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Anke Gerber[1]
Kirsten I.M. Rohde[2]

[1] *University of Hamburg, Germany;*
[2] *Erasmus School of Economics, Erasmus University Rotterdam, and Tinbergen Institute, The Netherlands.*

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1082 MS Amsterdam
The Netherlands
Tel.: +31(0)20 525 8579

Weighted Temporal Utility

Anke Gerber[*] Kirsten I.M. Rohde[‡ §]

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[*]Department of Economics, Hamburg University, Von-Melle-Park 5, 20146 Hamburg, Germany, e-mail: anke.gerber@wiso.uni-hamburg.de

[‡]Department of Economics, H12-01, Erasmus University, P.O. Box 1738, 3000 DR Rotterdam, the Netherlands, e-mail: rohde@ese.eur.nl

[§]Tinbergen Institute

Abstract

We propose a novel utility representation for preferences over risky timed outcomes. The weighted temporal utility model generalizes many well known utility functions for intertemporal decision making under risk. A decision maker with a weighted temporal utility function can have time consistent yet non-stationary preferences or stationary yet time inconsistent preferences. Thus, our model can explain the empirical evidence in Halevy (2012) which is at odds with standard models of intertemporal choice that assume non-linear time perception to be the sole driver of non-stationary and time-inconsistent behavior. We also propose a non-parametric approach to elicit a weighted temporal utility function.

Keywords: Intertemporal choice, stationarity, time consistency
JEL-Classification: D91, D81

1 Introduction

Virtually any decision we make involves an uncertain outcome at some point in the future. Not only investments and savings involve such future payoffs, but also daily decisions about, for instance, what to eat and whether or not to go to the gym. Empirical evidence shows that many decisions are time-inconsistent in the sense that the mere passage of time makes people change their plans (Frederick, Loewenstein, and O'Donoghue, 2002). Such time-inconsistencies can cause under-investment and unhealthy lifestyles, which impose a large cost on society. A good understanding of the drivers of these time-inconsistencies can help to provide solutions to overcome them and to reduce the associated costs.

The literature on intertemporal choice has focussed almost exclusively on one potential driver of time-inconsistencies: non-stationarity. Stationarity holds if a preference between outcomes to be received at different points in time is unaffected by a common additional delay of all outcomes. Deviations from stationarity are often thought to be driven by pure time preference, being the way people weight future points in time, irrespective of the outcomes received at these points in time. Hyperbolic discount models were proposed to accommodate such pure time preference (Loewenstein and Prelec, 1992; Harvey, 1986, 1995; Mazur, 1987; and Phelps and Pollak, 1968). These models can be given a psychological foundation by construal-level theory (Trope and Liberman, 2010) and the non-linear manner in which humans perceive temporal distance (Zauberman et al., 2009). To the extent that these non-linear perceptions of time are irrational, we can view deviations from stationarity caused by pure time preference as irrational.

Deviations from stationarity can induce time-inconsistencies. Yet, Halevy (2012) provided empirical evidence that non-stationary behavior can be time-consistent and that stationary behavior can be time-inconsistent. In his study only two-thirds of the subjects who exhibit time-consistency also exhibit stationarity and half of the subjects whose choices are time-inconsistent exhibit stationarity. These findings show that deviations from stationarity are not the sole drivers of irrationalities and they cast doubt on the extent to which such deviations are irrational.

This paper proposes an additional driver of non-stationarities, which can explain the data in Halevy (2012) and which – unlike non-linear time perception – need not be irrational at all. In our model deviations from stationarity are not only caused by pure time preference, but also by the time-dependence of the utility of an outcome. Such time-dependence naturally arises whenever the decision maker has some baseline consumption to which he adds any outcome he receives (cf. Noor, 2009, and Gerber and Rohde, 2010). If the decision maker expects his baseline consumption to change over time, then the utility of an outcome depends on its timing irrespective of pure time preference. This dependency, which can induce non-stationarity, can be viewed as foresight of future utility and, thereby, is not irrational as long as it is perfect foresight.

We introduce a weighted temporal utility model to account for time-dependent utility. This model evaluates an outcome to be received at a particular time with a particular probability as follows. First the time-dependent utility of the outcome is determined. Then this utility is discounted by a weight, which depends on the probability and the time at which the outcome is received. Thus, our model requires outcomes and probabilities to be separable, but allows for interactions between probabilities and time on the one hand and between outcomes and time on the other hand. Keren and Roelofsma (1995), Abdellaoui, Diecidue, and Öncüler (2011), and Baucells and Heukamp (2012) provide empirical evidence for probability and time not being separable. Their results show that the weight given to a probability depends on the timing of the outcome. The magnitude effect, which shows that larger outcomes are discounted at a lower rate than smaller outcomes, suggests that an outcome and its timing are not separable (Frederick, Loewenstein, and O'Donoghue, 2002).

Our weighted temporal utility model is similar in spirit to the probability time-tradeoff model of Baucells and Heukamp (2012). Like them, we consider preferences over single outcomes to be received with a particular probability at a particular point in time. Baucells and Heukamp (2012) assume that outcome and probability are separable at time 0 and make a specific assumption on how time and probability are interacting. We assume that outcome and probability are separable at every point in time, but do not need any assumption about the interaction of probability and time. We will show that the weighted temporal utility model can also accommodate the empirical findings supporting the model of Baucells and Heukamp (2012). Moreover, our model accommodates rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting as special cases.

As the timing of an outcome influences both its utility and the weight given to the probability that it will be received, measuring the weighting and utility functions of the weighted temporal utility model may seem difficult at first sight. We will show how this can be accomplished in a non-parametric way. This non-parametric approach does not require any assumption about the shape of the utility and weighting functions. In particular, it does not require an assumption of linear utility, which is often used in the literature.

The outline of this paper is as follows. Section 2 introduces the weighted temporal utility model and provides a characterization result. Section 3 shows how this model can accommodate all possible combinations of non-stationarity, time-inconsistency and time-invariance. Section 4 proposes a non-parametric approach to elicit a weighted temporal utility function and Section 5 concludes.

2 The Model

This paper considers preferences $\succcurlyeq$ over *risky timed outcomes* (x, p, t) which give *outcome* $x \in \mathbb{R}_+$ with *probability* $p \in [0, 1]$ at *time* $t \in \mathbb{R}_+$. We assume that $\succcurlyeq$ is a continuous weak order. Strict preference $\succ$ and indifference $\sim$ are defined as usual. We further assume that $(x, p, t) \sim (y, q, s)$ whenever $px = qy = 0$.

Impatience holds if for all x, p, s, t with $s < t$ and $px > 0$ we have $(x, p, s) \succ (x, p, t)$. *Monotonicity in probabilities* holds if for every outcome $x > 0$, every time t, and all probabilities $p > q$ we have $(x, p, t) \succ (x, q, t)$. *Monotonicity in outcomes* holds if for every probability $p > 0$, time t, and outcomes $x > y$ we have $(x, p, t) \succ (y, p, t)$. *Monotonicity* holds if both monotonicity in probabilities and monotonicity in outcomes holds.

Weighted temporal utility (WTU) holds if $\succcurlyeq$ can be represented by

$$V(x, p, t) = w(p, t)v(x, t),$$

where w is a *weighting function* and v is a *utility function*. Under WTU a decision maker evaluates a risky timed outcome (x, p, t) by first determining the utility $v(x, t)$ that outcome x will yield at time t, irrespective of the probability that it will be received, and then discounting this temporal utility by a weight $w(p, t)$, which can be viewed as a time-dependent probability weighting function.

WTU captures the two ways in which the time at which a risky outcome is received, can influence its evaluation. First of all, the instantaneous utility derived from outcome x may depend on time t. This will be the case, if, for instance, a decision maker expects to be much wealthier in the future and therefore expects €100 to generate much less utility in the future than now. Second, as the instantaneous utility is generated in the future and only with a probability p, it can be viewed as a psychologically distant utility. The weighting function $w(p, t)$ transforms the two components, p and t, of this psychological distance into a discount which is applied to the instantaneous utility $v(x, t)$.

In psychology construal level theory (Trope and Liberman, 2010) has been proposed as a theory which shows how psychological distance resulting from a.o. risk and time, influences decision making. Prelec and Loewenstein (1991) showed that there are many parallels between the impact of risk and time on decision making, which supports the idea that risk and time can be summarized into one variable: psychological distance. Our model, like the one of Baucells and Heukamp (2012), puts construal level theory into a (mathematical) weighting function. The weighting function w can be thought of as a function that first combines probability and delay into psychological distance, and then gives a weight to this distance. Keren and Roelofsma (1995), Abdellaoui, Diecidue, and Öncüler (2011), and Baucells and Heukamp (2012) provide empirical evidence for the non-separability of probability and time. Hence, we do not assume that $w(p, t)$ can be written as $w(p, t) = f(p)g(t)$ for some functions f and g. Yet, rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting are special cases of WTU.

WTU is an alternative to the probability and time tradeoff model of Baucells and Heukamp (2012), which is given by $V(x,p,t) = w(pe^{-r_x t})v(x)$. In their model the tradeoff between probability p and time t may depend on the outcome x. More precisely, they provide empirical evidence that the willingness to wait in exchange for a higher probability to receive a reward increases in the size of the reward. They use the term *subendurance* for this behavioral pattern and show that it can be rationalized by a weighting function w which also depends on the outcome x. In our model the dependency on outcomes of the tradeoff between probability and time is captured by the temporal utility function v. Hence, our model clearly separates attitudes towards psychological distance and attitudes towards outcomes, where the former are captured by the temporal weighting function w and the latter by the temporal utility function v. The following example shows that our model is compatible with all the empirical findings that Baucells and Heukamp (2012) use to justify their model.

Example 2.1 Consider a decision maker with WTU function $V(x,p,t) = w(p,t)v(x,t)$, where $w(p,t) = e^{-(-\ln(p)+0.023t)^{0.65}}$ for all p, t, $v(x,0) = \sqrt{100+x} - \sqrt{100}$ for all x, and $v(x,t) = \sqrt{105+x} - \sqrt{105}$ for all x and $t > 0$. Let time be denoted in weeks. The weighted temporal utilities of the prospects in Baucells and Heukamp (2012, Table 1) are summarized in Table 1. These utility levels yield modal choices as reported in Baucells and Heukamp (2012). Thus, WTU can account for their empirical findings.

In the remainder of this section we will provide a characterization of WTU and of a special case of WTU, where the time-dependence of utility is generated by time-dependent baseline consumption. In addition to impatience and monotonicity two conditions are necessary and sufficient for WTU to hold.

The *hexagon condition at time t* holds if for all outcomes $x, y, z > 0$ and all probabilities

	Prospect A	Prospect B	V(A)	V(B)
1.	(€9, 100%, now)	(€12, 80%, now)	0.4403	0.3998
2.	(€9, 10%, now)	(€12, 8%, now)	0.0789	0.0939
3.	(€9, 100%, 3 months)	(€12, 80%, 3 months)	0.2789	0.3014
4.	(fl.100, 100%, now)	(fl.110, 100%, 4 weeks)	2.0603	1.7820
5.	(fl.100, 100%, 26 weeks)	(fl.110, 100%, 30 weeks)	0.9867	1.0041
6.	(fl.100, 50%, now)	(fl.110, 50%, 4 weeks)	0.9369	0.9373
7.	(€100, 100%, 1 month)	(€100, 90%, now)	3.2930	3.2858
8.	(€5, 100%, 1 month)	(€5, 90%, now)	0.1951	0.1959

Table 1: Prospects and utility values for Example 2.1. $V(A)$ and $V(B)$ denote the weighted temporal utility of prospect A and B, respectively. *Note:* Rows 4-6 have outcomes denoted in Dutch Guilders. We transformed them into Euro by using the conversion rate at the introduction of the Euro: fl.100 is approximately €45.45 and fl.110 is approximately €50. We set 1 month equal to 4 weeks and 3 months equal to 12 weeks.

$p, q, l > 0$ we have that

$$\begin{array}{ccc} (y,p,t) \sim (x,q,t) & \text{and} & (z,p,t) \sim (y,q,t) \text{ and} \\ (y,q,t) \sim (x,l,t) & \text{imply} & \\ & & (z,q,t) \sim (y,l,t). \end{array}$$

The hexagon condition can be interpreted as follows. Assume that the tradeoff between p and q equals the tradeoff between q and l in the sense that they both offset the tradeoff between y and x at time t (the upper and lower left indifferences of the definition). If the tradeoff between p and q also offsets the tradeoff between z and y at time t, then the hexagon condition implies that the tradeoff between q and l offsets the tradeoff between z and y at time t as well (the upper and lower right indifferences of the definition). Thus, the hexagon condition allows us to claim that, at time t, the tradeoff between p and q equals the tradeoff between q and l, irrespective of the outcomes. Wakker (1989) showed that the hexagon condition is weaker than the often used Thomsen condition (Thomsen, 1927).

Probability-independent time-outcome tradeoff holds if for all outcomes $x, y, x_0, y_0 > 0$, all probabilities p, p_0, and every time t we have that

$$\begin{array}{lcl} (x,1,t) \sim (x_0,1,0) & \text{and} & (x,p,t) \sim (x_0,p_0,0) \text{ and} \\ (y,1,t) \sim (y_0,1,0) & \text{imply} & \\ & & (y,p,t) \sim (y_0,p_0,0). \end{array}$$

Probability-independent time-outcome tradeoff can be interpreted as follows. Assume that the tradeoff between x for sure and x_0 for sure equals the tradeoff between y for sure and y_0 for sure in the sense that they both offset the tradeoff between time t and time 0 (the upper and lower left indifferences of the definition). Assume that the tradeoff between x with probability p and x_0 with probability p_0 also offsets the tradeoff between time t and time 0. Then probability-independent time-outcome tradeoff implies that the tradeoff between y with probability p and y_0 with probability p_0 offsets the tradeoff between time t and time 0 as well (the upper and lower right indifferences of the definition).

The hexagon condition at time 0 and probability-independent time-outcome tradeoff imply WTU, as is shown in the following theorem. The proof is in the Appendix.

Theorem 2.2 *Under impatience and monotonicity the following statements are equivalent:*

(i) *Probability-independent time-outcome tradeoff and the hexagon condition at time* 0 *hold.*

(ii) *Preferences* $\succcurlyeq$ *can be represented by*

$$V(x,p,t) = w(p,t)v(x,t)$$

with $w(0,t) = v(0,t) = 0$ *for all* t*. Moreover, for all* x, p, t *we have* $w(p,t) \geq 0$ *and* $v(x,t) \geq 0$ *with* w *increasing in* p *and* v *increasing in* x*.*

Baseline Consumption

We will now consider a special case of WTU which naturally arises if the decision maker has a discounted expected utility function and adds any outcome to his baseline consumption at the time when the outcome is received. If b_t is baseline consumption and u is the decision maker's von Neumann-Morgenstern utility function, then the utility function over risky timed outcomes is of the form

$$V(x, p, t) = p\delta(t)\left(u(b_t + x) - u(b_t)\right), \tag{1}$$

where δ is the time discount function. The utility generated by receiving outcome x at time t, thereby, is the extra utility outcome x generates on top of the utility derived from baseline consumption at time t. Given the empirical evidence against the separability of probability and time (Keren and Roelofsma, 1995; Abdellaoui, Diecidue, and Öncüler, 2011; Baucells and Heukamp, 2012), we will focus on the following more general version of (1):

$$V(x, p, t) = w(p, t)\left(u(b_t + x) - u(b_t)\right), \tag{2}$$

where w is the weighting function that we already know from the general WTU model. The question is, under which conditions the temporal utility function $v(x, t)$ can be written as a utility difference $u(b_t + x) - u(b_t)$ for some b_t, i.e. under which condition the decision maker behaves as if he has a baseline consumption and evaluates an outcome by the extra utility it generates on top of the baseline consumption. It turns out that the following condition is vital.

Assume that preferences at time 0 can be represented by $V(x, p, 0) = w(p)u(x)$. *Baseline* *consumption* b_t *exists at time* $t > 0$ if for all $x, y, p, p_x, p_y, q_x, q_y$ with $x, y > 0$ we have that

$$\begin{aligned}
(b_t + x, p_x, 0) &\sim (b_t, 1, 0), \\
(b_t + y, p_y, 0) &\sim (b_t, 1, 0), \\
(b_t + x, q_x, 0) &\sim (x, p, t), \text{ and} \\
(b_t + y, q_y, 0) &\sim (y, p, t) \\
&\text{imply} \\
\frac{w(q_x)}{1 - w(p_x)} &= \frac{w(q_y)}{1 - w(p_y)}
\end{aligned}$$

The following theorem provides a characterization of (2). The proof is in the Appendix.

Theorem 2.3 *Under impatience, monotonicity, and the hexagon condition at time* 0 *the following statements are equivalent:*

(i) *Baseline consumption* b_t *exists for every* $t > 0$.

(ii) *Preferences* $\succcurlyeq$ *can be represented by*

$$V(x, p, t) = w(p, t)\,(u(b_t + x) - u(b_t))$$

with $b_0 = 0$ *and* $w(0, t) = u(0) = 0$ *for all* t.

Moreover, the $\{b_t\}_{t>0}$ *in* (i) *are the same as in* (ii).

3 Stationarity, Time Invariance and Time Consistency

Two points in time are crucial when choosing between risky timed outcomes: *consumption time* – the time at which the outcome is received - and *decision time* – the time at which the decision is made. So far we have only varied consumption time while the decision time was fixed at $t = 0$. In order to shed light on time inconsistencies this section will also consider changes in the decision time. Varying the decision time and the consumption time gives rise to three notions of consistent behavior depending on whether only one or both of them are varied. We will discuss the three notions of consistent behavior in terms of our preference model.

This section assumes that for every decision time τ the decision maker has a preference relation $\succcurlyeq^\tau$ over risky timed outcomes to be received from time τ onwards. By $\{\succcurlyeq^\tau\}_\tau$ we denote the set of preferences for all decision times τ. Strict preference $\succ^\tau$ and indifference $\sim^\tau$ are defined as usual. In line with Halevy (2012) we define stationarity, time invariance and time consistency as follows.

Definition 3.1 *Preferences* $\{\succcurlyeq^\tau\}_\tau$ *are **stationary** if for every* $x, y, p, q, \tau, s, t,$ *with* $0 \leq \tau \leq s, t,$ *and for every* $\Delta \geq 0$,

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t + \Delta) \sim^\tau (y, q, s + \Delta)$$

Stationarity means that preferences remain unchanged if the decision time remains unchanged and all consumption times are delayed by a common time interval.

Definition 3.2 *Preferences* $\{\succcurlyeq^\tau\}_\tau$ *are **time invariant** if for every* x, y, p, q, τ, s, t *with* $0 \leq \tau \leq s, t,$ *and for every* $\Delta \geq 0$,

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t + \Delta) \sim^{\tau+\Delta} (y, q, s + \Delta)$$

Time invariance means that preferences remain unchanged if the distance from consumption time to decision time remains unchanged.

Definition 3.3 *Preferences* $\{\succcurlyeq^\tau\}_\tau$ *are **time consistent** if for every* $x, y, p, q, \tau, \tau', s, t,$ *with* $0 \leq \tau, \tau' \leq s, t,$

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t) \sim^{\tau'} (y, q, s)$$

Time consistency means that the preference over a pair of risky timed outcomes is independent of the decision time.

It is straightforward to show that any two of the three properties (stationarity, time invariance, and time consistency) imply the third and hence either none or at least two of the properties must be violated. Thus, there are five possible preference types as summarized in Table 2.

Type	Stationary	Time Invariant	Time Consistent
I	Yes	Yes	Yes
II	Yes	No	No
III	No	Yes	No
IV	No	No	Yes
V	No	No	No

Table 2: The five possible preference types. "Yes" ("No") means that preferences have (do not have) the corresponding property.

Halevy (2012) provides experimental evidence for all five preference types in Table 2. In particular, in his experiment only two-thirds of the subjects who exhibit time consistency also exhibit stationarity and half of the subjects whose choices are time inconsistent exhibit stationarity. This shows that non-stationary behavior, e.g. due to decreasing impatience, is not equivalent to time inconsistency. We will now demonstrate that an extension of our preference model to arbitrary decision times $\tau \geq 0$ can account for all five preference types in Table 2. We summarize these findings in the following theorem which is proved in the Appendix.

Theorem 3.4 *Assume that time-τ preferences $\succcurlyeq^\tau$ are represented by the utility function*

$$V_\tau(x,p,t) = w_\tau(p,t)v_\tau(x,t)$$

for all x,p,t,τ, for some functions w_τ and v_τ. Then the following specifications of the functions w_τ and v_τ yield the preference types in Table 2.

1. ***Type I*** *If $v_\tau(x,t) = \bar{v}(x)$ for some function $\bar{v}$ and if $w_\tau(p,t) = \delta^{t-\tau}w(p)$ for some function w and some $\delta > 0$, then preferences are stationary, time invariant and time consistent.*

2. ***Type II*** *If $v_\tau(x,t) = \bar{v}_\tau(x)$ for some function $\bar{v}_\tau$ and if $w_\tau(p,t) = \delta^{t-\tau}w(p)$ for some function w and some $\delta > 0$, then preferences are stationary, but neither time invariant nor time consistent unless $\bar{v}_\tau$ satisfies*
$$\frac{\bar{v}_\tau(x)}{\bar{v}_\tau(y)} = \frac{\bar{v}_{\tau+\Delta}(x)}{\bar{v}_{\tau+\Delta}(y)} \tag{3}$$
for all $x,y,\tau,\Delta \geq 0$.

3. ***Type III*** *If $v_\tau(x,t) = \bar{v}(x)$ for some function $\bar{v}$ and if $w_\tau(p,t) = w(p)[1+\alpha(t-\tau)]^{-1}$ for some function w and some $\alpha > 0$, then preferences are time invariant, but neither stationary nor time consistent.*

4. ***Type IV*** *If $v_\tau(x,t) = \bar{v}(x,t)$ for some function $\bar{v}$ and if $w_\tau(p,t) = w(p)\delta^{t-\tau}$ for some function w and some $\delta > 0$, then preferences are time consistent, but neither stationary nor time invariant unless $\bar{v}$ satisfies*
$$\frac{\bar{v}(x,t)}{\bar{v}(y,s)} = \frac{\bar{v}(x,t+\Delta)}{\bar{v}(y,s+\Delta)} \tag{4}$$
for all $x,y,t,s,\Delta \geq 0$.

5. ***Type V*** *If $v_\tau(x,t) = \bar{v}_\tau(x)$ for some function $\bar{v}_\tau$ and if $w_\tau(p,t) = w(p)[1+\alpha(t-\tau)]^{-1}$ for some function w and some $\alpha > 0$, then preferences violate stationarity. Moreover, preferences are not time invariant unless $\bar{v}_\tau$ satisfies*
$$\frac{\bar{v}_\tau(x)}{\bar{v}_\tau(y)} = \frac{\bar{v}_{\tau+\Delta}(x)}{\bar{v}_{\tau+\Delta}(y)} \tag{5}$$

for all $x, y, \tau, \Delta \geq 0$. Finally, preferences are not time consistent unless $\bar{v}$ satisfies

$$\frac{\bar{v}_\tau(x)(1+\alpha(s-\tau))}{\bar{v}_\tau(y)(1+\alpha(t-\tau))} = \frac{\bar{v}_{\tau'}(x)(1+\alpha(s-\tau'))}{\bar{v}_{\tau'}(y)(1+\alpha(t-\tau'))} \tag{6}$$

for all x, y, τ, τ', s, t, with $0 \leq \tau, \tau' \leq s, t$.

Theorem 3.4 shows that WTU is sufficiently rich to cover all possible preference types concerning stationarity, time invariance and time consistency. Moreover, it clearly demonstrates that violations of stationarity are by no means the only source of time inconsistent behavior. Hence, we have to measure both w and v in order to get a complete picture of how a decision maker's preference responds to changes in the decision and in the consumption time. This is the topic of the following section.

4 Parameter-Free Elicitation of $V(x, p, t)$

In the following we present a parameter-free method for eliciting the weighting function $w(p, t)$ and the utility function $v(x, t)$ of WTU for a given continuous preference relation $\succcurlyeq$ over risky timed outcomes which satisfies impatience and monotonicity. We start with an elicitation of $w(p, 0)$.

Elicitation of $w(p, 0)$

Fix an arbitrary outcome $x > 0$, an arbitrary probability p_0 with $0 < p_0 < 1$, and a parameter κ with $0 < \kappa < 1$. Without loss of generality we can normalize w so that

$$w(p_0, 0) = \kappa \text{ and } w(1, 0) = 1.$$ [1]

Elicit y_1 such that

$$(x, p_0, 0) \sim (y_1, 1, 0) \tag{7}$$

[1] Observe that $V(x, p, t) = w(p, t)v(x, t)$ and $V'(x, p, t) = w'(p, t)v'(x, t)$ both represent $\succcurlyeq$ if and only if there exists $\alpha_1, \alpha_2, \beta > 0$ such that $w'(p, t) = (\alpha_1 w(p, t))^\beta$ and $v'(x, t) = (\alpha_2 v(x, t))^\beta$.

and p_1 such that

$$(x, p_1, 0) \sim (y_1, p_0, 0). \tag{8}$$

By monotonicity y_1 and p_1 are unique and satisfy $y_1 < x$ and $p_1 < p_0$. Indifference (7) is equivalent to

$$\kappa v(x, 0) = v(y_1, 0) \tag{9}$$

and (8) is equivalent to

$$w(p_1, 0)v(x, 0) = \kappa v(y_1, 0). \tag{10}$$

From (9) and (10) it follows that

$$w(p_1, 0) = \kappa^2.$$

We can continue like this and elicit y_i and p_i for $i = 2, 3, \ldots,$ such that

$$(x, p_{i-1}, 0) \sim (y_i, 1, 0) \tag{11}$$

and

$$(x, p_i, 0) \sim (y_i, p_{i-1}, 0). \tag{12}$$

It follows that

$$w(p_i, 0) = \kappa^{2^i} \text{ for all } i, \tag{13}$$

which can be shown as follows. For $i = 1$ we already verified that (13) holds. Now suppose that $w(p_{i-1}, 0) = \kappa^{2^{i-1}}$. From indifference (11) we have

$$w(p_{i-1}, 0)v(x, 0) = v(y_i, 0)$$

From indifference (12) we have

$$w(p_i, 0)v(x, 0) = w(p_{i-1}, 0)v(y_i, 0).$$

It follows that

$$w(p_i, 0) = (\kappa^{2^{i-1}})^2 = \kappa^{2^i}.$$

By choosing the starting point p_0 arbitrarily close to 1 we can make the grid on which we determine the weighting function $w(p, 0)$ arbitrarily fine. Finally, note that $w(0, 0) = 0$ since by assumption $\succcurlyeq$ satisfies monotonicity and $(x, 0, 0) \sim (0, p, 0)$ for all $x > 0$ and for all $p > 0$.

Elicitation of $v(x, 0)$

Given $w(p, 0)$ with $w(1, 0) = 1$ it is straightforward to elicit $v(x, 0)$. Fix an arbitrary outcome $x > 0$. Without loss of generality we can normalize v so that

$$v(x, 0) = 1.$$ [2]

Then, for any outcome y with $y < x$ elicit p such that

$$(y, 1, 0) \sim (x, p, 0).$$

Then we have $v(y, 0) = w(p, 0)$. Similarly, for any outcome y with $y > x$ elicit q such that

$$(x, 1, 0) \sim (y, q, 0).$$

It follows that $v(y, 0) = \frac{1}{w(q,0)}$.

Elicitation of $w(p, t)$ and $v(x, t)$ for $t > 0$

In order to elicit $w(p, t)$ and $v(x, t)$ for $t > 0$ we use the method in the proof of Theorem 2.2. First observe that $w(0, t) = v(0, t) = 0$ for all $t > 0$ since by assumption $\succcurlyeq$ satisfies monotonicity and $(x, 0, t) \sim (0, p, t)$ for all $x > 0$ and for all $p > 0$. For every $x > 0$ elicit $x_0(x, t)$ such that

$$(x, 1, t) \sim (x_0(x, t), 1, 0)$$

and define

$$v(x, t) = v(x_0(x, t), 0).$$

Fix $x > 0$. For every $p > 0$ elicit $p_0(p, t)$ such that

$$(x, p, t) \sim (x_0(x, t), p_0(p, t), 0)$$

and define

$$w(p, t) = w(p_0(p, t), 0).$$

[2]See Footnote 1.

If for every $t > 0$ there exists a baseline consumption b_t, we can elicit $w(p, 0)$ and $v(x, 0)$ as above and then follow the constructive proof of Theorem 2.3 in order to get the utility representation (2). Define $b_0 = 0$ and $u(x) = v(x, 0)$ for all outcomes x. For all $x > 0, p \geq 0, t > 0$, let $\pi(x, t)$ solve

$$(b_t + x, \pi(x, t), 0) \sim (b_t, 1, 0)$$

and let $\pi'(x, p, t)$ solve

$$(x, p, t) \sim (b_t + x, \pi'(x, p, t), 0).$$

Define

$$w(p, t) = \frac{w_0(\pi'(x, p, t))}{1 - w_0(\pi(x, t))}.$$

and observe that $w(p, t)$ is well-defined by the definition of baseline consumption b_t.

5 Conclusion

We introduced the weighted temporal utility model, which evaluates risky timed outcomes by the product of a time-dependent utility generated by this outcome and a time-dependent probability weight. The model is consistent with empirical findings suggesting that probability and time as well as outcome and time are not separable. For single outcomes to be received with a specific probability at a single point in time weighted temporal utility covers rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting as special cases.

Another special case of weighted temporal utility arises when the decision maker evaluates an outcome at a specific point in time by the extra utility it generates on top of the utility derived from baseline consumption. If baseline consumption is expected to change over time, then the utility generated by an outcome is indeed time-dependent.

We showed that the time-dependency of the utility generated by an outcome can give rise to non-stationarities even if probabilities are weighted linearly and time is discounted exponentially. We also showed that this type of non-stationarity does not necessarily induce time-inconsistent behavior. Our model can therefore explain the findings of Halevy (2012).

It is important to note that the deviations from stationarity which are induced by the time-dependence of utilities, are not necessarily irrational. If one, for instance, considers the special case with baseline consumption, then a perfect foresight of changes in baseline consumption induces deviations from stationarity. Yet these deviations are driven by perfect foresight, and, thereby, perfectly rational.

6 References

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Appendix

Lemma 1 *Under monotonicity the following statements are equivalent:*

(i) *The hexagon condition at time* 0 *holds.*

(ii) *Preferences at time* 0 *can be represented by*

$$V(x,p,0) = w_0(p)u_0(x).$$

with $w_0(p) \geq 0$ *and* $u_0(x) \geq 0$ *for all* x, p, *and* $w_0(0) = u_0(0) = 0$. *Moreover,* w_0 *and* u_0 *are increasing.*

Proof of Lemma 1: The fact that *(ii)* implies *(i)* can easily be shown and also follows directly from Theorem III.4.1 in Wakker (1989).

Now assume that *(i)* holds. Then Theorem III.4.1 in Wakker (1989) shows that we have a representation

$$V(x,p,0) = w_0(p)u_0(x)$$

for all positive outcomes and positive probabilities. Moreover, $w_0(p) > 0$ for all $p > 0$ and $u_0(x) > 0$ for all $x > 0$. We will first show that $w_0(p)$ goes to zero as p goes to zero. Suppose that this were not the case and that $w_0(p)$ would go to a positive number W as p goes to zero. Now consider any two outcomes $y > x > 0$ and a very small probability $\varepsilon > 0$. Then

$$(y,\varepsilon,0) \succ (x,\varepsilon,0) \succ (0,\varepsilon,0) \sim (y,0,0)$$

By continuity there must be a probability $\kappa > 0$ with

$$(x,\varepsilon,0) \sim (y,\kappa,0),$$

which implies

$$w_0(\varepsilon)u_0(x) = w_0(\kappa)u_0(y).$$

Note that $\kappa < \varepsilon$. Yet, when ε is small enough, we have that $\frac{w_0(\varepsilon)}{w_0(\kappa)} \approx \frac{W}{W} = 1$, which contradicts the fact that we can find such a κ with $w_0(\varepsilon)u_0(x) = w_0(\kappa)u_0(y)$ for all outcomes $y > x$. Thus, it must be the case that $w_0(p)$ goes to zero as p goes to zero. A similar argument shows that $u_0(x)$ goes to zero as x goes to zero.

Define $u_0(0) = 0$ and $w_0(0) = 0$. Consider $(x, p, 0) \succcurlyeq (y, q, 0)$. If x, y, p, and q are all positive then we have that

$$w_0(p)u_0(x) \geq w_0(q)u_0(y).$$

If $x = 0$ or $p = 0$ then we must have $y = 0$ or $q = 0$, which implies that $w_0(p)u_0(x) \geq w_0(q)u_0(y)$. If $y = 0$ or $q = 0$ and $x > 0$ and $p > 0$, then $w_0(p)u_0(x) \geq w_0(q)u_0(y)$ follows as well. This shows that preferences at time 0 can be represented by $V(x, p, 0) = w_0(p)u_0(x)$. Since $\succcurlyeq$ satisfies monotonicity and $w_0(p) > 0$ for all $p > 0$ and $u_0(x) > 0$ for all $x > 0$ it is straightforward to show that w_0 and u_0 are increasing. This proves the Lemma. □

Proof of Theorem 2.2

We first prove that *(i)* implies *(ii)*. Assume that probability-independent time-outcome tradeoff and the hexagon condition at time 0 hold. By Lemma 1 preferences at time 0 can be represented by $V(x, p, 0) = w_0(p)u_0(x)$. For every outcome $x \geq 0$ and time t define the outcome $x_0(x, t)$ by $(x, 1, t) \sim (x_0(x, t), 1, 0)$. By impatience and continuity $x_0(x, t)$ is always well-defined. For every $x > 0$ and every p, t define the probability $p_0(x, p, t)$ by $(x, p, t) \sim (x_0(x, t), p_0(x, p, t), 0)$. By impatience, monotonicity, and continuity $p_0(x, p, t)$ is always defined for $x > 0$. Probability-independent time-outcome tradeoff implies that $p_0(x, p, t) = p_0(y, p, t)$ for all $x, y > 0$. Thus, we define $p_0(p, t) = p_0(x, p, t)$.

Now we define

$$v(x, t) = u_0(x_0(x, t))$$

for all x, t with $x > 0$ and set $v(0, t) = 0$. Further we define

$$w(p, t) = w_0(p_0(p, t))$$

for all p, t. Then we have for $x, y > 0$

$$\begin{aligned} (x,p,t) &\succcurlyeq (y,q,s) \\ \iff (x_0(x,t), p_0(p,t), 0) &\succcurlyeq (x_0(y,s), p_0(q,s), 0) \\ \iff w_0(p_0(p,t))u_0(x_0(x,t)) &\geq w_0(p_0(q,s))u_0(x_0(y,s)) \\ \iff w(p,t)v(x,t) &\geq w(q,s)v(y,s). \end{aligned}$$

If $x = 0$ or $y = 0$ then $(x,p,t) \succcurlyeq (y,q,s)$ implies $w(p,t)v(x,t) \geq w(q,s)v(y,s)$ as well. Thus, $V(x,p,t) = w(p,t)v(x,t)$ represents $\succcurlyeq$.

Now we need to prove that *(ii)* implies *(i)*. Assume that preferences $\succcurlyeq$ can be represented by

$$V(x,p,t) = w(p,t)v(x,t).$$

The hexagon condition at time 0 follows from Lemma 1. Assume that $x, y > 0$ and $(x,1,t) \sim (x_0,1,0)$, $(x,p,t) \sim (x_0,p_0,0)$, and $(y,1,t) \sim (y_0,1,0)$. Then

$$\frac{v(x,t)}{v(x_0,0)} = \frac{v(y,t)}{v(y_0,0)}$$

and

$$\frac{v(x,t)}{v(x_0,0)} = \frac{w(p_0,0)}{w(p,t)}.$$

It follows that

$$\frac{v(y,t)}{v(y_0,0)} = \frac{w(p_0,0)}{w(p,t)}.$$

Thus, $(y,p,t) \sim (y_0,p_0,0)$. □

Proof of Theorem 2.3

We first prove that *(i)* implies *(ii)*. Under the assumptions of the theorem preferences at time 0 can be represented by $w_0(p)u_0(x)$, with $w_0(0) = u_0(0) = 0$, which follows from Lemma 1. Without loss of generality we can set $w_0(1) = 1$. Define $b_0 = 0$ and $u(x) = u_0(x)$ for all outcomes x. For all $x > 0$ and $t > 0$ define $\pi(x,t)$ by the indifference $(b_t +$

$x, \pi(x,t), 0) \sim (b_t, 1, 0)$. By monotonicity and continuity $\pi(x,t)$ is well defined. Note that this means that

$$w_0(\pi(x,t))u(b_t + x) = u(b_t).$$

For all $x > 0, t > 0$, and $p \geq 0$ define $\pi'(x,p,t)$ by the indifference $(x,p,t) \sim (b_t + x, \pi'(x,p,t), 0)$. By monotonicity, impatience and continuity $\pi'(x,p,t)$ is well defined. For all $t > 0$, $x > 0$, and $p \geq 0$ define

$$w(x,p,t) = \frac{w_0(\pi'(x,p,t))}{1 - w_0(\pi(x,t))}.$$

By the definition of baseline consumption b_t we have that

$$\frac{w_0(\pi'(x,p,t))}{1 - w_0(\pi(x,t))} = \frac{w_0(\pi'(y,p,t))}{1 - w_0(\pi(y,t))}$$

for all $x, y > 0$. It follows that $w(x,p,t)$ is independent of x. Thus, we define $w(p,t) = w(x,p,t)$ for all $t > 0$ and all p, and $w(p,0) = w_0(p)$ for all p.

Now we need to prove that $w(p,t)\,(u(b_t + x) - u(b_t))$ represents $\succcurlyeq$. For $x, y > 0$ we have that

$$\begin{aligned}
(x,p,t) &\succcurlyeq (y,q,s) \\
\iff (b_t + x, \pi'(x,p,t), 0) &\succcurlyeq (b_s + y, \pi'(y,q,s), 0) \\
\iff w(\pi'(x,p,t),0)u(b_t + x) &\geq w(\pi'(y,q,s),0)u(b_s + y) \\
\iff w(p,t)(1 - w_0(\pi(x,t)))u(b_t + x) &\geq w(q,s)(1 - w_0(\pi(y,s)))u(b_s + y) \\
\iff w(p,t)\,(u(b_t + x) - u(b_t)) &\geq w(q,s)\,(u(b_s + y) - u(b_s))\,.
\end{aligned}$$

For $x = 0$ we have $(x,p,t) \succcurlyeq (y,q,s)$ if and only if $y = 0$ or $q = 0$, which implies

$$w(p,t)\,(u(b_t + x) - u(b_t)) \geq w(q,s)\,(u(b_s + y) - u(b_s))\,,$$

as $\pi'(z,0,s) = 0$ for $z > 0$ implies $w(0,s) = 0$. Similarly, for $y = 0$ we have that $(x,p,t) \succcurlyeq (y,q,s)$ implies

$$w(p,t)\,(u(b_t + x) - u(b_t)) \geq w(q,s)\,(u(b_s + y) - u(b_s))\,.$$

This proves our result.

Now we need to prove that *(ii)* implies *(i)*. Assume that preferences $\succcurlyeq$ can be represented by

$$V(x,p,t) = w(p,t)\left(u(b_t + x) - u(b_t)\right)$$

where $b_0 = 0, u(0) = w(0,t) = 0$ for all t. Assume that for $x, y > 0$, and $t > 0$

$$(b_t + x, p_x, 0) \sim (b_t, 1, 0), \tag{14}$$

$$(b_t + y, p_y, 0) \sim (b_t, 1, 0), \tag{15}$$

$$(b_t + x, q_x, 0) \sim (x, p, t), \text{ and} \tag{16}$$

$$(b_t + y, q_y, 0) \sim (y, p, t). \tag{17}$$

Then (14) implies that

$$w(p_x, 0)u(b_t + x) = u(b_t)$$

and (15) implies that

$$w(p_y, 0)u(b_t + y) = u(b_t).$$

It follows that

$$u(b_t + x) - u(b_t) = \left(1 - w(p_x, 0)\right)u(b_t + x)$$

and

$$u(b_t + y) - u(b_t) = \left(1 - w(p_y, 0)\right)u(b_t + y).$$

(16) implies that

$$w(q_x, 0)u(b_t + x) = w(p,t)\left(u(b_t + x) - u(b_t)\right) = w(p,t)\left(1 - w(p_x, 0)\right)u(b_t + x)$$

Thus,

$$\frac{w(q_x, 0)}{1 - w(p_x, 0)} = w(p,t).$$

Similarly, (17) implies that

$$\frac{w(q_y, 0)}{1 - w(p_y, 0)} = w(p,t).$$

The result follows. □

Proof of Theorem 3.4

1. Let $V_\tau(x,p,t) = \delta^{t-\tau} w(p)\bar{v}(x)$. Then for every x,y,p,q,τ,s,t, with $0 \leq \tau \leq s,t$, and for every $\Delta \geq 0$,

$$\begin{array}{rcl} (x,p,t) & \sim^\tau & (y,q,s) \\ \iff \delta^{t-s}\dfrac{w(p)}{w(q)} & = & \dfrac{\bar{v}(y)}{\bar{v}(x)} \\ \iff (x,p,t+\Delta) & \sim^\tau & (y,q,s+\Delta) \end{array}$$

Hence, preferences are stationary. Moreover, for every x,y,p,q,τ,s,t with $0 \leq \tau \leq s,t$, and for every $\Delta \geq 0$,

$$\begin{array}{rcl} (x,p,t) & \sim^\tau & (y,q,s) \\ \iff \delta^{t-s}\dfrac{w(p)}{w(q)} & = & \dfrac{\bar{v}(y)}{\bar{v}(x)} \\ \iff (x,p,t+\Delta) & \sim^{\tau+\Delta} & (y,q,s+\Delta) \end{array}$$

Thus, preferences are time invariant. From stationarity and time invariance it then follows that preferences are also time consistent.

2. Let $V_\tau(x,p,t) = \delta^{t-\tau} w(p)\bar{v}_\tau(x)$. Then for every x,y,p,q,τ,s,t, with $0 \leq \tau \leq s,t$, and for every $\Delta \geq 0$,

$$\begin{array}{rcl} (x,p,t) & \sim^\tau & (y,q,s) \\ \iff \delta^{t-s}\dfrac{w(p)}{w(q)} & = & \dfrac{\bar{v}_\tau(y)}{\bar{v}_\tau(x)} \\ \iff (x,p,t+\Delta) & \sim^\tau & (y,q,s+\Delta) \end{array}$$

Hence, preferences are stationary. To see that preferences are not time-invariant in general, observe that

$$\begin{array}{rcl} (x,p,t) & \sim^\tau & (y,q,s) \\ \iff \delta^{t-s}\dfrac{w(p)}{w(q)} & = & \dfrac{\bar{v}_\tau(y)}{\bar{v}_\tau(x)} \end{array}$$

and

$$\begin{aligned} (x,p,t+\Delta) &\sim^{\tau+\Delta} (y,q,s+\Delta) \\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}_{\tau+\Delta}(y)}{\bar{v}_{\tau+\Delta}(x)} \end{aligned}$$

Therefore, preferences are time invariant if and only if for every $x, y, \tau, \Delta \geq 0$, (3) holds. If (3) is violated, then preferences are also not time consistent because otherwise, stationarity and time consistency would imply time invariance.

3. Let $V_\tau(x,p,t) = w(p)[1+\alpha(t-\tau)]^{-1}\bar{v}(x)$. Then for every x, y, p, q, τ, s, t with $0 \leq \tau \leq s, t$, and for every $\Delta \geq 0$,

$$\begin{aligned} (x,p,t) &\sim^{\tau} (y,q,s) \\ \iff \frac{w(p)\bar{v}(x)}{w(q)\bar{v}(y)} &= \frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \\ \iff (x,p,t+\Delta) &\sim^{\tau+\Delta} (y,q,s+\Delta) \end{aligned}$$

Hence, preferences are time invariant. To see that preferences are not stationary, observe that

$$\begin{aligned} (x,p,t) &\sim^{\tau} (y,q,s) \\ \iff \frac{w(p)\bar{v}(x)}{w(q)\bar{v}(y)} &= \frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \end{aligned}$$

and

$$\begin{aligned} (x,p,t+\Delta) &\sim^{\tau} (y,q,s+\Delta) \\ \iff \frac{w(p)\bar{v}(x)}{w(q)\bar{v}(y)} &= \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)} \end{aligned}$$

Since $\frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \neq \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)}$ for $\Delta > 0$, preferences are not stationary. Hence, preferences are also not time consistent, because time invariance and time consistency implies stationarity.

4. Let $V_\tau(x,p,t) = \delta^{t-\tau} w(p)\bar{v}(x,t)$. Then for every $x, y, p, q, \tau, \tau', s, t$, with $0 \leq \tau, \tau' \leq s, t$,

$$\begin{array}{rcl} (x,p,t) & \sim^\tau & (y,q,s) \\ \iff \delta^{t-s}\dfrac{w(p)}{w(q)} & = & \dfrac{\bar{v}(y,s)}{\bar{v}(x,t)} \\ \iff (x,p,t) & \sim^{\tau'} & (y,q,s) \end{array}$$

Hence, preferences are time consistent. To see that preferences are not stationary in general, observe that

$$\begin{array}{rcl} (x,p,t) & \sim^\tau & (y,q,s) \\ \iff \delta^{t-s}\dfrac{w(p)}{w(q)} & = & \dfrac{\bar{v}(y,s)}{\bar{v}(x,t)} \end{array}$$

and

$$\begin{array}{rcl} (x,p,t+\Delta) & \sim^\tau & (y,q,s+\Delta) \\ \iff \delta^{t-s}\dfrac{w(p)}{w(q)} & = & \dfrac{\bar{v}(y,s+\Delta)}{\bar{v}(x,t+\Delta)} \end{array}$$

Therefore, preferences are stationary if and only if for every $x, y, s, t, \Delta \geq 0$, (4) holds. If (4) is violated, then preferences are also not time invariant because otherwise, time consistency and time invariance would imply stationarity.

5. Let $V_\tau(x,p,t) = w(p)[1+\alpha(t-\tau)]^{-1}\bar{v}_\tau(x)$. To see that preferences are not stationary, observe that

$$\begin{array}{rcl} (x,p,t) & \sim^\tau & (y,q,s) \\ \iff \dfrac{w(p)\bar{v}_\tau(x)}{w(q)\bar{v}_\tau(y)} & = & \dfrac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \end{array}$$

and

$$\begin{aligned} (x,p,t+\Delta) &\sim^{\tau} (y,q,s+\Delta) \\ \Longleftrightarrow \frac{w(p)\bar{v}_\tau(x)}{w(q)\bar{v}_\tau(y)} &= \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)} \end{aligned}$$

Since $\frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \neq \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)}$ for $\Delta > 0$, preferences are not stationary. To see that preferences are not necessarily time invariant, observe that

$$\begin{aligned} (x,p,t) &\sim^{\tau} (y,q,s) \\ \Longleftrightarrow \frac{w(p)(1+\alpha(s-\tau))}{w(q)(1+\alpha(t-\tau))} &= \frac{\bar{v}_\tau(y)}{\bar{v}_\tau(x)} \end{aligned}$$

and

$$\begin{aligned} (x,p,t+\Delta) &\sim^{\tau+\Delta} (y,q,s+\Delta) \\ \Longleftrightarrow \frac{w(p)(1+\alpha(s-\tau))}{w(q)(1+\alpha(t-\tau))} &= \frac{\bar{v}_{\tau+\Delta}(y)}{\bar{v}_{\tau+\Delta}(x)} \end{aligned}$$

Therefore, preferences are time invariant if and only if for every $x, y, s, t, \Delta \geq 0$, (5) holds. To see that preferences are not necessarily time consistent, observe that

$$\begin{aligned} (x,p,t) &\sim^{\tau} (y,q,s) \\ \Longleftrightarrow \frac{\bar{v}_\tau(x)(1+\alpha(s-\tau))}{\bar{v}_\tau(y)(1+\alpha(t-\tau))} &= \frac{w(q)}{w(p)} \end{aligned}$$

and

$$\begin{aligned} (x,p,t) &\sim^{\tau'} (y,q,s) \\ \Longleftrightarrow \frac{\bar{v}_{\tau'}(x)(1+\alpha(s-\tau'))}{\bar{v}_{\tau'}(y)(1+\alpha(t-\tau'))} &= \frac{w(q)}{w(p)} \end{aligned}$$

Therefore, preferences are time consistent if and only if for every x, y, s, t, τ, τ', with $0 \leq \tau, \tau' \leq s, t$, (6) holds.

$\square$