Note on “An efficient approach for solving the lot-sizing problem with time-varying storage capacities”

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Abstract

In a recent paper Gutiérrez et al. (2008) show that the lot-sizing problem with inventory bounds can be solved in $\mathcal{O}(T \log T)$ time. In this note we show that their algorithm does not lead to an optimal solution in general.

Keywords: Inventory, Lot-sizing, Inventory bounds

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1 Introduction

The lot-sizing problem with inventory bounds (LSB) is described as follows. Given the (deterministic) demands for a finite planning horizon of length T, find a production plan at minimal cost such that all demand is satisfied, while the inventory level in each period should be no larger than the storage capacity. Toczylowski (1995) solved this problem in $\mathcal{O}(T^2)$ time. Recently, Gutiérrez et al. (2008) improved the running time to $\mathcal{O}(T \log T)$ time. In this note, we show that the algorithm of Gutiérrez et al. (2008) does not lead to an optimal solution in general.

This note is organized as follows. In Section 2 we formulate the problem. In Section 3 we briefly describe the geometric technique of Wagelmans et al. (1992), which is applied in Gutiérrez et al. (2008). Furthermore, in Section 4 we show why the algorithm of Gutiérrez et al. (2008) does not lead to an optimal solution in general. Finally, this note ends with some concluding remarks in Section 5.

2 Problem description

To describe the LSB, we use the same notation as Gutiérrez et al. (2008). For $t = 1, \ldots, T$ we let

- d_t: demand in period t with $d_{i,j} = \sum_{t=i}^{j} d_t$
- f_t: setup cost in period t
- p_t: unit production cost in period t
- h_t: unit holding cost in period t
- S_t: inventory bound in period t
- y_t: binary setup variable in period t
- x_t: production quantity in period t
- I_t: ending inventory in period t.

Given this notation, the problem is formulated as

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{T} (f_t y_t + p_t x_t + h_t I_t) \\
\text{s.t.} \quad & I_t = I_{t-1} + x_t - d_t \qquad && \text{for } t = 1, \ldots, T \\
& x_t \le d_{t,T} y_t && \text{for } t = 1, \ldots, T
\end{aligned}
$$

$$
\begin{aligned}
I_{t-1} + x_t \leq S_t & \qquad \text{for } t = 1, \ldots, T \\
I_t, x_t \geq 0, y_t \in \{0, 1\} & \qquad \text{for } t = 1, \ldots, T,
\end{aligned}
$$

where we assume that $I_0 = I_T = 0$. Note that the bound S_t is imposed on the starting inventory $I_{t-1} + x_t$ in period t. Using the inventory balance constraint, this constraint can also be written as $I_t \leq S_t - d_t$. For feasibility we need that $S_t \geq d_t$ for $t = 1, \ldots, T$.

3 Geometric technique of Wagelmans et al. (1992)

Because Gutiérrez et al. (2008) apply the geometric technique of Wagelmans et al. (1992), we briefly describe this technique. First, let $F(j)$ be the optimal cost for periods $j, \ldots, T$ in case of no inventory bounds. Using the zero-inventory property (Wagner and Whitin, 1958), the problem can be solved by the recursion

$$
\begin{aligned}
F(j) &= \min_{j < t \leq T+1} \{f_j + c_j d_{j,t-1} + F(t)\} \\
&= f_j + \min_{j < t \leq T+1} \{F(t) + c_j d_{j,t-1}\} \qquad (1)
\end{aligned}
$$

where we made the common substitution $c_j = p_j + \sum_{t=j}^{T} h_t$ and we let $F(T+1) = 0$. Note that the recursion and interpretation of $F(j)$ are only valid if $d_j > 0$ for $j = 1, \ldots, T$, which we assume for ease of exposition. Assume that we want to determine $F(j)$ and that the values $F(t)$ are known for $t = j+1, \ldots, T+1$. Then we can plot the points $(d_{t,T}, F(t))$ for $t = j+1, \ldots, T+1$ and determine the lower envelope of this set of points (see Figure 1). The points that contribute to the lower envelope are called efficient periods, while the non-contributing points are called non-efficient.

Given the lower envelope of these points, Wagelmans et al. (1992) show that the minimum in (1), and hence the value of $F(j)$, can be found in $\mathcal{O}(\log T)$ time by finding the point that is tangent to the line with slope c_j (see Figure 1). Furthermore, they show that adding the new point $(d_{j,T}, F(j))$ and updating the lower envelope takes $\mathcal{O}(T)$ time in the total execution of the algorithm. This means that the overall running time is $\mathcal{O}(T \log T)$.

4 Mistakes in Gutiérrez et al. (2008)

Gutiérrez et al. (2008) apply the technique of Wagelmans et al. (1992) to solve the problem with inventory bounds. To that end, they use the recursive variable $G(j)$, the optimal cost

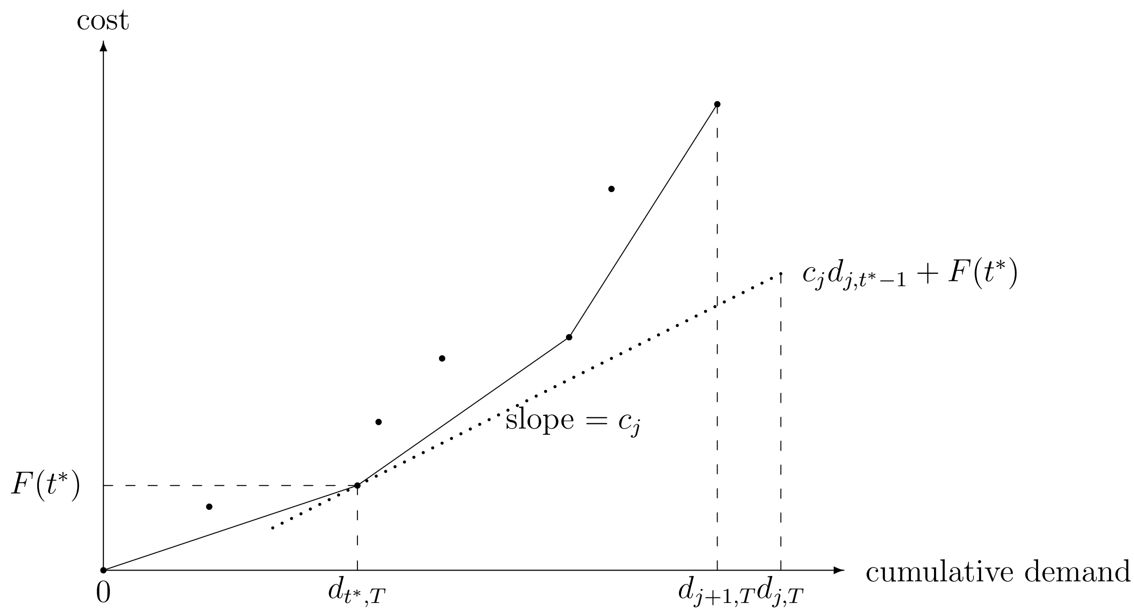

Figure 1: Illustration of the geometric technique of Wagelmans et al. (1992)

to satisfy the demands in periods $j, \ldots, T$. Before giving the recursion formula, we need some more notation. Let $\hat{x}_j$ be the optimal production quantity in period j corresponding to $G(j)$. Furthermore, let M_j be the maximum production quantity in period j, i.e., $M_j = \min\{d_{j,t-1} + S_t : j \leq t \leq T\}$. Finally, let R_j be the largest period that can be completely satisfied by production in period j, i.e., $R_j = \max\{t : j \leq t \leq T, d_{j,t} \leq M_j\}$.

For the problem starting from period j, Love (1973) and Gutiérrez et al. (2008) show that one of the following properties holds in an optimal solution: (i) the production quantity is equal to the sum of an integer number of consecutive demands starting at period j, or (ii) the production quantity is equal to the maximum production quantity M_j. Given these properties the LSB can be solved by the recursion (again we assume that $d_j > 0$ for ease of exposition)

$$\begin{aligned} G(j) &= \min \begin{cases} \min_{j<t\leq R_j+1}\{f_j + c_j d_{j,t-1} + G(t)\} \\ \min_{\{j<t\leq R_j+1 : \hat{x}_t \geq M_j - d_{j,t-1}\}}\{f_j + c_j M_j - c_t(M_j - d_{j,t-1}) + G(t)\} \end{cases} \\ &= f_j + \min \begin{cases} \min_{j<t\leq R_j+1}\{c_j d_{j,t-1} + G(t)\} \\ \min_{\{j<t\leq R_j+1 : \hat{x}_t \geq M_j - d_{j,t-1}\}}\{c_j d_{j,t-1} + G(t) + (c_j - c_t)(M_j - d_{j,t-1})\} \end{cases} \end{aligned} \tag{2}$$

Note that the term $(c_j - c_t)(M_j - d_{j,t-1})$ corresponds to the cost of producing an amount of $M_j - d_{j,t-1}$ in period j instead of in period t. This amount should be lower than the production quantity $\hat{x}_t$ and so the condition $\hat{x}_t \geq M_j - d_{j,t-1}$ is needed for feasibility.

Gutiérrez et al. (2008) now proceed as follows. To calculate $G(j)$, they utilize the lower envelope of the points $(d_{t,T}, G(t))$ for $t = j+1, \ldots, T+1$. Let $q(j)$ be the period corresponding to the point tangent to the line with slope c_j. Note that $q(j)$ is equal to period t^* in Figure 1 and equal to period r in Figure 2. Gutiérrez et al. (2008) use this period to determine the periods in which the minima of (2) are obtained.

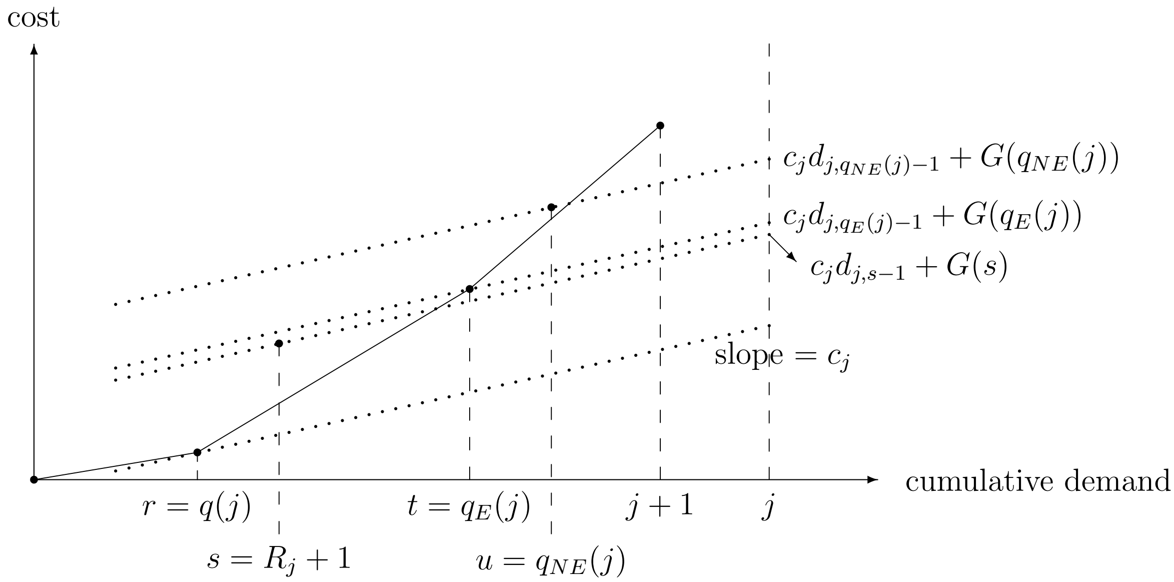

Figure 2: Minimum attained at a non-efficient period (for ease of notation, an index k at the horizontal axis represents the cumulative demand $d_{k,T}$)

We will now show that the algorithm does not necessarily find these minima in case $q(j) > R_j + 1$. Note that in this case it is not feasible to produce $d_{j,q(j)-1}$ units, because of the inventory bounds. Therefore, Gutiérrez et al. (2008) determine the largest efficient and non-efficient periods smaller than or equal to $R_j + 1$ with the smallest slope ratios, denoted by $q_E(j)$ and $q_{NE}(j)$, respectively. The slope ratio of some period t is the slope of the line between the point corresponding to period t and the point corresponding to the successor of period t in the lower envelope (note that a successor of some period is to the left of this period in Figure 2, since time is reversed on the horizontal axis). They claim that the value $G(j)$ is found by restricting the indices j in the minima of (2) to $q_E(j)$ and $q_{NE}(j)$. That is, the value $G(j)$ is found (i) by producing for a consecutive number of periods up to period $q_E(j) - 1$ or up to $q_{NE}(j) - 1$, or (ii) by producing M_j units in period j and to have the next production in period $q_E(j)$ or $q_{NE}(j)$.

The mistake in case (i) is that the period that minimizes the first term in (2) is not necessarily equal to period $q_E(j)$ or $q_{NE}(j)$. Since the feasible indices range from $j+1, \ldots, R_j+1$, the lower envelope of the points $(d_{t,T}, G(t))$ for $t = j+1, \ldots, R_j+1$ should be used (instead of the points $(d_{t,T}, G(t))$ for $t = j+1, \ldots, T+1$) to find the minimum. This means that in every iteration the left part of the lower envelope needs to be updated, which takes additional computation time. It follows from Figure 2 that the optimal period is neither equal to $q_E(j)$ nor to $q_{NE}(j)$ but equal to period s. Note that $q_{NE}(j) = u$ because the slope ratio of period u $((G(u) - G(t))/d_{u,t-1})$ is smaller than the slope ratio of period s $((G(s) - G(r))/d_{s,r-1})$. An issue for case (ii) is that Gutiérrez et al. (2008) do not check in their algorithm whether the condition $\hat{x}_t \geq M_j - d_{j,t-1}$ holds for $t \in \{q_E(j), q_{NE}(j)\}$. Hence, the periods $q_E(j)$ and $q_{NE}(j)$ may correspond to an infeasible solution. Furthermore, this means that the (feasible) periods $q_E(j)$ and $q_{NE}(j)$ cannot be easily found by binary search.

A more fundamental mistake is that Gutiérrez et al. (2008) try to find the minimum of the second term in (2) by utilizing the lower envelope of the points $(d_{t,T}, G(t))$ for $t = j+1, \ldots, T+1$. Because of the term $(c_j - c_t)(M_j - d_{j,t-1})$, the approach of Wagelmans et al. (1992) cannot be applied anymore. If this term is added to every value $G(j)$ at the start of an iteration, then the minimum can still be obtained by the geometric technique. However, this means that each point and the corresponding lower envelope should be recalculated in every iteration, which implies that the order of $\mathcal{O}(T \log T)$ running time cannot be achieved anymore. The following numerical example shows that the lower envelope of the points $(d_{t,T}, G(t))$, $t = j+1, \ldots, T+1$ cannot be used to identify the minimum of the second term in (2) (even if the left part of the lower envelope is updated). This implies that the algorithm of Gutiérrez et al. (2008) fails to find an optimal solution in general.

Example 1 Consider the 3-period problem instance of Table 1. It follows that $R_1 = 2$ and $R_2 = R_3 = 3$, which means that the problem starting from period 2 is uncapacitated. After execution of the algorithm (Gutiérrez et al., 2008, p. 690, Algorithm 1), we get the following values (we only present the most relevant values), where LE denotes the set of efficient periods in the lower envelope:

Initialization:	$G(4) = 0$, $LE = \{4\}$ (we assume that $c_4 = 0$)
Iteration 1:	$q(3) = 4$, $G(3) = 11$, $LE = \{4, 3\}$
Iteration 2:	$q(2) = 4$, $G(2) = 23$, $LE = \{4, 3, 2\}$
Iteration 3:	$q(1) = 4 > 3 = R_1 + 1$, $q_E(1) = 3$ and $c_1 < c_3$, $G(1) = 14$ (obtained from code line 19 in Algorithm 1)

t	1	2	3
d_t	2	2	2
f_t	0	3	7
c_t	1	5	2
$S_t = M_t$	5	4	2

Table 1: Problem instance corresponding to Example 1

The situation after iteration 2 is depicted in Figure 3. The point that is tangent to the line with slope 1 (the dotted lines have slope 1) is period 4 (so $q(1) = 4$). However, this point is not feasible since $4 > 3 = R_1 + 1$. This means that the next point with lowest slope ratio is selected, which is period 3 (so $q_E(1) = 3$). Since this is a feasible period, the solution obtained from the algorithm is: $x_1 = 5$ and $x_3 = 1$ with cost 14 ($= c_1 d_{1,2} + G(3) + (c_1 - c_3)(M_1 - d_{1,2})$). However, this is not the optimal solution. This solution can be found by the geometric technique after adding the terms $(c_1 - c_t)(M_1 - d_{1,t-1})$ to $G(t)$ for $t = 2, 3$, resulting in the lower envelope represented by the thick line in Figure 3. The point that is tangent to the line with slope 1 now corresponds to period 2. Therefore, the optimal solution is: $x_1 = 5$ and $x_2 = 1$ with cost 13 ($= c_1 d_{1,1} + G(2) + (c_1 - c_2)(M_1 - d_{1,1})$). In conclusion, the optimal solution is not found by Algorithm 1 of Gutiérrez et al. (2008).

5 Concluding remarks

The question remains open whether the lot-sizing problem with inventory bounds can be solved in $\mathcal{O}(T \log T)$ time. As follows from this note, to find the minimum in the first term of (2), one needs to find an algorithm that updates the left part of the lower envelope in $\mathcal{O}(T \log T)$ time. Furthermore, it seems that one needs another recursion to find the

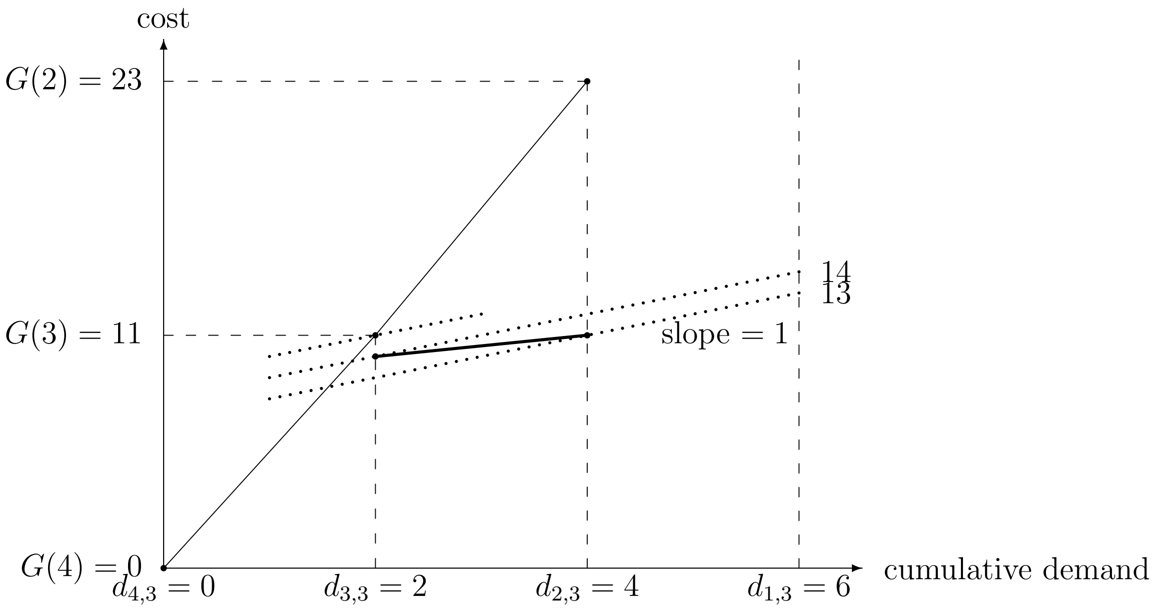

Figure 3: Graphical illustration of Example 1

minimum in the second term of (2) (in $\mathcal{O}(T \log T)$ time), as the current lower envelope does not provide the required information. As far as we know, there is no algorithm that solves the LSB problem in $\mathcal{O}(T \log T)$ time.

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