex expansions'. See also [22, 24, 25].

At first sight one might think that median graphs are quite esoteric. But in [12] a one-to-one correspondence was established between the class of connected triangle-free graphs and a special subclass of the class of median graphs. Hence, median graphs being triangle-free and connected, it was proved that "in the universe of all graphs, there are as many median graphs as there are connected triangle-free graphs".

To make full use of the Expansion Theorem and its consequences we require several concepts and notations. For an illustration of the definitions and notations see Figure 1. We refer the reader to [21, 22, 24, 25] for details and for the proofs of all the results that are summarized in this section.

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *union* $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and the *intersection* $G_1 \cap G_2$ is the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$. We write $G_1 \cap G_2 \neq \emptyset$ when $V_1 \cap V_2 \neq \emptyset$. The graph $G_1 - G_2$ is the subgraph of G_1 induced by the vertices in G_1 but not in G_2. A *proper cover* of a connected graph G consists of two subgraphs G_1 and G_2 such that $G_1 \cap G_2 \neq \emptyset$ and $G = G_1 \cup G_2$. Note that this implies that there are no edges between $G_1 - G_2$ and $G_2 - G_1$. If both G_1 and G_2 are convex, we say that G_1, G_2 is a *convex cover*. Note that G_1, G_2 is a convex cover if and only if $G_1 \cap G_2$ is convex. Every graph admits the *trivial cover* G_1, G_2 with $G_1 = G_2 = G$, which is of course convex. On the other hand a cycle of length at least four does not have a convex cover with two proper subgraphs.

Let G' be a connected graph and let G'_1, G'_2 be a convex cover of G' with $G'_0 = G'_1 \cap G'_2$. For $i = 1, 2$, let G_i be an isomorphic copy of G'_i, and let λ_i be an isomorphism from G'_i to G_i. We write $G_{0i} = \lambda_i[G'_0]$ and $u_i = \lambda_i(u')$, for u' in G'_0. The *convex expansion* of G' with respect to the convex cover G'_1, G'_2 is the graph G obtained from the disjoint union of G_1 and G_2 by inserting an edge between u_1 in G_{01} and u_2 in G_{02}, for each u' in G'_0. We denote the set of edges between G_{01} and G_{02} by F_{12}. Note that F_{12} induces an isomorphism between G_{01} and G_{02}.

It is straightforward to prove that the expansion of a median graph with respect to a convex cover is again a median graph. The hard part of the proof of the Expansion Theorem is to show that a median graph is always the convex expansion of a smaller one. We need some of the ideas and notations from this proof for the sequel.

Let G be a median graph, and let v_1v_2 be an edge in G. Let G_1 be the subgraph induced by all vertices closer to v_1 than to v_2 and let G_2 be the subgraph induced by all vertices closer to v_2 than to v_1. Since G is bipartite, it follows that G_1, G_2 is a vertex-partition of G. Let F_{12} be the set of edges between G_1 and G_2, and let G_{0i} be the subgraph induced by the ends of F_{12} in G_i, for $i = 1, 2$. Then it is proved in [21] (although not exactly in that order) that the following facts hold:

(i) F_{12} is a matching as well as a cutset (minimal disconnecting edge-set),

(ii) the subgraphs G_1, G_2, G_{01}, G_{02} are convex subgraphs of G,

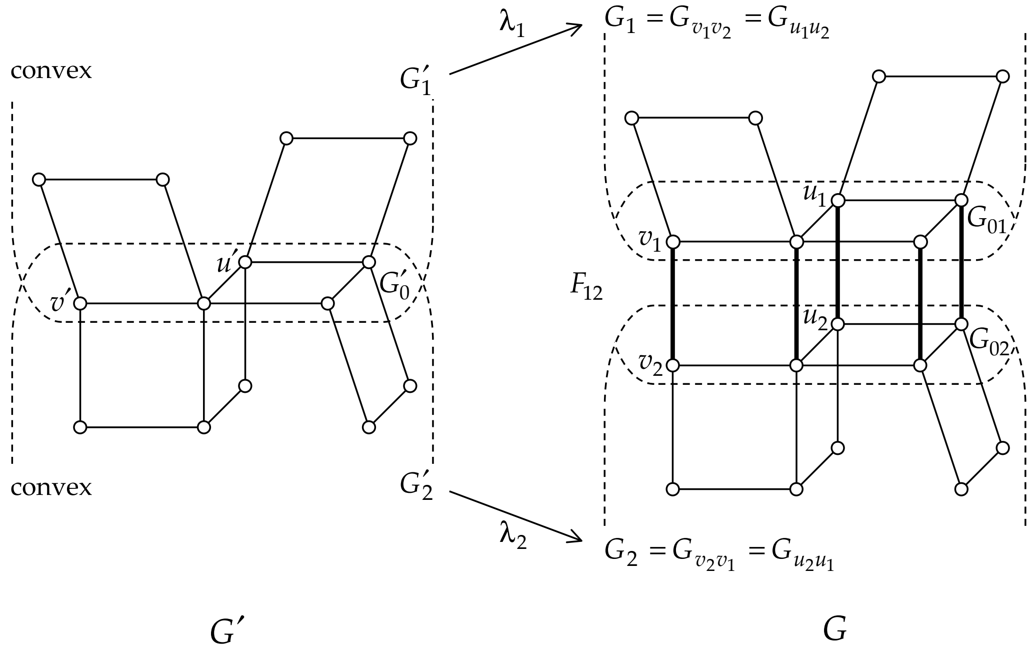

Figure 1: Expansion

(iii) the obvious mapping of G_{01} onto G_{02} defined by F_{12} (i.e. $u_1 \rightarrow u_2$, for any edge u_1u_2 in F_{12} with u_{0i} in G_{0i}) is an isomorphism,

(iv) for every edge u_1u_2 in F_{12} with u_i in G_{0i}, the subgraph G_1 consists of all vertices of G closer to u_1 than u_2, and the subgraph G_2 consists of all vertices of G closer to u_2 than u_1.

We call such a partition G_1, G_2 of G a *split*. Note that any edge in F_{12} defines the same split. The subgraphs G_1 and G_2 are the *sides* of the split. If we are in u_1 of an edge u_1u_2 of F_{12}, then G_1 is the *side* of u_1 and G_2 is the *opposide* of u_1, see [25] for the reason why opposide is spelled this way. We call u_1 and u_2 *mates* of each other.

The converse of expansion is *contraction*: for a split G_1, G_2 contract each edge in F_{12} so that mates are being identified. Then the resulting graph G' is again a median graph and the split G_1, G_2 is 'contracted' to a convex cover of G'. Moreover, the expansion with respect to that cover reproduces G. Thus the proof of the Expansion Theorem is complete. To obtain the median graph G from K_1, the expansions are order independent. The edge set E of G is the disjoint union of the resulting matchings. This provides us with a very strong tool: we can use induction on the number of splits. This is needed in the proof Theorem 1 and various other properties mentioned here.

If, at each expansion, one of the two parts in the cover is a single vertex, so that the other part is the whole graph, then a tree results. If at each expansion, each part of the cover is the entire graph, an n-dimensional hypercube results, where n is the number of expansions.

We need some more consequences of the structural characterization of median graphs in the sequel. As observed above, convex subgraphs of a median graph are gated. So the subgraphs G_1, G_2, G_{01} and G_{02} are all gated. Let v be a vertex in, say, G_1. Let u_1 be its gate in G_{01}. Then it follows from the above structural properties of a split that the mate u_2 of u_1 is the gate for v in G_{02} as well as G_2. Moreover, for any w in G_2, it follows that $u_1, u_2 \in I(v, w)$. We use this property of gates and mates in the next sections, and refer to it as the *mate-gate property.*

Let π be a profile on G, and let G_1, G_2 be a split of G. We denote the subprofile of π contained in G_i by π_i, for $i = 1, 2$. If $|\pi_1| > |\pi_2|$, then we call G_1 the *majority side* of the split. If $|\pi_1| = |\pi_2|$, then we call π *balanced* on the split G_1, G_2.

In the sequel we will use all the notation developed here for split G_1, G_2, edge u_1u_2, subprofiles π_1 and π_2, and so forth, and so forth, without further mention.

For any edge uv in G we denote by G_{uv} the subgraph of G induced by all vertices closer to u than to v. Then G_{uv}, G_{vu} is a split. Note that, for any edge u_1u_2 in F_{12}, we could write G_1 also as $G_{u_1u_2}$, and G_2 as $G_{u_2u_1}$, see Figure 1.

An important consequence of the Expansion Theorem was proved in [23] and [17]. The median set of a profile is always contained in the majority side of an unbalanced split, and it intersects both sides of a balanced split. This is made more precise in the following theorem, which is basic for almost all the proofs in this paper.

Theorem 1 *Let G be a median graph and let π be a profile on G. Then*

$$M(\pi) \ = \ \bigcap{}_{G_1,G_2 \ split \ with \ |\pi_1|>|\pi_2|} \ G_1.$$

Note that, split sides being convex, this means that median sets are necessarily convex. It is a well-known fact that odd profiles have a unique median in median graphs. It also follows easily from this theorem. Let π be an odd profile. Then there are no balanced splits for π. So, if u is a median vertex and v is a neighbor of u, then G_{uv} is a majority side and G_{vu} is a minority side in the split G_{uv}, G_{vu}. So v is not in $M(\pi)$. Since $M(\pi)$ is convex, it consist only of u. A special case of this theorem concerns intervals. Note that the next corollary was already proved in [28] without any reference to profiles and the median function. But now it is a simple corollary of Theorem 1. Recall that $I(x, y) = M(x, y)$ on any connected graph.

Corollary 2 *Let G be a median graph and let x and y be vertices of G. Then*

$$I(x,y) \ = \ \bigcap{}_{G_1,G_2 \ split \ with \ x,y\in G_1} G_1.$$

Hence intervals in median graphs are convex. We get this property for free here, but it can also be easily proved using the definition of a median graph. Another noteworthy corollary of this theorem is also needed. Note that, for any edge uv, we have the following fact. Edge uv has an end in each side of the split G_{uv}, G_{vu}, but for any other split G_1, G_2, the edge is contained in one of the two sides. In a way, split G_{uv}, G_{vu} distinguishes edge uv. Now, if u is a vertex in $M(\pi)$, for some profile π and v is a neighbor of u outside $M(\pi)$, then G_{uv} contains a majority of π, and necessarily $M(\pi) \subseteq G_{uv}$. We call this the *edge-distinguishing property.*

A 'counterpart' for Theorem 1 was also proved in [17].

Theorem 3 *Let G be a median graph, and let π be a profile on G that is balanced on all splits. Then $M(\pi) = G$.*

Note that we could merge Theorems 1 and 3 into one statement using the convention $\cap\emptyset = G$.

4 Medians of even profiles on median graphs

In this Section we extend the results from [17] on median sets of even profiles. In 1998, McMorris et al. [17] showed that axioms (A), (B) and (C) characterize the median function on 'cube-free median' graphs, where cube-free means that the 3-cube Q_3 does not occur as a subgraph. The surprising key result in [17] for cube-free median graphs was the following: any even profile π on a cube-free median graph admits a permutation such that it can be written as $(y_1, y_2, \ldots y_{2m-1}, y_{2m})$ with

$$M(\pi) = \cap_{1 \leq i \leq m} I(y_{2i-1}, y_{2i}).$$

So the median set of an even profile is the intersection of intervals between its elements. A simple example on the 3-cube showed that this is not true on arbitrary median graphs: take the profile of length four of the black vertices in Figure 2.

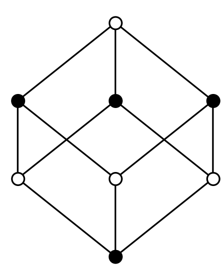

Figure 2: The 3-cube Q_3 with a profile

To extend the characterization in [17] to arbitrary median graphs, a fourth axiom was added in [17]: the less intuitively appealing 'convexity' axiom. For a profile $\pi = (x_1, x_2, \ldots, x_k)$, this axiom involves the vertex-deleted profiles $\pi - x_i$ for $1 \leq i \leq k$. This is the profile of length $|\pi| - 1$, where only the element x_i is removed.

(K) Convexity: Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile in G with $k \geq 2$. If $\cap_{i=1}^k L(\pi - x_i) = \emptyset$, then $L(\pi) = Con\big(\cup_{i=1}^k L(\pi - x_i)\big)$.

It is easy to check that for $L = M$, axiom (K) holds vacuously when π is an odd profile. The fact that (K) holds for the median function when π is an even profile on a median graph, is a key result in the above mentioned paper.

Theorem 4 [17] *Let $\pi = (x_1, \ldots, x_k)$ be an even profile on the median graph G. Then $M(\pi) = Con\big(\cup_{i=1}^k M(\pi - x_i)\big)$.*

We present an extension of this result. Note that, if $\pi = (x_1, \ldots, x_k)$ is an even profile, then the vertex-deleted profile $\pi - x_i$ is odd, so it has a unique median.

Theorem 5 *Let G be a median graph, and let $\pi = (x_1, \ldots, x_k)$ be an even profile on G. For $i = 1, \ldots, k$, let y_i be the median of the vertex-deleted profile $\pi - x_i$, and let $\pi' = (y_1, \ldots, y_k)$. Then $\pi' \subseteq M(\pi) = M(\pi')$.*

Proof. Since π is even, a majority side for π of a split remains a majority side for $\pi - x_i$. So y_i lies in $M(\pi)$. Hence we have $\pi' \subseteq M(\pi)$. By Theorem 1, any majority side for π of a split contains π', hence trivially is a majority side for π'. So $M(\pi') \subseteq M(\pi)$.

Take any balanced split G_1, G_2 for π. So exactly half of π is in G_1 and the other half of π is in G_2. Now, if x_i is in G_1, then the majority of $\pi - x_i$ is in G_2, so that y_i is in G_2. So, for $i = 1, \ldots, k$, the vertices x_i and y_i are always on opposite sides of G_1, G_2. Hence this split is also balanced for π'. Thus we have shown that the majority sides for π are precisely the majority sides for π'. By Theorem 1, we have $M(\pi) = M(\pi')$. ■

The example in Figure 2 shows that, for even $\pi = (x_1, \ldots, x_k)$, we can not always write $M(\pi)$ as the intersection of intervals between profile-elements as in the cube-free case. But it turns out that $M(\pi)$ is the interval between two well-chosen vertices that are determined by the profile. Take any profile element x_i. Then these two vertices are, loosely speaking, the vertex in $M(\pi)$ closest to x_i and the vertex in $M(\pi)$ farthest away from x_i. Clearly, the first vertex is the gate z_i in $M(\pi)$ for x_i. The vertex that is intuitively 'farthest away' is the median y_i of the vertex-deleted profile $\pi - x_i$. Another way of looking at these two vertices is: for the closest vertex we maximize the influence of x_i by taking the median of the vertex-added profile πx_i, for the vertex 'farthest away' we minimize the influence of x_i by taking the median of the vertex-deleted profile $\pi - x_i$.

Lemma 6 *Let G be a median graph, and let $\pi = (x_1, x_2, \ldots, x_k)$ be an even profile on G. Then, for $i = 1, \ldots, k$, the median z_i of the vertex-added profile πx_i is the gate for x_i in $M(\pi)$.*

Proof. By Theorem 1, the median set $M(\pi)$ of π is the intersection of the majority sides of the splits in G. Since π is even, these sides remain majority sides for the profile πx_i. So $M(\pi x_i) \subseteq M(\pi)$. For the splits that are balanced with respect to π, adding x_i

to the profile means that the balance is tipped towards the side containing x_i. So the majority sides of the vertex-added profile πx_i are the majority sides of the unbalanced splits for π and the sides containing x_i of the balanced splits for π. Since πx_i is odd, it has no balanced splits. As in the statement of the Lemma, we define z_i to be the unique vertex in $M(\pi x_i)$. Take any neighbor v of z_i in $M(\pi)$. Consider the split $G_{z_i v}, G_{v z_i}$. By the edge-distinguishing property, $G_{z_i v}$ is a majority side for πx_i. Both sides contain a vertex of $M(\pi)$, so π is balanced on this split. Since x_i tips the balance, it follows that x_i lies in $G_{z_i v}$. Hence x_i is closer to z_i than to v. By the neighbor-gate property, z_i is the gate for x_i in the gated set $M(\pi)$. ■

Theorem 7 *Let G be a median graph, and let $\pi = (x_1, x_2, \ldots, x_k)$ be an even profile on G. For $i = 1, \ldots, k$, let y_i be the median of the vertex-deleted profile $\pi - x_i$ and z_i be the median of the vertex-added profile πx_i. Then $M(\pi) = I(y_i, z_i)$.*

Proof. Since k is even, a majority side of a split for π remains a majority side when we add or delete a vertex from π. So, by Theorem 1, we have $M(\pi x_i) \subseteq M(\pi)$ as well as $M(\pi - x_i) \subseteq M(\pi)$.

Let G_1, G_2 be any balanced split for π with, say, x_i in G_1. Then G_1 is a majority side for the vertex-added profile πx_i and so z_i is in G_1. Moreover G_2 is a majority side for the vertex-deleted profile $\pi - x_i$ and y_i is in G_2. By Lemma 6, z_i is the gate for x_i in $M(\pi)$, so, trivially, z_i lies in $M(\pi)$. By Theorem 5, y_i lies in $M(\pi)$. Hence, by the convexity of $M(\pi)$, we have $I(y_i, z_i) \subseteq M(\pi)$.

Take any vertex w outside $I(y_i, z_i)$. Recall that an interval in a median graph is convex, hence gated. Let u be the gate for w in $I(y_i, z_i)$, and let v be a neighbor of u with $d(w, v) = d(w, u) - 1$. Then v is not in $I(y_i, z_i)$. Now w being closer to v than to u, we have, by definition, that w is in G_{vu}. Since v is not in $(I(y_i, z_i)$, both y_i and z_i are closer to u than to v, so they are both in G_{uv}. Now z_i being in G_{uv} means that G_{uv} contains at least half of πx_i, and y_i being in G_{uv} means that G_{uv} contains at least half of $\pi - x_i$. Hence G_{uv} must contain a majority of π. So w is not in $M(\pi)$. Thus we have shown that $M(\pi) \subseteq I(y_i, z_i)$, which completes the proof. ■

5 Consensus Functions satisfying (A), (B) and (C)

In this section we prove that axioms (A), (B) and (C) suffice for the consensus function to make it the median function.

Median sets in a median graph always reside at the majority side of a split. Our first lemma states that in fact this 'majority' property holds for any consensus function L on a median graph G provided it satisfies (A), (B) and (C). We will refer to this as our 'Majority Lemma'. We use the 'standard' notation developed above for the split G_1, G_2.

Lemma 8 (Majority Lemma) *Let $G = (V, E)$ be a median graph, let G_1, G_2 be a split of G, and let π be a profile on G with $|\pi_1| > |\pi_2|$. If $L : V^* \to 2^V - \emptyset$ is a consensus function satisfying the axioms (A), (B) and (C), then $L(\pi) \subseteq G_1$.*

Proof. Assume to the contrary that there exists a vertex v in $L(\pi) \cap G_2$. Let $\pi_2 = (x_1, x_2, \ldots, x_\ell)$ and let $\pi_1 = (x_{\ell+1}, x_{\ell+2}, \ldots, x_k)$, where $2\ell < k$. Let g_2 be the gate for v in G_{02}. Let g_1 be the mate of g_2 in G_{01}, so that g_1 is the gate for v in G_1,

By betweenness, we have $v \in I(v, g_1) = L(v, g_1)$. Thus, our assumption that v lies in $L(\pi)$, together with consistency implies that

$$v \in L(v, g_1) \cap \cdots \cap L(v, g_1) \cap L(\pi) = L((v, g_1)(v, g_1) \ldots (v, g_1)\, \pi),$$

where the intersection is taken over ℓ terms $L(v, g_1)$ and the pair (v, g_1) occurs ℓ times in the right hand side. Since $2\ell < k$ and $L(v, v, \ldots, v) = \{v\}$ we obtain

$$L((v, \ldots, v)(v, g_1) \ldots (v, g_1)\, \pi) = \{v\}, \tag{1}$$

where the v's are repeated $k - 2\ell$ times and the pair (v, g_1) is repeated ℓ times.

Since the mate g_2 of g_1 is the gate for g_1 in G_2, we have $g_2 \in I(g_1, x)$, for any $x \in G_2$. Hence

$$\{g_1, g_2\} \subseteq \cap_{i=1}^{\ell} I(g_1, x_i). \tag{2}$$

An immediate consequence of the mate-gate property is that, for any $x \in G_1$, the interval $I(v, x)$ contains both g_1 and g_2. Hence

$$\{g_1, g_2\} \subseteq \cap_{i=\ell+1}^{k} I(v, x_i). \tag{3}$$

Now let us reorganize the left-hand side of Equation (1) into the nonempty intersection of a collection of intervals. Note that by Equation (1) together with anonymity, we have

$$\begin{aligned} \{v\} &= L((v, \ldots, v)(v, g_1) \ldots (v, g_1)\, \pi_1\pi_2) \\ &= L((g_1, x_1)(g_1, x_2) \ldots (g_1, x_\ell)(v, x_{\ell+1})(v, x_{\ell+2}) \ldots (v, x_k)) \\ &= [\cap_{i=1}^{k} I(g_1, x_i)] \,\cap\, [\cap_{i=\ell+1}^{k} I(v, x_i)]. \end{aligned}$$

But this intersection of intervals contains both g_1 and g_2, by Equations (2) and (3). This impossibility settles the proof. ■

An immediate, but surprisingly strong consequence of the Majority Lemma and Theorem 1 is the following.

Corollary 9 *Let $G = (V, E)$ be a median graph and let $L : V^* \to 2^V - \emptyset$ be a consensus function satisfying the axioms (A), (B) and (C). Then $L(\pi) \subseteq M(\pi)$, for any profile π. In particular, $L(\pi) = M(\pi)$, when $|\pi|$ is odd.*

For the balanced case we need the following lemma. It might be considered as an extension of the edge-distinguishing property to the consensus function L. In the terminology of [14] we prove that L is $\frac{1}{2}$-condorcet, see that paper for details.

Lemma 10 *Let G be a median graph, and let π be a profile on G. If G_1, G_2 is a balanced split for π, then, for any edge u_1u_2 in F_{12}, either both u_1 and u_2 are in $L(\pi)$ or neither u_1 nor u_2 is in $L(\pi)$.*

Proof. Let u_1u_2 be an edge in F_{12}. Note that $I(u_1, u_2) = \{u_1, u_2\}$. Assume to the contrary that $u_1 \in L(\pi)$ and $u_2 \notin L(\pi)$. Let $\pi_1 = (x_1, x_2, ..., x_k)$ and let $\pi_2 = (y_1, y_2, ..., y_k)$. Let $(u_1, u_2)^k$ denote the profile (u_1, u_2) repeated k times. Then, by consistency and betweenness, we have $L((u_1, u_2)^k) = I(u_1, u_2) = \{u_1, u_2\}$. Hence, by consistency, we have $u_1 \in L(\pi) \cap L((u_1, u_2)^k) = L(\pi(u_1, u_2)^k)$. So $L(\pi(u_1, u_2)^k) = L(\pi) \cap \{u_1, u_2\} = \{u_1\}$.

Recall that u_2 is the gate for u_1 in G_2, and u_1 is the gate for u_2 in G_1. So both u_1 and u_2 are in $I(u_1, y_i)$ as well as in $I(u_2, x_i)$, for $i = 1, \ldots, k$. By betweenness, consistency and anonymity, we have

$$\begin{aligned} \{u_1, u_2\} &\subseteq [\cap_{i=1}^{k} I(u_1, y_k)] \cap [\cap_{i-1}^{k} I(u_2, x_i)] \\ &= [\cap_{i=1}^{k} L(u_1, y_k)] \cap [\cap_{i-1}^{k} L(u_2, x_i)] \\ &= L((u_1, y_1)(u_1, y_2)...(u_1, y_k)(u_2, x_1)(u_2, x_2)...(u_2, x_k)) \\ &= L(\pi(u_1, u_2)^k). \end{aligned}$$

Thus we get a contradiction, which completes the proof. ■

We are now ready to prove our main result.

Theorem 11 *Let L be a consensus function on a median graph G. Then L satisfies (A), (B) and (C) if and only if $L = M$.*

Proof. Recall that the median function M satisfies (A), (B) and (C) on any connected graph.

To prove the converse, let $G = (V, E)$ be a median graph and let $L : V^* \to 2^V - \emptyset$ be a consensus function on G satisfying the axioms (A), (B) and (C). Assume by way of contradiction that L is not identical to the median function M. Let $\pi = (x_1, \ldots, x_k)$ be a profile on G for which $L(\pi) \neq M(\pi)$. By Corollary 9, we have $L(\pi) \subseteq M(\pi)$ and π must be even. Since $M(\pi)$ is connected, being convex, there is an edge uv in $M(\pi)$ such that u is in $L(\pi)$ and v is not. Consider the split G_{uv}, G_{vu}. Since both sides contain a vertex of $M(\pi)$, this split must be balanced by Theorem 1. Hence Lemma 10 tells us that either both u and v are in $L(\pi)$ or neither is in $L(\pi)$. This contradicts the choice of uv, which completes the proof. ■

6 Concluding Remarks

In [17] the median function on cube-free median graphs has been characterized as the unique consensus function satisfying the three simple and appealing axioms Anonymity, Betweenness and Consistency. To prove this, a nice characterization of the median sets of even profiles was used: such a median set can be written as the intersection of intervals between profile elements. To characterize the median function on arbitrary median graphs an extra 'heavy duty' axiom Convexity was needed to deal with the case of even profiles. In [14] the Convexity axiom was replaced by another 'heavy duty' axiom: the $\frac{1}{2}$-condorcet axiom. In this paper we have shown that the three basic axioms suffice to characterize the median function on arbitrary median graphs. Moreover the results in [17] on median sets of even profiles have been extended in Section 4. Specifically, an elegant characterization of $M(\pi)$ for even $\pi = (x_1, x_2, \ldots, x_k)$ is that, for any x_i, the set $M(\pi)$ is the interval between the median of the vertex-deleted profile $\pi - x_i$ and the median of the vertex-added profile πx_i.

Loosely speaking, a network is a graph in which the edges are assigned a length (positive real number) and interior points of edges are also allowed as location for the facility. In [18] the case for cube-free median graphs from [17] was extended to cube-free *median networks*. Basically this was done by proving that the important properties of splits is carried over to the network case. Then the characterization of the median function defined on profiles of vertices on the network followed easily from the result in [17]. This can also be done in our case. So on median networks the median function is characterized by the three basic axioms (A), (B) and (C) as well. The results and proofs in [18] are straightforward but they need a lot of technical details. Therefore we omit these and just refer the reader to [18] and leave the proof of the general case as an exercise.

So far the only location functions that have been characterized axiomatically are the center function and the ℓ_p-functions with p a positive integer, and the antimedian function, see [19, 27, 17, 35, 11, 18, 15, 16, 5]. The antimedian function has been characterized on paths only. All the other functions have been characterized on trees only, except for the median function $M = \ell_1$. The reason for this exception is that the median function behaves nicely on median graphs and that a rich structure theory on median graphs is available.

There are still many intriguing questions. For instance, is there any other class of graphs on which the median function is characterized by axioms (A), (B) and (C)? Can the median function be characterized on other classes by adding extra axioms? And of course one would like to have axiomatic characterizations of the other location functions on graphs other then trees, or of other consensus functions.

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