

# **A One Line Derivation of EGARCH**

**Michael McAleer**

Department of Quantitative Finance  
National Tsing Hua University

Taiwan

and

Econometric Institute  
Erasmus School of Economics  
Erasmus University Rotterdam

and

Tinbergen Institute  
The Netherlands

and

Department of Quantitative Economics  
Complutense University of Madrid

**Christian M. Hafner**

Institut de statistique, biostatistique et sciences actuarielles  
Université catholique de Louvain

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## Abstract

One of the most popular univariate asymmetric conditional volatility models is the exponential GARCH (or EGARCH) specification. In addition to asymmetry, which captures the different effects on conditional volatility of positive and negative effects of equal magnitude, EGARCH can also accommodate leverage, which is the negative correlation between returns shocks and subsequent shocks to volatility. However, there are as yet no statistical properties available for the (quasi-) maximum likelihood estimator of the EGARCH parameters. It is often argued heuristically that the reason for the lack of statistical properties arises from the presence in the model of an absolute value of a function of the parameters, which does not permit analytical derivatives or the derivation of statistical properties. It is shown in this paper that: (i) the EGARCH model can be derived from a random coefficient complex nonlinear moving average (RCCNMA) process; and (ii) the reason for the lack of statistical properties of the estimators of EGARCH is that the stationarity and invertibility conditions for the RCCNMA process are not known.

**Keywords:** Leverage, asymmetry, existence, random coefficient models, complex nonlinear moving average process.

**JEL classifications:** C22, C52, C58, G32.

## 1. Introduction

In the world of univariate conditional volatility models, the ARCH model of Engle (1982) and the generalization to the GARCH model by Bollerslev (1986) are the two most widely estimated symmetric models of time-varying conditional volatility, where symmetry refers to the identical effects on volatility of positive and negative shocks of equal magnitude.

The asymmetric effects on conditional volatility of positive and negative shocks of equal magnitude can be captured in different ways by the exponential GARCH (or EGARCH) model of Nelson (1990, 1991), and the GJR (alternatively, asymmetric or threshold) model of Glosten, Jagannathan and Runkle (1992). These are the two most widely estimated asymmetric univariate models of conditional volatility.

A special case of asymmetry is that of leverage. As defined by Black (1976) in terms of the debt-to-equity ratio, leverage is associated with increases in volatility for negative shocks to returns and decreases in volatility for positive shocks to returns. In short, leverage captures the negative correlation between returns shocks and subsequent shocks to volatility. The EGARCH model is capable of capturing leverage, depending on appropriate restrictions on the parameters of the model.

Although the GJR model can capture asymmetry, leverage is not possible, unless the short run persistence effect (that is, the ARCH parameter) is negative. Such a restriction is not consistent with the standard sufficient condition for conditional volatility to be positive.

The univariate GARCH model has been extended to its multivariate counterpart in, for example, the BEKK model of Baba et al. (1985) and Engle and Kroner (1995), and the VARMA-GARCH model of Ling and McAleer (2003). The GJR model has a multivariate counterpart in the VARMA-AGARCH model of McAleer, Hoti and Chan (2009). However, the EGARCH model has not yet been developed formally for multivariate processes, with appropriate regularity conditions.

Although it is not essential, leverage is an attractive feature of the EGARCH model. Since EGARCH is derived as a discrete-time approximation to a continuous-time stochastic volatility process, and is expressed in logarithms, the exponential operator is required to obtain conditional volatility, which is guaranteed to be positive. Therefore, no restrictions are required for conditional volatility to be positive.

However, there are as yet no statistical properties available for the (quasi-) maximum likelihood estimator of the EGARCH parameters. It is often argued heuristically that the reason for the lack of statistical properties arises from the presence in the model of an absolute value of a function of the parameters, which does not permit analytical derivatives or the derivation of statistical properties.

It is shown in this paper that the EGARCH model can be derived from a random coefficient complex nonlinear moving average (RCCNMA) process, and that the reason for the lack of statistical properties of the estimators of EGARCH is that the stationarity and invertibility conditions for the RCCNMA process are not known.

The remainder of the paper is organized as follows. In Section 2, the EGARCH model is discussed. Section 3 presents a RCCNMA process, from which EGARCH is derived in Section 4. Some concluding comments are given in Section 5.

## 2. EGARCH

Let the conditional mean of financial returns be given as:

$$y_t = E(y_t | I_{t-1}) + \varepsilon_t \tag{1}$$

where  $y_t = \Delta \log P_t$  represents the log-difference in stock prices ( $P_t$ ),  $I_{t-1}$  is the information set at time  $t-1$ , and  $\varepsilon_t$  is conditionally heteroskedastic.

The EGARCH specification of Nelson (1990, 1991) is given as:

$$\log h_t = \omega + \alpha |\eta_{t-1}| + \gamma \eta_{t-1} + \beta \log h_{t-1}, \quad |\beta| < 1 \quad (2)$$

where the standardized shocks,  $\eta_t$ , are given by  $\eta_t = \varepsilon_t / \sqrt{h_t}$ ,  $\eta_t \sim iid(0, \omega)$ , and  $|\beta| < 1$  is the stability condition when  $\log h_{t-1}$  is included in the model. Asymmetry exists if  $\gamma \neq 0$ , while leverage exists if  $\gamma < 0$  and  $\gamma < \alpha < -\gamma$ .

### 3. Random Coefficient Complex Nonlinear Moving Average Process

Consider a random coefficient complex nonlinear moving average (RCCNMA) process given by:

$$\varepsilon_t = \phi_t \sqrt{|\eta_{t-1}|} + \psi_t \sqrt{\eta_{t-1}} + \eta_t \quad (3)$$

where  $\sqrt{\eta_{t-1}}$  is a complex-valued function of  $\eta_{t-1}$ ,  $\phi_t \sim iid(0, \alpha)$ , and  $\psi_t \sim iid(0, \gamma)$ .

If  $\eta_t$  is a standard normal variable, then  $E\sqrt{|\eta_t|} = c$ , which is a known constant. Moreover:

$$E\sqrt{\eta_t} = E\sqrt{\eta_t} \times I(\eta_t \geq 0) + E\sqrt{\eta_t} \times I(\eta_t < 0)$$

$$= c/2 + ic/2 = (1+i)c/2,$$

where  $i = \sqrt{-1}$  and

$$I(\eta_t \geq 0) = I(\eta_t < 0) = 0.5$$

is an indicator function. As the mean of the complex-valued function is a finite constant, it follows that both the unconditional and conditional means of  $\varepsilon_t$  in equation (3) are zero.

As the RCCNMA process given in equation (3) is not in the class of random coefficient linear moving average models examined in Marek (2005), the stationarity and invertibility conditions of the RCCNMA process are not known.

#### 4. One Line Derivation of EGARCH

It follows from equation (3) that:

$$h_t = E(\varepsilon_t^2 | I_{t-1}) = \omega + \alpha |\eta_{t-1}| + \gamma \eta_{t-1}. \quad (4)$$

The use of an infinite lag for the RCCNMA process in equation (3) would yield the EGARCH model in equation (2).

It is worth noting that the transformation of  $h_t$  in equation (4) is not logarithmic, but the approximation given by:

$$\log h_t = \log(1 + (h_t - 1)) \approx h_t - 1$$

can be used to replace  $h_t$  in equation (4) with  $1 + \log h_t$ .

The interpretation of leverage, whereby  $\gamma < 0$  and  $\gamma < \alpha < -\gamma$ , is lost in the above derivation as both  $\alpha$  and  $\gamma$  are required to be positive.

More importantly, in terms of interpreting the model, the derivation of EGARCH, albeit without the logarithmic transformation, in equation (4) shows that the statistical properties of the model

do not exist because the model is based on a RCCNMA process, for which the stationarity and invertibility conditions are not known.

## 5. Conclusion

The paper was concerned with one of the most popular univariate asymmetric conditional volatility models is the exponential GARCH (or EGARCH) specification. The EGARCH model is popular, among other reasons, as it can capture both asymmetry, namely the different effects on conditional volatility of positive and negative effects of equal magnitude, and leverage, which is the negative correlation between returns shocks and subsequent shocks to volatility.

There are as yet no statistical properties available for the (quasi-) maximum likelihood estimator of the EGARCH parameters. Although is often argued heuristically that the reason for the lack of statistical properties arises from the presence in the model of an absolute value of a function of the parameters, which does not permit analytical derivatives or the derivation of statistical properties, it was shown in the paper that the EGARCH model could be derived from a random coefficient complex nonlinear moving average (RCCNMA) process, and that the reason for the lack of statistical properties of the estimators of EGARCH is that the stationarity and invertibility conditions for the RCCNMA process are not known.

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