A simple axiomatization of the median procedure on median graphs

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Abstract

A profile \( \pi = (x_1, \ldots, x_k) \), of length \( k \), in a finite connected graph \( G \) is a sequence of vertices of \( G \), with repetitions allowed. A median \( x \) of \( \pi \) is a vertex for which the sum of the distances from \( x \) to the vertices in the profile is minimum. The median function finds the set of all medians of a profile. Medians are important in location theory and consensus theory. A median graph is a graph for which every profile of length 3 has a unique median. Median graphs are well studied. They arise in many arenas, and have many applications.

We establish a succinct axiomatic characterization of the median procedure on median graphs. This is a simplification of the characterization given by McMorris, Mulder and Roberts [17] in 1998. We show that the median procedure can be characterized on the class of all median graphs with only three simple and intuitively appealing axioms: anonymity, betweenness and consistency. We also extend a key result of the same paper, characterizing the median function for profiles of even length on median graphs.

Keywords: median graph, median, median function, location function, consensus function, consensus axiom
1 Introduction

Facility location problems involve a set of ‘clients’ at various locations. One seeks a set of locations acceptable for the provision of a given service. Graphs are a natural model for the locations and interconnections. Hundreds of papers have been written about location problems on graphs using the geodesic metric, see for example the reference lists in [8, 20, 30]. Let $G = (V, E)$ be a graph. Each client is represented by its preferred location in the graph, so by a vertex. Thus the set of clients may be represented by a sequence, or profile $\pi = (x_1, x_2, \ldots, x_k)$. Note that $\pi$ being a sequence, repetitions of vertices are allowed, by which clients having the same preferred location can be represented. Let $V^*$ be the set of all finite sequences of vertices. The location problem is then modelled by a location function $L : V^* \rightarrow 2^V - \emptyset$, where $2^V$ is the power set of $V$. An appropriate objective function depends on the specific application. To locate a site for an emergency service, one might seek to minimize the greatest distance to any client: hence the center is a good choice. For a facility designed for the delivery of goods, one might want to minimize the average distance to the clients. Here the median set is appropriate. Many versions of ‘central’ subgraphs have been considered on various classes of graphs, see [13, 37, 38, 32, 33, 31].

In consensus theory, a finite set, or profile, of voters (users, clients) provides a list of preferences for the outcomes of a decision procedure. One seeks ‘consensus’, namely a set of outcomes which best satisfy the voters. See the list of references in [36, 3, 4] for surveys of such social choice functions. The theory of consensus is widely used in e.g. social choice theory, voting theory, economic theory and biomathematics.

In both settings, that of consensus and that of location, numerous researchers have addressed the issue of identifying an objective function via a succinct ‘wish list’ of desired properties. The goal here is to identify functions for which this list, or something close, gives a characterization. This method allows one to argue in favor of a particular set of locations (or particular consensus) as being precisely that satisfying certain desirable properties. Another perspective is that one requires that consensus be achieved in a rational way, that is, the objective function should satisfy certain rational rules or ‘consensus axioms’. In 1951 Arrow [2] initiated this axiomatic approach for consensus functions by showing that certain sets of axioms could not be satisfied. For a recent survey of this axiomatic approach with an extensive list of references see [6].

Three location functions have been studied axiomatically: the center function, the median function and the mean function. The latter two functions are special instances of the $\ell_p$-function, viz. for $p = 1$ and $p = 2$, respectively. This function was introduced in [16]: here $\|\pi\|_p = \sqrt[p]{\sum_{i=1}^k |d(x, x_i)|^p}$ is minimized, where $\pi = (x_1, x_2, \ldots, x_k)$. For the center function [19, 27] and the mean function [11, 35, 15, 16], characterizations have been obtained only on trees and tree networks. Characterizations beyond trees seem to be very difficult for these functions.

The median function is more promising. This function satisfies three simple and basic axioms, viz. (A) Anonymity: the clients are anonymous, (B) Betweenness: any location
strictly between two clients minimizes the sum of the distances to these two clients, and

\( (C) \) Consistency: if two sets of clients both prefer location \( x \), then the union of all these clients also prefers location \( x \). It is an easy and well-known result that \((A), (B)\) and \((C)\) are satisfied by the median function for all graphs (in fact for all metric spaces). On most graphs these axioms are not sufficient to characterize the median function. Hence the question arises: \emph{On which graphs is the median function characterized by these three basic axioms?}

Recently it was proved that the median function is characterized by the three basic axioms on hypercubes, see [26]. From results in [17] it follows that it is also characterized by the three basic axioms on trees. In 1990 [24] the first author proposed a ‘meta-conjecture’, which reads as follows:

\textbf{Meta-conjecture [Mulder, 1990]} Any ‘reasonable’ property shared by trees and hypercubes is shared by all median graphs.

So, in view of the characterization of the median function on trees and hypercubes, this meta-conjecture suggests that the median function should be characterized by the three basic axioms on the class of median graphs. But the proof techniques in [26] were quite specific for hypercubes and could not be generalized.

A median graph is a graph in which any profile of three vertices has a unique median vertex. Median graphs were independently introduced by Avann [1], who called them ‘unique ternary distance graphs’, by Nebeský [29] and by Mulder and Schrijver [28]. Median graphs are now well studied: see [22, 14, 25, 9] for survey articles. They are important because of the role they play in ternary algebras, ordered sets, discrete lattices, Helly hypergraphs, product graphs and so forth. They have been used in applications in such diverse fields as dynamic search, location theory, social choice theory, biomathematics, mathematical chemistry, computer science, mathematical economics and literary history. Classical examples are trees, hypercubes, and grid graphs.

There is a rich structure theory for median graphs. Notably, in 1978, Mulder [21] showed that every median graph can be obtained from \( K_1 \) by a series of ‘convex expansions’. Because we make extensive use of the ideas underlying this operation, we will describe it in detail in the sequel. Trees and hypercubes arise as extreme cases of this procedure.

McMorris et al. [17] characterized the median function, by means of \((A), (B)\) and \((C)\), on ‘cube-free median’ graphs, which include trees. The obstacle in [17] to extend this result to all median graphs were formed by the profiles of even length. By proving some nice and surprising results on median sets of such profiles in median graphs, the same authors [17] could extend the result to all median graphs: A fourth, less intuitively appealing, ‘convexity’ axiom was needed to deal with the even profiles. This characterization was not shown to be tight, however. It remained an open question whether or not the basic axioms of anonymity, betweenness and consistency in fact imply the more complicated axiom of convexity. In this paper we will settle this open question: surprisingly, the three basic axioms suffice to characterize the median function on all
median graphs. Our first approach was to prove some more nice and surprising results for median sets of even profiles. We include these results in Section 4. But in the end we found a direct proof that \((A), (B)\) and \((C)\) characterize the median function by making extensive use of the structure theory developed in \([22, 17, 25]\).

The paper is organized as follows: In Section 2 we present the background on consensus functions on graphs. Section 3 focuses on median graphs, including necessary notation and results. In Section 4 we present our results on median sets of even profiles, which extend those of McMorris et al. \([17]\). In Section 5 we prove our main result.

## 2 Preliminaries

Throughout this paper \(G = (V, E)\) is a connected graph. All subgraphs considered are induced. Therefore, we may use the same symbol to denote a subgraph as well as its vertex set, equating subgraph \(H\) with its vertex set. For any \(u, v \in V\), we denote the distance between \(u\) and \(v\) by \(d(u, v)\). The interval between \(u\) and \(v\) in \(G\) is the set

\[
I(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\},
\]

in other words the set of all vertices 'between' \(u\) and \(v\).

Let \(W\) be a subset of \(V\). Then \(W\) is convex in \(G\) if it contains all shortest paths between pairs of vertices, that is, \(I(u, v)\) is contained in \(W\), for any two vertices \(u, v\) in \(W\). Trivially, the intersection of two convex subsets is again convex. If \(W\) is a subset of \(V\) then the convex closure \(\text{Con}(W)\) is the smallest convex set containing \(W\). A subgraph of \(G\) is convex if it is induced by a convex set in \(G\).

Let \(v\) be a vertex of \(G\). If there is a unique vertex \(x\) in \(W\) such that \(x\) lies in \(I(v, w)\) for all \(w\) in \(W\), then \(x\) is a gate for \(v\) in \(W\). The concept of gate was introduced by Dress \([7]\), see also \([8, 34]\). Note that, if \(v\) has a gate in \(W\), then it is the unique vertex in \(W\) closest to \(v\). The converse need not be true. Clearly, if \(v\) lies in \(W\), then \(v\) is its own gate. A subset \(W\) of \(V\) is called gated if each vertex \(v\) of \(G\) has a gate in \(W\). A subgraph is gated if it is induced by a gated set. A gated set is necessarily convex. For arbitrary graphs the converse is not true. The following property for gated sets probably belongs to folklore, and it is an easy exercise to prove: If \(W\) is a gated set, and \(v\) is a vertex outside \(W\), then \(x\) in \(W\) is the gate for \(v\) if and only if \(v\) is closer to \(x\) than to all neighbors of \(x\) in \(W\). We refer to this property as the neighbor-gate property.

A profile \(\pi\) on \(G\) of length \(k\) is a nonempty sequence \(\pi = (x_1, x_2, \ldots, x_k)\) of vertices of \(V\) with repetitions allowed. We denote its length by \(k = |\pi|\). When \(|\pi|\) is odd, we call \(\pi\) an odd profile, otherwise an even profile. Let \(V^*\) be the set of all profiles of finite length. The concatenation of profiles \(\pi_1\) and \(\pi_2\) is denoted by \(\pi_1 \pi_2\). We write the concatenation of a profile \(\pi\) and a single element profile \((v)\) as \(\pi v\) rather than \(\pi(v)\). We refer to a profile \(\pi\) whose elements are contained in a subgraph \(H\) as a profile contained in \(H\), and, abusing notation slightly, we write \(\pi \subseteq H\).
A consensus function on $G$ is a function $L : V^* \to 2^V - \emptyset$. For convenience, we write $L(x_1, x_2, \ldots, x_k)$ instead of $L((x_1, x_2, \ldots, x_k))$ for any function $L$ defined on profiles, but will keep the brackets where needed.

A median of a profile $\pi = (x_1, x_2, \ldots, x_k)$ is a vertex $x$ in $V$ minimizing the distance sum $\sum_{i=1}^{k} d(x, x_i)$. The median set $M(\pi)$ of $\pi$ is the set of all medians of $\pi$. Note that, since $G$ is connected, this defines a consensus function, namely the median function $M : V^* \to 2^V - \emptyset$. Trivially, we have $M(x) = \{x\}$, and $M(x, y) = I(x, y)$. Moreover, if $I(u, v) \cap I(v, w) \cap I(w, u) \neq \emptyset$, then $M(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u)$.

The three basic axioms for our consensus functions are

(A) **Anonymity:** For any profile $\pi = (x_1, x_2, \ldots, x_k)$ on $V$ and any permutation $\sigma$ of $\{1, 2, \ldots, k\}$, we have $L(\pi) = L(\pi^{\sigma})$, where $\pi^{\sigma} = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)})$.

(B) **Betweenness:** $L(u, v) = I(u, v)$, for $u, v \in V$.

(C) **Consistency:** If $L(\pi) \cap L(\rho) \neq \emptyset$ for profiles $\pi$ and $\rho$, then $L(\pi \rho) = L(\pi) \cap L(\rho)$.

It is a simple exercise to prove that the median function satisfies these three basic axioms, see e.g. [17, 25].

Let $L$ be a consensus function satisfying (B) and (C). Take any vertex $v$ in $V$. Since $L(v) \neq \emptyset$, we have $L(v) \cap L(v) \neq \emptyset$. Hence, by consistency and betweenness, we have $L(v) = L(v) \cap L(v) = L(v, v) = I(v, v)$. So $L(v) = \{v\}$. This property is known as Faithfulness. Then consistency and faithfulness imply that $L(u, u, \ldots, u) = \{u\}$, where $(u, u, \ldots, u)$ is the constant profile containing only $u$’s. This property is called Unanimity.

It belongs to folklore that (B) and (C) are independent. Here are two examples to show independence. The consensus function $L$ on $V$ defined by $L(\pi) = V$, for all $\pi$, clearly is anonymous and consistent, but does not satisfy betweenness unless $G$ is $K_1$ or $K_2$. The consensus function $L$ on $V$ defined by $L(x, y) = I(x, y)$ for any $x$ and $y$, and $L(\pi) = V$ for all other $\pi$, satisfies anonymity and betweenness, but not consistency, again, unless $G$ is $K_1$ or $K_2$.

## 3 Median graphs

A median graph is a graph $G$ for which $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$, for any three vertices $u, v, w$ in $G$. Clearly, median graphs are connected. It is easily seen that they are bipartite. In median graphs convex sets are gated. This follows easily from the definition of median graph, using results from [22]. Median graphs possess a beautiful structure and elegant characterizations abound, see e.g. the surveys in [14, 25]. One such characterization is that they are precisely the graphs in which every profile of length 3 has a unique median. The most useful and insightful characterization of median graphs might be the Expansion Theorem in [21]: A graph $G$ is a median graph if and only if
$G$ can be obtained from the one-vertex graph $K_1$ by successive ‘convex expansions’. See also [22, 24, 25].

At first sight one might think that median graphs are quite esoteric. But in [12] a one-to-one correspondence was established between the class of connected triangle-free graphs and a special subclass of the class of median graphs. Hence, median graphs being triangle-free and connected, it was proved that “in the universe of all graphs, there are as many median graphs as there are connected triangle-free graphs”.

To make full use of the Expansion Theorem and its consequences we require several concepts and notations. For an illustration of the definitions and notations see Figure 1. We refer the reader to [21, 22, 24, 25] for details and for the proofs of all the results that are summarized in this section.

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the union $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and the intersection $G_1 \cap G_2$ is the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$. We write $G_1 \cap G_2 \neq \emptyset$ when $V_1 \cap V_2 \neq \emptyset$.

The graph $G_1 - G_2$ is the subgraph of $G_1$ induced by the vertices in $G_1$ but not in $G_2$. A proper cover of a connected graph $G$ consists of two subgraphs $G_1$ and $G_2$ such that $G_1 \cap G_2 \neq \emptyset$ and $G = G_1 \cup G_2$. Note that this implies that there are no edges between $G_1 - G_2$ and $G_2 - G_1$. If both $G_1$ and $G_2$ are convex, we say that $G_1, G_2$ is a convex cover. Note that $G_1, G_2$ is a convex cover if and only if $G_1 \cap G_2$ is convex. Every graph admits the trivial cover $G_1, G_2$ with $G_1 = G_2 = G$, which is of course convex. On the other hand a cycle of length at least four does not have a convex cover with two proper subgraphs.

Let $G'$ be a connected graph and let $G'_1, G'_2$ be a convex cover of $G'$ with $G'_0 = G'_1 \cap G'_2$. For $i = 1, 2$, let $G_i$ be an isomorphic copy of $G'_i$, and let $\lambda_i$ be an isomorphism from $G'_i$ to $G_i$. We write $G_0i = \lambda_i[G'_0]$ and $u_i = \lambda_i(u')$, for $u'$ in $G'_0$. The convex expansion of $G'$ with respect to the convex cover $G'_1, G'_2$ is the graph $G$ obtained from the disjoint union of $G_1$ and $G_2$ by inserting an edge between $u_1$ in $G_{01}$ and $u_2$ in $G_{02}$, for each $u'$ in $G'_0$. We denote the set of edges between $G_{01}$ and $G_{02}$ by $F_{12}$. Note that $F_{12}$ induces an isomorphism between $G_{01}$ and $G_{02}$.

It is straightforward to prove that the expansion of a median graph with respect to a convex cover is again a median graph. The hard part of the proof of the Expansion Theorem is to show that a median graph is always the convex expansion of a smaller one. We need some of the ideas and notations from this proof for the sequel.

Let $G$ be a median graph, and let $v_1v_2$ be an edge in $G$. Let $G_1$ be the subgraph induced by all vertices closer to $v_1$ than to $v_2$ and let $G_2$ be the subgraph induced by all vertices closer to $v_2$ than to $v_1$. Since $G$ is bipartite, it follows that $G_1, G_2$ is a vertex-partition of $G$. Let $F_{12}$ be the set of edges between $G_1$ and $G_2$, and let $G_0i$ be the subgraph induced by the ends of $F_{12}$ in $G_i$, for $i = 1, 2$. Then it is proved in [21] (although not exactly in that order) that the following facts hold:

(i) $F_{12}$ is a matching as well as a cutset (minimal disconnecting edge-set),

(ii) the subgraphs $G_1, G_2, G_{01}, G_{02}$ are convex subgraphs of $G$, 

6
(iii) the obvious mapping of $G_{01}$ onto $G_{02}$ defined by $F_{12}$ (i.e. $u_1 \rightarrow u_2$, for any edge $u_1u_2$ in $F_{12}$ with $u_0$ in $G_{0i}$) is an isomorphism,

(iv) for every edge $u_1u_2$ in $F_{12}$ with $u_i$ in $G_{0i}$, the subgraph $G_1$ consists of all vertices of $G$ closer to $u_1$ than $u_2$, and the subgraph $G_2$ consists of all vertices of $G$ closer to $u_2$ than $u_1$.

We call such a partition $G_1, G_2$ of $G$ a split. Note that any edge in $F_{12}$ defines the same split. The subgraphs $G_1$ and $G_2$ are the sides of the split. If we are in $u_1$ of an edge $u_1u_2$ of $F_{12}$, then $G_1$ is the side of $u_1$ and $G_2$ is the opposite of $u_1$, see [25] for the reason why opposite is spelled this way. We call $u_1$ and $u_2$ mates of each other.

The converse of expansion is contraction: for a split $G_1, G_2$ contract each edge in $F_{12}$ so that mates are being identified. Then the resulting graph $G'$ is again a median graph and the split $G_1, G_2$ is ‘contracted’ to a convex cover of $G'$. Moreover, the expansion with respect to that cover reproduces $G$. Thus the proof of the Expansion Theorem is complete. To obtain the median graph $G$ from $K_1$, the expansions are order independent. The edge set $E$ of $G$ is the disjoint union of the resulting matchings. This provides us with a very strong tool: we can use induction on the number of splits. This is needed in the proof Theorem 1 and various other properties mentioned here.

If, at each expansion, one of the two parts in the cover is a single vertex, so that the other part is the whole graph, then a tree results. If at each expansion, each part of the
cover is the entire graph, an \( n \)-dimensional hypercube results, where \( n \) is the number of expansions.

We need some more consequences of the structural characterization of median graphs in the sequel. As observed above, convex subgraphs of a median graph are gated. So the subgraphs \( G_1, G_2, G_{01} \) and \( G_{02} \) are all gated. Let \( v \) be a vertex in, say, \( G_1 \). Let \( u_1 \) be its gate in \( G_{01} \). Then it follows from the above structural properties of a split that the mate \( u_2 \) of \( u_1 \) is the gate for \( v \) in \( G_{02} \) as well as \( G_2 \). Moreover, for any \( w \) in \( G_2 \), it follows that \( u_1, u_2 \in I(v, w) \). We use this property of gates and mates in the next sections, and refer to it as the mate-gate property.

Let \( \pi \) be a profile on \( G \), and let \( G_1, G_2 \) be a split of \( G \). We denote the subprofile of \( \pi \) contained in \( G_i \) by \( \pi_i \), for \( i = 1, 2 \). If \( |\pi_1| > |\pi_2| \), then we call \( G_1 \) the majority side of the split. If \( |\pi_1| = |\pi_2| \), then we call \( \pi \) balanced on the split \( G_1, G_2 \).

In the sequel we will use all the notation developed here for split \( G_1, G_2 \), edge \( u_1u_2 \), subprofiles \( \pi_1 \) and \( \pi_2 \), and so forth, without further mention.

For any edge \( uv \) in \( G \) we denote by \( G_{uv} \) the subgraph of \( G \) induced by all vertices closer to \( u \) than to \( v \). Then \( G_{uv}, G_{vu} \) is a split. Note that, for any edge \( u_1u_2 \) in \( F_{12} \), we could write \( G_1 \) also as \( G_{u_1u_2} \), and \( G_2 \) as \( G_{u_2u_1} \), see Figure 1.

An important consequence of the Expansion Theorem was proved in [23] and [17]. The median set of a profile is always contained in the majority side of an unbalanced split, and it intersects both sides of a balanced split. This is made more precise in the following theorem, which is basic for almost all the proofs in this paper.

**Theorem 1** Let \( G \) be a median graph and let \( \pi \) be a profile on \( G \). Then

\[
M(\pi) = \bigcap_{G_1, G_2 \text{ split with } |\pi_1| > |\pi_2|} G_1.
\]

Note that, split sides being convex, this means that median sets are necessarily convex. It is a well-known fact that odd profiles have a unique median in median graphs. It also follows easily from this theorem. Let \( \pi \) be an odd profile. Then there are no balanced splits for \( \pi \). So, if \( u \) is a median vertex and \( v \) is a neighbor of \( u \), then \( G_{uv} \) is a majority side and \( G_{vu} \) is a minority side in the split \( G_{uv}, G_{vu} \). So \( v \) is not in \( M(\pi) \). Since \( M(\pi) \) is convex, it consist only of \( u \). A special case of this theorem concerns intervals.

Note that the next corollary was already proved in [28] without any reference to profiles and the median function. But now it is a simple corollary of Theorem 1. Recall that \( I(x, y) = M(x, y) \) on any connected graph.

**Corollary 2** Let \( G \) be a median graph and let \( x \) and \( y \) be vertices of \( G \). Then

\[
I(x, y) = \bigcap_{G_1, G_2 \text{ split with } x, y \in G_1} G_1.
\]

Hence intervals in median graphs are convex. We get this property for free here, but it can also be easily proved using the definition of a median graph. Another noteworthy corollary of this theorem is also needed. Note that, for any edge \( uv \), we have the following
fact. Edge $uv$ has an end in each side of the split $G_{uv}, G_{vu}$, but for any other split $G_1, G_2$, the edge is contained in one of the two sides. In a way, split $G_{uv}, G_{vu}$ distinguishes edge $uv$. Now, if $u$ is a vertex in $M(\pi)$, for some profile $\pi$ and $v$ is a neighbor of $u$ outside $M(\pi)$, then $G_{uv}$ contains a majority of $\pi$, and necessarily $M(\pi) \subseteq G_{uv}$. We call this the edge-distinguishing property.

A ‘counterpart’ for Theorem 1 was also proved in [17].

**Theorem 3** Let $G$ be a median graph, and let $\pi$ be a profile on $G$ that is balanced on all splits. Then $M(\pi) = G$.

Note that we could merge Theorems 1 and 3 into one statement using the convention $\cap \emptyset = G$.

### 4 Medians of even profiles on median graphs

In this Section we extend the results from [17] on median sets of even profiles. In 1998, McMorris et al. [17] showed that axioms $(A)$, $(B)$ and $(C)$ characterize the median function on ‘cube-free median’ graphs, where cube-free means that the 3-cube $Q_3$ does not occur as a subgraph. The surprising key result in [17] for cube-free median graphs was the following: any even profile $\pi$ on a cube-free median graph admits a permutation such that it can be written as $(y_1, y_2, \ldots, y_{2m-1}, y_{2m})$ with

$$M(\pi) = \cap_{1 \leq i \leq m} I(y_{2i-1}, y_{2i}).$$

So the median set of an even profile is the intersection of intervals between its elements. A simple example on the 3-cube showed that this is not true on arbitrary median graphs: take the profile of length four of the black vertices in Figure 2.

![Figure 2: The 3-cube $Q_3$ with a profile](image)

To extend the characterization in [17] to arbitrary median graphs, a fourth axiom was added in [17]: the less intuitively appealing ‘convexity’ axiom. For a profile $\pi = (x_1, x_2, \ldots, x_k)$, this axiom involves the vertex-deleted profiles $\pi - x_i$ for $1 \leq i \leq k$. This is the profile of length $|\pi| - 1$, where only the element $x_i$ is removed.
(K) Convexity: Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile in $G$ with $k \geq 2$. If $\cap_{i=1}^{k} L(\pi - x_i) = \emptyset$, then $L(\pi) = \text{Con}(\cup_{i=1}^{k} L(\pi - x_i))$.

It is easy to check that for $L = M$, axiom (K) holds vacuously when $\pi$ is an odd profile. The fact that (K) holds for the median function when $\pi$ is an even profile on a median graph, is a key result in the above mentioned paper.

**Theorem 4** [17] Let $\pi = (x_1, \ldots, x_k)$ be an even profile on the median graph $G$. Then $M(\pi) = \text{Con}(\cup_{i=1}^{k} M(\pi - x_i))$.

We present an extension of this result. Note that, if $\pi = (x_1, \ldots, x_k)$ is an even profile, then the vertex-deleted profile $\pi - x_i$ is odd, so it has a unique median.

**Theorem 5** Let $G$ be a median graph, and let $\pi = (x_1, \ldots, x_k)$ be an even profile on $G$. For $i = 1, \ldots, k$, let $y_i$ be the median of the vertex-deleted profile $\pi - x_i$, and let $\pi' = (y_1, \ldots, y_k)$. Then $\pi' \subseteq M(\pi) = M(\pi')$.

**Proof.** Since $\pi$ is even, a majority side for $\pi$ of a split remains a majority side for $\pi - x_i$. So $y_i$ lies in $M(\pi)$. Hence we have $\pi' \subseteq M(\pi)$. By Theorem 1, any majority side for $\pi$ of a split contains $\pi'$, hence trivially is a majority side for $\pi'$. So $M(\pi') \subseteq M(\pi)$.

Take any balanced split $G_1, G_2$ for $\pi$. So exactly half of $\pi$ is in $G_1$ and the other half of $\pi$ is in $G_2$. Now, if $x_i$ is in $G_1$, then the majority of $\pi - x_i$ is in $G_2$, so that $y_i$ is in $G_2$. So, for $i = 1, \ldots, k$, the vertices $x_i$ and $y_i$ are always on opposite sides of $G_1, G_2$. Hence this split is also balanced for $\pi'$. Thus we have shown that the majority sides for $\pi$ are precisely the majority sides for $\pi'$. By Theorem 1, we have $M(\pi) = M(\pi')$.

The example in Figure 2 shows that, for even $\pi = (x_1, \ldots, x_k)$, we cannot always write $M(\pi)$ as the intersection of intervals between profile-elements as in the cube-free case. But it turns out that $M(\pi)$ is the interval between two well-chosen vertices that are determined by the profile. Take any profile element $x_i$. Then these two vertices are, loosely speaking, the vertex in $M(\pi)$ closest to $x_i$ and the vertex in $M(\pi)$ furthest away from $x_i$. Clearly, the first vertex is the gate $z_i$ in $M(\pi)$ for $x_i$. The vertex that is intuitively ‘furthest away’ is the median $y_i$ of the vertex-deleted profile $\pi - x_i$. Another way of looking at these two vertices is: for the closest vertex we maximize the influence of $x_i$ by taking the median of the vertex-added profile $\pi x_i$, for the vertex ‘furthest away’ we minimize the influence of $x_i$ by taking the median of the vertex-deleted profile $\pi - x_i$.

**Lemma 6** Let $G$ be a median graph, and let $\pi = (x_1, x_2, \ldots, x_k)$ be an even profile on $G$. Then, for $i = 1, \ldots, k$, the median $z_i$ of the vertex-added profile $\pi x_i$ is the gate for $x_i$ in $M(\pi)$.

**Proof.** By Theorem 1, the median set $M(\pi_1)$ of $\pi$ is the intersection of the majority sides of the splits in $G$. Since $\pi$ is even, these sides remain majority sides for the profile $\pi x_i$. So $M(\pi x_i) \subseteq M(\pi)$. For the splits that are balanced with respect to $\pi$, adding $x_i$
to the profile means that the balance is tipped towards the side containing \( x_i \). So the
majority sides of the vertex-added profile \( \pi x_i \) are the majority sides of the unbalanced
splits for \( \pi \) and the sides containing \( x_i \) of the balanced splits for \( \pi \). Since \( \pi x_i \) is odd, it
has no balanced splits. As in the statement of the Lemma, we define \( z_i \) to be the unique
vertex in \( M(\pi x_i) \). Take any neighbor \( v \) of \( z_i \) in \( M(\pi) \). Consider the split \( G_{z,v}, G_{v,z} \). By
the edge-distinguishing property, \( G_{z,v} \) is a majority side for \( \pi x_i \). Both sides contain a
vertex of \( M(\pi) \), so \( \pi \) is balanced on this split. Since \( x_i \) tips the balance, it follows that
\( x_i \) lies in \( G_{z,v} \). Hence \( x_i \) is closer to \( z_i \) than to \( v \). By the neighbor-gate property, \( z_i \) is
the gate for \( x_i \) in the gated set \( M(\pi) \). 

\[ \square \]

**Theorem 7** Let \( G \) be a median graph, and let \( \pi = (x_1, x_2, \ldots, x_k) \) be an even profile on
\( G \). For \( i = 1, \ldots, k \), let \( y_i \) be the median of the vertex-deleted profile \( \pi - x_i \) and \( z_i \) be the
median of the vertex-added profile \( \pi x_i \). Then \( M(\pi) = I(y_i, z_i) \).

**Proof.** Since \( k \) is even, a majority side of a split for \( \pi \) remains a majority side when
we add or delete a vertex from \( \pi \). So, by Theorem 1, we have \( M(\pi x_i) \subseteq M(\pi) \) as well as
\( M(\pi - x_i) \subseteq M(\pi) \).

Let \( G_1, G_2 \) be any balanced split for \( \pi \) with, say, \( x_i \) in \( G_1 \). Then \( G_1 \) is a majority
side for the vertex-added profile \( \pi x_i \) and so \( z_i \) is in \( G_1 \). Moreover \( G_2 \) is a majority side
for the vertex-deleted profile \( \pi - x_i \) and \( y_i \) is in \( G_2 \). By Lemma 6, \( z_i \) is the gate for \( x_i \)
in \( M(\pi) \), so, trivially, \( z_i \) lies in \( M(\pi) \). By Theorem 5, \( y_i \) lies in \( M(\pi) \). Hence, by the
convexity of \( M(\pi) \), we have \( I(y_i, z_i) \subseteq M(\pi) \).

Take any vertex \( w \) outside \( I(y_i, z_i) \). Recall that an interval in a median graph is convex, hence gated. Let \( u \) be the gate for \( w \) in \( I(y_i, z_i) \), and let \( v \) be a neighbor of \( u \)
with \( d(w, v) = d(w, u) - 1 \). Then \( v \) is not in \( I(y_i, z_i) \). Now \( w \) being closer to \( v \) than to
\( u \), we have, by definition, that \( w \) is in \( G_{vu} \). Since \( v \) is not in \( I(y_i, z_i) \), both \( y_i \) and \( z_i \)
are closer to \( u \) than to \( v \), so they are both in \( G_{uv} \). Now \( z_i \) being in \( G_{uv} \) means that \( G_{uv} \)
contains at least half of \( \pi x_i \), and \( y_i \) being in \( G_{uv} \) means that \( G_{uv} \) contains at least half
of \( \pi - x_i \). Hence \( G_{uv} \) must contain a majority of \( \pi \). So \( w \) is not in \( M(\pi) \). Thus we have
shown that \( M(\pi) \subseteq I(y_i, z_i) \), which completes the proof. 

\[ \square \]

5 Consensus Functions satisfying \( (A) \), \( (B) \) and \( (C) \)

In this section we prove that axioms \( (A) \), \( (B) \) and \( (C) \) suffice for the consensus function
to make it the median function.

Median sets in a median graph always reside at the majority side of a split. Our first
lemma states that in fact this ‘majority’ property holds for any consensus function \( L \) on
a median graph \( G \) provided it satisfies \( (A) \), \( (B) \) and \( (C) \). We will refer to this as our
‘Majority Lemma’. We use the ‘standard’ notation developed above for the split \( G_1, G_2 \).
Lemma 8 (Majority Lemma) Let $G = (V, E)$ be a median graph, let $G_1, G_2$ be a split of $G$, and let $\pi$ be a profile on $G$ with $|\pi_1| > |\pi_2|$. If $L : V^* \rightarrow 2^V - \emptyset$ is a consensus function satisfying the axioms (A), (B) and (C), then $L(\pi) \subseteq G_1$.

Proof. Assume to the contrary that there exists a vertex $v$ in $L(\pi) \cap G_2$. Let $\pi_2 = (x_1, x_2, \ldots, x_\ell)$ and let $\pi_1 = (x_{\ell+1}, x_{\ell+2}, \ldots, x_k)$, where $2\ell < k$. Let $g_2$ be the gate for $v$ in $G_{02}$. Let $g_1$ be the mate of $g_2$ in $G_{01}$, so that $g_1$ is the gate for $v$ in $G_1$.

By betweenness, we have $v \in I(v, g_1) = L(v, g_1)$. Thus, our assumption that $v$ lies in $L(\pi)$, together with consistency implies that

$$v \in L(v, g_1) \cap \cdots \cap L(v, g_1) \cap L(\pi) = L((v, g_1)(v, g_1)\ldots(v, g_1)\pi),$$

where the intersection is taken over $\ell$ terms $L(v, g_1)$ and the pair $(v, g_1)$ occurs $\ell$ times in the right hand side. Since $2\ell < k$ and $L(v, v, \ldots, v) = \{v\}$ we obtain

$$L((v, \ldots, v)(v, g_1)\ldots(v, g_1)\pi) = \{v\},$$

(1)

where the $v$'s are repeated $k-2\ell$ times and the pair $(v, g_1)$ is repeated $\ell$ times.

Since the mate $g_2$ of $g_1$ is the gate for $g_1$ in $G_2$, we have $g_2 \in I(g_1, x)$, for any $x \in G_2$. Hence

$$\{g_1, g_2\} \subseteq \cap_{i=1}^\ell I(g_1, x_i).$$

(2)

An immediate consequence of the mate-gate property is that, for any $x \in G_1$, the interval $I(v, x)$ contains both $g_1$ and $g_2$. Hence

$$\{g_1, g_2\} \subseteq \cap_{i=\ell+1}^k I(v, x_i).$$

(3)

Now let us reorganize the left-hand side of Equation (1) into the nonempty intersection of a collection of intervals. Note that by Equation (1) together with anonymity, we have

$$\{v\} = L((v, \ldots, v)(v, g_1)\ldots(v, g_1)\pi_1\pi_2) = L((g_1, x_1)(g_1, x_2)\ldots(g_1, x_\ell)(v, x_{\ell+1})\ldots(v, x_k)) = [\cap_{i=1}^\ell I(g_1, x_i)] \cap [\cap_{i=\ell+1}^k I(v, x_i)].$$

But this intersection of intervals contains both $g_1$ and $g_2$, by Equations (2) and (3). This impossibility settles the proof.

An immediate, but surprisingly strong consequence of the Majority Lemma and Theorem 1 is the following.

Corollary 9 Let $G = (V, E)$ be a median graph and let $L : V^* \rightarrow 2^V - \emptyset$ be a consensus function satisfying the axioms (A), (B) and (C). Then $L(\pi) \subseteq M(\pi)$, for any profile $\pi$. In particular, $L(\pi) = M(\pi)$, when $|\pi|$ is odd.
For the balanced case we need the following lemma. It might be considered as an extension of the edge-distinguishing property to the consensus function $L$. In the terminology of [14] we prove that $L$ is $\frac{1}{2}$-condorcet, see that paper for details.

**Lemma 10** Let $G$ be a median graph, and let $\pi$ be a profile on $G$. If $G_1, G_2$ is a balanced split for $\pi$, then, for any edge $u_1u_2$ in $F_{12}$, either both $u_1$ and $u_2$ are in $L(\pi)$ or neither $u_1$ nor $u_2$ is in $L(\pi)$.

**Proof.** Let $u_1u_2$ be an edge in $F_{12}$. Note that $I(u_1, u_2) = \{u_1, u_2\}$. Assume to the contrary that $u_1 \in L(\pi)$ and $u_2 \notin L(\pi)$. Let $\pi_1 = (x_1, x_2, ..., x_k)$ and let $\pi_2 = (y_1, y_2, ..., y_k)$. Let $(u_1, u_2)^k$ denote the profile $(u_1, u_2)$ repeated $k$ times. Then, by consistency and betweenness, we have $L((u_1, u_2)^k) = I(u_1, u_2) = \{u_1, u_2\}$. Hence, by consistency, we have $u_1 \in L(\pi) \cap L((u_1, u_2)^k) = L(\pi(u_1, u_2)^k)$. So $L(\pi(u_1, u_2)^k) = L(\pi) \cap \{u_1, u_2\} = \{u_1\}$.

Recall that $u_2$ is the gate for $u_1$ in $G_2$, and $u_1$ is the gate for $u_2$ in $G_1$. So both $u_1$ and $u_2$ are in $I(u_1, y_i)$ as well as in $I(u_2, x_i)$, for $i = 1, \ldots, k$. By betweenness, consistency and anonymity, we have

$$
\{u_1, u_2\} \subseteq [\cap_{i=1}^{k} I(u_1, y_k)] \cap [\cap_{i=1}^{k} I(u_2, x_i)]
$$

$$
= [\cap_{i=1}^{k} L(u_1, y_k)] \cap [\cap_{i=1}^{k} L(u_2, x_i)]
$$

$$
= L((u_1, y_1)(u_1, y_2)...(u_1, y_k)(u_2, x_1)(u_2, x_2)...(u_2, x_k))
$$

$$
= L(\pi(u_1, u_2)^k).
$$

Thus we get a contradiction, which completes the proof.

We are now ready to prove our main result.

**Theorem 11** Let $L$ be a consensus function on a median graph $G$. Then $L$ satisfies (A), (B) and (C) if and only if $L = M$.

**Proof.** Recall that the median function $M$ satisfies (A), (B) and (C) on any connected graph.

To prove the converse, let $G = (V, E)$ be a median graph and let $L : V^* \to 2^V - \emptyset$ be a consensus function on $G$ satisfying the axioms (A), (B) and (C). Assume by way of contradiction that $L$ is not identical to the median function $M$. Let $\pi = (x_1, \ldots, x_k)$ be a profile on $G$ for which $L(\pi) \neq M(\pi)$. By Corollary 9, we have $L(\pi) \subseteq M(\pi)$ and $\pi$ must be even. Since $M(\pi)$ is connected, being convex, there is an edge $uv$ in $M(\pi)$ such that $u$ is in $L(\pi)$ and $v$ is not. Consider the split $G_{uv}, G_{vu}$. Since both sides contain a vertex of $M(\pi)$, this split must be balanced by Theorem 1. Hence Lemma 10 tells us that either both $u$ and $v$ are in $L(\pi)$ or neither is in $L(\pi)$. This contradicts the choice of $uv$, which completes the proof.
6 Concluding Remarks

In [17] the median function on cube-free median graphs has been characterized as the unique consensus function satisfying the three simple and appealing axioms Anonymity, Betweenness and Consistency. To prove this, a nice characterization of the median sets of even profiles was used: such a median set can be written as the intersection of intervals between profile elements. To characterize the median function on arbitrary median graphs an extra ‘heavy duty’ axiom Convexity was needed to deal with the case of even profiles. In [14] the Convexity axiom was replaced by another ‘heavy duty’ axiom: the $\frac{1}{2}$-condorcet axiom. In this paper we have shown that the three basic axioms suffice to characterize the median function on arbitrary median graphs. Moreover the results in [17] on median sets of even profiles have been extended in Section 4. Specifically, an elegant characterization of $M(\pi)$ for even $\pi = (x_1, x_2, \ldots, x_k)$ is that, for any $x_i$, the set $M(\pi)$ is the interval between the median of the vertex-deleted profile $\pi - x_i$ and the median of the vertex-added profile $\pi x_i$.

Loosely speaking, a network is a graph in which the edges are assigned a length (positive real number) and interior points of edges are also allowed as location for the facility. In [18] the case for cube-free median graphs from [17] was extended to cube-free median networks. Basically this was done by proving that the important properties of splits is carried over to the network case. Then the characterization of the median function defined on profiles of vertices on the network followed easily from the result in [17]. This can also be done in our case. So on median networks the median function is characterized by the three basic axioms (A), (B) and (C) as well. The results and proofs in [18] are straightforward but they need a lot of technical details. Therefore we omit these and just refer the reader to [18] and leave the proof of the general case as an exercise.

So far the only location functions that have been characterized axiomatically are the center function and the $\ell_p$-functions with $p$ a positive integer, and the antimedian function, see [19, 27, 17, 35, 11, 18, 15, 16, 5]. The antimedian function has been characterized on paths only. All the other functions have been characterized on trees only, except for the median function $M = \ell_1$. The reason for this exception is that the median function behaves nicely on median graphs and that a rich structure theory on median graphs is available.

There are still many intriguing questions. For instance, is there any other class of graphs on which the median function is characterized by axioms (A), (B) and (C)? Can the median function be characterized on other classes by adding extra axioms? And of course one would like to have axiomatic characterizations of the other location functions on graphs other then trees, or of other consensus functions.
References


[16] F.R. McMorris, H.M. Mulder, O. Ortega, Axiomatic characterization of the $\ell_p$-function on trees, to appear in *Networks*.


