On the Bass diffusion theory, empirical models and out-of-sample forecasting

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On the Bass diffusion theory, empirical models and out-of-sample forecasting*

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Abstract

The Bass (1969) diffusion theory often guides the construction of forecasting models for new product diffusion. To match the model with data, one needs to put forward a statistical model. This paper compares four empirical versions of the model, where two of these explicitly incorporate autoregressive dynamics. Next, it is shown that some of the regression models imply multi-step ahead forecasts that are biased. Therefore, one better relies on the simulation methods, which are put forward in this paper. An empirical analysis of twelve series (Van den Bulte and Lilien 1997) indicates that one-step ahead forecasts substantially improve by including autoregressive terms and that simulated two-step ahead forecasts are quite accurate.

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1 Introduction and outline

The theory for understanding the diffusion process of new products, as introduced in Bass (1969), is often considered for practical forecasting. It makes adoption and cumulative adoption a function of innovation and imitation effects. The theory also preserves the typically observed sigmoid pattern, which levels off to a maturity level for cumulative adoption.

The basic theory in Bass (1969) is phrased in terms of a continuous process. In practice, of course, one has only discretely observed data. Hence, for calibration of the theory in order to retrieve the structural parameters, one needs to translate the theory into an estimable regression model. A basic aspect of this translation concerns the source of randomness, or, in econometric terms, the stochastic nature of the error process, see also Putsis and Srinivasan (2000).

In this paper I discuss various empirical versions of the Bass diffusion theory, and evaluate their merits for forecasting. As a by-product, I show that the original Bass regression model leads to biased multi-step ahead forecasts. To overcome this problem, one needs to resort to the simulation techniques outlined below. An empirical illustration for the twelve series in Van den Bulte and Lilien (1997) shows that forecasts can be rather different across models, although a few generalizing conclusions can be drawn.

2 Translating theory to practice

The Bass (1969) theory starts with a population of m potential adopters. For each of these, the time to adoption is a random variable with a distribution function $F(\tau)$ and density $f(\tau)$, such that the hazard rate satisfies

$$\frac{f(\tau)}{1 - F(\tau)} = p + qF(\tau), \tag{1}$$

where τ refers to continuous time. The parameters p and q are associated with innovation and imitation, respectively. The cumulative number of adopters at time τ, $N(\tau)$, is a random variable with mean $\bar{N}(\tau) = E[N(\tau)] = mF(\tau)$. The function

$\bar{N}(\tau)$ satisfies the differential equation

$$\bar{n}(\tau) = \frac{d\bar{N}(\tau)}{d\tau} = p[m - \bar{N}(\tau)] + \frac{q}{m}\bar{N}(\tau)[m - \bar{N}(\tau)]. \tag{2}$$

The solution of this differential equation for cumulative adoption is

$$\bar{N}(\tau) = mF(\tau) = m\left[\frac{1 - e^{-(p+q)\tau}}{1 + \frac{q}{p}e^{-(p+q)\tau}}\right], \tag{3}$$

and for adoption itself it is

$$\bar{n}(\tau) = mf(\tau) = m\left[\frac{p(p+q)^2 e^{-(p+q)\tau}}{(p + qe^{-(p+q)\tau})^2}\right]. \tag{4}$$

Analyzing these two functions of τ reveals that $\bar{N}(\tau)$ indeed has a sigmoid pattern.

In practice one of course has only discretely observed data. Denote X_t as the adoptions and N_t as the cumulative adoptions, where t is often months or years, with $t = 1, 2, .., n$. There are now various ways to translate the continuous time theory to models for the data on X_t and N_t.

Bass (1969) proposes to consider the regression model

$$\begin{aligned} X_t &= p(m - N_{t-1}) + \frac{q}{m}N_{t-1}(m - N_{t-1}) + \varepsilon_t & (5)\\ &= \alpha_1 + \alpha_2 N_{t-1} + \alpha_3 N_{t-1}^2 + \varepsilon_t, & (6) \end{aligned}$$

where it is assumed that ε_t is an independent and identically distributed error term with mean zero and common variance σ_2. Note that (p, q, m) must be obtained from $(\alpha_1, \alpha_2, \alpha_3)$, but that for out-of-sample forecasting one can use (6), and hence apply ordinary least squares (OLS).

Recently, Boswijk and Franses (2002) have proposed a modification of this Bass regression model. It is based on the notion that $\bar{N}(\tau)$ can be viewed as an equilibrium path, around which the actual cumulative adoptions may fluctuate. With some additional assumption on the error process, Boswijk and Franses (2002) (BF) arrive at

$$X_t = \beta_1 + \beta_2 N_{t-1} + \beta_3 N_{t-1}^2 + \beta_4 X_{t-1} + \varepsilon_t, \tag{7}$$

with ε_t having the same properties as above. Note that this model adds the regressor X_{t-1} to the original Bass regression model.

Another empirical version of the Bass theory, a version which is often used in practice, is proposed in Srinivasan and Mason (1986). These authors recognize that the Bass (1969) formulation above may introduce aggregation bias, as X_t is simply taken as the discrete representative of $n(\tau)$. Therefore, Srinivasan and Mason (1986) (SM) propose to apply nonlinear least-squares (NLS) to

$$X_t = m[F(t;\theta) - F(t-1;\theta)] + \varepsilon_t, \tag{8}$$

where θ collects p and q. In the analysis below, I also consider the case where ε_t is an autoregression of order 1, for the same arguments why BF modifies the Bass model.

3 Generating forecasts

The Bass-theory-based diffusion models are often used for out-of-sample forecasting. A quick scan of the literature suggests that it apparently is common knowledge how such forecasts are constructed, as no explicit mention is made of how this is done. Still, it might be relevant to make this a bit more explicit, where I only focus on point forecasts.

The SM model seems to imply the most easy to construct forecasts. Suppose one aims to predict X_{n+h}, where n is the forecast origin and h is the horizon, then, given the assumption on the error term, the forecast is

$$\hat{X}_{n+h} = \hat{m}[F(n+h;\hat{\theta}) - F(n-1+h;\hat{\theta})], \tag{9}$$

where $\hat{m}$ and $\hat{\theta}$ are obtained for the sample $t = 1, 2, ..., n$. When the error term is AR(1), straightforward modifications of this formula should be made. If the error term has an expected value equal to zero, then these forecasts are unbiased. for any h.

This is in contrast with the Bass regression model, and also its Boswijk-Franses modification. For one-step ahead, the true observation at $n + 1$ is

$$X_{n+1} = \alpha_1 + \alpha_2 N_n + \alpha_3 N_n^2 + \varepsilon_{n+1}. \tag{10}$$

The forecast from origin n, based on conditional expectations will then be

$$\hat{X}_{n+1} = \hat{\alpha}_1 + \hat{\alpha}_2 N_n + \hat{\alpha}_3 N_n^2 \tag{11}$$

and the squared forecast error is σ^2. This forecast is unbiased.

For two steps ahead, matters become different. The true observation is equal to

$$X_{n+2} = \alpha_1 + \alpha_2 N_{n+1} + \alpha_3 N_{n+1}^2 + \varepsilon_{n+2}, \tag{12}$$

which, as $N_{n+1} = N_n + X_{n+1}$, equals

$$X_{n+2} = \alpha_1 + \alpha_2 (X_{n+1} + N_n) + \alpha_3 (X_{n+1} + N_n)^2 + \varepsilon_{n+2}. \tag{13}$$

Upon substituting X_{n+1}, this becomes

$$\begin{aligned} X_{n+2} = {} & \alpha_1 + \alpha_2(\alpha_1 + \alpha_2 N_n + \alpha_3 N_n^2 + \varepsilon_{n+1} + N_n) \\ & + \alpha_3(\alpha_1 + \alpha_2 N_n + \alpha_3 N_n^2 + \varepsilon_{n+1} + N_n)^2 + \varepsilon_{n+2}. \end{aligned} \tag{14}$$

Hence, the two-step ahead forecast error is based on

$$\hat{X}_{n+2} - X_{n+2} = \varepsilon_{n+2} + \alpha_2 \varepsilon_{n+1} + \alpha_3 (2\alpha_1 \varepsilon_{n+1} + 2(\alpha_2 + 1) N_n \varepsilon_{n+1} + 2\alpha_3 N_n^2 \varepsilon_{n+1} + \varepsilon_{n+1}^2). \tag{15}$$

This shows that the expected forecast error is

$$E(\hat{X}_{n+2} - X_{n+2}) = \alpha_3 \sigma^2. \tag{16}$$

It is straightforward to derive that for h is 3 or more, this bias grows exponentially with h. Naturally, the size of the bias depends on α_3 and σ^2, which both can be small. As the sign of α_3 is always negative, the forecast is downward biased.

To obtain unbiased forecasts for the Bass (and BF) regression models for $h = 2, 3...$, one needs to resort to simulation techniques. Consider again the Bass regression, where it is now written as

$$X_t = g(Z_{t-1}; \pi) + \varepsilon_t, \tag{17}$$

where Z_{t-1} contains 1, N_{t-1} and N_{t-1}^2, and π concerns p, q and m. A simulation-based one-step ahead forecast is now given by

$$X_{n+1,i} = g(Z_n; \hat{\pi}) + e_i, \tag{18}$$

where e_i is a random draw from the $N(0, \hat{\sigma}^2)$ distribution. Based on I such draws, an unbiased forecast can be constructed as

$$\hat{X}_{n+1} = \frac{1}{I} \sum_{i=1}^{I} X_{n+1,i}. \tag{19}$$

Note that a convenient by-product of this approach is the full distribution of the forecasts. Indeed, for more than one-step ahead, this distribution is tedious to derive analytically for the Bass regression model, as can be understood from (14).

A two-step simulation-based forecast can be based on the average value of

$$X_{n+2,i} = g(Z_n, X_{n+1,i}; \hat{\pi}) + e_i, \tag{20}$$

again for I draws. Taking this further to h steps, an easy recursion formula can be given and programmed in, for example, Eviews or SPSS.

4 Empirical illustration

In this section I consider the four Bass-theory-based regression models to evaluate their relative forecasting performance. Additionally, I examine whether simulation-based forecasts can lead to improvement. The number of simulated forecasts (I) is set at 10000.

A natural approach to compare the models would be to rely on extensive Monte Carlo simulation experiments. The experiments would lead to the conclusion that simulation-based forecasts for the Bass regression models are to be preferred, in particular for longer horizons. On the other hand, these experiments would not be helpful to compare the four different models, as, most likely, the model which used for creating the artificial data would perform best in terms of forecasting. Indeed, the Bass and SM models are not nested, and hence, comparison using simulations is cumbersome.

Therefore, I decided to use the real-life data reported in Van den Bulte and Lilien (1997). Of each time series, I saved the last two observations for out-of-sample forecast evaluation. The Bass and BF model parameters are estimated using OLS, while those of the SM models with and without autocorrelation, are estimated using NLS.

The results for one-step ahead forecasts are displayed in Table 1. For these forecasts, no simulation-based extrapolation techniques are required, and hence this table contains only four columns. A first observation from Table 1 is that the squared forecast errors can differ widely across models, see for example the corn (1943), corn (1948), foreign language and accelerated program series, where the squared errors can differ by a factor of 30 or even 50. When the focus is on the average rank of the methods in the next panel of Table 1, one can observe that autoregressive dynamics improve forecasts and that the Boswijk and Franses (2002) model performs best. Finally, the last panel shows that all methods improve on the Bass model, and that the SM method with autoregressive dynamics improves the SM method without dynamics.

Table 2 contains the two-step ahead forecast errors for the four models, as well as for the simulation-based methods. Again the relative differences can be quite large. Furthermore, it appears that the original Bass model performs best, whereas the simulation-based Bass forecasts come as second. Also, the addition of autoregressive dynamics does not seem to add much to forecast performance. The final panel emphasizes that most methods do not improve on the Bass model, while the basic Bass model (with and without simulated forecasts) clearly improves on the SM method.

5 Conclusion

This paper dealt with out-of-sample forecasts from Bass-theory-based regression models. Four of these models, two of which contained autoregressive dynamics, were compared using twelve real-life diffusion series. It was found that these autoregressive dynamics improved one-step ahead forecasts but did not substantially improve forecasts beyond the one-period horizon.

This paper also demonstrated that the Bass regression model implies biased multi-step ahead forecasts. Therefore, a simulation-based procedure was outlined, which should deliver unbiased forecasts. The empirical results in this paper did not indicate substantive improvement, which is due to the small size of the adjustment and the fact that only two-step ahead forecasts were considered. It might well be that for other series and longer horizons, the expected improvement could be observed. This extension is considered as an interesting topic for further research.

Table 1: One-step ahead squared forecast errors for the Bass regression model (Bass), the Boswijk-Franses version of the Bass model (Bass-AR), the Srinivasan-Mason regression model (SM) and the SM model with first-order autoregressive errors (SM-AR)

Variable	Bass	Bass-AR	SM	SM-AR
Air conditioner	0.720	0.650	0.666	0.680
Clothes dryer	0.271	0.225	0.214	0.156
Color television	1.566	0.281	1.462	0.972
Corn (1948)	3.222	2.030	66.484	57.379
Corn (1943)	1.658	1.400	21.305	21.956
Tetracycline	3.547	1.086	0.753	0.031
Ultrasound	40.221	103.812	64.303	104.999
Mammography	6.125	22.293	1.279	7.293
CT scanner	2590.823	1286.777	1516.289	849.561
Foreign language	11.869	0.175	0.099	0.002
Accelerated program	3.093	0.215	10.094	9.332
Compulsory school	2.773	2.315	1.655	1.619
Average rank	3.083	2.167	2.500	2.250
Median % improvement over Bass		26.985	14.467	21.743
Median % improvement over SM	-17.372	-1.369		10.622

Table 2: Two-step ahead squared forecast errors for the Bass regression model (Bass), the Bass model with simulated forecasts, (Bass-SIM), the Boswijk-Franses version of the Bass model (Bass-AR), the Bass-AR-SIM model, the Srinivasan-Mason regression model (SM) and the SM model with first-order autoregressive errors (SM-AR)

Variable	Bass	Bass-SIM	Bass-AR	Bass-AR-SIM	SM	SM-AR
Air conditioner	0.163	0.157	0.152	0.150	0.141	0.180
Clothes dryer	0.017	0.017	0.020	0.018	0.043	0.065
Color television	5.658	5.660	2.147	2.170	6.956	4.614
Corn (1948)	0.234	0.470	0.510	0.943	19.649	17.581
Corn (1943)	0.092	0.174	0.016	0.044	33.642	37.501
Tetracycline	0.845	0.964	1.046	1.045	0.022	0.025
Ultrasound	59.881	60.250	118.904	118.302	116.389	141.854
Mammography	0.164	0.224	4.251	4.487	0.499	0.052
CT scanner	3111.083	3121.012	3356.176	3391.121	2211.715	1859.692
Foreign language	0.055	0.053	2.755	2.771	6.708	5.320
Accelerated program	0.085	0.337	8.666	12.820	0.001	0.072
Compulsory school	0.735	0.818	0.565	0.675	0.351	0.379
Average rank	2.833	3.333	3.750	3.917	3.417	3.750
Median % impr. over Bass		-5.954	-20.717	-16.335	-58.654	2.432
Median % impr. over SM	33.606	33.433	-4.981	-4.013		-12.554

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