Financial Markets Analysis by Probabilistic Fuzzy Modelling

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Abstract

For successful trading in financial markets, it is important to develop financial models where one can identify different states of the market for modifying one's actions. In this paper, we propose to use probabilistic fuzzy systems for this purpose. We concentrate on Takagi–Sugeno (TS) probabilistic fuzzy systems that combine interpretability of fuzzy systems with the statistical properties of probabilistic systems. We start by recapitulating the general architecture of TS probabilistic fuzzy rule-based systems and summarize the corresponding reasoning schemes. We mention how probabilities can be estimated from a given data set and how a probability distribution can be approximated by a fuzzy histogram. We apply our methodology for financial time series analysis and demonstrate how a probabilistic TS fuzzy system can be identified, assuming that a linguistic term set is given. We illustrate the interpretability of such a system by inspecting the rule bases of our models.

Keywords

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1 Introduction

Complex systems such as financial markets are characterized by changing process dynamics, which manifest themselves in various ways like regime shifts and volatility variations. In the specific case of financial markets, it is important to recognize the 'state-of-the-market', so that the market participants' decisions (e.g. trading decisions) can be adapted to the prevailing market conditions in order to safeguard success in the markets. Consequently, many financial models try to capture the changes in the market conditions. An example of such a model is the so-called GARCH (Generalized Auto Regressive Conditional Heteroskedasticity) model [2], which assumes that the volatility of the market returns is dependent on the past volatility.

The GARCH model is an example of probabilistic models, which are almost always used in finance. Probabilistic models deal only with probabilistic uncertainty regarding the market developments (e.g. return series). There is, however, often other types of uncertainty present, such as fuzziness in the definitions of concepts and the linguistic uncertainty, which are related to the perception of market participants. These other types of uncertainty should best be modelled explicitly by using paradigms other than probabilistic modelling. Financial models should then ideally combine different paradigms in order to deal with different types of uncertainty. The advantages of this approach are two-fold. First, explicit modelling of different types of uncertainty separates quantities that are conceptually different. Thereby, it improves the interpretability of the models, since conceptually different quantities are treated separately. Second, the adaptability of the models can be improved, since different types of information can be used for the modelling purposes.

In this paper, we propose to use probabilistic fuzzy systems for financial modelling in general, and for the analysis of financial time series in particular. Fuzzy systems (FSs) are widely applied in fields like classification, decision support, process simulation, and control ([5], [3]). Financial and marketing applications have also been reported regularly ([10], [11]). Original applications of FSs have concentrated on their design from expert knowledge ([6], [7]). In the past decade, however, data-driven techniques for designing FSs have gained much attention, partly due to the availability of large amounts of data from modern sensory, measurement and computer systems. One important advantage of fuzzy inference systems is their linguistic interpretability, whereby the results from the data-driven approach can be combined with or compared to the knowledge available from experts. When applying FSs, one usually focusses on this aspect by modelling fuzziness and linguistic vagueness using membership functions. However, one has often ignored the probabilistic uncertainty, which is often also present. *Probabilistic fuzzy systems* (PFSs) combine both types of uncertainty in order to provide "the best of the two worlds."

PFSs combine *interpretability* of fuzzy systems with the *statistical properties* of probabilistic systems. We consider PFSs where the rules describe a stochastic mapping from the antecedent space to the consequent space ([9], [8]). These PFSs can be considered as a generalization of deterministic rule-based fuzzy systems. In this paper, we concentrate on probabilistic Takagi–Sugeno fuzzy systems and their design from data. We demonstrate how these systems can be applied to financial time series modelling and illustrate how the resulting model can be analyzed and interpreted.

The rest of the paper is structured as follows. In Section 2, we recapitulate the general architecture of TS probabilistic fuzzy rule-based systems and summarize the corresponding reasoning schemes. In Section 3, we mention relevant results from the theory of mathematical statistics on fuzzy sets in order to be able to estimate *probabilities on fuzzy sets*. We also illustrate how a probability distribution can be approximated by a fuzzy histogram. In Section 4, we apply the proposed methodology for financial time series analysis and demonstrate how a probabilistic TS fuzzy system can be identified, assuming that a linguistic term set is given. We illustrate the interpretability of such a system by inspecting the rule bases of our models. Finally, the conclusions and a short discussion are given in Section 5.

2 Probabilistic Fuzzy Systems

For the scope of this paper, we concentrate on zero-order Takagi–Sugeno PFSs, although extensions to other types of fuzzy systems are also possible. The heart of a zero-order Takagi–Sugeno probabilistic fuzzy system consists of a *probabilistic fuzzy rule-base* which is made up of a set of probabilistic fuzzy rules, together with an appropriate *inference mechanism* for reasoning. The probabilistic fuzzy rules have the general form [8]:

$$\begin{aligned}\text{Rule } R_q\text{: If } \mathbf{x} \text{ is } A_q \text{ then}&\\ \underline{y} = y_{q1} &\text{ with } \Pr(y_1|A_q) \text{ and}\\ \underline{y} = y_{q2} &\text{ with } \Pr(y_2|A_q) \text{ and} \ldots \text{ and}\\ \underline{y} = y_{qN} &\text{ with } \Pr(y_N|A_q),\end{aligned} \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_M) \in X$ is an M-dimensional input vector, A_q is an antecedent linguistic value defined by a fuzzy membership function $\mu_q(\mathbf{x})$, $\underline{y}$ is the stochastic consequent variable being equal to one of the values $y_{q1},\ y_{q2},\ldots,y_{qN}$. The selection of this consequent value is done proportionally to the conditional probabilities $\Pr(y_{q1}|A_q),\ldots$, $\Pr(y_{qN}|A_q)$, with $\forall j : \Pr(y_{qj}|A_q) = \Pr(\underline{y} = y_{qj}|\mathbf{x} \text{ is } A_q)$.

In this paper, we use fuzzy rules (1), where the consequent values y_{qj} are the same for all rules. Mathematically expressed, we assume that

$$\forall j, q, q' : y_{qj} = y_{q'j} = y_j. \tag{2}$$

Hence, each rule describes a probabilistic mapping from a fuzzy antecedent to the same set of consequents. The rules differ in the probabilistic mapping that they describe. This assumption is not restrictive if the consequents are chosen such that they can be used to characterize the whole system output (or equivalently consequent) space.

The reasoning in probabilistic systems essentially performs an interpolation as in many fuzzy systems. The following paragraphs summarize two reasoning schemes as derived in [8].

2.1 Probabilistic fuzzy reasoning I

In this scheme, we begin by estimating the conditional probabilities $\Pr(y_j|\mathbf{x})$ for arbitrary $\mathbf{x}$ and then calculate the regression hyperplane y on $\mathbf{x}$. First, the conditional probabilities $\Pr(y_j|\mathbf{x})$ are calculated by using a weighted sum of conditional probabilities $\Pr(y_j|A_q)$,

$$\Pr(y_j|\mathbf{x}) = \sum_{q=1}^{Q} \phi_q \Pr(y_j|A_q) = \frac{\sum_{q=1}^{Q} \Pr(A_q)\mu_q(\mathbf{x}) \Pr(y_j|A_q)}{\sum_{q=1}^{Q} \Pr(A_q)\mu_q(\mathbf{x})}, \tag{3}$$

with $\phi_q = \Pr(A_q)\mu_q(\mathbf{x})/\sum_{q=1}^{Q} \Pr(A_q)\mu_q(\mathbf{x})$. The weight factors ϕ_q take into account both the membership to the fuzzy antecedent A_q and the probability of the fuzzy event A_q. Note that equation (3) actually implements a *stochastic mapping* $X \to Y$: for each arbitrary input vector $\mathbf{x}$, the conditional probability distribution $\Pr(y_j|\mathbf{x}), (j = 1, 2, \ldots)$ is given by (3).

In practice, one often wants to know the *expected behavior* as described by a regression curve, i.e. the regression hyperplane of y on X. This is defined as the location of the mathematical expectations $E(y|\mathbf{x})$ [4], and it can be calculated according to

$$y = E(\underline{y}|\mathbf{x}) = \sum_{j=1}^{N} y_j \Pr(y_j|\mathbf{x}). \tag{4}$$

2.2 Probabilistic fuzzy reasoning II

In this reasoning scheme, we start by calculating the expectations $E(\underline{y}|A_q), q = 1, 2, \ldots, Q$, according to

$$E(\underline{y}|A_q) = \sum_{j=1}^{N} y_j \Pr(y_j|A_q). \tag{5}$$

Next, we estimate y (as a function of $\mathbf{x}$) by the weighted sum of expectations $E(\underline{y}|A_q), q = 1, 2, \ldots, Q$ according to

$$y = \sum_{q=1}^{Q} \phi_q E(\underline{y}|A_q) = \frac{\sum_{q=1}^{Q} \Pr(A_q)\mu_q(\mathbf{x}) E(\underline{y}|A_q)}{\sum_{q=1}^{Q} \Pr(A_q)\mu_q(\mathbf{x})}, \tag{6}$$

with $\phi_q = \Pr(A_q)\mu_q(\mathbf{x}) / \sum_{q=1}^{Q} \Pr(A_q)\mu_q(\mathbf{x})$. Hence, equation (6) calculates the expected output of the probabilistic fuzzy system given the expected output of each rule. Again, the weight factors ϕ_q take into account both the membership to the fuzzy antecedent A_q and the probability of the fuzzy event A_q. Note that (6) involves an interpolation procedure, just like (3). Note also that equations (4) and (6) describe the same hyperplane [8].

3 Mathematical Statistics on Fuzzy Sets

In this section, we describe how the probabilities in Section 2 can be computed from data. Furthermore, we discuss the approximation of probability density functions by using fuzzy histograms.

3.1 Probability estimation

Given a set of S samples $\mathbf{x}_s, (s = 1, \ldots, S)$ in a 'well-defined' [9] sample space X, the probability $\Pr(A_c)$ describing the probability of the 'fuzzy event' '$\mathbf{x}$ is A_c', can be estimated according to

$$\Pr(A_c) \approx \tilde{f}_{A_c} = \frac{f_{A_c}}{S} = \frac{1}{S} \sum_{\mathbf{x}_s} \mu_{A_c}(\mathbf{x}_s) = \hat{\mu}_{A_c}. \tag{7}$$

Here, $\tilde{f}_{A_c}$ denotes the relative frequency and f_{A_c} the absolute frequency of the fuzzy sample values $\mu_{A_c}(\mathbf{x}_s)$ for fuzzy class A_c. In addition, conditional probabilities on fuzzy sets can be assessed according to

$$\Pr(A_c|A_b) = \frac{\Pr(A_c \bigcap A_b)}{\Pr(A_b)} \approx \frac{\sum_{\mathbf{x}_s} \mu_{A_b}(\mathbf{x}_s)\mu_{A_c}(\mathbf{x}_s)}{\sum_{\mathbf{x}_s} \mu_{A_b}(\mathbf{x}_s)}. \tag{8}$$

In Section 2, we mentioned expressions of type $\Pr(y_j|A_q)$ describing the probability of a crisp event $\underline{y} = y_j$, given the occurrence of fuzzy event $\mathbf{x}$ is A_q. Having a training set of data pairs $(\mathbf{x}_s, y_s), s = 1, \ldots, S$, such a conditional probability can be calculated by means of an adapted version of (8),

$$\Pr(y_j|A_q) \approx \frac{\sum_{(\mathbf{x}_s, y_s)} \chi_j(y_s)\mu_{A_q}(\mathbf{x}_s)}{\sum_{\mathbf{x}_s} \mu_{A_q}(\mathbf{x}_s)}, \tag{9}$$

with $\chi_j(y)$ defined as

$$\chi_j(y) = \begin{cases} 1 & \text{if } y = y_j \\ 0 & \text{if } y \neq y_j. \end{cases} \tag{10}$$

3.2 Fuzzy histograms

The technique for estimating a probability density function (pdf) using (crisp) histograms is well-known. By appropriately partitioning the domain of sample space X in a set of Q disjunct classes C_q, each "column" $f_q(\mathbf{x})$, $(q = 1, 2, \ldots, Q)$ of the histogram is defined by the functions

$$f_q(\mathbf{x}) = \begin{cases} \frac{\Pr(C_q)}{c_q} & \text{if } \mathbf{x} \in C_q \\ 0 & \text{if } \mathbf{x} \notin C_q, \end{cases} \tag{11}$$

where the probability $\Pr(C_q)$ is estimated in the usual way (using the relative frequency of samples $\mathbf{x}_s \in C_q$) and where the scaling scalar c_q equals the size of class C_q (which in the one-dimensional case, equals the length of the interval C_q). The complete pdf $f(\mathbf{x})$ is approximated by a summation of the functions $f_q(\mathbf{x})$ according to

$$f(\mathbf{x}) \approx f_{app}(\mathbf{x}) = \sum_q f_q(\mathbf{x}). \tag{12}$$

Probability density functions defined on a sample space X that is partitioned fuzzily can also be estimated by using a fuzzy histogram. To do so, we need a generalization of the above-given crisp approach. Let X be fuzzily partitioned in a set of Q fuzzy classes A_q described by membership functions $\mu_{A_q}()$, then the probability $f_q(\mathbf{x})$ for the fuzzy class A_q can be estimated according to

$$f_q(\mathbf{x}) = \frac{\Pr(A_q)\mu_{A_q}(\mathbf{x})}{\int_{-\infty}^{\infty} \mu_{A_q}(\mathbf{x})d\mathbf{x}}, \tag{13}$$

with $\int_{-\infty}^{\infty} \cdot d\mathbf{x}$ representing an M-fold integral.

The numerator in (13) describes a probability weighted with membership functions. The denominator of (13) is a scaling factor representing the fuzzified size of class C_q. The complete pdf $f(\mathbf{x})$ is again approximated by a summation of the functions $f_q(\mathbf{x})$ according to

$$f(\mathbf{x}) \approx f_{app}(\mathbf{x}) = \sum_q f_q(\mathbf{x}) = \sum_q \frac{\Pr(A_q)\mu_{A_q}(\mathbf{x})}{\int_{-\infty}^{\infty} \mu_{A_q}(\mathbf{x})d\mathbf{x}}. \tag{14}$$

Finally, we mention here that definition (14) guarantees that, like in the crisp case, the approximation $f_{app}(\mathbf{x})$ is properly defined in the sense that

$$\int_{-\infty}^{\infty} f_{app}(\mathbf{x})d\mathbf{x} = 1. \tag{15}$$

The proof of this observation is obtained by using (14), so that

$$\begin{aligned}
\int_{-\infty}^{\infty} f_{app}(\mathbf{x})d\mathbf{x} &= \int_{-\infty}^{\infty} \sum_q \frac{\Pr(A_q)\mu_{A_q}(\mathbf{x})}{\int_{-\infty}^{\infty} \mu_{A_q}(\mathbf{x})d\mathbf{x}} & (16)\\
&= \sum_q \Pr(A_q)\frac{\int_{-\infty}^{\infty} \mu_{A_q}(\mathbf{x})d\mathbf{x}}{\int_{-\infty}^{\infty} \mu_{A_q}(\mathbf{x})d\mathbf{x}} = \sum_q \Pr(A_q) = 1. & (17)
\end{aligned}$$

4 Analysis of Financial Time Series

In this section, we give examples of analysis of financial time series by using probabilistic fuzzy systems. In Section 4.1 an artificial time series generated by a GARCH system is studied. It is shown that a probabilistic TS system can be used to discover some basic properties of the underlying data generating system without making extensive assumptions about the structure of this system. Afterwards, we study high frequency Dow Jones data and discuss the results of our proposed method.

4.1 GARCH modelling

GARCH (Generalized Auto Regressive Conditional Heteroskedasticity) models are often used in financial literature to describe the volatility behavior of asset return series [2]. Being able to infer something about the volatility of tomorrow from today's volatility has important implications for the valuation of many financial contracts, more particularly for the *contingent claims*. Typically, the value of such contract depends on the probability that the price $\underline{S}$ of some underlying asset attains a pre-specified level. We define the asset return $\underline{u}(t)$ at time t as the instantaneous relative price change: $\forall t : \underline{u}(t) = \ln(\underline{S}(t)/\underline{S}(t-1))$. Then $\underline{\sigma}(t)$ is the volatility of the return $\underline{u}(t)$, i.e. the standard deviation over a given previous period. This *local* volatility $\underline{\sigma}(t)$ is assumed to move around the constant *global* volatility $\overline{\sigma}$.

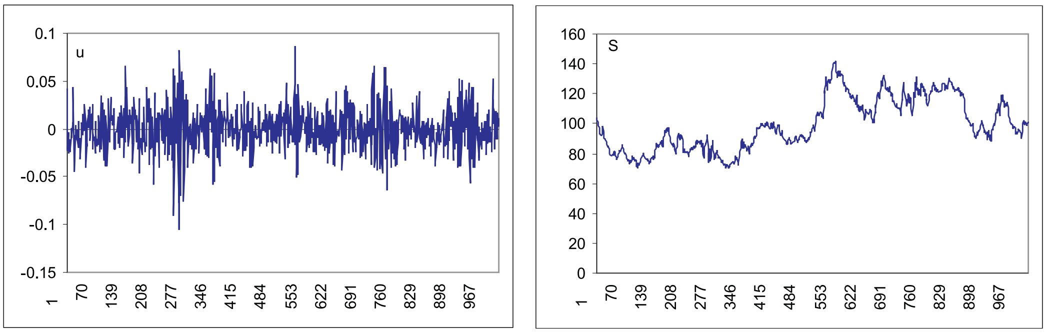

Figure 1: **(left)** Return path **(right)** Price path from a simulated GARCH process

For purposes of our study, we generate data according to a GARCH(1,1) process, which is characterized as follows.

1. Each return $\underline{u}(t)$ is drawn from a normal distribution with a constant mean μ and with a standard deviation equal to the local volatility $\underline{\sigma}(t)$: $\underline{u}(t) \sim N(\mu, \underline{\sigma}(t))$.

2. Each period, the local volatility estimate is updated by using $\underline{\sigma}^2(t) = \gamma\overline{\sigma}^2 + \alpha\underline{u}^2(t-1) + \beta\underline{\sigma}^2(t-1)$.

3. The parameter values used are in line with those found empirically in stock return series: $\overline{\sigma} = 0.03$, $\gamma = 0.02$, $\alpha = 0.2$ and $\beta = 0.78$. The series is initiated with $\sigma_0 = \overline{\sigma}$.

In Figure 1: we show simulation results for 1000 consecutive samples. The return series in the left graph exhibit volatility clusters that are typical for the process. The right graph shows the price development that, starting with $S_0 = 100$, is calculated from the instantaneous return as $S(t) = S(t-1)\, e^{u(t)}$.

4.1.1 Characterizing the input space

The left panel of Figure 2: shows a scatter plot of the product space $\underline{u}(t-1) \times \underline{u}(t)$ of the antecedents and the consequent. In the probabilistic fuzzy rule base, we consider three antecedent linguistic values A_q, defined by fuzzy membership functions $\mu_{A_q}(u), q = 1, 2, 3$, (see the right panel of Figure 2:) . The corresponding linguistic values “Low”, “Average” and “High” respectively describe return values in linguistic terms. Using (7), we have estimated the corresponding probabilities yielding $\Pr(\underline{u}(t-1)$ is “Low”$) = 0.0594$, $\Pr(\underline{u}(t-1)$ is “Average”$) = 0.8722$, and $\Pr(\underline{u}(t-1)$ is “High”$) = 0.0684$.

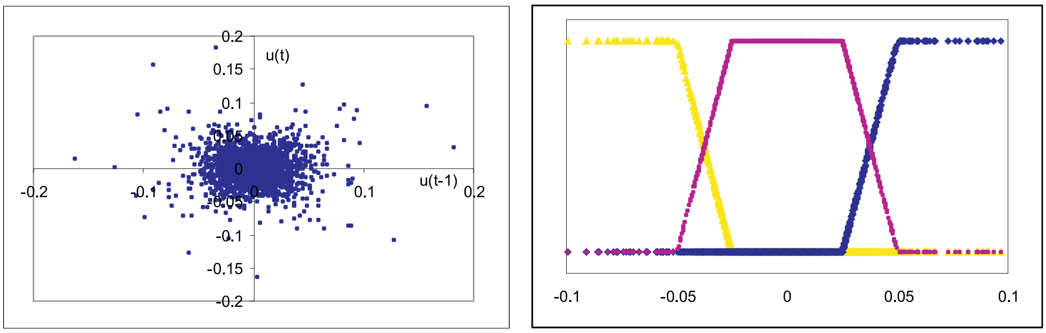

Figure 2: **(left)** Scatter of $\underline{u}(t-1)$ against $\underline{u}(t)$, **(right)** memberships value for $\underline{u}(t-1)$

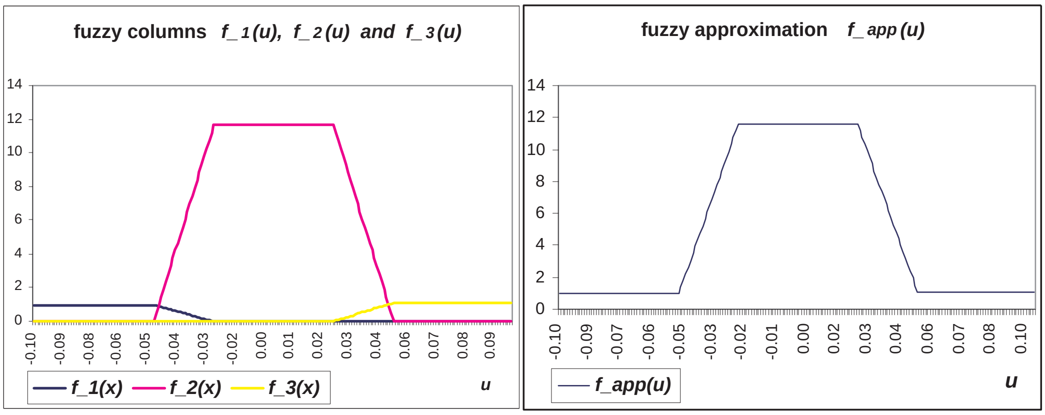

Figure 3: **(left)** Fuzzy histogram, **(right)** the fuzzy approximation $f_{app}(u)$.

We can also approximate the pdf $f(u)$ by using a fuzzy histogram based on the fuzzy partition of $\underline{u}(t-1)$ from Figure 2:. In the left panel of Figure 3:, the fuzzy histogram on the input space is shown computed according to (13). The calculations can be summarized as

$$f_1 = \frac{0.0594 * \mu_{A_1}(u)}{0.0625}, \quad f_2 = \frac{0.8722 * \mu_{A_2}(u)}{0.0750}, \quad f_3 = \frac{0.0684 * \mu_{A_3}(u)}{0.0625}, \tag{18}$$

where the membership functions $\mu_{A_q}(u)$ are given by the functions in the right panel of Figure 2:.

In the right panel of Figure 3:, the fuzzy approximation of the pdf $f(u)$, defined on the input space according to (14), is shown. Note that the antecedent space is partitioned very roughly in only three partitions, but the approximation is already indicative of the distribution also observed in Figure 2:. Fuzzy histograms defined on properly partitioned sample spaces show better approximation results than crisp histograms. This phenomenon can be explained by the fact that the fuzzy approximation of the pdf is obtained by means of an interpolation procedure according to (14).

Future return	very low (-0.05)	low (-0.025)	average (0)	high (0.025)	very high (0.05)	Prob
Current return						
All	*0.0550*	*0.2265*	*0.4435*	*0.2140*	*0.0610*	*1.0000*
Low	0.1271	0.2084	0.2954	0.2302	0.1390	0.0594
Average	0.0437	0.2293	0.4666	0.2136	0.0468	0.8722
High	0.1374	0.2077	0.2808	0.2063	0.1679	0.0684

Table 1: *Unconditional and conditional probabilities* $\Pr(u_j)$ *and* $\Pr(u_j|A_q)$.

4.1.2 Characterizing the output space

In order to keep this illustrative example simple, we chose (without further optimization) five equidistant crisp consequent values $u_1 = -0.050$, $u_2 = -0.025$, $u_3 = 0.000$, $u_4 = 0.025$, $u_5 = 0.050$ to describe the future returns. We labeled these arithmetic values with the linguistic terms "very low", respectively "low", "average", "high", "very high". Then, each return value from the time series is classified according to the nearest prototype value using the Euclidian norm. By simply counting all u-values and determining the relative score, we make an estimate of the (unconditional) output probability distribution of $\Pr(u_j) = \Pr(\underline{u}(t) = u_j), j = 1, 2, \ldots, 5$. The results of these calculations are shown in the (emphasized) row (labeled *'All'*) of Table 1:.

4.1.3 Characterizing the probabilistic fuzzy input-output mapping

By using (9), we can also calculate $\Pr(u_j|A_q), j = 1, 2, 3, 4, 5; q = 1, 2, 3$. It concerns probabilities like "the probability that the future return is high given that the current return is Low". These conditional probabilities are summarized in Table 1:.

It becomes clear after analyzing these results that for low current returns, the probability for very high or very low future returns is higher than the overall probability. A similar conclusion can also be drawn for high current returns. For low or high current returns, the deviation of low and high future returns from the overall probability distribution is also visible, although to a lesser extent. If we attach linguistic values to the magnitude of the difference between the conditional probability and the overall probability (e.g.: More than 5 percent is "very likely" or "very unlikely" and more than 2 percent is "rather likely" or "rather unlikely"), then the above

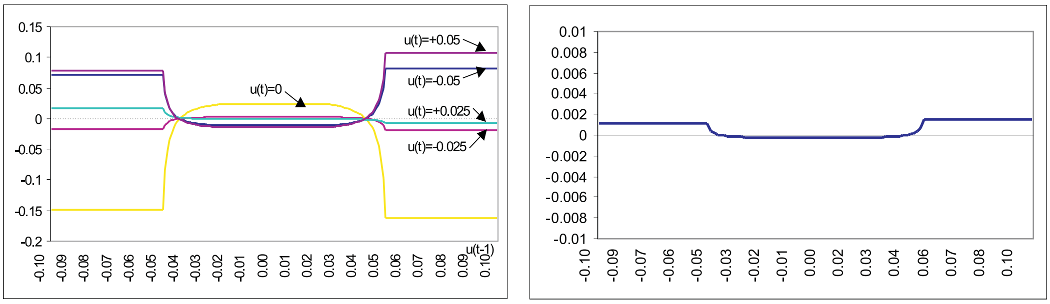

Figure 4: **(left)** $\Pr(u_j|u(t-1)) - \Pr(u_j)$, **(right)** regression line of $u(t)$ on $u(t-1)$

results can thus be summarized as

> If *current* return is Low or the *current* return is High, then a low or high *future* return is rather likely, and a very low or very high *future* return is very likely.

This is a pretty good intuitive description of the GARCH process that has generated the data, where periods of high returns are correlated to periods of high volatility.

Finally, we show two additional results. First, we have plotted the regression line of $u(t)$ on $u(t-1)$ (estimated according to (4)) in the right panel of Figure 4:. As expected for this problem, we found that $u(t) \approx 0$. In the left panel of the same figure, we show the difference between the conditional probabilities $\Pr(u_j|u(t-1))$ and the unconditional probability $\Pr(u_j)$, for $j = 1, 2, 3, 4, 5$. If current returns are Average, we observe that all conditional probabilities are almost equal to the unconditional one. However, if current returns are Low or High, we observe differences in the probability distribution of the future returns $u(t)$, i.e. average future returns are less dominating under those conditions, while lower and higher future returns are more probable, indicating a high volatility regime.

4.2 Analysis of high frequency return series

In this section, we apply a similar analysis as in Section 4.1 to a real return time series. The goal is to illustrate briefly what kind of information the probabilistic TS fuzzy models can provide when the underlying process that has generated the data is not known.

The data consists of half-hourly samples of the most recent tick of the Dow Jones index of the New York Stock Exchange. The samples run from 31.12.95 19:30 through 31.12.96 19:00, in total 17567 individual data points. Of these, we have used only the samples from the opening

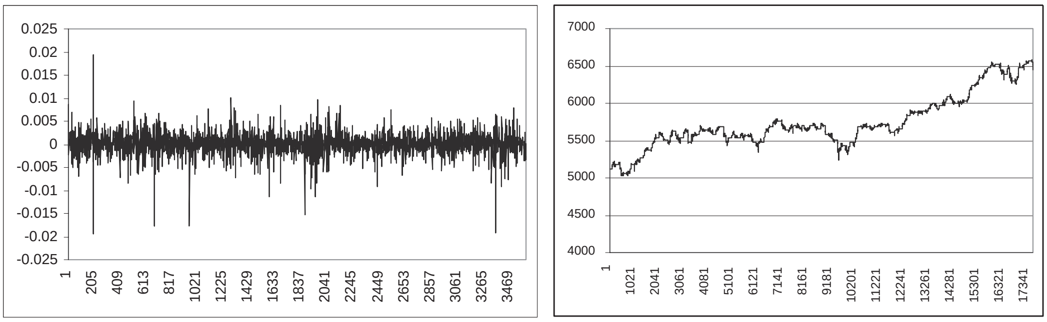

Figure 5: **(left)** Half-hourly returns **(right)** Corresponding Dow Jones index.

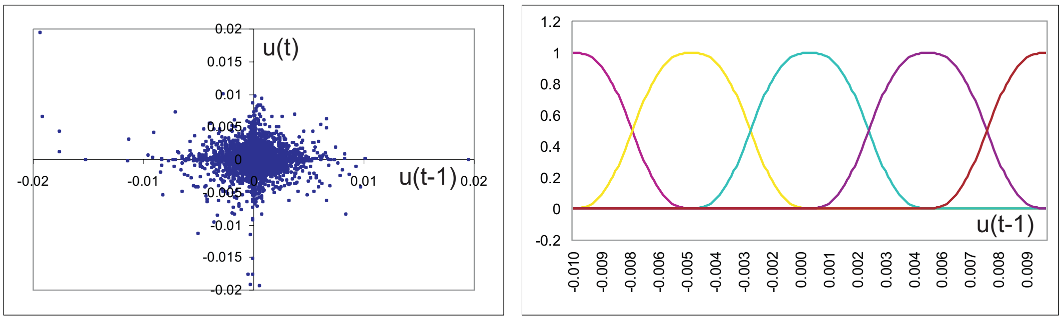

Figure 6: **(left)** Scatter of $\underline{u}(t-1)$ against $\underline{u}(t)$, **(right)** memberships values for $\underline{u}(t-1)$

hours of the market, which has reduced the number of data points to 3594. The half-hourly returns and the index values are depicted in Figure 5:.

We have studied the behavior of one-step ahead returns $\underline{u}(t)$ conditional on $\underline{u}(t-1)$. The scatter plot of $\underline{u}(t)$ against $\underline{u}(t-1)$ is depicted in the left panel of Figure 6:. The right panel of Figure 6: shows the fuzzy partitioning of the antecedent space $\underline{u}(t-1)$, where the fuzzy sets are placed evenly over the antecedent space. The fuzzy approximation for the distribution of the half-hourly returns is depicted in Figure 7:. Note the 'fat-tail-like' phenomenon observed in the figure.

For the consequent space, we use five discrete values, -0.01 (very low), -0.005 (low), 0 (average), 0.005 (high) and 0.01 (very high). Note that these values are same as the cores of the membership functions used for partitioning the antecedent space. Assuming that each half-hourly return is classified to the nearest discrete value, we can compute the unconditional probability distribution $\Pr(u_j) = \Pr(\underline{u}(t) = u_j), j = 1, 2, \ldots, 5$. The results of these calculations are shown in the (emphasized) row (labeled *'All'*) of Table 2:. The conditional probability distributions are computed by using (9), and they are shown in Table 2:.

data.

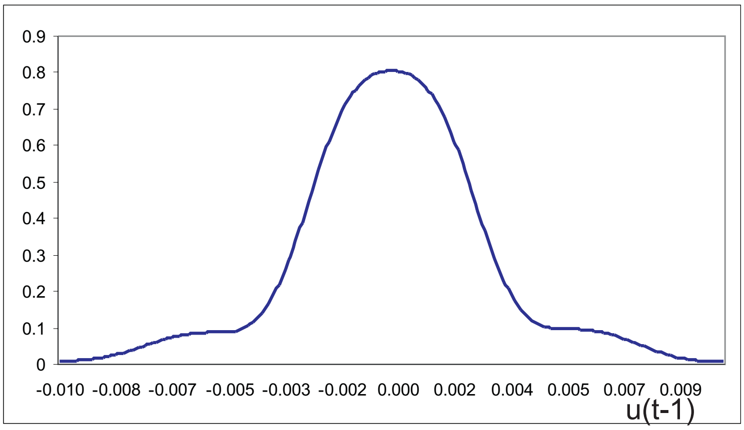

Figure 7: Fuzzy approximation $f_{app}(u)$ of the probability distribution for half-hourly returns.

Future return	very low (-0.01)	low (-0.005)	average (0)	high (0.005)	very high (0.01)
Current return					
All	*0.004*	*0.073*	*0.839*	*0.081*	*0.003*
Very Low	0.000	0.056	0.683	0.220	0.041
Low	0.010	0.083	0.828	0.076	0.003
Average	0.004	0.071	0.840	0.081	0.003
High	0.001	0.079	0.846	0.074	0.000
Very High	0.062	0.127	0.774	0.037	0.000

Table 2: *Unconditional and conditional probabilities* $\Pr(u_j)$ *and* $\Pr(u_j|A_q)$ for the Dow Jones data.

The resulting model can be interpreted by studying the deviations in the rows of Table 2: from the first row of Table 2:. We observe the following. At low current return values $\underline{u}(t-1)$, we have a somewhat higher probability of (very) low future return values $\underline{u}(t)$. This may indicate some GARCH-like behaviour, although the probability of high subsequent values is not increased and the picture is different for high current return values. Far more striking is the behavior at very low and very high current return values. Here, we observe a clear reversal in the sense that very low values have an increased probability to be followed by very high values. For very high values there is even a greater probability of a subsequent very low value. Such reversal behavior has been reported in literature for different sample rates, mostly for long term sampled data such as weekly, monthly or yearly data [1].

5 Conclusions and Discussion

In this paper, we have described zero-order Takagi–Sugeno (TS) probabilistic fuzzy systems, which implement a *stochastic input-output mapping*. If desired, the stochastic mapping can be converted in a *deterministic input-output mapping* describing the expected behavior. For both types of mappings, appropriate reasoning schemes are presented, which, unlike classical fuzzy systems, take the statistical properties of the data explicitly into account. Further, we have shown a technique for representing fuzzy histograms. We illustrated our theoretical observations by analyzing a simulated GARCH type of financial time series data and by analyzing high-frequency Dow Jones index data.

The findings presented in this paper constitute only a first step. Nevertheless, we already believe that the PFSs as presented in this paper will turn out to be a very fruitful paradigm for combining fuzzy and statistical uncertainty and that this framework provides tools for getting "the best of the two worlds". This also enhances the adaptation power of our models to different types of uncertainty present in real-world problems. Extensions of the proposed approach are under construction, such as the design of appropriate probabilistic fuzzy reasoning schemes for other types of FSs. At the same time, we are working on applications, most importantly in the area of financial time series analysis.

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