Guides and Shortcuts in Graphs

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Abstract

The geodesic structure of a graphs appears to be a very rich structure. There are many ways to describe this structure, each of which captures only some aspects. Ternary algebras are for this purpose very useful and have a long tradition. We study two instances: signpost systems, and a special case of which, step systems. Signpost systems were already used to characterize graph classes. Here we use these for the study of the geodesic structure of a spanning subgraph F with respect to its host graph G. Such a signpost system is called a guide to (F, G). Our main results are: the characterization of the step system of a cycle, the characterization of guides for spanning trees and hamiltonian cycles.

Keywords: geodesic structure, signpost system, step system, guide, shortcut, spanning tree, hamiltonian cycle
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1 Introduction

The notion of distance in a graph G is a basic concept in graph theory. The distance $d(u, v)$ between two vertices u and v is simply defined as being the length of a shortest u, v-path or u, v-geodesic. Thus, V being the vertex set, (V, d) is an instance of a discrete metric space. As such there is a big difference with Euclidean space, that being a continuous metric space. But there is a more important difference: In a connected graph there may be many u, v-geodesics. The study of distance captures only one aspect of the set of geodesics. It turns out that the set of all geodesics represents a surprisingly rich structure with many subtleties. A classical tool to study this structure is the notion of interval: the interval $I(u, v)$ between u and v is the set of vertices on the u, v-geodesics. The first systematic study of the interval function I was [16]. Many, many papers using the interval function have appeared since. Another way to study the geodesic structure is to use ternary algebras. Already in the early fifties of the last century Sholander [31, 32, 33] used ternary algebras to study betweenness, and using this Avann [2] studied graphs in 1961. By now there are many ways of using the algebraic approach to study the geodesic structure of a graph, not to mention other, non-algebraic approaches. We can only give a few examples here: [3, 4, 11, 10, 15, 16, 20, 22, 12, 30]. None of these approaches by itself captures all aspects of the geodesic structure. So it appears that we need all these different approaches, and quite certainly even some more.

In this paper we single out two ternary algebras that capture different aspects of the geodesic structure: a *signpost system*, introduced in [18], and a special instance of signpost systems, the *step system of a graph*, introduced in [22]. See also [22, 27, 28, 29]. Loosely speaking a step in a graph is used to describe how to get one step closer from u to v. A signpost at u for getting to v directs us to a point x "closer to v", where we look for a new signpost that will lead us again closer to v. Note that a step system is defined on a graph, whereas a signpost system may be defined without reference to a graph. At first sight there are many similarities between the two systems. But a closer look reveals that the differences between the two systems allows us to capture different aspects of the geodesic structure.

A *ternary algebra* on a finite set V is a set $S \subseteq V \times V \times V$ of ordered triples from V that satisfies certain *axioms*. These axioms are such that they capture essential features of the structure to be studied. This approach is in the Sholander tradition of [31, 32, 33]. In various ways one can define the underlying graph of a ternary algebra, a tradition that probably originates with Avann [1, 2]. Most of the papers on ternary algebras focus on graphs having an additional structure, such as median graphs and generalizations, see e.g. [2, 21, 16, 13, 5, 6, 7, 18, 14].

In Section 2 we study signpost systems and their underlying graphs. In this graph an edge ux is defined by the fact that (u, x, v) is a signpost, that is, to get to v from u our first step is to move to x. The underlying graph need not be connected. A point of focus is which axioms will guarantee that the underlying graph is in fact connected. We search for axioms that are in what we call standard form. The reason for this is the following. In [25] a striking and unexpected impossibility result was obtained: using first order logic the second author was able to prove that certain axiomatic characterizations of the induced path function do not exist. For this it was necessary to have the axioms in such standard form. See [17] for details of this problem.

In Section 4 we study a special type of signpost system, which is not a step system. To explain the concepts of this section we use the following metaphor. Take a city, say Prague. The street plan of the old city with its overwhelming amount of beautiful historical buildings, sites, streets, bridges and squares may be represented by a connected graph G. A foreign tourist, say the first author, visits the city. Using the main touristic routes he will see many beautiful things, but he will miss many others and has to walk in crowded streets all day long. These main routes are represented by a spanning subgraph F of G. A local guide, say the second author, knows many shortcuts existing in G but not in F, to get from one place to another, and on the way many hidden treasures can be seen as well. Loosely speaking, a *guide* to the pair (F, G) is a signpost system that describes and studies these shortcuts. We believe that this concept of guide, that highlights the geodesic structure of a spanning subgraph with respect to its host graph, will shed new light on the study of the geodesic structure of graphs. In Sections 5 and 6 we study guides where the spanning subgraph is a tree or a hamiltonian cycle, respectively. As a preparation for the hamiltonian section we study the step system of a cycle in Section 3, This section can also be viewed independently as an instance of studying systematically the step systems of special classes of graphs, see [22, 27] for other instances. In all the mentioned cases we obtain characterizations of the relevant signpost (step) systems involving various sets of axioms in standard form.

2 Signpost systems and their underlying graphs

Let V be a finite nonempty set. A *ternary system* $S = (V, R)$ on V consists of a set V and a ternary relation $R \subseteq V \times V \times V$ on V. We use the following convention: instead of $(v, w, x) \in R$ we write $vwSx$ and instead of $(v, w, x) \notin R$ we write $\neg vwSx$. Note that a similar convention was used in [28, 29].

Let $G = (V, E)$ be a finite, simple, connected graph. For any u, v in V, we denote the geodesic *distance* between u and v by $d(u, v)$. It is the length of a shortest u, v-path, or *u, v-geodesic*. If w lies on a u, v-geodesic, then we say that w is *between* u and v. This way of viewing w as between u and v has been have been phrased in many different guises and languages. An important one is that of intervals: the *interval* between v and x in G is the set

$$I(v,x) = \{w \mid d(v,w) + d(w,x) = d(v,x)\},$$

in other words all vertices 'between' v and x, see [16, 19]. Many other guises involve ternary systems, see e.g. [31, 32, 33, 10, 18, 3, 4]. We present two from the literature. The *geodesic betweenness* of G is the ternary system S defined as follows:

$$vwSx \text{ if and only if } d(v,w) + d(w,x) = d(v,x).$$

Note that in this case v, w, x need not be distinct.

The *step system* of G is the ternary system S defined as follows:

$$vwSx \text{ if and only if } d(v,w) = 1 \text{ and } d(w,x) = d(v,x) - 1.$$

This was introduced in [22]. Note that now v is necessarily distinct from w and x, but w and x may be the same vertex.

At first sight the two ternary systems seem to be quite similar. Of course this should be the case, because they both describe the same thing. But a closer look at the two systems reveals important differences. When one studies properties of both systems, especially, when one wants to characterize the betweenness relation and the step system using axioms on S, then axioms and proofs for both systems show essential differences. See [22] for step systems, and see e.g. [31, 32, 16, 30, 23, 18, 26, 19] for various systems to describe the interval function or geodesic betweenness. We want to stress here the fact that the structure of the shortest paths in a graph is so rich that we need various ways to model this to be able to capture as many aspects of it as possible.

In [18] the notion of signpost system was introduced, which combines certain elements of the above two ternary systems, but is different from both. It captures yet some other aspects of the geodesic structure, see also [28, 29]. Its name is derived from the fact that it reflects the structure of signposting in a road network. A *signpost system* is a ternary system $S = (V, R)$ satisfying the following three simple axioms:

(a1) if $vwSx$, then $wvSv$, for $v, w, x \in V$,

(a2) if $vwSx$, then $\neg wvSx$, for $v, w, x \in V$,

(a3) if $v \neq x$, then there exists a $w \in V$ such that $vwSx$, for $v, x \in V$.

The triples in R are called *signposts*. We can interpret these axioms as follows. If $vwSx$, then there is a signpost at v that tells us that, to get to x, we need to go to w first, and then at w look for another signpost towards x. Loosely speaking, axiom $(a1)$ guarantees us that we can find our way back. Axiom $(a2)$ prevents us from getting stuck in a loop. Axiom $(a3)$ guarantees that at any point there are signposts to all other points. In real life situations, this last fact may be realized only by combining the existing signpost system along the roads with a map of the region. Note that axiom $(a2)$ implies the following basic fact: if $vwSx$, then v is distinct from w. Moreover, $(a1)$ and $(a2)$ together imply that v is distinct from x. So we have

$$vwSx \;\Rightarrow\; v \neq w,\; v \neq x. \tag{1}$$

Clearly, a step system of a graph also satisfies these three axioms. But a signpost system is a broader concept, as we will see below. The geodesic betweenness satisfies trivially axioms $(a1)$ and $(a3)$. But it does not satisfy axiom $(a2)$, because in the geodesic betweenness we have $uuSx$, and even $uuSu$, for any u and x in V. This is just one of the subtle differences between the step approach and the betweenness approach. In the present paper we also discuss signpost systems that are different from the step system of a connected graph.

Note that $(a1)$ also implies that

$$\text{if } vwSw, \text{ then } wvSv, \text{ for } v, w \in V.$$

In terms of a signposting $vwSw$ means that there is a *direct connection* between v and w: if we are at v and follow the signpost to w, then we do not encounter any other signpost before reaching w. This observation motivates the following definition, see [18]. If $S = (V, R)$ is a signpost system, then the *underlying graph* of S is the graph $G_S = (V, E_S)$ defined as follows:

$$v \text{ and } w \text{ are adjacent in } G_S \text{ if and only if } vwSw, \text{ for any } v, w \in V.$$

Note that this implies that, if $vwSx$ for some x, then v and w are adjacent in G_S. When no confusion arises, we write G instead of G_S and E instead of E_S. We want to stress that, in the literature, the underlying graphs of other ternary systems have been defined differently. Because of our notation for ternary systems, we denote the edges of a graph as follows: if v and w are joined by an edge, then we denote this edge by $\{v, w\}$, instead of a more usual form vw. For clarification of some features of signpost systems we refer the reader to Figure 1.

Example 1. Let $S = (V, R)$ be a signpost system with $V = \{v_1, v_2, v_3, x, y\}$ and let G denote the underlying graph of S. Assume that $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}, \{v_1, x\}, \{x, y\}\}$, see Figure 1. The edges give already some signposts. The other signposts are: v_1v_2Sx, v_1v_2Sy, v_2v_3Sx, v_2v_3Sy, v_3v_1Sx, v_3v_1Sy, xv_1Sv_2, xv_1Sv_3, $yxSv_1$, $yxSv_2$, $yxSv_3$. We see that some signposts in S represent paths in G: a path from x to v_1 up to v_3, and a path from y to v_1 up to v_3. But other signposts in S show a vicious circle in G: "illusory" paths from v_1 via v_2 and v_3 to x or to y.

Let $G = (V, E)$ be a graph. By definition, G is connected if and only if

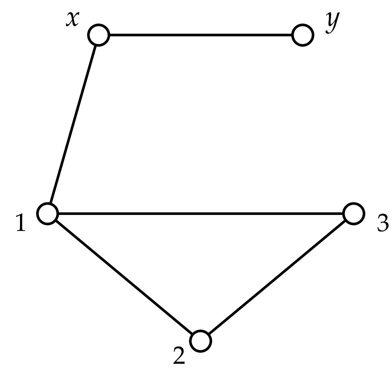

Figure 1: Example 1

(c) for any two distinct vertices u and v in V, there exists a positive integer n and vertices $x_0, x_1, \ldots, x_n$ such that $u = x_0$, $x_n = v$, and $\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\}$ are edges in E.

The statement (c) contains the variables u and v in V, the variable n from the set of positive integers, and $n+1$ variables $x_0, \ldots, x_n$ in V. From the logical point of view this axiom is rather complicated: it contains variables from two different sets, one of which is infinite. Moreover, the number of variables from the finite set V is *not* fixed. Axioms $(a1)$, $(a2)$ and $(a3)$ for signpost systems have a different form: each of these axioms contains a *fixed* number of variables, all from the same *finite* set V. We call such a form for an axiom the *standard form*. Why would one want axioms in standard? In [25] a striking and unexpected result was obtained. Using first order logic the impossibility of certain axiomatic characterizations of the induced path function was proved. See [17] for more details of this problem. For such a type of result axioms should be in standard form. Therefore, our goal is to use only axioms in standard form.

For studying the step system of a connected graph the reader is referred to [22, 23, 24]. In [24] a set $\mathcal{A}$ of four axioms in standard form is given, and the following theorem is proved: Let S be a signpost system, and let G be the underlying graph of S. Then S is the step system of G if and only if the conditions (1) and (2) hold:

(1) S satisfies each of the axioms in $\mathcal{A}$,

(2) G is connected.

It is an interesting open problem whether there exists a finite set of axioms, all in standard form, such that condition (2) can be omitted in this characterization. All our axioms below for signpost systems will be of standard form.

For our purposes we need the following additional axioms:

(a4) if $vwSx$, $vwSz$, and $xySz$, then $vwSy$ for $v, w, x, y, z \in V$,

(a5) if $vwSx$ and $xySw$, then $vwSy$ and $xySv$ for $v, w, x, y \in V$,

(a6) if $vwSx$, $w \neq x$, $xySy$, and $\neg vwSy$, then $yxSw$ for $v, w, x, y \in V$,

(a7) there exist exactly two u such that $vuSu$ for $v \in V$.

Axiom $(a4)$ plays an important role in studying signpost systems. As was proved in [26], the step system of every cycle and every median graph satisfies axiom $(a4)$. On the other hand, the graph $K_{2,3}$ does not satisfy this axiom. Axiom $(a5)$ is also important in studying signpost systems: the step system of every connected graph satisfies this axiom. It combines two of the axioms in the above mentioned set $\mathcal{A}$ from [24]. The step system of every cycle, but also of every tree, satisfies axiom $(a6)$. It is obvious that the step system of every cycle satisfies axiom $(a7)$.

For convenience, we introduce the following convention. Let S be a signpost system with underlying graph G, and let $x_0, x_1, \ldots, x_n, y \in V$, with $n \geq 2$. Instead of

$$x_0x_1Sy,\ x_1x_2Sy,\ \ldots\ ,\ x_{n-1}x_nSy$$

we write

$$x_0x_1 \ldots x_nSy.$$

Note that, by the observations above, x_i and x_{i+1} are adjacent in G, for $i = 0, \ldots, n-1$. So we have a *walk* in G on the vertices $x_0, x_1, \ldots, x_n$, which we denote by $(x_0, x_1, \ldots, x_n)$. Other trivial but important consequences of this notation are that

$$x_1 \ldots x_nSy,\ x_0x_1Sy \ \Rightarrow\ x_0x_1 \ldots x_nSy,$$

$$x_0x_1 \ldots x_nSy \ \Rightarrow\ x_i \ldots x_nSy, \quad \text{for } 1 \leq i \leq n-1.$$

As was proved in [28], if S is a signpost system satisfying axiom $(a4)$, then the underlying graph of S is connected. Below we present an alternative proof of this result and some other results from [28] using the techniques and ideas developed in this paper.

Lemma 1 *Let S be a signpost system on V satisfying axiom $(a4)$, and let $x_0, x_1, \ldots,$ $x_m, z \in V$, with $m \geq 2$, such that*

$$x_0x_1 \ldots x_mSz.$$

Then $x_0x_1Sx_k$ for each $k = 1, \ldots, m$.

Proof. We proceed by induction on k. By definition $x_0x_1 \dots x_mSz$ implies that x_0x_1Sz and thus, by axiom $(a1)$, $x_1x_0Sx_0$, and by axiom $(a1)$ again, $x_0x_1Sx_1$. This settles the basis of the induction. Now let $2 \le k \le m$. By the induction hypothesis, $x_0x_1Sx_{k-1}$. Recall that x_0x_1Sz and $x_{k-1}x_kSz$. Axiom $(a4)$ implies that $x_0x_1Sx_k$, which completes the proof. □ □ □

Note that in the following lemma we have numbered the vertices x_i in reverse order to simplify the proof of the lemma.

Lemma 2 *Let S be a signpost system satisfying axiom $(a4)$, and let $x_0, x_1, \dots, x_m, z \in V$, with $m \ge 2$, such that $x_mx_{m-1} \dots x_0Sz$. Then $x_mx_{m-1} \dots x_0Sx_0$.*

Proof. We proceed by induction on m. First let $m = 1$. Then x_1x_0Sz. By applying axiom $(a1)$ twice, we get $x_1x_0Sx_0$. Let $m \ge 2$. By the induction hypothesis, we have $x_{m-1}x_{m-2} \dots x_0Sx_0$. Lemma 1 applied to $x_mx_{m-1} \dots x_0Sz$ gives us $x_mx_{m-1}Sx_0$. Hence $x_mx_{m-1} \dots x_0Sx_0$. □ □ □

Proposition 3 (Cf. Corollary 1 in [28]). *Let S be a signpost system on V satisfying axiom $(a4)$, and let $x_0, x_1, \dots, x_m, z \in V$, with $m \ge 2$, such that $x_0x_1 \dots x_mSz$. Then*

$$(x_0, x_1, \dots, x_m) \tag{2}$$

is a path in the underlying graph of S.

Proof. Let G be the underlying graph of S. As observed above $(x_0, x_1, \dots, x_m)$ is a walk in G. Suppose, to the contrary, that $(x_0, x_1, \dots, x_m)$ is not a path in G. Then there exist i and j, with $0 \le i < j \le m$, such that $x_j = x_i$. Obviously, $x_ix_{i+1} \dots x_jSz$, and thus, by Lemma 2, $x_ix_{i+1} \dots x_jSx_j$. Since $x_j = x_i$, we have $x_ix_{i+1} \dots x_jSx_i$. By definition, this implies that $x_ix_{i+1}Sx_i$. This contradicts the fact that, in $vwSx$, vertex v should be distinct from x, see (1). Thus the lemma is proved. □ □ □

Corollary 4 (Cf. Theorem 3 in [28]). *Let S be a signpost system on V satisfying axiom $(a4)$, and let G be the underlying graph of S. Then G is connected.*

Proof. Consider arbitrary distinct vertices u and v in G. Combining the fact that V is finite together with axiom $(a3)$ and Proposition 3, we see that there exists a u,v-path in G. Thus G is connected. □ □ □

Next we consider signpost systems satisfying axiom $(a5)$.

Lemma 5 *Let S be a signpost system on V satisfying axiom $(a5)$, and let $x_0, x_1, \ldots,$ $x_k \in V$, with $k \geq 1$. If $x_0x_1 \ldots x_kSx_k$, then*

$$x_kx_{k-1} \ldots x_iSx_i \textit{ for each } i = 0, \ldots, k-1. \tag{3}$$

Proof. We proceed by induction on k. First let $k = 1$. This translates $x_0x_1 \ldots x_kSx_k$ into $x_0x_1Sx_1$ Hence, by axiom $(a1)$, we have $x_1x_0Sx_0$, by which the basis of the induction is established.

Let $k \geq 2$. By the induction hypothesis, we have

$$x_{k-1}x_{k-2} \ldots x_iSx_i \text{ for each } i = 0, \ldots, k-2. \tag{4}$$

Next we prove that $x_kx_{k-1}Sx_i$ for each $i = 0, \ldots, k-1$. Put $j = k - i - 1$. This means that we have to prove

$$x_kx_{k-1}Sx_{k-j-1} \text{ for each } j = 0, \ldots, k-1. \tag{5}$$

We prove (5) by induction on j. By definition $x_0x_1 \ldots x_kSx_k$ implies that $x_{k-1}x_kSx_k$. By axiom $(a1)$, we have $x_kx_{k-1}Sx_{k-1}$. This settles (5) for $j = 0$. Let $1 \leq j \leq k-1$. Note that $k - j = k - (j-1) - 1$. By the induction hypothesis, $x_kx_{k-1}Sx_{k-j}$. By definition $x_0x_1 \ldots x_kSx_k$ implies that $x_{k-j-1}x_{k-j}Sx_k$. So now we have $x_{k-j-1}x_{k-j}Sx_k$ and $x_kx_{k-1}Sx_{k-j}$. By axiom $(a5)$ with $v = x_{k-j-1}$, $w = x_{k-j}$, $x = x_k$, and $y = x_{k-1}$, we have $x_kx_{k-1}Sx_{k-j-1}$. Hence (5) is proved.

Combining (4) with $x_kx_{k-1}Sx_i$ (for each $i = 0, \ldots, k-1$. Put $j = k - i - 1$), we have

$$x_kx_{k-1} \ldots x_iSx_i \text{ for each } i = 0, \ldots, k-2.$$

Moreover, we have $x_kx_{k-1}Sx_{k-1}$. This concludes the proof of the lemma. □ □ □

With this Lemma in hand we have a reformulation of Lemma 2 form [23] as a simple corollary.

Corollary 6 (Cf. Lemma 2 in [23]). *Let S be a signpost system on V satisfying axiom $(a5)$, and let $x_0, x_1, \ldots, x_k \in V$, where $k \geq 1$. If $x_0x_1 \ldots x_kSx_k$, then $x_kx_{k-1} \ldots x_0Sx_0$.*

Proof. The corollary follows immediately from Lemma 5. □ □ □

Corollaries 4 and 6 will be used in the next section.

3 The step system of a cycle

In this section an axiomatic characterization of the step system of a cycle is given. First we recall the axiomatic characterization of the step system of a tree proved in [29]. Let$S$ be a signpost system on V. We need the following axioms:

(at1) if $u \neq v$, then there exists at most one w such that $uwSv$, for $u, v \in V$,

(at2) if $uvSv$, then $uvSw$ or $vuSw$, for $u, v, w \in V$.

The axiomatic characterization of the step system of a tree in [29] reads as follows.

Theorem A (Cf. [29]) *Let S be a signpost system, and let G be the underlying graph of S. Then G is a tree and S is the step system of G if and only if S satisfies axioms $(at1)$ and $(at2)$.*

In [24] a related characterization of a tree as a finite groupoid is given. In some sense the "opposite" of a tree is a cycle. So it would be nice to have an axiomatic characterization of the step system of a cycle as well. We prepare the ground by proving some lemmas first.

Lemma 7 *Let S be a signpost system on V satisfying axioms $(a4)$ and $(a7)$, and let G be the underlying graph of S. Then G is a cycle.*

Proof. As follows from axiom $(a7)$, G is a regular graph of degree two. Since S satisfies axiom $(a4)$, Corollary 4 implies that G is connected. Hence, G is a cycle. □ □ □

Let C_p be the cycle with p vertices. Note that $p \geq 3$. In the sequel we assume that t_0, t_1, ..., t_{p-2}, t_{p-1} are the vertices and $\{t_0, t_1\}$, $\{t_1, t_2\}$, ..., $\{t_{p-2}, t_{p-1}\}$, $\{t_{p-1}, t_0\}$ are the edges of C_p. Moreover, we use the following notation:

$$t_p = t_0, t_{p+1} = t_1 \ldots t_{2p-2} = t_{p-2} \text{ and } t_{2p-1} = t_{p-1}.$$

The next two lemmas describe simple properties of shortest paths in cycles in terms of signpost systems. Loosely speaking, if we are in v on a cycle, then there are two ways to get to x on the cycle, either to the left or to the right. If the shortest way is to the left, then all the intermediate steps from v to x are to the left as well. This is what the next lemma is about.

Lemma 8 *Let S be a signpost system on V and let C_p be the underlying graph of S. Let k be an integer with $1 \leq k \leq p-1$. If $t_0t_1St_k$, then $t_0t_1 \ldots t_kSt_k$.*

Proof. The case when $k = 1$ is trivial. So let $k \geq 2$. We will prove that $t_i t_{i+1} S t_k$, for every i with $i = 0, \ldots, k-1$. We proceed by induction on i. The case $i = 0$ is the assumption in the lemma. Let $1 \leq i \leq k-1$. By the induction hypothesis, we have $t_{i-1} t_i S t_k$. By axiom $(a3)$ there exists an x in V such that $t_i x S t_k$. Clearly, $\{t_i, x\}$ is an edge of the cycle. Hence $x = t_{i-1}$ or $x = t_{i+1}$. But the case $t_i t_{i-1} S t_k$ contradicts axiom $(a2)$. Hence, $t_i t_{i+1} S t_k$, which completes the proof. $\square$ $\square$ $\square$

Take a shortest v, x-path P on the cycle. If we shift the whole path l steps clockwise along the cycle, then it remains a shortest path between its ends. This is what the next lemma is about.

Lemma 9 *Let S be a signpost system satisfying axioms $(a5)$ and $(a6)$, and let C_p be the underlying graph of S. Let k be an integer with $1 \leq k \leq p-1$. If $t_0 t_1 S t_k$, then*

$$t_l t_{1+l} \ldots t_{k+l} S t_{k+l} \text{ and } t_{k+l} t_{k+l-1} \ldots t_l S t_l. \tag{6}$$

for $l = 0, 1, \ldots, p-2, p-1$.

Proof. Since C_p is the underlying graph of S, it follows from the definition of underlying graph that $t_l t_{1+l} S t_{l+1}$ as well as $t_{1+l} t_l S t_l$, for all l. This settles the case $k = 1$.

So let $k \geq 2$. We proceed by induction by l. First let $l = 0$. By the assumption in the lemma, we have $t_0 t_1 S t_k$. Lemma 8 implies that $t_0 t_1 \ldots t_k S t_k$. By Corollary 6, we have $t_k t_{k-1} \ldots t_0 S t_0$. Thus (6) is proved for $l = 0$.

Let now $1 \leq l \leq p-1$. By the induction hypothesis, we have

$$t_{l-1} t_l \ldots t_{k+l-1} S t_{k+l-1} \text{ and } t_{k+l-1} t_{k+l-2} \ldots t_{l-1} S t_{l-1}.$$

We distinguish two cases.

Case 1. $t_{l-1} t_l S t_{k+l}$.
Now Lemma 8 implies that $t_{l-1} t_l \ldots t_{k+l} S t_{k+l}$. So we also have $t_l t_{l+1} \ldots t_{k+l} S t_{k+l}$. By Corollary 6, we have $t_{k+l} t_{k+l-1} \ldots t_l S t_l$. Hence (6) follows.

Case 2. $\neg t_{l-1} t_l S t_{k+l}$.
Since $1 \leq l \leq p-1$, we have $t_{k+l-1} \neq t_l$. Since $\{t_{k+l-1}, t_{k+l}\}$ is an edge, we have $t_{k+l-1} t_{k+l} S t_{k+l}$. The induction hypothesis states that $t_{l-1} t_l \ldots t_{k+l-1} S t_{k+l-1}$, in particular, we have $t_{l-1} t_l S t_{k+l-1}$.

Recapitulating, we have $t_{l-1} t_l S t_{k+l-1}$, $t_{k+l-1} \neq t_l$, $t_{k+l-1} t_{k+l} S t_{k+l}$, $\neg t_{l-1} t_l S t_{k+l}$. Then axiom $(a6)$, with $v = t_{l-1}$, $w = t_l$, $x = t_{k+l-1}$, and $y = t_{k+l}$, implies that $t_{k+l} t_{k+l-1} S t_l$.

By induction, we have $t_{l-1} t_l \ldots t_{k+l-1} S t_{k+l-1}$, in particular we have $t_l \ldots t_{k+l-1} S t_{k+l-1}$. Hence, by Corollary 6, we have $t_{k+l-1} t_{k+l-2} \ldots t_l S t_l$. Together with $t_{k+l} t_{k+l-1} S t_l$

from the previous paragraph, we get $t_{k+l}t_{k+l-1}\dots t_lSt_l$. By Corollary 6, we have $t_lt_{1+l}\dots t_{k+l}St_{k+l}$. Hence (6) follows, by which the lemma is proved. □ □ □

We carry our study one step further.

Proposition 10 *Let S be a signpost system on V such that its underlying graph G is a cycle. Then S is the step system of G if and only if S satisfies axioms $(a4)$, $(a5)$, $(a6)$, and $(a7)$.*

Proof. It is straightforward to verify that, since G is a cycle and S is the step system of G, that S satisfies axioms $(a4)$, $(a5)$, $(a6)$, $(a7)$.

Conversely, let S satisfy axioms $(a4)$, $(a5)$, $(a6)$, $(a7)$. Without loss of generality we may assume that $G = C_p$. Let S^+ denote the step system of G. We wish to prove that $S = S^+$. Assume to the contrary that $S \neq S^+$. We distinguish two cases.

Case 1. There exist x, y, z in V with $xySz$ and $\neg xyS^+z$.
Without loss of generality we assume that $x = t_0$, $y = t_1$, and $z = t_k$ for some $k \leq p-1$. Thus we have $t_0t_1St_k$. Then it follows from Lemma 9 with $l = p-1$ that,

$$t_{p-k}t_{p-k+1}\dots t_pSt_p.$$

Moreover, by Lemma 8, we have $t_0t_1\dots t_kSt_k$ and, by Corollary 6, we have $t_kt_{k-1}\dots t_0St_0$. Since $\neg t_0t_1S^+t_k$, and S^+ is the step system of C_p, it follows that the path from t_0 to t_k via t_1 along the cycle is not the shortest path from t_0 to t_k. This means that we have $k > p-k$. So from $t_kt_{k-1}\dots t_0St_0$ we deduce

$$t_{p-k+1}t_{p-k}St_0.$$

Recall that $t_p = t_0$. So from $t_{p-k}t_{p-k+1}\dots t_pSt_p$ we deduce that $t_{p-k}t_{p-k+1}St_0$. According to axiom $(a2)$ we cannot have $t_{p-k+1}t_{p-k}St_0$ and $t_{p-k}t_{p-k+1}St_0$ at the same time. This impossibility settles Case 1.

Case 2. If $xySz$, then xyS^+z, for x, y, z in V.

Since $S \neq S^+$, there exist x, y, z such that xyS^+z and $\neg xySz$. Without loss of generality, we have $\neg t_0t_{p-t}St_k$ and $t_0tp-1S^+t_k$. The last signpost implies that there is a shortest path in G from t_0 to t_k via t_{p-1}, so that $k \geq \frac{1}{2}p$. Axiom $(a3)$ implies the existence of a vertex u with t_0uSt_k. Since $\neg t_0t_{p-1}St_k$, it follows that $u = t_1$. But $t_0t_1St_k$ implies that $t_0t_1S^+t_k$. So also the path from t_0 to t_k via t_1 is a geodesic, so that $k \leq \frac{1}{2}p$. Hence p must be even and $k = \frac{1}{2}p$. Putting $l = k$ in Lemma 9 we get $t_{2k}t_{2k-1}\dots t_kSt_k$. In particular we have $t_{2k}t_{2k-1}t_kSt_k$, which also reads as $t_pt_{p-1}\dots t_kSt_k$. Recall that $t_o = t_p$. So we have $t_0t_{p-1}\dots t_kSt_k$. But this is impossible, since $\neg t_0t_{p-t}St_k$. This settles Case 2, and completes the proof. □ □ □

All we did in this section culminates in the main result of this section.

Theorem 11 *Let S be a signpost system on V, and let G be the underlying graph of S. Then G is a cycle and S is the step system of G if and only if S satisfies axioms $(a4)$, $(a5)$, $(a6)$, and $(a7)$.*

Proof. As above it is straightforward to verify that, if G is a cycle and S is the step system of G, then S satisfies axioms $(a4)$, $(a5)$, $(a6)$ and $(a7)$.

Conversely, let S satisfy the four axioms. By Lemma 8, G is a cycle. By Proposition 10, S is the step system of G. □ □ □

4 Guides to graphs and shortcuts

In this section we consider a connected graph $G = (V, E)$ and a connected spanning subgraph $F = (V, E_F)$ of G. We call F a *factor* of G. So a factor of G contains all vertices of G but not necessarily all edges. Clearly, we have $d_G(u, v) \leq d_F(u, v)$, for any u, v in V. We want to study the geodesic structure of F with respect to G. More specifically, if P a u, v-geodesic in F, is it possible to find a shorter u, v-path Q in G vertex-contained in P? By vertex-contained we mean here that the vertex-sequence of Q is a subsequence of the vertex-sequence of P. Note that Q may use edges that are not in P. To help our thinking we call such a path Q in G a *shortcut*, and the u, v-geodesic P in F a *detour* in G. Note that in the literature a u, v-detour was already introduced: as a *longest* u, v-path, see [8]. But we think that our usage of the term here is justified. Moreover, we use it here only as a way of visualizing the concepts of shortcut and guide. Clearly, for any edge $\{u, v\}$ in F, there is no shorter path between u and v in G. Therefore, we call such an edge a *strong edge* in G. On the other hand, for any edge $\{u, v\}$ in $G - F$, there exists a detour between u and v in F of length at least two, by the connectedness of F. So we could delete $\{u, v\}$ from G without affecting connectedness. Therefore we call such an edge a *weak edge* of G.

Now we make the definition of shortcut more precise. Let G be a graph and let F be a connected factor of G. We say that a path $(x_0, x_1, \ldots, x_k)$ in G is an (F, G)-*shortcut* if there exists a geodesic

$$(y_0, y_1, \ldots, y_l)$$

in F such that there are integers $i_0, i_1, \ldots, i_k$ with the property that $0 = i_0 < i_1 < \ldots < i_k = l$ and $x_j = y_{i_j}$ for every $j = 0, ...k$. Under condition that T is a spanning tree of a graph G, the (T, G)-shortcuts were studied in [9], but the terminology was different there: (T, G)-shortcuts were called T-paths in [9].

Note that a geodesic in G need not be a shortcut for F, see the following example.

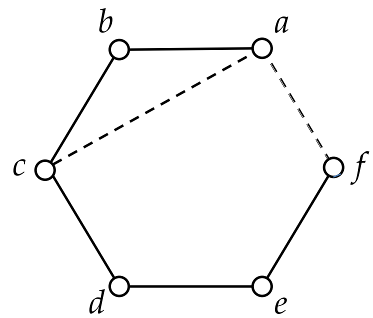

Figure 2: Example 2

Example 2. Let $G = (V, E)$ be the graph of order 6 with the vertex set $V = \{a, b, c, d, e, f\}$ as in Figure 2. The graph has eight edges, two of which are drawn as dashed lines. Let F be the spanning subgraph consisting of the path (a, b, c, d, e, f). So $E - E_F$ consists of the two edges $\{a, c\}$ and $\{a, f\}$, the ones drawn as dashed lines. There exist exactly two (F, G)-shortcuts connecting a and e, namely (a, b, c, d, e) and (a, c, d, e), and there exists exactly one a, e-geodesic in G, namely (a, f, e).

In the sequel we want to translate these notions into the language of signpost systems. The reader is asked to bear this in mind when we introduce terminology for signpost systems below. The signpost systems below are different from step systems. In a step system S with underlying graph G, the signpost $vwSz$ signifies that the edge $\{v, w\}$ brings us from v one step closer to x. So it reflects only the geodesic structure of G. To get a grip on the difference between G and a connected factor F of G, a signpost $vwSx$ now still means that $\{v, w\}$ is an edge in G, but if $\{v, w\}$ is not in F, then $vwSx$ still tells us that we get closer to x in F as follows: w is on a v, x-geodesic in F. This is made more precise as follows. Let $G = (V, E)$ be a connected graph, and let F be a connected factor of G. Let d_F denote the distance function of F. The *guide* to (F, G) is ternary system S on V defined as follows:

$$vwSx \text{ if and only if } vw \in E \text{ and } d_F(v, w) + d_F(w, x) = d_F(v, x),$$

for v, w, x in V. For the special case when F is a spanning tree of G, the concept of the guide to (F, G) was introduced in [29]. It is easy to see that the guide to (F, G) satisfies axioms $(a1)$, $(a2)$ and $(a3)$, so it is a signpost system. Our goal is to characterize signpost systems that are guides to particular factors by signpost axioms. So we concentrate now on signpost systems.

First we present one other axiom for signpost systems S on V.

(a0) if $vwSw$ and there exists an $x \neq w$ such that $vxSw$, then there exists a $y \neq v$ such that $wySv$, for $v, w \in V$.

In terms of a signposting, axiom $(a0)$ reads as follows: $vwSw$ means that there is a direct connection between v and w. Now $vxSw$ with $x \neq w$ means that we can also get from v to w via a 'detour' through x. If such a detour exists, then it should also be possible to get back from w to v via a detour, say through y. Below we present an example that there need not always exist such a detour back. But first we introduce some terminology in connection with existence of non-existence of such detours.

Let S be a signpost system on V, and let G denote its underlying graph of S. We say that α is an *edge* of S if and only if α is an edge of G. Let S be a signpost system, and $\alpha = \{v, w\}$ be an edge of S. We say that α is *strong* if $\neg vxSw$ for every $x \in V$, $x \neq w$, and $\neg wySv$ for every $y \in V(S)$, $y \neq v$. So in terms of detours: there are no detours back and forth between the ends v and w of the edge. We say that α is *weak* if there exist vertices x and y of S such that $x \neq w$, $y \neq v$, $vxSw$, and $wySv$. So in terms of detours: there is a detour back *and* forth between the ends of the edge. Then axiom $(a0)$ translates into: any edge of s is strong or weak. We call a signpost system *balanced* if it satisfies axiom $(a0)$. The next example shows that $(a0)$ need not always be fulfilled.

Example 3. Consider the ternary system S on $V = \{a, b, c\}$ with $xySz$ if and only if $x = a, y = z = b$; or $x = b, y = z = a$; or $x = b, y = z = c$; or $x = c, y = z = b$; or $x = a, y = z = c$; or $x = c, y = x = a$; or $x = a, y = b, z = c$. It is clear that S is a signpost system and edge $\{a, b\}$ of S is neither weak nor strong. So S is not balanced.

In the sequel we also need the next two axioms.

(a8) if $vwSx$ and $vySw$, then $vySx$ for $u, v, w, x \in V$,

(a9) if $vwSw$, $vxSw$, $wySv$, $x \neq w$, and $y \neq v$, then ($vxSz$ and $ywSz$ if and only if $vwSz$), for $v, w, x, y, z \in V$.

Axiom $(a8)$ reads as follows: if there is a signpost at v directing us to w to get to x, then any detour from v to w will still get us to x. Axiom $(a9)$ reads as follows: if there is a direct connection between v and w, and there is a detour from v to w via x and one from w to v via y. Note that axioms $(a0)$, $(a8)$ and $(a9)$ are standard form.

If S is a balanced signpost system, then we denote by $W(S)$ the set of all weak edges in S. The characterizations of guides S to (F, G), for specific instances of F, will use induction on the number of weak edges. To establish the induction we need the next lemma. Note that axiom $(a9)$ does not appear anywhere in the proof. But this lemma is needed for the proof of Theorems 14 and 15. There axiom $(a9)$ is an essential part of the characterization. Therefore we include it here in the lemma.

Lemma 12 *Let S be a balanced signpost system on V satisfying axioms $(a8)$, and $(a9)$, and let $\{u_0, u_1\}$ be a weak edge of S. Let S^- be the ternary system on V with*

$$z_1 z_2 S^- z_3 \text{ if and only if } z_1 z_2 S z_3 \text{ and } \{z_1, z_2\} \neq \{u_0 1, u_1\}, \text{ for } z_1, z_2, z_3 \in V.$$

Then S^- is a balanced signpost system satisfying axioms $(a8)$, and $(a9)$.

Proof. Obviously, S^- satisfies axioms $(a1)$ and $(a2)$. Consider $v, x \in V$ with $v \neq x$. By axiom $(a3)$ for S, there exists a w in V such that $vwSx$. If $\{v, w\} \neq \{u_0, u_1\}$, then, by the definition of S^-, we have vwS^-x. Assume that $\{v, w\} = \{u_0, u_1\}$. Then $\{v, w\}$ is a weak edge in S. By definition, there exists a $y \in V$, with $y \neq v$ such that $vySw$. Since $vwSx$, axiom $(a8)$ for S implies that $vySx$. Obviously, $\{v, y\} \neq \{u_0, u_1\}$. Hence vyS^-x. Thus S^- satisfies axiom $(a3)$, and S^- is a signpost system.

Consider v, w, x in V such that vwS^-w, $x \neq w$, and vxS^-w. Then $vwSw$ and $vxSw$. Since S is a balanced signpost system, there exists a y in V such that $y \neq v$ and $wySv$. If $\{w, y\} \neq \{u_0, u_1\}$, then wyS^-v. Assume that $\{w, y\} = \{u_0, u_1\}$. Then $\{w, y\}$ is a weak edge in S and thus there exist z in V, $z \neq y$, such that $wzSy$. By axiom $(a8)$ for S, we have $wzSv$. Since $\{w, z\} \neq \{u_0, u_1\}$, we have wzS^-y. Hence S^- is a balanced signpost system.

It is easy to see that S^- satisfies axioms $(a8)$, and $(a9)$. $\square\ \square\ \square$

Let F be a connected factor of a graph $G = (V, E)$, and let S denote the guide to (F, G). Consider $x_0, x_1, \ldots x_n \in V$. It is easy to show that $x_0x_1 \ldots x_nSx_n$ if and only if $(x_0, x_1, \ldots, x_n)$ is an (F, G)-shortcut.

Proposition 13 *Let G be a graph, and let F be connected factor of G, and let S denote the guide to (F, G). Then S is a balanced signpost system and the set of strong edges of S equals to the set of edges of F.*

Proof. As observed above, F being a connected factor, it follows straightforward that S is a signpost system. Obviously, every edge of F is a strong edge of S. So we only have to verify that S satisfies $(a0)$.

Let v and w be vertices such that $vwSw$, so $\{v, w\}$ is an edge in G. If this is an edge in F, then there is no vertex x distinct from v, w such that $d_F(v, x) + d_F(x, w) = d_F(v, w) = 1$. So $(a0)$ is satisfied. If $\{v, w\}$ is not in F, then let $R = (v, x, \ldots, y, w)$ be a detour in F. Now we have $vxSw$. But we also have $wySv$. So again $(a0)$ is satisfied. $\square\ \square\ \square$

5 Tree Guides

Let S be a balanced signpost system. We say that S is a *tree guide* if S is the guide to (T, G), where G is a connected graph and T is a spanning tree of G. An axiomatic characterization of a tree guide will be given. We need two more axioms for signpost systems S on V. It is easy to see that these axioms are 'tree-like'.

(at1s) if $u \neq v$, then there exists at most one w such that $\{u, w\}$ is a strong edge of S and $uwSv$, for $u, v \in V$,

(at2s) if $\{u, v\}$ is a strong edge of S, then $uvSw$ or $vuSw$, for $u, v, w \in V$.

In the characterization of tree guides both axioms $(a8)$ and $(a9)$ are needed. Note that in the proof below axiom $(a8)$ occurs only hidden: in the induction hypothesis Lemma 12 is needed, and in the proof of this lemma axiom $(a8)$ played an essential role.

Theorem 14 *Let S be a balanced signpost system. Then the following statements are equivalent:*

- (i) *S is a tree guide,*
- (ii) *S satisfies axioms $(at1s)$, $(at2s)$, $(a8)$, and $(a9)$.*

Proof. First we prove that (i) implies (ii). Let S be a tree guide. Then there exist a connected graph G and a spanning tree T of G such that S is the guide to (T, G). By Proposition 13, the set of strong edges of S equals to the set of edges of T. Hence it is obvious that S satisfies of axioms $(at1s)$, $(at2s)$ and $(a8)$.

Note that, since T is a tree, we have $pqSr$ if and only if $\{p, q\}$ is an edge in G and q lies on the unique p, r-path in T. To prove that S satisfies $(a9)$, let v, w, x, y, z be vertices in V with $vwSw$, $vxSw$ and $wySv$ with $x \neq w$ and $y \neq v$. Now $\{v, w\}$, $\{v, x\}$ and $\{w, y\}$ are edges in G. Moreover, $vxSw$ implies that x lies on the unique v, w-path P in T. Since $x \neq w$, the edge $\{v, w\}$ is not in T. The signpost $wySv$ implies that y also lies on P. So P begins in v, passes through x and y and ends in w. Note that we may have $x = y$ or x before y or x after y on P. We have to prove that $vxSz$ and $ywSz$ if and only if $vwSz$. Let Q be the unique w, z-path in T. Both signposts $ywSz$ and $vwSz$ imply that P and Q have w as only vertex in common. From this we deduce that axiom $(a9)$ holds.

We now prove that (ii) implies (i). So let S be a signpost system on V satisfying axioms $(at1s)$, $(at2s)$, $(a8)$, and $(a9)$. We denote by G the underlying graph of S. We want to prove that there exists a spanning tree T of G such that S is the guide to (T, G). We proceed by induction on $|W_S|$.

For the basis of the induction let $|W(S)| = 0$. Then every edge in S is strong and therefore S satisfies axioms $(at1)$ and $(at2)$. Theorem A implies that G is a tree and S is the step system of G. Put $G = T$. It is obvious that S is the guide to (T, G).

Now assume that $|W(S)| > 0$. Consider a weak edge $\{v, w\}$ in S. First note that, $\{v, w\}$ being an edge in G, we have $vwSw$. We denote by S^- the ternary system on V defined as follows:

$$z_1 z_2 S^- z_3 \text{ if and only if } z_1 z_2 S z_3 \text{ and } \{z_1, z_2\} \neq \{v, w\}, \text{ for } z_1, z_2, z_3 \in V.$$

By Lemma 12, S^- is a balanced signpost system satisfying axioms ($a8$), and ($a9$). (Recall that here axiom ($a8$) has been used.) Moreover, it is easy to verify that S^- satisfies axioms ($at1s$) and ($at2s$). We denote by G^- the graph obtained from G by deleting edge $\{v, w\}$. Obviously, $|W(S^-)| = |W(S)| - 1$. By the induction hypothesis, S^- is the guide to (T, G^-). Let d denote the distance function of T. Let P be the v, w-geodesic in T, so that the length of P is $d(v, w)$. Since $\{v, w\}$ is a weak edge, we have $d(v, w) \geq 2$. Let x be the neighbor of v on P and let y be the neighbor of w on P. Note that $x \neq w$ and $y \neq v$. Then we have $d(x, w) = d(v, w) - 1$ and $d(y, v) = d(w, v) - 1$, hence $d(v, w) = d(v, x) + d(x, w)$ and $d(w, v) = d(w, y) + d(y, v)$. Since S^- is the guide to (T, G^-), it follows that vxS^-w and wyS^-, whence $vxSw$ and $wySv$. Thus we have found vertices v, w, x, y that satisfy the "if"-condition in axiom ($a9$).

Choose any z in V. We have to show that $vwSz$ if and only if $d(v, w) + d(w, z) = d(v, z)$. Let Q be the w, z-geodesic in T.

First assume that $d(v, w) + d(w, z) = d(v, z)$. Then the concatenation $P \to Q$ of P and Q is the v, z-geodesic in T, and x is the neighbor of v on $P \to Q$. So vxS^-z holds. Moreover y followed by Q is the y, z-geodesic in T. So also ywS^-z holds. Hence we have also $vxSz$ and $ywSz$. By axiom ($a9$) we have $vwSz$.

Conversely assume that $vwSz$. By axiom ($a9$) for S, we have $vxSz$ and $ywSz$. Then, by the definition of S^-, we have vxS^-z and ywS^-z. The signpost ywS^-z means that in T we have $d(y, z) = 1 + d(w, z)$. So y followed by Q actually is the y, z-geodesic in T. Since P is the v, w-geodesic in T and P contains y, it follows from the properties of a tree that $P \to Q$ is the v, z-geodesic in T. So we have $d(v, w) + d(w, z) = d(v, z)$ in T. Thus we have proved that S is the guide to (T, G). $\square$ $\square$ $\square$

In a different form tree guides already appeared in [29, 9]. An earlier characterization of a tree guide can be found in [29]. In our eyes the above characterization is much more intuitively appealing than the characterization in [29].

6 Hamiltonian guides

Let S be a balanced signpost system. We say that S is a *hamiltonian guide* if S is the guide to (C, G), where G is a graph and C is a hamiltonian cycle of G. In this section, we present an axiomatic characterization of hamiltonian guides.

Let $G = (V, E)$ be a connected graph, and let C be a hamiltonian cycle of G. Let S be the guide to (C, G) Without loss of generality $C = C_p$, for some p. Recall the notation for C_p from Section **??**. Obviously, there exists some n such that $p = 2n$ or $p = 2n + 1$. Let $0 < j \leq n$. Note that, if $\{t_0, t_j\}$ is an edge in G, then $t_0 t_j S t_k$ if and only if $j \leq k \leq n$.

As could be expected, for hamiltonian guides we need axioms involving strong edges that are very similar to $(a6)$ and $(a7)$, the axioms crucial in characterizing the step system of a cycle. These axioms are:

(a6s) if $vwSx$, $w \neq x$, $\neg vwSy$, and $\{x, y\}$ is a strong edge of S, then $yxSw$, for $v, w, x, y \in V$,

(a7s) there exists exactly two u such that $\{v, u\}$ is a strong edge of S, for $v \in V$.

Again each of axioms $(a6s)$ and $(a7s)$ is in standard from.

Recall that, by Lemma 13, a hamiltonian guide is a balanced.

Theorem 15 *Let S be a balanced signpost system on V. Then the following statements are equivalent:*

(i) *S is a hamiltonian guide,*

(ii) *S satisfies axioms* $(a4)$, $(a5)$, $(a6s)$, $(a7s)$, $(a8)$, *and* $(a9)$.

Proof. We first prove that (i) implies (ii). Let S be a hamiltonian guide. So there exist a graph $G = (V, E)$ and a hamiltonian cycle C of G such that S is the guide to (C, G). Without loss of generality we may assume that $C = C_p$. Set $n = \lfloor \frac{1}{2} p \rfloor$. In the axioms the variables v, w, x, y, z are used. Without loss of generality we may assume that $v = t_0$, and $w = t_j$ with $1 \leq j \leq n$. Below we use the following additional notation: $x = t_k$, $y = t_l$, $z = t_m$.

$(a4)$: Assume that $t_0t_jSt_k, t_0t_kSt_m$, and $t_kt_lSt_m$. Then $k < l \leq m \leq n$ and so $t_0t_jSt_l$.

$(a5)$: Let $t_0t_jSt_k$ and $t_kt_lSt_j$. Then $0 < j \leq k \leq n$ and $k > l \leq j$. Thus we have $t_0t_jSt_l$ and $t_kt_lSt_0$.

$(a6s)$: Let $t_0t_jSt_k$, $j \neq k$, $\neg t_0t_jSt_l$, and $|k - l| = 1$. Then $l = k + 1$. Thus we have $t_lt_kSt_j$.

$(a7s)$: Obvious.

$(a8)$: Let $t_0t_jSt_k$ and $t_0t_lSt_j$. Then $0 < j \leq k \leq n$ and $0 < l \leq j$. Thus we have $t_0t_lSt_k$.

$(a9)$: Let $t_0t_jSt_j$, $t_0t_kSt_j$, and $t_jt_lSt_0$, $k \neq j$, $l \neq 0$.

Recall that $1 \leq j \leq n$. Then $t_0t_kSt_j$ implies that $1 \leq k \leq j$. With $k \neq j$, we get $1 \leq k < j$. Also, $t_jt_lSt_0$ implies that $0 \leq l \leq j$. With $l \not 0$ we get $0 < l \leq j$.

First assume that $t_0t_kSt_m$ and $t_lt_jSt_m$. Since $0 \leq l \leq j$, the signpost $t_lt_jSt_m$ implies that $0 < l \leq j \leq m$. So we have $t_0t_jSt_m$.

Next assume that $t_0t_jSt_m$. Then we have $0 < j \leq m$. With $1 \leq k < j$ we get $0 < k < m$, so $t_0t_kSt_m$. Moreover, with $0 < l \leq j$ we get $l < j \leq m$, so $t_lt_lSt_m$. This settles axiom $(a9)$, by which (ii) is proved.

Next we now prove that (ii) implies (i). Let $G = (V, E)$ be the underlying graph of S. We want to prove that there exists a spanning cycle C of G such that S is the guide to (C, G). We proceed by induction on $|W(S)|$.

For the basis of the induction let $|W(S)| = 0$. Then every edge in S is strong and therefore S satisfies axioms $(a6)$. and $(a7)$. By assumption S satisfies axioms $(a4)$, and $(a5)$. So, from Theorem 11 it follows that G is a cycle and S is the step system of G. Put $G = C$. It is obvious that S is the guide to (C, G).

Now assume that $|W(S)| > 0$. Consider a weak edge $\{v, w\}$ in S. We denote by S^- the ternary system on V defined as follows:

$$z_1 z_2 S^- z_3 \text{ if and only if } z_1 z_2 S z_3 \text{ and } \{z_1, z_2\} \neq \{v, w\}, \text{ for } z_1, z_2, z_3 \in V.$$

By Theorem 12, S^- is a balanced signpost system satisfying axioms $(a8)$, and $(a9)$. Moreover, it is easy to verify that S^- satisfies axioms $(a4)$, $(a5)$, $(a6s)$, and $(a7s)$. We denote by G^- the graph obtained from G by deleting edge $\{u_0, u_1\}$. Note that deleting a weak edge from G results in a connected graph. Obviously, $|W(S^-)| = |W(S)| - 1$. By the induction hypothesis, there exists a hamiltonian cycle C in G such that S^- is the guide to (C, G^-). Let d denote the distance function of C.

Recall that $\{v, w\} \in E$. We need to prove that

$$vwSx \text{ if and only if } d(v, w) + d(w, x) = d(v, x), \text{ for } x \in V. \tag{7}$$

Let $C = C_p$, for some p. Set $n = \lfloor \frac{1}{2} p \rfloor$. Without loss of generality we assume that $v = t_0$ and $w = t_k$ with $2 \leq k \leq n$. Note that $d(v, w) = k$. Since $\{t_0, t_1\}$ and $\{t_k, t_{k-1}\}$ are strong edges of S, we have $t_0 t_1 S t_k$ and $t_k t_{k-1} S t_1$. Put $x = t_m$ with $0 \leq m \leq n$. By axiom $(a9)$, we have $t_0 t_k S t_m$ if and only if $t_0 t_1 S t_m$ and $t_{k-1} t_k S t_m$. Hence $t_0 t_k S t_m$ is a signpost of S. So we get $1 < k \leq m \leq n$. This implies that $d(w, x) = k - m$ and $d(v, x) = m$. Thus (7) is proved. $\Box$ $\Box$ $\Box$

7 Concluding remarks

The geodesic structure of a connected graph appears to be surprisingly rich. To capture all subtleties we need many different ways to describe and study this geodesic structure. Various approaches exist in the literature, for instance the interval function I, see e.g. [16, 19], betweenness structures, see e.g. [31, 32, 12, 30, 15], ternary algebras, see e.g. [2, 3, 4, 6, 16, 18, 21, 22, 26, 28, 29]. Each of these approaches provides us with a concept that can be defined independently of graphs. In most cases an underlying graph can be defined in a natural way, thus the connection with a geodesic structure is restored, see e.g. [2, 6, 16, 18]. It is our believe that all these different approaches are in some sense necessary to capture all aspects of this rich geodesic structure, that even more, new approaches may be needed to capture new, not yet discovered aspects.

Above we have focussed on signpost systems and the special case of step systems. A signpost system is useful also to study the geodesic structure of a spanning subgraph with respect to its host graph. This was initiated in [29] and carried further above by studying guides with respect to spanning trees and hamiltonian cycles.

The striking result in [25] was our motivation to restrict ourselves to axioms in standard form. A main open problem is the characterization of connected graphs by step systems or signpost systems using axioms in standard form only.

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