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Return-Volatility Relationship: Insights from Linear and Non-Linear Quantile Regression

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Abstract

The purpose of this paper is to examine the asymmetric relationship between price and implied volatility and the associated extreme quantile dependence using linear and non linear quantile regression approach. Our goal in this paper is to demonstrate that the relationship between the volatility and market return as quantified by Ordinary Least Square (OLS) regression is not uniform across the distribution of the volatility-price return pairs using quantile regressions. We examine the bivariate relationship of six volatility-return pairs, viz. CBOE-VIX and S&P-500, FTSE-100 Volatility and FTSE-100, NASDAQ-100 Volatility (VXN) and NASDAQ, DAX Volatility (VDAX) and DAX-30, CAC Volatility (VCAC) and CAC-40 and STOXX Volatility (VSTOXX) and STOXX. The assumption of a normal distribution in the return series is not appropriate when the distribution is skewed and hence OLS does not capture the complete picture of the relationship. Quantile regression on the other hand can be set up with various loss functions, both parametric and non-parametric (linear case) and can be evaluated with skewed marginal based copulas (for the non linear case). Which is helpful in evaluating the non-normal and non-linear nature of the relationship between price and volatility. In the empirical analysis we compare the results from linear quantile regression (LQR) and copula based non linear quantile regression known as copula quantile regression (CQR). The discussion of the properties of the volatility series and empirical findings in this paper have significance for portfolio optimization, hedging strategies, trading strategies and risk management in general.

Keywords: Return-Volatility relationship, quantile regression, copula, copula quantile regression, volatility index, tail dependence.

1 Introduction

Quantifiction of relationship between the change in stock index return and changes in the volatility index serves as the basis for hedging. This relationship is mostly quantified as being asymmetric (Badshah, 2012; Dennis, Mayhew & Stivers, 2006; Fleming, Ostdiek, & Whaley, 1995; Giot, 2005; Hibbert, Daigler, & Dupoyet, 2008; Low, 2004; Whaley, 2000; Wu, 2001). Asymmetric relationship means that the negative change in the stock market has higher impact on the volatility index than a positive change or vice versa. The asymmetric volatility-return relationship has been pointed out in two hypothesis i.e., the leverage hypothesis (Black, 1976; Christie, 1982) and the volatility feedback hypothesis (Campbell and Hentchel, 1992).

In a call/put option contract time to maturity and strike price form its basic characteristics, the other inputs viz., risk free rate and divident payout can be decided easily (Black and Scholes, 1973). When pricing an option the expected volatility over the life of the option becomes a critical input, it is also the only input which is not directly observed by market participants. In an actively traded market volatility can be calculated by inverting the chosen option pricing formula for the observed market price of the option. This volatility calculated by inverting the option pricing formula is known as implied volatility. With increasing focus on risk modelling in modern finance modelling and predicting asset volatility along with its dependence with the underlying asset class has become an important research topic.

The change in volatility leads to the movement in the stock market prices. For example an expected rise in volatility will lead to a decline in stock market prices. The volatility indices are used for option pricing and hedging calculations and the change in them gets reflected on the corresponding stock markets. Financial risk is mostly composed of rare or extreme events which results in high risk and lies in the tail of the return distribution. In option pricing rare or extreme events results in volatility skew patterns (Liu, Pan and Wang, 2005).

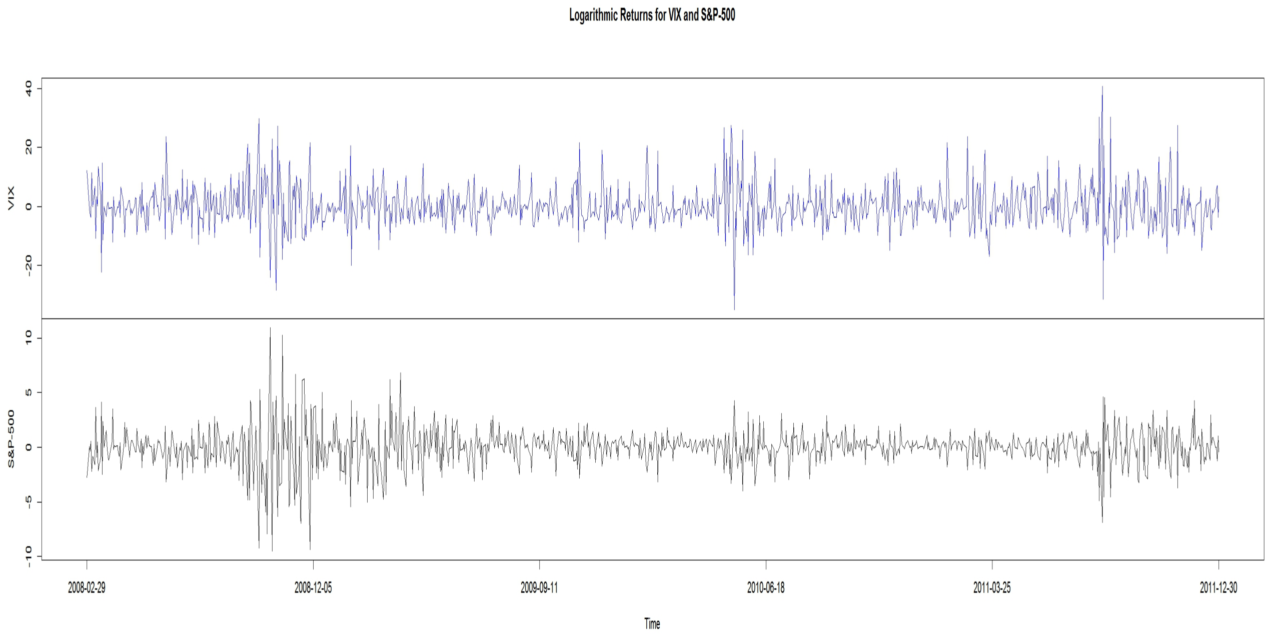


Figure 1: Time Series Plot VIX and S&P 500

Ordinary least squares regression method is the most widely used method for quantifying a relationship between two class of assets or return distribution in finance literature. Figure-1, shows the logarithmic return series of VIX and S&P-500 stock indices from year 2008-2011. The time series plot shows that the VIX index change according to the change in S&P-500. We employ two cases of quantile regression (linear and non-linear) to evaluate the asymmetric volatility-return relationship between change in the volatility index (VIX, VFTSE,VXN, VDAX, VSTOXX and VCAC) and corresponding stock index return (S&P-500, FTSE-100, NASDAQ, DAX-30, STOXX and CAC-40). We focus on the daily asymmetric return-volatility relation in this study.

Giot (2005), Hibbert et al. (2008) and Low (2004) use OLS in their study of asymmetric return-volatility relationship across implied volatility (IV) change distribution. OLS as it is evaluation is based on the deviations from the mean of the distribution underestimates the extreme quantile relationships. Badshah (2012) extends the past studies using LQR to estimate the negative asymmetric return-volatility relationship between stock index return (S&P-500, NASDAQ, DAX-30,STOXX) and changes in volatility index return (VIX, VXN, VDAX, VSTOXX) for lower and upper quantiles which give negative and positive returns . In his study Badshah (2012) found that negative returns has higher impacts than the positive returns using linear quantile regression framework. Kumar (2012), used LQR to examine the statistical properties of volatility index of India and its relationship with Indian stock market.

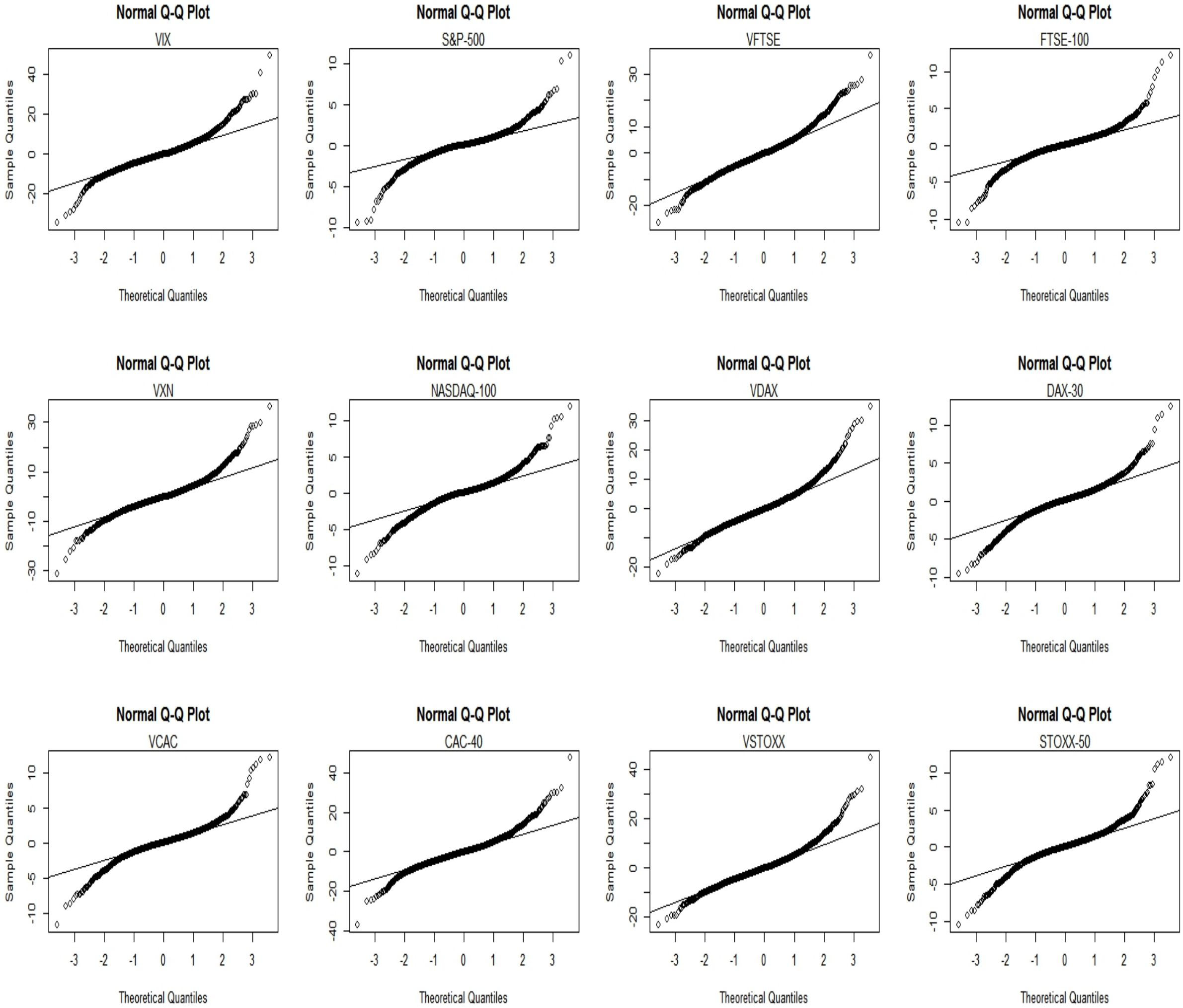


Figure 2: Q-Q Plots

Figure-2, gives the quantile-quantile plots for our data, none of the data series show normality in its distribution. When the data distribution is not normal, QR can provide more efficient estimates for return-volatility relationship (Badshah, 2012). QR can not only be used linearly but can also be evaluated for non-linear relationships using Copula based models. The only comprehensive study (Badshah, 2012) done using QR on return-volatility relationship till now focus on linear case of the relationship. We extend this study by considering the non-linear nature of the relationship using copula based non linear quantile regression models, CQR.

The rest of the paper is designed as follows; in section-2 we give details about linear quantile regression LQR, followed by non-linear quantile regression using copula CQR in section-3. In Section-4 we describe our data together with our research design and methodology. We discuss the results in section-5 and conclude in section-6.

2 Quantile Regression

Regression analysis is undoubtedly the most widely used technique in market risk modelling, from factor models to model returns to autocorrelated models to model volatility in time series. All models are based on regression analysis with different approaches.

A simple linear regression model can be written as:

$$Y = \alpha + \beta X + \varepsilon, \tag{1}$$

which represents the dependent variable, Y as a linear function of one or more independent variable, X, subject to a random 'disturbance' or 'error' term, ε which is assumed to be $i.i.d$ and independent of X.

A bivariate normal distribution is assumed between a dependent and independent variable in simple linear regression. It estimates the mean value of the dependent variable for given levels of the independent variables. For this type of regression, where we want to understand the central tendency in a dataset, OLS is an effective method. OLS loses its effectiveness when we try to go beyond the median value or towards the extremes of a data set (see; Allen, Singh and Powell, 2010; Allen, Gerrans, Singh and Powell, 2009; Barnes and Hughes, 2002). Specifically in the case of an unknown or arbitrary joint distribution (X, Y), OLS does not provide all the necessary information required to quantify the conditional distribution of the dependent variable. As given in descriptive statistics (section-4.1) the dataset used in this analysis is not normal and hence quantile regression can be a better choice.

Quantile Regression is modelled as an extension of classical OLS (Koenker and Bassett, 1978). In Quantile Regression the estimation of conditional mean as estimated by OLS is extended to similar estimation of an ensemble of models of various conditional quantile functions for a data distribution. In this fashion Quantile Regression can better quantify the conditional distribution of $(Y|X)$. The central special case is the median regression estimator that minimises a sum of absolute errors. The estimates of remaining conditional quantile functions are obtained by minimizing an asymmetrically weighted sum of absolute errors, where weights are the function of the quantile of interest. This makes Quantile Regression a robust technique even in presence of outliers. Taken together the ensemble of estimated conditional quantile functions of $(Y|X)$ offers a much more complete view of the effect of covariates on the location, scale and shape of the distribution of the response variable.

For parameter estimation in Quantile Regression, quantiles as proposed by Koenker and Bassett (1978) can be defined through an optimisation problem. To solve an OLS regression problem a sample mean is defined as the solution of the problem of minimising the sum of squared residuals, in the same way the median quantile (0.5%) in Quantile Regression is defined through the problem of minimising the sum of absolute residuals.

The symmetrical piecewise linear absolute value function assures the same number of observations above and below the median of the distribution.

The other quantile values can be obtained by minimizing a sum of asymmetrically weighted absolute residuals, (giving different weights to positive and negative residuals). Solving

$$min\xi\varepsilon R\sum\rho_\tau(y_i-\xi) \tag{2}$$

Where $\rho_\tau(\textbf{.})$ is the tilted absolute value function as shown in Figure 2.4, which gives the τth sample quantile with its solution. Taking the directional derivatives of the objective function with respect to ξ (from left to right) shows that this problem yields the sample quantile as its solution.

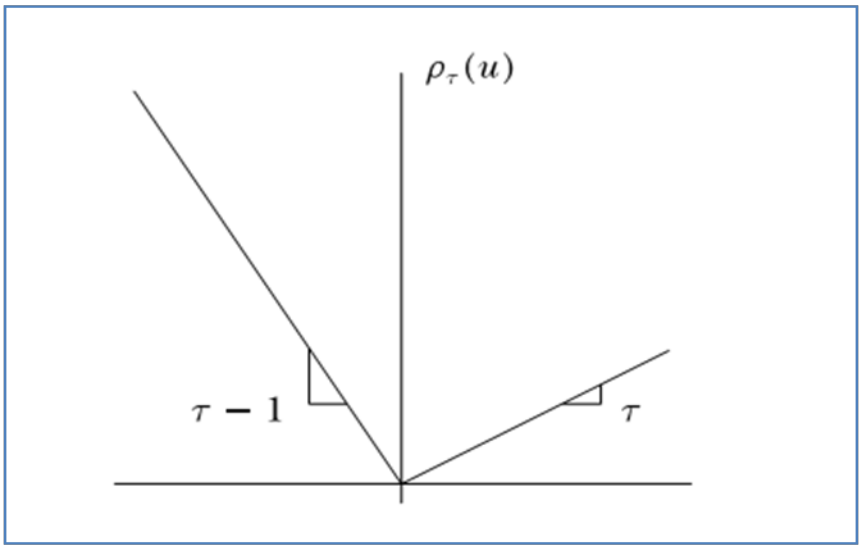


Figure 3: Quantile Regression ρ Function

After defining the unconditional quantiles as an optimisation problem, it is easy to define conditional quantiles similarly. Taking the least squares regression model as a base to proceed, for a random sample, $y_1, y_2, \ldots, y_n$, we solve

$$min_{\mu\varepsilon R}\sum_{i=1}^{n}(y_i-\mu)^2 \tag{3}$$

which gives the sample mean, an estimate of the unconditional population mean, EY. Replacing the scalar μ by a parametric function $\mu(x,\beta)$ and then solving

$$min_{\mu\varepsilon R^p}\sum_{i=1}^{n}(y_i-\mu(x_i,\beta))^2 \tag{4}$$

gives an estimate of the conditional expectation function *E(Y|x)*.

Proceeding the same way for Quantile Regression, to obtain an estimate of the conditional median function, the scalar ξ in the first equation is replaced by the parametric function $\xi(x_t, \beta)$ and τ is set to $1/2$. The estimates of the other conditional quantile functions are obtained by replacing absolute values by $\rho_\tau(.)$ and solving

$$min_{\mu \varepsilon R^p} \sum \rho_\tau(y_i - \xi(x_i, \beta)) \tag{5}$$

The resulting minimization problem, when $\xi(x, \beta)$ is formulated as a linear function of parameters, and can be solved very efficiently by linear programming methods. Further insight into this robust regression technique can be obtained from Koenkar and Bassett's Quantile Regression monograph (2005) or a text book introduction to Quantile Regression as can be found in Alexandar (2008).

Quantile regression has been frequently tested in research work over the past decade in various areas of econometric analysis, financial modelling and socio-economic research. The studies include Buchinsky and Leslie (1997) who analyse changing US wage structures. Buchinsky and Hunt (1999) analyse the earning mobility and factors affecting the transmission of earnings across generations. Eide and Showanter (1998) study the effect of school quality on education. Financial research work using Quantile Regression includes Engle and Manganelli (2004) and Morillo (2000) quantifying VaR using Quantile Regression and studying option pricing using Monte Carlo simulations. Barnes and Hughes (2002) applied Quantile Regression to study CAPM in their work on cross sections of stock market returns. Chan and Lakonishok (1992) applied Quantile Regression to robust measurement of size and book to market effects. Gowlland, Xiao and Zeng (2009) investigate book to market effect beyond central tendency. Allen, Singh and Powell (2011), apply Quantile Regression to test the Fama-French factor model in the DJIA-30 stocks which focuses on the applicability of better estimates of factor based risk factors across quantiles.

Other than Badshah (2012) and Kumar (2012) there is no prior work done on investigating the return-volatility relationship between volatiltiy indices and corresponding market indices using quantile regression. We not only apply the LQR model to evaluate the return-volatility relationship but we also test the non-linear case of CQR to examine the relationship.

3 Non-Linear Quantile Regression (CQR)

Bouyé and Salmon (2009) extended Koenker and Basset's (1978) idea of regression quantiles and introduced a general approach to non linear quantile regression modelling using copula functions. Copula functions are used to define the dependence structure between the dependent and independent variables of interest. We first give a brief introduction to copula followed by the introduction to the concept of CQR.

3.1 Copula

Modelling dependency structure within assets is a key issue in risk measurement. The most common measure for dependency, correlation, loses its effect when a dependency measure is required for distribution which deviates from the mean or are not normally distributed. Examples of deviations from normality are the presence of kurtosis or fat tails and skewness in univariate distributions. Deviation from normality also occurs in multivariate distributions given by asymmetric dependence, which infers that assets show different level of correlation during different market conditions (Erb et al., 1994; Longin and Solnik, 2001; Ang and Chen, 2002 and Patton, 2004). Modelling dependence with correlation is not inefficient when the distribution follow the strict assumptions of normality and constant dependence across the quantiles. But as now it is well known in financial risk modelling, return distribution does not necessarily follow normality across quantiles, we need more sophisticated tools for modelling dependence than correlation and Copulas provide one such measure.

The statistical tool which is used to model the underlying dependence structure of a multivariate distribution is the copula function. The capability of copula to model and estimate multivariate distributions comes from *Sklar's Theorem,* according to which each joint distribution can be decomposed into its marginal distributions and a copula C responsible for the dependence structure. Here we define Copula with Sklar's theorem along with some important types of copula, adapted from Franke, Härdle and Hafner (2008).

A function $C : [0,1]^d \to [0,1]$ is a d dimensional copula if it satisfies the following conditions for every $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$ and $j \in \{1, \ldots, d\}$

1. if $u_j = 0$ then $C(u_1, \ldots, u_d) = 0$
2. $C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$
3. for every $\upsilon = (\upsilon_1, \ldots, \upsilon_d)^\top \in [0,1]^d, \upsilon_j \leq u_j$

$$V_C(u, \upsilon) \geq 0$$

where $V_C(u, \upsilon)$ is given by

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\ldots+i_d} C(g_{1i_1}, \ldots, g_{di_d})$$

Properties 1 and 3 state that copulae are grounded functions and that all d-dimensional boxes with vertices in $[0,1]^d$ have non-negative C-volume. Property second shows that the copulae have uniform marginal distributions.

Sklar's Theorem

Consider a d-dimensional distribution function F with marginals $F_1 \ldots, F_d$. Then for every $x_1, \ldots, x_d \in \overline{\mathbb{R}}$, a copula, C can exist with

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\} \tag{6}$$

C is unique if $F_1 \ldots, F_d$ are continuous. If $F_1, \ldots, F_d$ are distributions then the function F is a joint distribution function with marginals $F_1, \ldots, F_d$.

For a joint distribution F with continuous marginals $F_1, \ldots, F_d$, for all $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$ the unique copula C is given as

$$C(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\} \tag{7}$$

Copula can be divided into two broad types, Elliptical Copulae-Gaussian Copula and Student's t-copula and Archimedean Copulae-Gumbel copula and Clayton copula and Frank Copula .

Normal or Gaussian Copula

The copula derived from the n-dimensional multivariate and univariate standard normal distribution functions, $\boldsymbol{\Phi}$ and Φ, is called a normal or gaussian copula. The normal copula can be defined as

$$C(u_1, \ldots, u_n; \boldsymbol{\Sigma}) = \boldsymbol{\Phi}\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\right) \tag{8}$$

where correlation matrix($\boldsymbol{\Sigma}$) is the parameter for normal copula and $u_i = F_i(x_i)$ is the marginal distribution function.

The normal copula density is given by

$$c(u_1, \ldots, u_n; \boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{\xi}'(\boldsymbol{\Sigma^{-1}} - \mathbf{I})\boldsymbol{\xi}\right) \tag{9}$$

where $\boldsymbol{\Sigma}$ is the correlation matrix, $|\boldsymbol{\Sigma}|$ is its determinant. $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)'$, where ξ_i is the u_i quantile of the standard normal variable X_i.

Figure- 4 gives the density plot for a bivariate gaussian copula with a correlation of 0.5. As shown in the figure, normal copula is a symmetric copula.

Student's t-Copula

Similar to gaussian copula, t-copula models the dependence structure of multivariate t-distributions. The parameters for student's t-copula are correlation matrix and degrees of freedom. Student's t-copula show symmetrical dependence but are higher than those

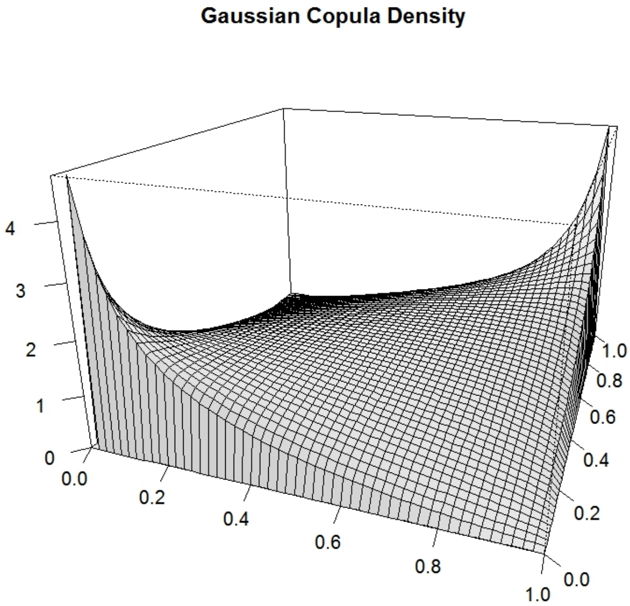


Figure 4: Density of the Gaussian copula

in gaussian copula as shown in figure-5. Kindly refer Alexandar (2008) for the density functions and quantile functions of student-t copula.

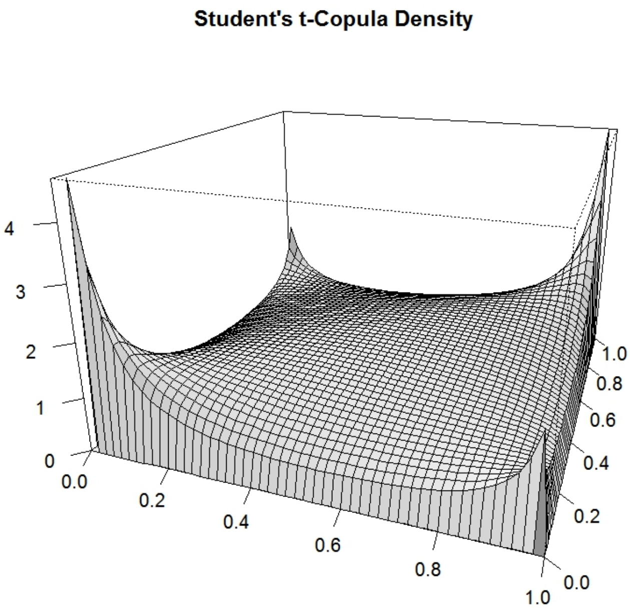


Figure 5: Density of t-copula

Archimedean Copulae

Archimedean copulae are family of copulae which are build on a generator function, with some restrictions. There can be various copulae in this family of copulae due to various generator functions available (see Nelson (1999)). For a generator function ϕ the Archimedean copula can be defined as

$$C(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \ldots + \phi(u_n)) \tag{10}$$

The density function is given by

$$c(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \ldots + \phi(u_n)) \prod_{i=1}^{n} \phi'(u_i) \tag{11}$$

Clayton Copula

The Clayton copula, as introduced by Clayton (1978) has a generator function;

$$\phi(u) = \alpha^{-1}(u^{-\alpha} - 1), \quad \alpha \neq 0 \tag{12}$$

the inverse generator function is

$$\phi^{-1}(x) = (\alpha x + 1)^{-1/\alpha}$$

With variation in parameter α the Clayton copulas capture a range of dependence. Clayton copula is particularly helpful in capturing positive lower tail dependence. Figure-6 gives a density plot for bivariate clayton copula with $\alpha = 0.5$, the asymmetric lower tail dependence is evident from the figure.

Like Normal and Student-t copula, Archimedean copulae can also be used for CQR[1]. Here we use only Normal and Student-t copula for our analysis as they capture both positive and negative dependnece, the clayton copula captures only positive lower tail dependence and hence its left out.

We will not further discuss the types of copula in detail but rather refer to Joe (1997) and Nelsen (1999), Alexandar (2008) and Cheung (2009) who give a useful overview of copula for financial practitioners. The quantile functions of the copulas used in the CQR are reported in the following discussion of copula quantile regression. The quantile function of Clayton is also given for the completeness.

3.2 Copula Quantile Regression (CQR)

Bouyé and Salmon (2009) in their work has discussed copula quantile regression in detail by highlighting the properties of quantile curves. They also gave the simple closed forms of

[1] The example of Clayton Copula with its quantile function is given in next subsection.

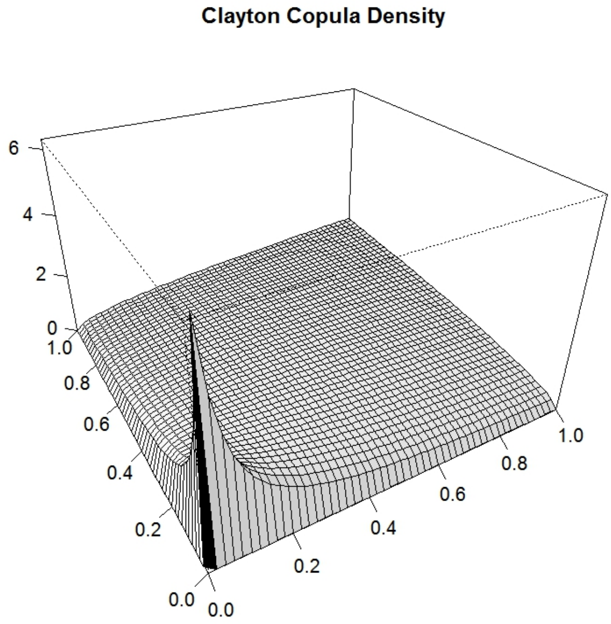


Figure 6: Density of Clayton Copula

the quantile curve for major copula (normal, Student t, Joe-Clayton and Frank) which are used in the linear quantile regression model (equation-5) to calculate non-linear regression quantiles. Here we will just give the closed form of the four copula quantile curve for the sake of brevity, please refer to the original paper by Bouyé and Salmon (2009) for detailed discussion. Alexandar (2008) also gives a breif introduction of non-linear copula based quantile regressions and also give some empirical examples using excel work books.

The non-linear quantile regression model is formed by replacing linear quantile regression model (5) with the quantile curve of a copula. Every copula has a quantile curve which may be decomposed in an explicit function.

If we have two marginals $F_X(x)$ and $F_Y(y)$ of x and y, with their estimated distribution parameters. We can then define a bivariate copula with certain parameters $\boldsymbol{\theta}$.

Normal CQR

The bivariate normal copula has one parameter, the correlation ϱ, its quantile curve can be written as

$$y = F_Y^{-1}\left[\Phi\left(\varrho\Phi^{-1}(F_X(x)) + \sqrt{1-\varrho^2}\Phi^{-1}(q)\right)\right] \tag{13}$$

Student-t CQR

The Student-t copula has two parameters, the degree of freedom ν and the correlation ϱ. The quantile curve of Student t copula is given by

$$y = F_Y^{-1}\left[t_\nu\left(\varrho t_\nu^{-1}(F_X(x)) + \sqrt{(1-\varrho^2)(\nu+1)^{-1}\left(\nu + t_\nu^{-1}(F_X(x))^2\right)}t_{\nu+1}^{-1}(q)\right)\right] \tag{14}$$

Clayton CQR

Clayton copula is a member of Archimedean Copulae with a generator function having parameter α. The quantile curve of Clayton copula is given by

$$y = F_Y^{-1}\left[\left(1+F_X(\mathrm{x})^{-\alpha}\left(q^{-\alpha/(1+\alpha)}-1\right)\right)^{-1/\alpha}\right] \tag{15}$$

To evaluate non-linear quantile regression using copula, for a given sample $\{(x_t, y_t)\}_{t=1}^{T}$ the q (or τ) quantile regression curve can be defined as $y_t = \xi(x_t, q; \hat{\boldsymbol{\theta}}_{\boldsymbol{q}})$. The parameters $\hat{\boldsymbol{\theta}}_{\boldsymbol{q}}$are found by solving the following optimization problem.

$$min_{\mu \varepsilon R^p} \sum \rho_q(y_t - \xi(x_t, q; \boldsymbol{\theta})) \tag{16}$$

This optimization problem can be solved by using Quantreg package of statistical software R after defining the copula using copula related packages.

In this study we use LQR and CQR with normal or gaussian and Student-t copula to evaluate the return-volatility relationship. We will now discuss the data and methodology implemented in the following section.

4 Data and Methodology

4.1 Description of Data

In this empirical analysis we use daily price data for market and volatility indices of six volatility-return pairs viz., VIX and S&P-500, VFTSE and FTSE 100, VXN and NASDAQ, VDAX and DAX-30, VCAC and CAC-40 and VSTOXX and STOXX. We obtained daily prices from Datastream for a period of approximately 10 years from 2/02/2001 to 31/12/2011. Daily percentage logarithmic returns are used for the analysis. Table-1, gives the descriptive statistics for our dataset. All the dataseries show excess kurtosis indicating fat tails, the Jarque-Berra test statistics in table-1 for normal distribution strongly rejects the presence of normal distribution in the series. With the descriptive statistics we can conclude that all the return time series (for market and volatility series) exhibit fat tails and are not normally distributed. The ADF test statistics also rejects

	VIX	**S&P-500**	**VFTSE**	**FTSE-100**	**VXN**	**NASDAQ**	**VDAX**	**DAX-30**	**VCAC**	**CAC-40**	**VSTOXX**	**STOXX-50**
Observations	2845	2845	2845	2845	2845	2845	2845	2845	2845	2845	2845	2845
Minimum	-35.0588	-9.4695	-26.7893	-10.5381	-31.3049	-11.1149	-22.2296	-9.6010	-11.7370	-37.1866	-23.4362	-10.4552
Quartile 1	-3.5950	-0.5659	-3.6566	-0.6881	-2.9385	-0.8043	-3.1912	-0.8592	-0.8263	-3.3499	-3.4479	-0.8517
Median	-0.3063	0.0273	-0.1913	0.0448	-0.1797	0.0462	-0.2621	0.0701	0.0313	-0.1943	-0.3179	0.0183
Arithmetic Mean	0.0022	-0.0025	0.0092	-0.0021	-0.0304	-0.0029	0.0287	0.0074	-0.0099	0.0249	0.0263	-0.0130
Quartile 3	2.9640	0.5964	3.1760	0.7634	2.4648	0.8377	2.7955	0.9174	0.8808	2.8021	2.8650	0.8792
Maximum	49.6008	10.9572	37.1670	12.2189	36.2851	11.8493	34.5301	12.3697	12.1434	47.5537	44.6046	11.9653
SE Mean	0.1177	0.0255	0.1142	0.0288	0.0973	0.0343	0.0986	0.0338	0.0331	0.1116	0.1066	0.0335
LCL Mean (0.95)	-0.2284	-0.0525	-0.2147	-0.0585	-0.2213	-0.0701	-0.1646	-0.0589	-0.0748	-0.1939	-0.1828	-0.0787
UCL Mean (0.95)	0.2329	0.0475	0.2331	0.0543	0.1605	0.0644	0.2220	0.0738	0.0550	0.2438	0.2353	0.0527
Variance	39.3801	1.8481	37.0906	2.3561	26.9609	3.3484	27.6513	3.2562	3.1183	35.4488	32.3380	3.1923
Stdev	6.2754	1.3595	6.0902	1.5350	5.1924	1.8299	5.2585	1.8045	1.7659	5.9539	5.6866	1.7867
Skewness	0.7148	-0.1798	0.5494	-0.1176	0.6461	0.0327	0.7517	-0.0518	0.0205	0.5678	0.8619	-0.0214
Kurtosis	4.8375	8.0012	2.2572	7.9493	4.4896	4.2954	3.2289	4.4147	5.8562	4.4866	3.8149	5.0474
ADF	-58.9219	-58.7627	-56.6697	-54.9693	-55.6025	-56.9344	-52.6907	-53.3281	-54.0247	-57.2406	-55.0843	-53.8264
JarqueBera	3022.9229	7618.9607	749.2105	7511.9282	2593.1979	2192.8800	1507.5219	2317.0224	4074.3108	2544.8816	2082.1950	3026.9972

Table 1: Descriptive Statistics

the presence of unit root in the time series.

4.2 Methodology

With this empirical exercise we evaluate the volatility-return relationship which can be represented by the following:

$$V_t = \alpha + \beta R_t + \varepsilon \tag{17}$$

where V_t is the daily logarithmic return of the volatility index and R_t gives the daily logarithmic return of the market index. α, β and ε gives the intercept , the coefficient which represent the degree of association and the error term respectively.

We will use three regression techniques in this study, the basic linear regression or OLS, linear quantile regression and non-linear copula quantile regression to quantify the return-volatility relationship for our six return-volatility pairs. The relationship quantified by OLS is around the mean of the distribution and hence does not quantify the tail regions. In this study we examine if the relationship quantified by the quantile regression are different from OLS and if they are different across the various quantiles in the distribution.

The major results from the study are discussed in the following section.

5 Discussion of the Results

5.1 Linear Regression-OLS

We will first evaluate the volatility-return relationship using OLS. As mentioned before OLS gives the relationship around the mean of the distribution and hence leaves out the extreme cases, like when the market is in crisis or when it is performing well. The relationship quantified by OLS gives the relationship between the average of volatility and return series.

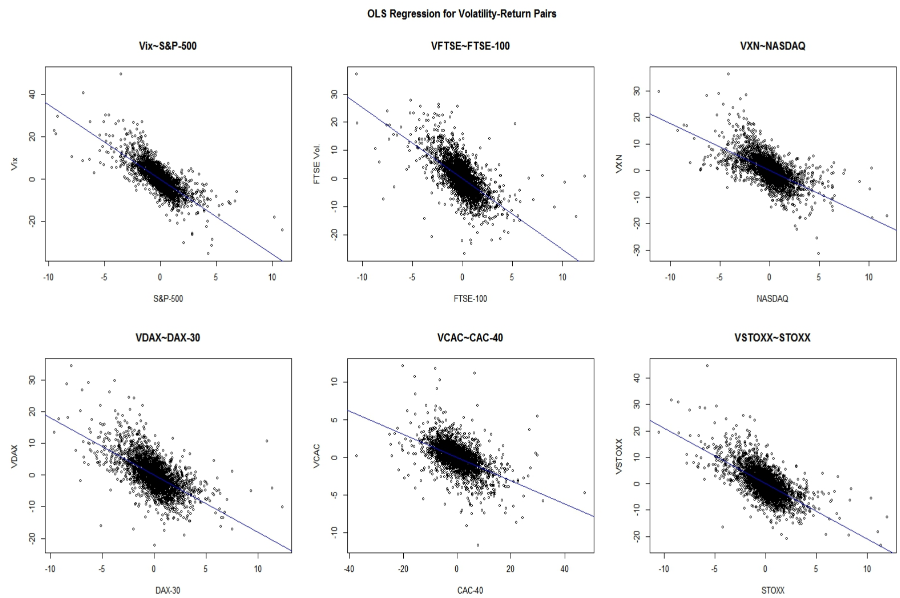


Figure 7: OLS Regression for Volatility-Return Pairs

Figure-7 gives the plot of OLS regression fit with the actual volatility-return data. The common observation in all the figures is that the regression line runs through the mean of the observations. As the regression line represent the mean behaviour, the estimated values are around the mean of the distribution and is unable to quantify the tails or other quantiles diverting from mean. Table-2 gives the point estimates of the intercept and regression coefficient for all the volatility-return pairs, the values of the regression coefficient indicate an inverse volatility return relationship. These results confirms the earlier research work.

	α	**P-Value**	β	**P-Value**
VIX-S&P	-0.0065	0.9325	-3.5147	0.0000
VFTSE-FTSE	0.0039	0.9646	-2.5387	0.0000
VXN-NASDAQ	-0.0355	0.6419	-1.7651	0.0000
VDAX-DAX	0.0421	0.5862	-1.8059	0.0000
VCAC-CAC	-0.0060	0.8304	-0.1549	0.0000
VSTOXX-STOXX	-0.0011	0.9893	-2.1028	0.0000

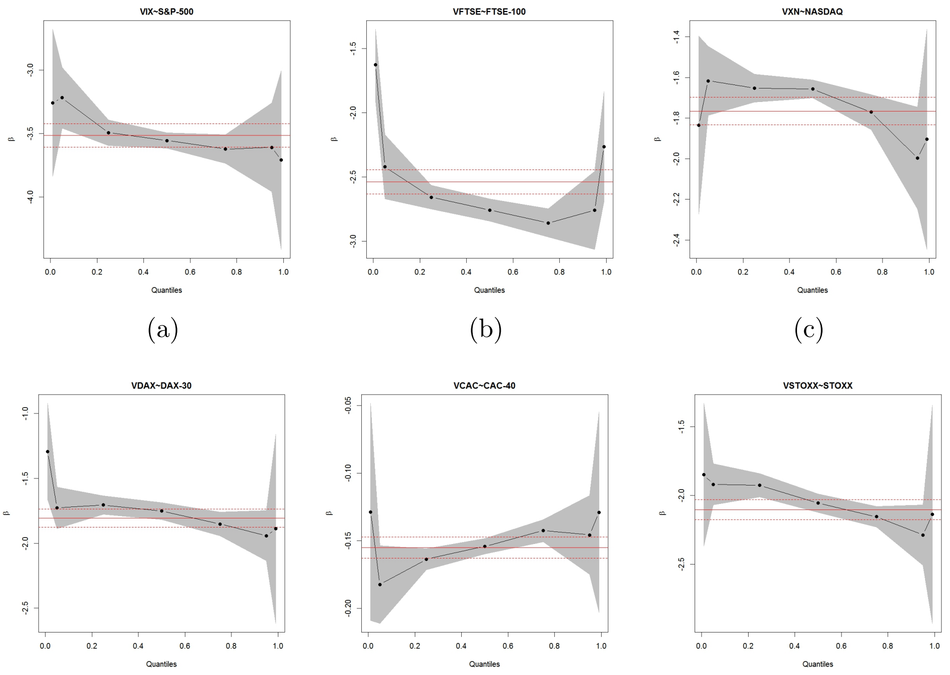
Table 2: OLS Regression Results

All the β values are significant at 99% level in the results

5.2 Linear Quantile Regression (LQR)

In financial risk measurement quantification of tails plays an important role in risk modelling. OLS estimates quantifies the relationship around the mean of the distribution but QR on the other hand can be used to quantify the relationship across various quantiles. We use LQR to model the volatility-return relationship across quantiles, we focus particularly on lower quantiles which represent high negative returns and represent the risk in the market. We evaluate volatility-return relationship across seven quantiles of interest $q = \{0.01, 0.05, 0.25, 0.5, 0.75, 0.95, 0.99\}$ which includes median as well as two extremes, lower 1% and higher 99% quantile.

Figure-8 gives the plots for the LQR coefficient (β) for all the volatility-return pairs, it is evident from the figure that these coefficients are different across the quantiles and hence the relationship also changes.



(a) (b) (c)

(d) (e) (f)

Figure 8: Volatility-Return Coefficient (β) Estimates Across Quantiles

Table-3 gives the estimates for the LQR model with intercept α and coefficient β which measures the dependence of volatility on market return. The dependence coefficient (β) values are significant across the quantiles and are also not same. The results clearly indicate that the volatility-return relationship changes across quantiles and it is also statistically significant.

	Quantile Regression Estimates													
	$\alpha_{0.01}$	$\beta_{0.01}$	$\alpha_{0.05}$	$\beta_{0.05}$	$\alpha_{0.25}$	$\beta_{0.25}$	$\alpha_{0.5}$	$\beta_{0.5}$	$\alpha_{0.75}$	$\beta_{0.75}$	$\alpha_{0.95}$	$\beta_{0.95}$	$\alpha_{0.99}$	$\beta_{0.99}$
VIX-S&P	-9.92	-3.26	-5.72	-3.22	-2.31	-3.49	-0.09	-3.55	1.95	-3.62	6.59	-3.61	12.06	-3.71
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.13	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VFTSE-FTSE	-11.87	-1.63	-6.84	-2.42	-2.63	-2.66	-0.15	-2.76	2.45	-2.86	7.51	-2.76	13.31	-2.26
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VXN-NASDAQ	-9.75	-1.83	-6.07	-1.62	-2.28	-1.65	-0.16	-1.66	2.00	-1.77	6.90	-2.00	12.22	-1.90
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VDAX-DAX	-10.12	-1.29	-5.96	-1.73	-2.43	-1.71	-0.06	-1.75	2.26	-1.85	6.56	-1.94	12.60	-1.89
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.43	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VCAC-CAC	-4.48	-0.13	-2.23	-0.18	-0.72	-0.16	0.02	-0.15	0.71	-0.14	2.18	-0.15	4.10	-0.13
p-value	0.00	0.01	0.00	0.00	0.00	0.00	0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VSTOXX-STOXX	-9.62	-1.85	-6.44	-1.92	-2.58	-1.93	-0.15	-2.05	2.34	-2.15	6.86	-2.29	12.41	-2.14
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 3: LQR Results

p-value of ≤ 0.05 *shows significance at 95% level or higher*

5.3 Copula Quantile Regression (CQR)

LQR quantifies linear volatility-return relationship but CQR can be used to quantify this relationship in a non-linear framework. In CQR the non linear volatility-return relationship is quantified by the copula quantile functions of the respective copula. We use Normal and Student-t Copula in this part of the analysis.

The marginals for the bivariate CQR are assumed to be Student-t distribution. The data is first transformed to marginals by fitting it to the standard Student-t distribution. The estimates are calculated using the Quantreg package in R.

Table-4 gives the ϱ estimates for the seven quantiles for Normal and Student-t copula. In most of the pairs the negative dependence is greater for low and high quantiles. Also the lower tail negative dependence is higher than the upper tail negative dependence.

Figure-9 plots the estimates for Student-t CQR for all the volatility-return pairs across the quantiles. The figure shows that the graph of the estimates have an approximate inverted U shape except for VIX-S&P 500. The inverted U shape (higher dependence across tails) is most prominent for VCAC-CAC 40 pair.

	Normal CQR						
	$\varrho_{0.01}$	$\varrho_{0.05}$	$\varrho_{0.25}$	$\varrho_{0.5}$	$\varrho_{0.75}$	$\varrho_{0.95}$	$\varrho_{0.99}$
VIX-S&P	-0.8645	-0.8913	-0.8861	-0.8916	-0.9099	-0.8696	-0.8602
VFTSE-FTSE	-0.7735	-0.8114	-0.7925	-0.8137	-0.8287	-0.7891	-0.7497
VXN-NASDAQ	-0.8066	-0.8164	-0.7647	-0.7045	-0.8047	-0.7831	-0.7718
VDAX-DAX	-0.7584	-0.7937	-0.7322	-0.7106	-0.7841	-0.7785	-0.7479
VCAC-CAC	-0.6915	-0.7755	-0.7067	-0.5631	-0.6759	-0.7691	-0.6970
VSTOXX-STOXX	-0.8015	-0.8055	-0.7517	-0.7496	-0.8122	-0.7999	-0.7469
	Student-t CQR						
	$\varrho_{0.01}$	$\varrho_{0.05}$	$\varrho_{0.25}$	$\varrho_{0.5}$	$\varrho_{0.75}$	$\varrho_{0.95}$	$\varrho_{0.99}$
VIX-S&P	-0.8726	-0.8920	-0.8814	-0.8857	-0.9030	-0.8691	-0.8617
VFTSE-FTSE	-0.7815	-0.8058	-0.7780	-0.7818	-0.8048	-0.7827	-0.7915
VXN-NASDAQ	-0.8207	-0.8074	-0.7500	-0.6930	-0.7916	-0.7806	-0.7794
VDAX-DAX	-0.7696	-0.8016	-0.7159	-0.6911	-0.7578	-0.7782	-0.7663
VCAC-CAC	-0.7868	-0.7608	-0.6436	-0.4888	-0.5959	-0.7667	-0.7661
VSTOXX-STOXX	-0.8190	-0.8065	-0.7404	-0.7322	-0.7990	-0.8027	-0.7458

Table 4: Normal and Student-t CQR Estimates

All the estimates given in the table are found to be statistically significant.

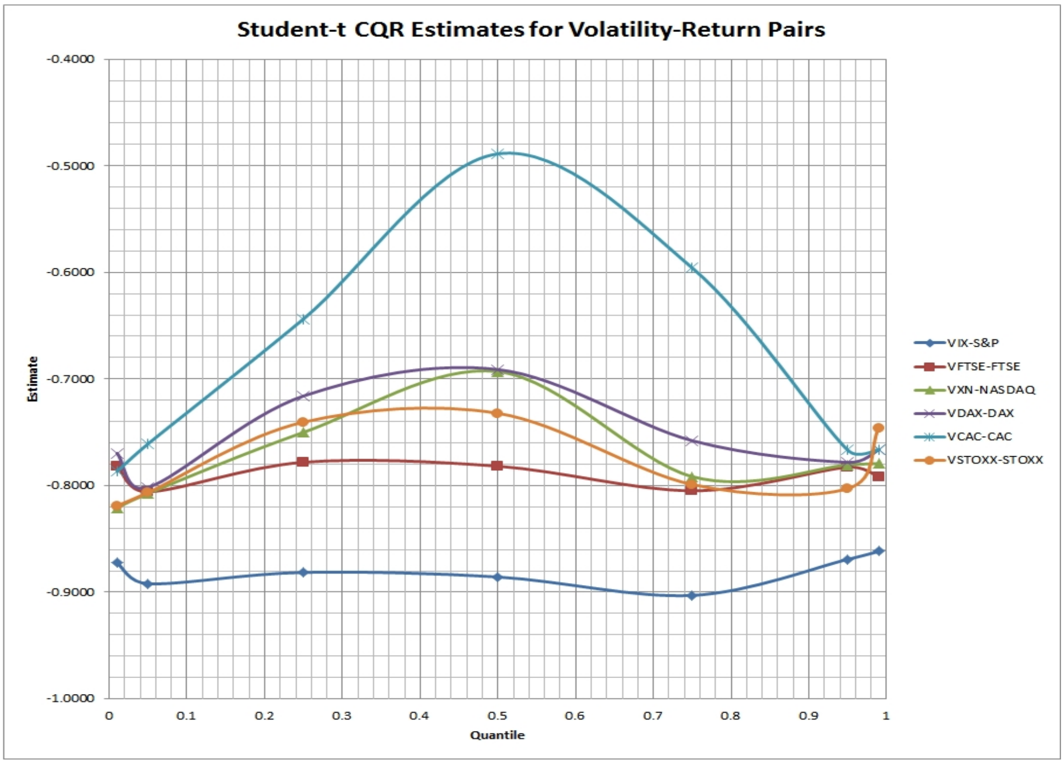
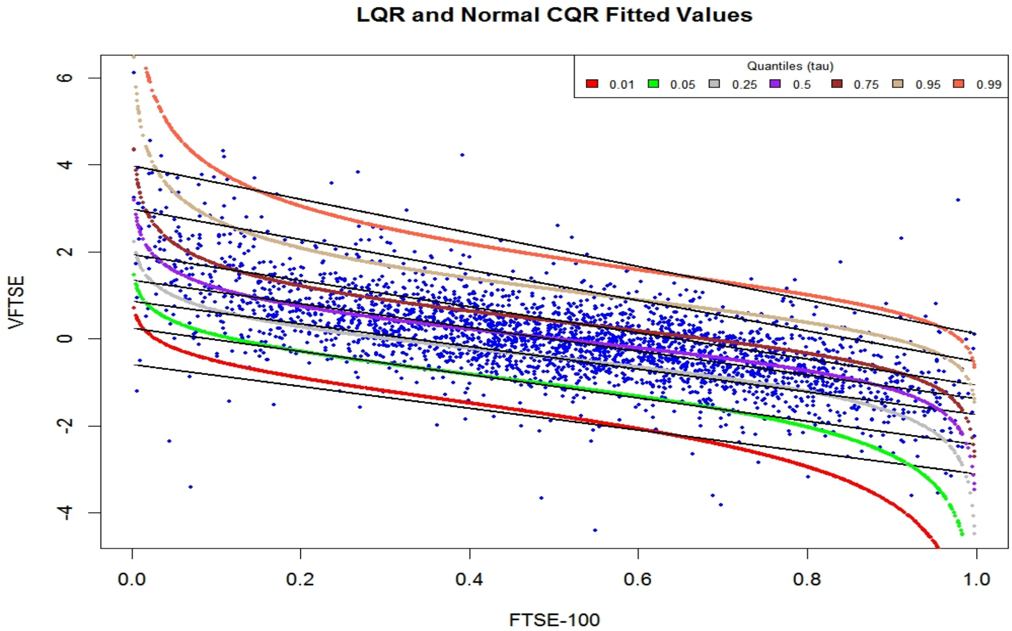
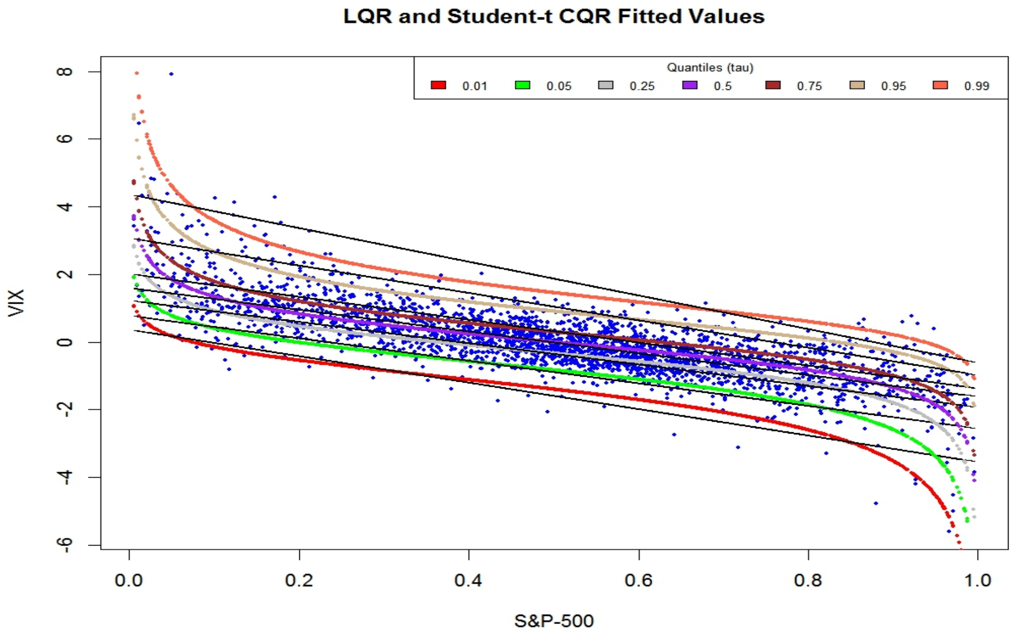


Figure 9: Student-t CQR Estimates

Another point of analysis is to see how well the estimates from LQR and CQR fit to the data. Figure-10 plots the LQR and CQR fitted values across the quantiles over the marginal data. Figure-10(a) plots the VFTSE-FTSE pair fitted values estimated from Normal CQR and LQR and figure-10(b) plots the VIX-S&P pair fitted values estimated from Student-t CQR and LQR. The figures show that we can model the non-linear relationship with the help of copula in quantile regression framework.



(a) VFTSE-FTSE Normal CQR



(b) VIX-S&P Student-t CQR

Figure 10: Fitted Values from CQR and LQR

6 Conclusion

The empirical analysis in this paper demonstrated the application of both linear and non linear quantile regression models. We used LQR and CQR to model the inverse volatility-return relationship for six volatility-return pairs. The study focussed on the use of copula to model non linear quantile regression which facilitates the quantification of bivariate non linear correlation within the quantiles of the distribution. Linear regression quantifies the relationship between a dependent and independent variable(s) around the mean of the distribution and hence does not quantify the relationship for the quantiles across the distribution. Quantile regression is a very useful tool to quantify the relationship across various quantiles in a distribution.

Tails of the return distribution are of immense interest in financial risk modelling as they represent the risk associated with the asset or the financial instrument. Volatility-return relationship and its quantification has importance for hedging as the change in volatility leads to the change in market prices. In this analysis we used OLS to quantify the linear volatility-return relationship around the mean which as quantified by LQR is not consistent for quantiles across the distribution. CQR is yet another useful tool for quantifying nonlinear bivariate relationship across quantiles. The analysis conducted in this paper demonstrated that CQR fits better to the actual data than LQR as it is capable of capturing non linear nature of the volatility-return relationship. The results from this analysis also supports the asymmetric volatility-return relationship for majority of the index pairs.

The empirical analysis of this paper has significance for hedging, portfolio management or risk modelling in general. The empirical analysis in this paper can be furthered by including more copula models like Frank Copula, Joe-Clayton copula etc, in CQR model.

Acknowledgements

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