COMMENT
Risk Aversion and Skewness Preference

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ERIM REPORT SERIES *RESEARCH IN MANAGEMENT*	
ERIM Report Series reference number	ERS-2003-009-F&A
Publication status / version	2003
Number of pages	16
Email address corresponding author	gtpost@few.eur.nl
Address	Erasmus Research Institute of Management (ERIM) Rotterdam School of Management / Faculteit Bedrijfskunde Rotterdam School of Economics / Faculteit Economische Wetenschappen Erasmus Universiteit Rotterdam PoBox 1738 3000 DR Rotterdam, The Netherlands Phone: # 31-(0) 10-408 1182 Fax: # 31-(0) 10-408 9640 Email: info@erim.eur.nl Internet: www.erim.eur.nl

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RESEARCH IN MANAGEMENT

BIBLIOGRAPHIC DATA AND CLASSIFICATIONS		
Abstract	Empirically, co-skewness of asset returns seems to explain a substantial part of the cross-sectional variation of mean return not explained by beta. This finding is typically interpreted in terms of a risk averse representative investor with a cubic utility function. This comment questions this interpretation. We show that the empirical tests fail to impose risk aversion and the implied utility function takes an inverse S-shape. Unfortunately, the first-order conditions are not sufficient to guarantee that the market portfolio is the global maximum for an inverse S-shaped utility function, and our results suggest that the market portfolio is more likely to represent the global minimum than the global maximum. In addition, if we impose risk aversion, then co-skewness has minimal explanatory power.	
Library of Congress Classification (LCC)	5001-6182	Business
	5601-5689	Accountancy, Bookkeeping
	4001-4280.7	Finance Management, Business Finance, Corporation Finance
	HB 141	Econometric Models
Journal of Economic Literature (JEL)	M	Business Administration and Business Economics
	M 41	Accounting
	G 3	Corporate Finance and Governance
	C 19	Econometric and Statistical methods: Other
European Business Schools Library Group (EBSLG)	85 A	Business General
	225 A	Accounting General
	220 A	Financial Management
	220 T	Quantitative methods for financial management
Gemeenschappelijke Onderwerpsontsluiting (GOO)		
Classification GOO	85.00	Bedrijfskunde, Organisatiekunde: algemeen
	85.25	Accounting
	85.30	Financieel management, financiering
	85.33	Beleggingsleer
Keywords GOO	Bedrijfskunde / Bedrijfseconomie	
	Accountancy, financieel management, bedrijfsfinanciering, besliskunde	
	Portfolio-analyse, Risico's, Econometrische modellen	
Free keywords	Asset pricing, Risk aversion, Skewness preference, Representative investor, Three-moment model	

Risk Aversion and Skewness Preference

A Comment

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Risk Aversion and Skewness Preference

A Comment

Empirically, co-skewness of asset returns seems to explain a substantial part of the cross-sectional variation of mean return not explained by beta. This finding is typically interpreted in terms of a risk averse representative investor with a cubic utility function. This comment questions this interpretation. We show that the empirical tests fail to impose risk aversion and the implied utility function takes an inverse S-shape. Unfortunately, the first-order conditions are not sufficient to guarantee that the market portfolio is the global maximum for this utility function, and our results suggest that the market portfolio is more likely to represent the global minimum. In addition, if we impose risk aversion, then co-skewness has minimal explanatory power.

THE TRADITIONAL MEAN-VARIANCE CAPITAL ASSET PRICING MODEL (MV CAPM) predicts an exact linear relationship between mean return and beta, i.e., standardized covariance with the market portfolio. Kraus and Litzenberger (1976), Friend and Westerfield (1980), and Harvey and Siddique (2000), among others, analyze a three-moment capital asset pricing model (3M CAPM) that adds gamma, i.e., standardized coskewness. Interestingly, gamma seems to explain a substantial part of the cross-sectional variation of mean return not explained by beta. Specifically, there is evidence for a substantial gamma premium, i.e., securities that increase market skewness earn low ‘abnormal’ average returns. The estimates for the annualized premium range from 2.5 percent (Kraus and Litzenberger, Friend and Westerfield) to 3.6 percent (Harvey and Siddique).

In our opinion, too little attention has been given to the economic meaning of the gamma premium. The 3M CAPM is typically motivated by a representative investor model with a cubic utility function (or a third-order Taylor series approximation to the true utility function). Specifically, an exact linear relationship between mean, beta and gamma represents the first-order condition for maximizing the expectation of such a utility function. In this comment, we provide theoretical and empirical arguments against this interpretation. Our arguments relate to the regularity condition of risk aversion or concavity for the utility function. We will demonstrate that the empirical tests generally fail to impose this condition. In fact, the estimation results severely violate risk aversion and they imply an inverse S-shaped utility function with risk seeking beyond a return level. If concavity is imposed, then the annualized gamma premium is roughly one half percent, which is only a small fraction of the total market risk premium of roughly six percent. By contrast, if concavity is not imposed, then the gamma premium is about three percent per annum. However, for the inverse S-shaped utility function, the first-order conditions are no longer sufficient for a global maximum. In fact, our empirical results suggest that the market portfolio is more likely to represent the global minimum. These findings lead us to question the theoretical interpretation of the gamma premium.

One motivation for our analysis is Post's (2003) finding that the value-weighted market portfolio is significantly inefficient in terms of second-order stochastic dominance (SSD) relative to benchmark portfolios formed on market capitalization and book-to-market equity ratio. Since the SSD criterion accounts for all concave utility functions, this finding suggests that a concave cubic utility function (3M CAPM) cannot explain what a concave quadratic utility function (MV CAPM) leaves unexplained.

The remainder of this comment is structured as follows. Section I recaptures the representative investor model behind the 3M CAPM. Next, Section II gives our theoretical objections against the interpretation of the gamma premium in terms of this model. Finally, Section III illustrates our objections by means of an empirical application of the 3M CAPM to well-known US stock market data.

I. 3M CAPM

The 3M CAPM is typically motivated by a single-period, portfolio-based, representative investor model that satisfies the following assumptions:

1. The investment universe consists of N risky assets and a riskless asset. The excess returns of the risky assets are denoted by $\boldsymbol{x} \in \Re^N$.[1]

2. The excess returns are random variables with a continuous joint cumulative distribution function (CDF) $G : D^N \to [0,1]$, with $D \subseteq \Re$ for the return domain.

3. The representative investor constructs a portfolio by choosing portfolio weights $l \in \Re^N$, so as to maximize the expectation of a utility function that is defined over portfolio return. We use the (standardized) two-parameter cubic utility function $u(x|q) \equiv x + q_1 x^2 + q_2 x^3$, with $q \equiv (q_1\ q_2)^{\mathrm{T}}$.[2] The utility function is well-behaved only if it satisfies the following three regularity conditions: (RC1) non-satiation, i.e., $u'(x|q) > 0 \quad \forall x \in D$, (RC2) risk aversion, i.e., $u''(x|q) \leq 0 \quad \forall x \in D$, and (RC3) non-increasing absolute risk aversion (NIARA), for which $u'''(x|q) \geq 0 \quad \forall x \in D$ is a necessary condition. We assume throughout the text that condition (RC1) is satisfied, and we analyze the role of conditions (RC2) and (RC3).

Under these assumptions, the representative investor solves the constrained optimization problem $\max_{l \in \Re^N} \int u(\boldsymbol{x}^{\mathrm{T}} l \,|\, q) dG(\boldsymbol{x})$. The value-weighted market portfolio of risky assets, say $t \in \Re_+^N$, must equal the optimal solution to this problem. The well-known Euler equation gives the first-order conditions for optimization:

$$E[u'(\boldsymbol{x}^{\mathrm{T}} t \,|\, q)\boldsymbol{x}] = \int u'(\boldsymbol{x}^{\mathrm{T}} t \,|\, q) x dG(\boldsymbol{x}) = \boldsymbol{0}. \qquad \textbf{(1)}$$

For cubic utility, the Euler equation implies an exact linear relationship between mean, beta and gamma (standardized coskewness):

$$m \equiv E[\boldsymbol{x}], \qquad \textbf{(2)}$$

$$b \equiv \frac{E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)(\boldsymbol{x} - m)]}{E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^2]} \qquad \textbf{(3)}$$

$$g \equiv \frac{E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^2 (\boldsymbol{x} - m)]}{E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^3]}. \qquad \textbf{(4)}$$

Specifically, using $u'(\boldsymbol{x}^{\mathrm{T}} t \,|\, q) = u'(m^{\mathrm{T}} t \,|\, q) + u''(\boldsymbol{x}^{\mathrm{T}} t \,|\, q)(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)$ $+ \frac{1}{2} u'''(\boldsymbol{x}^{\mathrm{T}} t \,|\, q)(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^2$, we may reformulate equation (2) as the following Security Market Plane:

$$m = r_1 b + r_2 g, \qquad \textbf{(5)}$$

with the following risk premiums:

$$r_1 \equiv \frac{-E[u''(\boldsymbol{x}^{\mathrm{T}} t \,|\, q)] E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^2]}{E[u'(\boldsymbol{x}^{\mathrm{T}} t \,|\, q)]} = \frac{-(2q_1 + 6q_2 m^{\mathrm{T}} t) E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^2]}{1 + 2q_1 m^{\mathrm{T}} t + 3q_2 E[(\boldsymbol{x}^{\mathrm{T}} t)^2]}, \qquad \textbf{(6)}$$

$$r_2 \equiv \frac{-\frac{1}{2} E[u'''(\boldsymbol{x}^{\mathrm{T}} t \,|\, q)] E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^2]}{E[u'(\boldsymbol{x}^{\mathrm{T}} t \,|\, q)]} = \frac{-3q_2 E[(\boldsymbol{x}^{\mathrm{T}} t - m^{\mathrm{T}} t)^3]}{1 + 2q_1 m^{\mathrm{T}} t + 3q_2 E[(\boldsymbol{x}^{\mathrm{T}} t)^2]}. \qquad \textbf{(7)}$$

Since the market portfolio has unity beta and unity gamma, the market risk premium equals the sum of the beta premium and the gamma premium, i.e., $m^{\mathrm{T}} t = r_1 + r_2$.

II. Theoretical objections

Empirical 3M CAPM studies typically do not estimate the utility parameters but rather directly estimate the beta and gamma premiums, and they do not report the implied utility parameters. Still, there are compelling theoretical arguments to expect that the implied utility function takes an inverse S-shaped form with risk aversion up to some return level and risk seeking beyond that level:

1. The regularity conditions (RC1), (RC2) and (RC3) are frequently mentioned as desirable properties in empirical 3M CAPM studies (see, e.g., Kraus and Litzenberger (1976), p.1086). However, the studies do not actually impose or test these conditions! The studies typically use two simple restrictions. First, the beta premium r_1 should be non-negative. Using (6), this restriction can be reformulated in terms of the utility parameters. Since $E[(\boldsymbol{x}^{\mathrm{T}}t - m^{\mathrm{T}}t)^2] > 0$ and $E[u'(\boldsymbol{x}^{\mathrm{T}}t|q)] > 0$ (recall (RC1)), we find that $r_1 \geq 0$ if and only if

$$2q_1 + 6q_2 m^{\mathrm{T}}t \leq 0. \quad \textbf{(8)}$$

   Second, securities that increase market skewness should earn a non-positive premium. Assuming that the market portfolio is negatively skewed (as is true in our analysis), this means that the gamma premium r_2 should be non-negative or, using (7) and (RC1),

$$q_2 \geq 0. \quad \textbf{(9)}$$

   Since $u'''(x|q) = 6q_2$, this gamma condition is equivalent to the NIARA condition (RC3). However, it is easily verified that the beta condition (8) is not equivalent to the risk aversion condition (RC2); the beta condition offers a weak necessary condition for risk aversion only, and it allows for an inverse S-shaped utility function.

2. As argued by Levy (1969), a cubic utility function cannot be concave over an unbounded range, i.e., if $D = \Re$. Even the smallest possible non-zero value for q_2 suffices to make utility convex over a range. Marginal utility $u'(x|q) = 1 + 2q_1 x + 3q_2 x^2$ is a quadratic function and hence it is increasing over a range. Therefore, if we impose concavity for an unbounded range, then the cubic term q_2 and the gamma premium r_2 must equal zero and gamma does not explain asset prices.

3. Of course, utility can be concave over a bounded sample range, say $D = [b_-, b_+]$ with $-\infty < b_- < 0$ for the sample minimum and $\infty > b_+ > 0$ for the sample maximum. However, Tsiang (1972) demonstrates that a quadratic function is likely to give a good approximation for any (continuously differentiable) concave utility function over the typical sample range, and that higher-order polynomials (including cubic functions) are unlikely to improve the

fit. Hence, if we impose the regularity conditions for the sample range, then the gamma premium generally will be very small for realistic return distributions, and gamma is unlikely to help explain asset prices.

Using these theoretical arguments, the sizable gamma premium found in empirical studies suggests that the underlying utility function is not concave (even over the sample range), but rather takes an inverse S-shape. The empirical results in Section III further support this view. The inverse S-shape introduces two complications:

1. We may ask if (local) risk seeking is economically meaningful. Of course, risk aversion is not a law of nature and there are several arguments to support (local) risk seeking. For example, Markowitz (1952) argues that the willingness to purchase both insurance and lottery tickets (the Friedman-Savage puzzle) implies that marginal utility is increasing for gains. Also, 'seemingly' risk seeking behavior may arise if irrational investors subjectively overweigh the true probability of extremely high returns, as in, e.g., the Cumulative Prospect Theory (Kahneman and Tversky (1992)). Still, we may object to assuming risk aversion in the theoretical section of a study and neither imposing that assumption in the empirical analysis nor reporting whether the empirical results satisfy that assumption.

2. Another problem associated with relaxing concavity, is that the first-order conditions are no longer sufficient conditions for optimality. We may wrongly classify a minimum or a local maximum (which will also satisfy the first-order conditions) as the global maximum. There exist various multivariate global optimization methods for locating the global maximum if the (known) objective function is not concave (see, e.g., Horst and Pardalos (1995)). Unfortunately, these methods generally are computationally more complex than checking the first-order condition, and the computational burden becomes prohibitive if the problem dimensionality is high (i.e., many assets are included). In addition, we do not see a simple solution for the case where the objective function is not known but rather has to be estimated empirically. However, it is relatively simple to test some weak necessary conditions (in addition to the first-order conditions) for a global maximum. For example, if the market portfolio is the global maximum, then its expected utility must exceed that of all individual assets. Section III provides empirical evidence that this condition is severely violated and that the market portfolio does not maximize the expectation of the inverse S-shaped utility function implied by the 3M CAPM test results, even if the first-order conditions are satisfied.

III. Empirical illustration

A. Methodology

In empirical applications, the CDF $G(\boldsymbol{x})$ generally is not known. Rather, information is typically limited to a discrete set of time-series observations, say $X \equiv (\boldsymbol{x}_1 \cdots \boldsymbol{x}_\mathrm{T})$, with $\boldsymbol{x}_t \equiv (x_{1t} \cdots x_{Nt})^\mathrm{T}$, which are here assumed serially independently and identically distributed random draws from $G(\boldsymbol{x})$. Using these observations, we can construct the following sample equivalent of the first-order condition (1):

$$\boldsymbol{m}(q) \equiv \frac{1}{T}\sum_{t=1}^{T} u'(\boldsymbol{x}_t^\mathrm{T} t \,|\, q)\boldsymbol{x}_t = 0\,. \qquad \textbf{(10)}$$

We may estimate the unknown parameters q using the generalized method of moments (GMM). The GMM estimator selects parameter estimates so that the pricing errors $\boldsymbol{m}(q)$ are as close to zero as possible, as defined by the criterion function

$$J \equiv \min_{q}\ \boldsymbol{m}(q)^\mathrm{T}\boldsymbol{W}\boldsymbol{m}(q)\,, \qquad \textbf{(11)}$$

where $\boldsymbol{W}$ is a weighting matrix.[3] In this study, the weighting matrix is set equal to the inverse of the covariance matrix of the pricing errors, i.e., $\boldsymbol{W} \equiv (\boldsymbol{m}(q)\boldsymbol{m}(q)^T)^{-1}$, and we use the continuous-updating method, which continuously alters $\boldsymbol{W}$ as q is changed in the minimization; see, e.g., Hansen, Heaton and Yaron (1996).

This approach focuses on estimating the utility parameters q rather than the risk premiums r_1 and r_2. However, we can compute the implied risk premiums from the utility parameters by using equations (6) and (7). The advantage of this approach is that linear inequality restrictions suffice to impose and test the utility conditions that are of interest here. We have already seen that the typical restrictions on the beta and gamma premiums can be formulated equivalently as linear restrictions on the utility parameters, i.e., (8) and (9). In empirical applications, we may use the following empirical equivalent of (8):

$$2q_1 + 6q_2 \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_t^\mathrm{T} t \le 0 \qquad \textbf{(12)}$$

In addition, we can also impose the exact risk aversion condition over the sample range, i.e., $u''(x|q) \le 0 \quad \forall x \in [b_-, b_+]$, by means of an inequality restriction. Specifically, if the NIARA condition (RC3) or (9) is satisfied, then the second-order derivative $u''(x|q) = 2q_1 + 6q_2 x$ is an increasing linear function and hence risk aversion over the sample range is equivalent to

$$2q_1 + 6q_2 b_+ \leq 0 . \qquad \textbf{(13)}$$

Apart from the above conditions, we wish to bound marginal utility, so as to impose nonsatiation and to avoid too extreme risk avoidance or risk seeking. For this purpose, we bound marginal utility $u'(x|q) = 1 + 2q_1 x + 3q_2 x^2$ from above by $\infty > a > 1$ and from below by $1/a$, i.e., $u'(x|q) \in [1/a, a] \quad \forall x \in [b_-, b_+]$. This guarantees that marginal utility is strictly positive and, in addition, that marginal utility can not be more than a times as high, or a times as low, as marginal utility at x=0 (recall that $u'(0|q) = 1$)). If the risk aversion condition is satisfied, then marginal utility is non-increasing, and hence it suffices to impose two simple restrictions at the boundaries of the sample range:

$$1 + 2q_1 b_- + 3q_2 b_-^2 \leq a , \qquad \textbf{(14)}$$

$$1 + 2q_1 b_+ + 3q_2 b_+^2 \geq 1/a . \qquad \textbf{(15)}$$

We start our analysis by using the bound a=10. Since we have few prior arguments to determine what is ‘extreme’ and what is not, we subsequently analyzed the sensitivity of our results to changing the value of a.

B. Data

We will use the Fama and French market portfolio, which is the value-weighted average of all NYSE, AMEX, and NASDAQ stocks. Further, we use the one-month US Treasury bill as the riskless asset. Finally, for the individual risky assets, we use the ten Fama and French decile portfolios based on market capitalization (ME). We use data on monthly returns (month-end to month-end) from July 1963 to December 2001 (462 months) obtained from the data library on the homepage of Kenneth French.[4]

Table I shows some descriptive statistics for the excess returns of the market portfolio and the benchmark portfolios. All benchmark portfolios have a negatively skewed return distribution. Interestingly, this negative skewness is not 'diversified away'; de market portfolio is more negatively skewed than most of the individual benchmark portfolios. Apparently, investing in the market portfolio yields a relatively small reduction in downside risk (relative to the individual benchmark portfolios) at the cost of a relatively large reduction in upside potential. The negative sign of market skewness also implies that gamma, i.e., co-skewness standardized by market skewness, has a positive sign; assets that increase (decrease) the skewness of the market portfolio have a negative (positive) gamma. At first glance, the descriptives suggest that skewness may help explain asset prices.

Specifically, the benchmark portfolios with a high Treynor ratio (mean to beta have) generally have a high gamma and hence strongly lower the skewness of the market portfolio, making them less attractive for investors who prefer positive skewness. For example, the ‘high yield’ small cap portfolio ME1 (with a Treynor ratio of 0.645) has a high gamma of 1.653, while the ‘low yield’ big cap portfolio ME10 (with a Treynor ratio of 0.227) has a high gamma of 1.829.

[Insert Table 1 about here]

C. Results

We estimate the Euler equation (10) for two different models. The first model imposes no restrictions on the utility parameters, apart from the bounding conditions (14) and (15). The second model adds the condition of risk aversion over the sample range, i.e., (13). Table II reports the estimated utility parameters for both models and the associated risk premiums, as computed from (6) and (7). Further, the table reports the *J*-statistics and the associated *p*-values. For illustration, Figure 1 shows the estimated utility function of both models. In addition, Figure 2 displays the Security Market Plane for both models.

[Insert Table 2 about here]

[Insert Figure 1 about here]

[Insert Figure 2 about here]

For the model that imposes risk aversion, the estimate for the cubic parameter q_2 is approximately zero, and the cubic utility function effectively takes a quadratic form. This illustrates the argument by Tsiang (1972) that a cubic utility function is unlikely to improve the fit relative to a quadratic utility function if we maintain the assumption of risk aversion. The small value for q_2 implies an annualized gamma premium of roughly one half percent, which is a very small fraction of the total market premium of 5.65 percent (12 times 0.472). The *J*-statistic is not significantly different from zero and we cannot reject the first-order conditions. Since the first-order conditions suffice for a global maximum for concave utility functions, this implies that we cannot reject the null that the market portfolio is the global maximum.

For the model without the risk aversion condition, the estimated utility function takes an inverse S-shaped form with an inflection point around zero. Interestingly, the estimated utility parameters satisfy conditions (8) and (9) and the beta premium and gamma premium (now roughly three percent per annum) are both positive. These findings illustrate the point that the typical 3M CAPM restrictions on the risk premiums do not suffice to guarantee risk aversion and that the empirical results of 3M CAPM studies generally violate this condition. Again, the *J*-statistic is not significantly different from zero and hence the market portfolio does not significantly violate the first-order conditions for the estimated utility functions. However, the first-order conditions are not sufficient to guarantee a global maximum for inverse S-shaped utility functions, and the market portfolio may represent a minimum or a local maximum for expected utility. Table III displays the sample mean of utility for the market portfolio and the benchmark portfolios. Interestingly, mean utility for the market portfolio is less than that for each of the ten benchmark portfolios! This suggests that the market portfolio is more likely to represent the global minimum than the global maximum of expected utility.

[Insert Table 3 about here]

Roughly speaking, investors who prefer positive skewness assign a relatively high weight to upside potential, and the reduction in downside risk associated with holding the market portfolio does not sufficiently compensate them for the reduction in upside potential. Rather than holding a well-diversified portfolio, such investors will hold a less diversified portfolio with large upside potential.[5] Since investors (both individual and institutional) actually hold highly undiversified portfolios (see, e.g., Levy (1978)), actual investors may indeed exhibit skewness preference (although there are several alternative explanations for 'underdiversification'). However, a representative investor with skewness preference is unlikely to explain asset prices, as such an investor would have to invest in the market portfolio.

Recall from Section IIIA that we bounded marginal utility by setting a=10, so as to avoid extreme risk avoidance or risk seeking. For the model without risk aversion, this restriction is binding for large losses, and loosening (tightening) the bound a will improve (reduce) the goodness-of-fit of the model. Still, the shape of the utility functions, the sign of the risk premiums and the relative goodness of the models is not significantly affected by the choice of a.

Finally, we focus on estimating the utility parameters here, and we may ask if directly estimating the risk premiums gives comparable results. In our case, OLS estimation of equation (5) gives an estimated beta premium of $r_1 = 0.318\ (0.000)$ and a gamma premium of $r_2 = 0.217\ (0.001)$, and R^2 equals roughly 92 percent. These results are very similar to our results *without* the risk aversion condition, and they again imply a reverse S-shaped utility function. Unfortunately, we cannot present the results for the case *with* the risk aversion condition, for the simple reason that we cannot impose this conditions on the risk premiums in a straightforward manner. In fact, this provides our motivation for estimating the utility parameters rather than the risk premiums.

III. Conclusions

The usual 3M CAPM tests fail to impose the standard regularity condition of concavity or risk aversion. We present theoretical and empirical evidence that the empirical results severely violate this condition. If we impose risk aversion, then gamma has minimal explanatory power for asset prices. If we do not impose risk aversion, then the implied utility function takes an inverse S-shape with risk aversion up to some return level and risk seeking beyond that level. Unfortunately, the first-order conditions are not sufficient to guarantee that the market portfolio is the global maximum for the expectation of this utility function. In fact, our empirical results suggest that the market portfolio is more likely to represent the global minimum. This may reflect the empirical fact that stocks generally are more strongly correlated in falling market than in rising markets. Consequently, investing in the market portfolio yields a relatively small reduction in downside risk at the cost of a relatively large reduction in upside potential. For investors with skewness preference, the small reduction in downside risk does not sufficiently compensate for the large reduction in upside potential. Put differently, skewness preference cannot explain why the market portfolio is mean-variance inefficient, because the market portfolio has a relatively high negative skewness relative to less diversified portfolios.

Our results lead us to believe that the usual theoretical interpretation of the gamma premium is not valid. We do not deny the statistical association between mean, beta and gamma, and results of, e.g., the thorough empirical study by Harvey and Siddique remain fascinating. However, we do call into question the causal interpretation that is typically given to this association. Specifically, the large gamma premium is unlikely to represent the price that investors are willing to pay for assets that increase the skewness of the market portfolio; if investors are risk averse, then the gamma premium will be very small, and if investors are risk seeking, then they will not hold the market portfolio (and gamma is not an appropriate risk measure). Rather, the large gamma premium suggests that gamma serves as a proxy for omitted variables that do explain asset prices. Further research on asset pricing models with heterogeneous investors and skewness preference, e.g., along the lines of Levy (1978), may help to identify these omitted variables.

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Table I

Descriptive Statistics Fama and French Portfolios

Monthly excess returns (month-end to month-end) for the value-weighted Fama and French market portfolio and the ten Fama and French decile portfolios based on market capitalization. Descriptive statistics are computed for the full sample from July 1963 to December 2001. Excess returns are computed from the raw return observations by subtracting the return on the one-month US Treasury bill. All data are obtained from the data library of Kenneth French.

	Mean	St. Dev.	Beta	Skew	Gamma	Min	Max
Market	0.471	4.467	1.000	-0.502	1.000	-23.09	16.05
ME 1	0.703	6.494	1.090	-0.083	1.653	-29.30	28.72
ME 2	0.697	6.322	1.184	-0.236	1.595	-30.60	27.79
ME 3	0.712	6.023	1.178	-0.452	1.506	-29.70	25.50
ME 4	0.682	5.852	1.163	-0.507	1.475	-30.05	24.18
ME 5	0.711	5.544	1.136	-0.607	1.431	-28.27	24.33
ME 6	0.596	5.323	1.110	-0.563	1.302	-26.89	20.52
ME 7	0.621	5.161	1.098	-0.449	1.155	-26.69	22.09
ME 8	0.604	5.048	1.087	-0.417	1.092	-24.90	18.00
ME 9	0.523	4.632	1.009	-0.338	0.905	-22.99	17.70
ME10	0.439	4.288	0.932	-0.297	0.829	-20.33	17.43

Table II

Estimation Results Three-Moment CAPM

We estimate the unknown parameters q_1 and q_2 of the cubic utility function $u(x|q) \equiv x + q_1 x^2 + q_2 x^3$ by means of GMM. The orthogonality conditions are given by the Euler equation (1). We analyze the efficiency of the Fama and French market portfolio relative to the ten size portfolios over the sample period July 1963-December 2001. We compare the model that imposes the condition of risk aversion over the sample range, i.e., (13), with the model without this restriction. The table shows the GMM estimates for the parameters (*p*-values within brackets), the associated estimates for the risk premiums r_1 and r_2, and the *J*-statistics (*p*-values within brackets).

Risk aversion	q_1	q_2	r_1	r_2	J
Yes	-0.0123 (0.008)	0.0003 (0.459)	0.475	0.034	0.011 (0.748)
No	-0.0104 (0.212)	0.0020 (0.140)	0.270	0.242	0.009 (0.839)

Table III

Sample Mean Utility of Fama and French Portfolios

The table shows the sample mean of the cubic utility function $u(x|q) \equiv x + q_1 x^2 + q_2 x^3$ for the Fama and French market and the ten size portfolios over the period July 1963 - December 2001. The utility parameters q_1 and q_2 are taken from the GMM estimation results for (i) the model with the risk aversion condition and (ii) the model without the risk aversion condition; see Table II.

	Concave utility function (i)	Inverse S-shaped utility function (ii)
Market portfolio	0.218	0.229
ME1	0.199	0.395
ME2	0.207	0.327
ME3	0.254	0.289
ME4	0.247	0.260
ME5	0.317	0.312
ME6	0.234	0.231
ME7	0.286	0.317
ME8	0.285	0.322
ME9	0.257	0.298
ME10	0.211	0.248

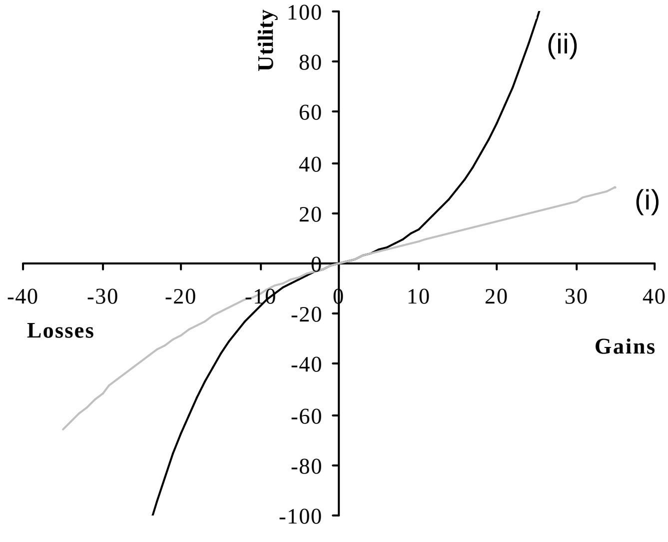
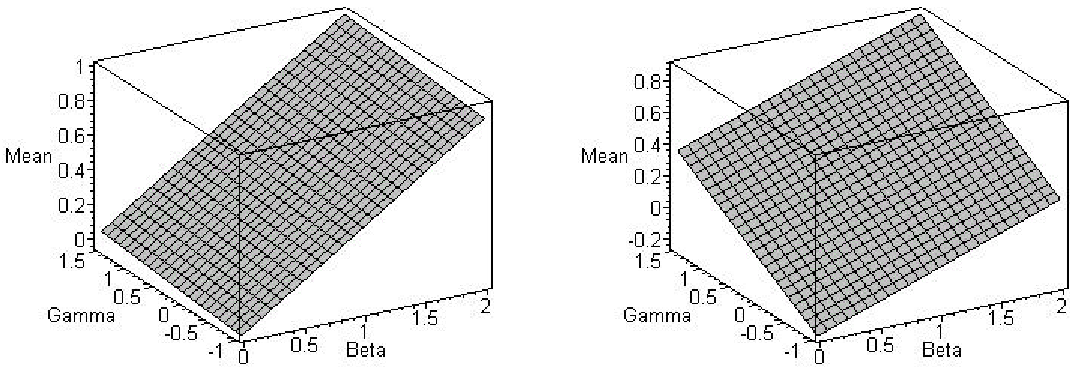


Figure 1: Optimal Cubic Utility Function. The figure shows the cubic utility function $u(x|q) \equiv x + q_1 x^2 + q_2 x^3$ for (i) the case with risk aversion condition (13) and (ii) the case without the risk aversion condition. The values for q_1 and q_2 are obtained by means of GMM estimation of the Euler equation (10) for the Fama and French market portfolio and the ten size portfolios, using data from July 1963 to December 2001; see Table II.

(i) With risk-averse condition **(ii) Without risk-averse condition**



$m = 0.475b + 0.034g$ $m = 0.270b + 0.242g$

Figure 2: Security market Plane. The figure shows the Security Market Plane as defined in (5): $m = r_1 b + r_2 g$ for (i) case with risk-aversion and (ii) the case without risk aversion. The risk premiums r_1 and r_2 are obtained by means of GMM estimation of the Euler equation; see Table II.

Footnotes

[1] Throughout the text, we will use $\Re^N$ for an N-dimensional Euclidean space, and $\Re^N_+$ denotes the positive orthant.
[2] This utility function is standardized such that $u(0|q) = 0$ and $u'(0|q) = 1$. Since utility functions are unique up to a linear transformation, this standardization does not affect our results.
[3] We may use the J-statistic to test if the Euler equation holds (i.e., the pricing errors $\boldsymbol{m}(q)$ are equal to zero). Specifically, under the null that the (N-2) overidentifying restrictions are satisfied, the J-statistic times the number of regression observations, i.e., JT, asymptotically obeys a chi-squared distribution with degrees of freedom equal to the number of overidentifying restrictions, i.e., (N-2) degrees of freedom.
[4] Similar results were obtained for the two non-overlapping subsamples from July 1963 to September 1982 and from October 1982 to December 2001. In addition, similar results were obtained for the Fama and French decile portfolios based on market-to-book equity ratio and the Fama and French portfolios based on industry classification. Results are available upon request from the authors.
[5] Simkowitz and Beedles (1979, p. XXX) already made this point: 'Many investors hold less than perfectly diversified portfolios, a phenomenon in contradiction with frequently offered advice. [...] If positive skewness is a desirable characteristic of return distributions, then the fact that the simple act of diversification destroys skew is a likely explanation of observed behavior.'

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