# Axiomatic characterization of the interval function of a block graph


Kannan Balakrishnan
Department of Computer Applications
Cochin University of Science and Technology, Cochin, India - 682022
email: kannanb@cusat.ac.in

Manoj Changat*
Department of Futures Studies
University of Kerala, Trivandrum - 695034, India
e-mail: mchangat@gmail.com

Anandavally K. Lakshmikuttyamma
Mar Ivanios College
Thiruvananthapuram - 695015, India
e-mail: anandavallykl@yahoo.com

Joseph Mathews
C-GRAF, Department of Futures Studies
University of Kerala, Trivandrum - 695034, India
e-mail:jose_chingam@yahoo.co.in

Henry Martyn Mulder†
Econometrisch Instituut, Erasmus Universiteit
P.O. Box 1738 3000 DR Rotterdam, The Netherlands
e-mail: hmmulder@ese.eur.nl

Prasanth G. Narasimha-Shenoi
Department of Mathematics
Government College, Chittur Palakkad - 678104, India
e-mail: prasanthgns@gmail.com

N. Narayanan
Department of Mathematics
IIT Madras, Chennai-600 036, India
e-mail: narayana@gmail.com



20 August 2014

Econometric Institute Report EI 2014-17

This report is a preprint. It is not a formal publication in any way, and it will be published elsewhere.



*Research supported by NBHM DAE under file NBHM/R.P/71/2013/425.

†This research work was initiated when this author was visiting University of Kerala under the Erudite Scheme of Government of Kerala during January 4-16, 2011.

**Abstract**

In 1952 Sholander [25] formulated an axiomatic characterization of the interval function of a tree with a partial proof. In 2011 Chvátal et al. [9] gave a completion of this proof. In this paper we present a characterization of the interval function of a block graph using axioms on an arbitrary transit function $R$. From this we deduce two new characterizations of the interval function of a tree.

**Keywords**: block graph, tree, interval function, transit function.

**MCS**: primary 05C05, 05C75, 05C99.

# 1 Introduction

One of the fundamental notions of metric graph theory is that of the *interval function* $I : V \times V \to 2^V$ of a connected graph $G = (V, E)$, where $I(u, v)$ is the set of vertices on shortest paths between $u$ and $v$ in $G$. The term interval function was coined in [15], which is the first extensive study of this function. The notion already existed long before. We do not know for sure, but one of the first occurrences might be the thesis of W.D. Duthie [10] of 1940 on "Segments in Ordered Sets", see also [11]. He characterized distributive lattices by postulates or 'axioms' on segments. This work was pursued by Sholander [25, 26] in the early 1950's. Sholander studied median semilattices using segments. Median semilattices can also be studied as graphs: the Hasse diagram of a median semilattice is precisely a median graph (and vice versa). This was done for the first time by Avann in 1963 (who called the graph a 'unique ternary distance graph'), and later independently by Nebeský in 1971, and Mulder [14, 18, 15] in 1978 - 1980. Sholander also presented a set of axioms on segments that characterizes the segments (intervals) of a tree. But he gave a partial proof for this characterization. Only recently, in 2011, a completion of this proof was presented by Chvátal, Rautenbach & Schäfer [9].

Sholander also pursued another line of study in his papers [25, 26], viz. that of betweenness, in the language of ternary relations. This generalized results by Pitcher & Smiley [24]. Sholander used this notion of betweenness to characterize median betweenness, a structure that is equivalent to median semilattices and median graphs, see e.g. [19, 15, 20]. Amongst the results in [25] was a characterization of a tree betweenness. A new characterization was obtained recently by Burigana [1], with a short new proof by Chvátal et al. [9].

The focus of Sholander was on sets of axioms with as few axioms as possible. This was also the approach of later authors, see [19, 9]. In this approach the axioms are necessarily of rather complex nature. In [15] and later work a different approach was taken: here the choice has been to find axioms that are as elementary as possible, and also such that they are applicable in the most general setting, not that of only very well-structured graphs or ordered sets (such as median graphs and median semilattices), see e.g. [21, 22, 23, 5]. In [15] five simple and elementary properties of the interval function were given that are now known as the 'five classical' axioms for the interval function. In [17] the interval function of a connected graph is characterized by a set of axioms that includes these five classical, elementary axioms. The approach in [17] was as follows. First as much as possible was deduced using the five classical axioms only. Then the road blocks were determined that prevented any further consequence. From this two more axioms were inferred that, together with the five classical axioms, characterize the interval function of a connected graph. These two extra axioms were more complicated, but still minimal in the sense that weaker axioms would not do the trick.

In [13] a similar approach for betweenness was chosen: using axioms as simple as possible to study betweenness in a broad context. As opposed to the above idea of betweenness as a ternary relation, a betweenness in [13] was formulated in terms of a function $R : V \times V \to 2^V$. One advantage of this approach is that now it could be used in other contexts. In [16] a unifying approach for moving around in discrete structures such as graphs and partially ordered sets was presented: a *transit function* $R : V \times V \to 2^V$ satisfying three elementary axioms. It includes all of the above functions, but also other so-called path functions, like the *induced path function* $J$, see [5, 6], where $J(u, v)$ consists of the vertices on induced paths between $u$ and $v$. Recently, in [3], characterizations of some graph classes were obtained using betweenness axioms on the interval function and on the induced path transit function.

In this paper we return to the interval function. Above we mentioned the Sholander characterization (with a proof by Chvátal et al.) by a set of axioms with as few axioms as possible. Here we choose the other approach (from [15, 13, 17]): try to find a set of axioms that are each as simple and elementary as possible. We present a characterization of the interval function of a block graph. All but one of the axioms are simple and elementary in the sense that these are the above axioms for a betweenness. As corollaries we obtain two new characterizations in the case of trees. Here our sets of axioms have one axiom in common with the Sholander set for trees. In one of our characterizations the remaining axioms form an actually weaker set than those in the Sholander characterization. We also investigate the independence of the axioms in our various characterizations. We present our results in the context of transit functions. Besides this we present a characterization of the interval function of block graphs with at most one cut vertex. As corollaries we obtain characterizations for special classes of trees, e.g. the paths and the stars.

## 2 Axioms on Transit Functions

Throughout this paper $V$ is a finite nonempty set. A *transit function* on $V$ is a function $R : V \times V \to 2^V$, where $2^V$ is the power set of $V$, satisfying the following three axioms.

**(t1):** $u \in R(u, v)$, for all $u, v$ in $V$.

**(t2):** $R(u, v) = R(v, u)$, for all $u, v$ in $V$.

**(t3):** $R(u, u) = \{u\}$, for all $u$ in $V$.

The third axiom could be deleted. It is usually added to exclude degenerate cases. For instance, the function $F(u, v) = V$, for all $u, v$ in $V$, satisfies the first two axioms, but will not enlighten us about any aspect of an underlying structure. In the sequel we will see that in many relevant cases $(t3)$ follows from other axioms. If $G = (V, E)$ is a graph with vertex set $V$, then we say that $R$ is a transit function on $G$. The underlying graph $G_R$ of a transit function $R$ is the graph with vertex set $V$, where two distinct vertices $u$ and $v$ are joined by an edge if and only if $R(u, v) = \{u, v\}$. Note that, in general, $G$ and $G_R$ need not be isomorphic graphs.

The notion of transit function was introduced in [16] as a unifying concept for many functions on graphs that have been studied so far, e.g. the (geodesic) interval function $I$, the induced path function $J$, see [5, 6], the triangle-path function $T$, see [4, 8], the all-paths function $A$, see [2], and so forth, and so forth. It was also meant to create a framework for new problems and ideas. The four mentioned functions are all so-called *path transit functions*, because they are defined in terms of paths in $G$. See [16] and [7] for further information on path transit functions. In [16] the problem is proposed to characterize any transit function in terms of *transit axioms*, that is, axioms in terms of the function only, independent of the graph on which the function is defined. Nebeský [20] obtained a very interesting impossibility result: there does not exist a characterization of the induced path function $J$ of a connected graph using transit axioms only.

Our focus in this paper is on the interval function. Let $G = (V, E)$ be a graph with distance function $d$, where $d(u, v)$ is the length of a shortest $u, v$-path or $u, v$-geodesic. Then the interval function $I_G$ of $G$ is defined by

$$I_G(u, v) = \{\ x \mid d(u, x) + d(x, v) = d(u, v)\},$$

that is, the set of vertices lying on shortest paths between $u$ and $v$. When no confusion arises, we usually write $I$ instead of $I_G$.

The geodesic intervals $I(u, v)$ in $G$ inherently have the structure of a betweenness (defined below), but arbitrary transit functions may not have these properties. The following *betweenness axioms* were introduced in [13] to capture basic aspects of the idea of betweenness. The first of these tells us that, if $x$ is between $u$ and $v$ but distinct from $v$, then $v$ is *not* between $u$ and $x$. The second tells us that, if $x$ is between $u$ and $v$ and $y$ is between $u$ and $x$, then $y$ is between $u$ and $v$.

**(b1):** $x \in R(u, v), x \neq v \Longrightarrow v \notin R(u, x)$, for all $u, v$ in $V$.

**(b2):** $x \in R(u, v) \Longrightarrow R(u, x) \subseteq R(u, v)$, for all $u, v$ in $V$.

A *betweenness* in the sense of [13] is a function $R : V \times V \to 2^V$ satisfying $(t1)$, $(t2)$ and these two betweenness axioms. Below we will see that this notion is weaker than the betweenness considered by Sholander [25, 26] and Chvátal et al. [9]. The idea behind the betweenness, in the sense of [13], is that it is applicable to other transit functions as well. For instance, in [13, 5, 6], the case is studied for which graphs the induced path function $J$ is a betweenness, that is, satisfies the axioms $(b1)$ and $(b2)$. Note that axioms $(t1)$ and $(b1)$ imply axiom $(t3)$. So a betweenness is a transit function.

In the first extensive study of the interval function [15], five simple properties of the interval function $I(u, v)$ of any connected graph were presented. In [17] these properties were coined as the five *classical axioms* on $I$. These five transit axioms are $(t1)$ and $(t2)$, the betweenness axiom $(b2)$, and the following two axioms.

**(c4):** $x \in R(u, v) \Rightarrow R(u, x) \cap R(x, v) = \{x\}$, for all $u, v$ in $V$.

**(c5):** $x \in R(u, v), y \in R(u, x) \Rightarrow x \in R(y, v)$, for all $u, v$ in $V$.

Obviously, axioms $(t1)$ and $(c4)$ imply $(t3)$. So any function satisfying the five classical axioms is a transit function.

**Proposition 1** *Axioms* $(t1)$*,* $(t2)$ *and* $(c4)$ *imply axiom* $(b1)$*.*

**Proof.** Let $R : V \times V \to 2^V$ be a function on $V$ satisfying axioms $(t1)$, $(t2)$ and $(c4)$. Let $x$ be in $R(u, v)$ with $x \neq v$. By $(c4)$, we have $R(u, x) \cap R(x, v) = \{x\}$. From $(t1)$ and $(t2)$, it follows that $v$ lies in $R(x, v)$. Since $v \neq x$, we have that $v$ is not in $R(u, x)$. □ □ □

From this proposition it follows that any transit function satisfying the five classical axioms is a betweenness in our sense. Axiom ($b1$) is strictly weaker than axiom ($c4$), as the following example shows.

**Example 2** [A betweenness $R$ that does not satisfy ($c4$)]
The *k-fan* $F_k$ consists of a path $P$ on $k$ vertices and an additional vertex $y$ adjacent to all vertices on the path. Take $k \geq 5$. We consider the induced path function $J$ on $F_k$. It is straightforward to check that $J$ is a betweenness on this fan (satisfies axioms ($t1$), ($t2$), ($b1$) and ($b2$)). But this also follows trivially from any of the main results in [6]. Let $u$ and $v$ be the end vertices of $P$, and let $x$ be a vertex on $P$ that is not adjacent to $u$ or $v$. Then $y$ belongs to both $J(u,x)$ and $J(x,v)$. Obviously, $J(u,v) = V$. So this choice of vertices $u, v, x$ does not satisfy axiom ($c4$). □ □ □

As an additional observation we would like to add here that the induced path function of the 4-fan satisfies ($c4$) but not ($c5$): now take $x$ to be the vertex on $P$ adjacent to $v$. Then $y$ is in $J(u,x)$, but $x$ is not in $J(y,v) = \{y, v\}$. This example, Example 2 and Proposition 1 were already given in [12].

In [17] a characterization of the interval function of a connected graph is given involving the five classical axioms, see the Introduction for more information on this.

Already as early as 1952, Sholander [25] gave a characterization of the interval function of a tree, although without a complete proof. In his paper intervals were still called segments. The completion of the proof was presented by Chvátal et al. in [9]. Sholander's axioms were the following three axioms.

**(S):** There exists an $x$ such that $R(u,v) \cap R(v,w) = R(x,v)$, for all $u, v, w$ in $V$.

**(T):** $R(u,v) \subseteq R(u,w) \Rightarrow R(u,v) \cap R(v,w) = \{v\}$, for all $u, v, w$ in $V$.

**(U):** $R(u,x) \cap R(x,v) = \{x\} \Rightarrow R(u,x) \cup R(x,v) = R(u,v)$, for all $u, v$ in $V$.

Sholander [25] proved that his axioms ($S$) and ($T$) imply the five classical axioms. So, any function $R : V \times V \to 2^V$ satisfying ($S$) and ($T$) is a betweenness in our sense. Proposition 1 and Example 2 show that our concept of a betweenness is an essentially weaker concept than a Sholander function $R$ satisfying axioms ($S$) and ($T$).

## 3 The Interval Function of a Block Graph

First we recall some definitions. A graph is separable if it contains a cut vertex, that is, a vertex, the removal of which increases the number of components. A *block* in a connected graph is a maximal non-separable subgraph. Hence a block is either a $K_2$ or a maximal 2-connected subgraph. A connected graph $G$ is a *block graph* if every block in $G$ is a complete graph. Loosely speaking, it is a tree-like structure of cliques. Trivially, complete graphs and trees are block graphs. In this section we characterize the interval function of a block graph.

In [6] the following lemma is proved. Unfortunately, the use of some of the axioms was not made explicit. Hence, and also for the sake of completeness, we give a full proof of the lemma here. Note that, in [6] it was used to study the question for which graphs the induced path function $J$ is a betweenness. So this lemma applies to more functions than just the interval function $I$.

**Lemma 3** *Let $R$ be a betweenness on $V$. Then the underlying graph $G_R$ of $R$ is connected.*

**Proof.** Let $u$ and $v$ be any two distinct vertices in $G_R$. We prove the existence of a $u,v$-path in $G_R$ by induction on $|R(u,v)|$. Note that, by $(t1)$ and $(t2)$, we have $u, v \in R(u,v)$, so $|R(u,v)| \geq 2$. If $|R(u,v)| = 2$, then $R(u,v) = \{u,v\}$. So, by the definition of $G_R$, there is an edge between $u$ and $v$, which constitutes a $u,v$-path.

Assume that $|R(u,v)| = n > 2$. Then there is a vertex $x$ in $R(u,v)$ distinct from $u$ and $v$. By $(b1)$, we have $v \notin R(u,x)$. By $(b2)$, we have $R(u,x) \subseteq R(u,v)$. So $|R(u,x)| < |R(u,v)|$. By induction, there is a $u,x$-path. Similarly, by $(t1)$, $(t2)$, $(b1)$ and $(b2)$, we have $|R(x,v)| < |R(u,v)|$. Hence, by induction, there is also an $x,v$-path. These two paths together form a $u,v$-walk, which contains a $u,v$-path, and we are done. □ □ □

Note that in the proof of Lemma 3 we need both betweenness axioms $(b1)$ and $(b2)$ to make the induction work. To characterize the interval function of a block graph we introduce the following axiom, which is weaker than Sholander's axiom $(U)$.

**(U*):** $R(u,x) \cap R(x,v) = \{x\} \Rightarrow R(u,v) \subseteq R(u,x) \cup R(x,v)$, for all $u, v$ in $V$.

The 3-fan is usually denoted by $K_4 - e$, it is obtained from $K_4$ by deleting one edge. For any path $P$, the vertex set of $P$ is denoted by $V(P)$. Now we are ready to prove our main result.

**Theorem 4** *Let $R : V \times V \to 2^V$ be a function on $V$. Then $R$ satisfies axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and $(U^*)$ if and only if $G_R$ is a block graph and $R = I_{G_R}$.*

**Proof.** First let $R$ be the interval function of a block graph $G$. Clearly we have $G_R = G$. Moreover, $R$ being an interval function, $R$ satisfies axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$. Since $G$ is a block graph, $R(u,v) = V(P)$, where $P$ is the unique shortest $u,v$-path. Assume that $R(u,x) \cap R(x,v) = \{x\}$. Then there are two possibilities. First, $x$ is on $P$. In this case $R(u,x) \cup R(x,v) = R(u,v)$. Second, $x$ is adjacent to two consecutive vertices $y$ and $z$ on $P$. In this case $R(u,v) = V(P) = [R(u,x) \cup R(x,v)] - \{x\}$. So axiom $(U^*)$ is satisfied.

Conversely, assume that $R : V \times V \to 2^V$ is a betweenness satisfying axiom $(U^*)$. Note that $G_R$ is connected by Lemma 3. So, if $d$ is the distance function of $G_R$, then $d(u,v)$ is finite, for any two vertices $u$ and $v$ in $V$. By axioms $(t1)$ and $(t2)$, we have $R(u,v) = R(v,u)$ and $u, v \in R(u,v)$. Moreover, a betweenness satisfies $(t3)$. We use these facts in the sequel without mention. We split the proof in a number of claims.

**Claim 1.** *If $P$ is an induced $u,v$-path in $G_R$, then $R(u,v) \subseteq V(P)$.*
We use induction on the length $\ell(P)$ of $P$. If $\ell(P) = 0$, then $u = v$, and $R(u,u) = \{u\} = V(P)$. If $\ell(P) = 1$, then $u$ and $v$ are adjacent. So, by definition, $R(u,v) = \{u,v\} = V(P)$. Now assume that $\ell(P) \geq 2$. Let $x$ be the neighbor of $v$ on $P$, and let $P'$ be the subpath of $P$ between $u$ and $x$. By induction, we have $R(u,x) \subseteq V(P')$. Hence $v$ is not in $R(u,x)$. So $R(u,x) \cap R(x,v) = R(u,x) \cap \{x,v\} = \{x\}$. By axiom $(U^*)$, we have $R(u,v) \subseteq R(u,x) \cup R(x,v) \subseteq V(P') \cup \{x,v\} = V(P)$.

**Claim 2.** *$G_R$ does not contain an induced cycle of length at least 4.*
Assume the contrary, and let $C$ be an induced cycle of length at least 4. Take two non-adjacent vertices $u$ and $v$ on $C$. Then we have two internally disjoint induced paths $P$ and $Q$ in $C$ between $u$ and $v$. By Claim 1, we have $R(u,v) \subseteq V(P)$ as well as $R(u,v) \subseteq V(Q)$. This implies that $R(u,v)$ can not contain any internal vertex of $P$ and also not any internal vertex of $Q$. So we have $R(u,v) = \{u,v\}$. But this is impossible, since $u$ and $v$ are not adjacent. This settles Claim 2.

**Claim 3.** *$G_R$ does not contain an induced $K_4 - e$.*
Assume the contrary. Let $u$ and $v$ be the two non-adjacent vertices, and let $x$ and $y$ be other two vertices. By Claim 1, we have $R(u, v) \subseteq \{u, x, v\}$ and $R(u, v) \subseteq \{u, y, v\}$. Since $u$ and $v$ are not adjacent, we have a contradiction.

**Claim 4.** *$G_R$ is a block graph.*
By Claims 2 and 3, every block in $G_R$ is a complete graph. Hence, $G_R$ being connected, it is a block graph.

**Claim 5.** $R = I_{G_R}$.
Write $I = I_{G_R}$. Since $G_R$ is a block graph, there is a unique shortest path between any two vertices in $G_R$. So, by Claim 1, we have $R(u, v) \subseteq I(u, v)$. We prove that $R(u, v) = I(u, v)$ by induction on $d(u, v)$. First, $R(u, u) = \{u\} = I(u, u)$. If $d(u, v) = 1$, then, by definition, we have $R(u, v) = \{u, v\} = I(u, v)$. If $d(u, v) = 2$ with $x$ the common neighbor of $u$ and $v$, then, by Claim 1, we have $R(u, v) \subseteq \{u, x, v\}$. But we also have $R(u, v) \neq \{u, v\}$. So $R(u, v) = \{u, x, v\} = I(u, v)$. Now let $d(u, v) \geq 3$, and let $P$ be the shortest $u, v$-path. By Claim 1, we have $R(u, v) \subseteq I(u, v) = V(P)$. Since $u$ and $v$ are not adjacent, there must be a vertex $z$ on $P$ distinct from $u$ and $v$ that is in $R(u, v)$. Assume that $R(u, v) \neq I(u, v) = V(P)$. Then there must be a vertex $y$ on $P$ that is not in $R(u, v)$. We may choose $z$ and $y$ to be adjacent on $P$. Without loss of generality, $y$ is between $u$ and $z$ on $P$. By axiom $(b2)$, we have $R(u, z) \subseteq R(u, v)$. So $y$ does not belong to $R(u, z)$. Now, $z$ being an internal vertex of the shortest $u, v$-path $P$, we have $d(u, z) < d(u, v)$. So, by induction, $R(u, z) = I(u, z)$. But, $y$ being on the shortest path between $u$ and $z$, we have that $y$ is in $I(u, z)$. This yields a contradiction, and settles Claim 5, by which the proof is complete. □ □ □

## 4 The interval function of a tree

In this section, we present two new characterizations of the interval function of a tree. These are corollaries of Theorem 4. As an intermediate result, we characterize the interval function of a graph that is a tree or a complete graph. We consider the following new axiom. It is just in between our axiom $(U^*)$ and Sholander's axiom $(U)$.

**(U′):** $R(u, x) \cap R(x, v) = \{x\}, R(u, v) \neq \{u, v\} \Rightarrow R(u, x) \cup R(x, v) = R(u, v)$, for all $u, v$ in $V$.

It is straightforward to check that, if $R$ is the interval function of a triangle $K_3$, then it satisfies $(U')$. So we can expect a broader class than just the trees. The graph consisting of a triangle and an extra vertex adjacent to exactly one vertex of the triangle is called a *paw*.

**Theorem 5** *Let $R : V \times V \to 2^V$ be a function on $V$. Then $R$ satisfies axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and $(U')$ if and only if $G_R$ is a tree or a complete and $R = I_{G_R}$.*

**Proof.** First let $R$ be the interval function of a graph $G$ that is a tree or a complete graph. Clearly we have $G_R = G$. Moreover, $R$ being an interval function, $R$ satisfies axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$. If $G$ is a tree, then $R(u, v) = V(P)$, where $P$ is the unique $u, v$-path. So $R(u, x) \cap R(x, v) = \{x\}$ holds if and only if $x$ is on $P$. Hence $R(u, x) \cup R(x, v) = R(u, v)$. Now let $G$ be a complete graph. Then $R(u, v) = \{u, v\}$, for any two distinct vertices $u$ and $v$. So axiom $(U')$ is trivially satisfied.

Conversely, assume that $R : V \times V \to 2^V$ is a betweenness satisfying axiom $(U')$. By Theorem 4, $G_R$ is a block graph, and $R$ is the interval function of $G_R$. Assume that $G_R$

contains an induced paw $S$. Let $u$ be the vertex of degree 1 in $S$, let $w$ be the vertex of degree 3 in $S$, and let $x$ and $v$ be the vertices of degree 2 in $S$. Then we have $R(u,x) = \{u,w,x\}$ and $R(x,v) = \{x,v\}$ and $R(u,v) = \{u,w,v\}$. Clearly, the vertices $u,x,v$ violate axiom $(U')$. So $G_R$ does not contain an induced paw. This implies that $G_R$ is a either a tree or a complete graph with at least three vertices. □ □ □

Now we present two new characterizations of the interval function of a tree. Both involve axiom $(U)$, and some of the five elementary classical axioms.

**Theorem 6** *Let $R : V \times V \to 2^V$ be a function on $V$. Then $R$ satisfies axioms $(t1),(t2),(b1),(b2)$ and $(U)$ if and only if $G_R$ is a tree and $R = I_{G_R}$.*

**Proof.** If $G_R$ is a tree and $R$ is the interval function of $G_R$, then it is straightforward to check that $R$ is a betweenness satisfying axiom $(U)$.

For the converse note that axiom $(U')$ is weaker than axiom $(U)$. Hence, by Theorem 5, the underlying graph $G_R$ of $R$ is either a tree or a complete graph and $R = I_{G_R}$. But $(U)$ clearly forbids a triangle in $G_R$. So $G_R$ is a tree. □ □ □

Note that our Example 2 shows that axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$ are weaker than axioms $(S)$ and $(T)$. So Theorem 6 is actually a new characterization of the interval function of a tree. For our second characterization we need another lemma. It turns out that we can replace the two betweenness axioms $(b1)$ and $(b2)$ by the single classical axiom $(c4)$.

**Lemma 7** *Axioms $(c4)$ and $(U)$ imply axiom $(b2)$.*

**Proof.** Let $R : V \times V \to 2^V$ be a function on $V$ satisfying axioms $(c4)$ and $(U)$. Take $x$ in $R(u,v)$. By $(c4)$, we have $R(u,x) \cap R(x,v) = \{x\}$. Hence, by $(U)$, we have $R(u,x) \cup R(x,v) = R(u,v)$. Therefore $R(u,x) \subseteq R(u,v)$. □ □ □

Using Lemmata 1 and 7 the following Theorem is an immediate corollary of Theorem 6.

**Theorem 8** *Let $R : V \times V \to 2^V$ be a function on $V$. Then $R$ satisfies axioms $(t1),(t2),(c4)$ and $(U)$ if and only if $G_R$ is a tree and $I_{G_R} = R$.*

From what we have so far we deduce another characterization of the interval function of a tree that involves some of the axioms and a condition on the underlying graph $G_R$. So it is not a fully axiomatic characterization.

**Proposition 9** *Let $R : V \times V \to 2^V$ be a function satisfying the three axioms $(t1)$, $(t2)$ and $(U)$. Then each component of $G_R$ is a tree, and $R = I_H$ on each component $H$ of $G_R$.*

**Proof.** We only give a sketch of the proof, because many of the arguments are similar to those in the proof of Theorem 4. The first step is to prove that, for any induced $u,v$-path $P$, we have $R(u,v) = V(P)$. This can be done by induction on the length of $P$ similar as in Claim 1. Next we prove that $G_R$ does not contain an induced cycle of length at least 4, using the same arguments as in Claim 2. By $(U)$, it is trivial that $G_R$ does not contain a triangle. Hence each component is a tree. By the the first step, we have that $R$ is the interval function on each component. □ □ □

Note that in the last step of this proof we did not need axiom $(b2)$. That $R$ is the interval function on each component just follows from Step 1 and the fact that the component is a tree. In the proof of Claim 5 in Theorem 4, we really needed axiom $(b2)$. With this Proposition in hand, we can replace axioms $(b1)$ and $(b2)$ in Theorem 6 and axiom $(c4)$ in Theorem 8 by the condition that $G_R$ is connected.

**Theorem 10** *Let $R : V \times V \to 2^V$ be a function on $V$. Then $G_R$ is connected and $R$ satisfies axioms $(t1), (t2)$ and $(U)$ if and only if $G_R$ is a tree and $I_{G_R} = R$.*

There are now three axiomatic characterizations available that involve axioms only: Sholander's from 1952 with a full proof in [9], and our two above. In all three axiom $(U)$ is used. This axiom is rather strong, because in itself it almost forces that there be a unique path between any two vertices. To explore the reach of the axiom we consider the following example.

**Example 11** Let $C = u_1 \to u_2 \to \ \ldots \to u_k \to u_1$ be a directed cycle on $k$ vertices with $k \geq 3$. Write $V = \{u_1, u_2, \ \ldots \ , u_k\}$. We define the function $R : V \times V \to 2^V$ as follows. For vertices $u$ and $v$, the set $R(u, v)$ is the set of vertices on the directed path from $u$ to $v$ in $C$. Then $R$ satisfies axioms $(t1)$, $(t3)$, $(b1)$, $(b2)$, $(c4)$, $(c5)$ and $(U)$ but not $(t2)$. It also satisfies the axiom $(t1')$, viz. $v$ lies in $R(u, v)$. In the directed cycle there is a unique directed path between any two vertices. But, clearly, $C$ is not a tree. □ □ □

Two questions arise. First, by replacing axiom $(t2)$ by $(t1')$ as in Example 11, we could develop results on such functions, with a directed graph as underlying graph. What can be done in this case? We will not pursue this question here. Second, is there a characterization of the interval function of a tree that does not involve $(U)$? Otherwise formulated, could we replace $(U)$ by one or more much axioms in the characterization of the interval function of a tree? Again we leave this as an open problem.

We make one observation here. Loosely speaking, axiom $(U)$, together with $(t1)$ and $(t2)$, forces the underlying graph to be cycle-free. But the converse does not hold. In the following example $G_R$ is cycle-free but $R$ does not satisfy $(U)$. Note that $G_R$ is not connected.

**Example 12** Let $V = \{u, v, w, x\}$. Define the transit function $R : V \times V \to 2^V$ as follows: $R(u, v) = \{u, v\}$, $R(v, w) = \{v, w\}$, $R(u, w) = \{u, x, w\}$, $R(u, x) = \{u, x, v\}$, $R(w, x) = \{w, x, v\}$, $R(v, x) = \{u, v, w, x\}$. Clearly, $G_R$ is cycle free: it contains only the edges $uv$ and $vw$. On the other hand, $R(u, v) \cap R(v, w) = \{v\}$, but $R(u, v) \cup R(v, w) = \{u, v, w\} \neq \{u, x, w\} = R(u, w)$. □ □ □

## 5 Independence of Axioms

In Section 2 we observed some implications among the axioms. For instance, axioms $(t1)$ and $(b1)$ imply $(t3)$, and also axioms $(t1)$ and $(c4)$ imply $(t3)$. In Lemmata 1 and 7 we deduced two other implications. In this section we try to establish independence of the axioms in our results.

First, define the function $R : V \times V \to 2^V$ by $R(u, v) = \emptyset$, for all $u, v$ in $V$. Then $R$ trivially does not satisfy $(t1)$, but, also trivially, satisfies the axioms $(t2)$, $(b1)$, $(b2)$, $(c4)$ and $(U)$. So axiom $(t1)$ is independent form the others. Second, Example 11 shows that axiom $(t2)$ is independent from the other axioms in Theorem 6. Hence it is also independent from the other axioms in Theorem 4.

Take any connected graph that is not a block graph. Its interval function is a betweenness, and also satisfies axiom $(c4)$. By Theorems 4, 19 and 6, axioms $(U^*)$, $(U')$ and $(U)$ are trivially independent from the other axioms in our theorems. The following example shows that axiom ($b1$) is independent.

**Example 13** [$R$ does not satisfy ($b1$)]
Let $V$ be a set with $|V| \geq 3$. Define the function $R : V \times V \to 2^V$ by $R(u,v) = V$, for all distinct $u, v \in V(G)$, and $R(u,u) = \{u\}$, for all $u$ in $V$. Then $R$ trivially satisfies ($t1$), ($t2$) and ($b2$). Now, for any $x$ distinct from $u$ and $v$, we have $R(u,x) \cap R(x,v) = V \neq \{x\}$. So in this case ($U$) is trivially satisfied. If $x = u$, then $R(u,u) \cup R(u,v) = \{u\} \cup R(u,v) = R(u,v)$. So again ($U$) is satisfied. Similarly, if $x = v$, axiom ($U$) is satisfied. Now take distinct $u, v$. Then $R(u,v) = V$. So it contains a vertex $x$ distinct from $u$ and $v$. Hence $R(u,x) = V = R(u,v)$, so that axiom ($b1$) is not satisfied. □ □ □

Our next example shows trivially that axiom ($t1$) is independent.

**Example 14** [$R$ does not satisfy ($t1$)]
Let $V$ be a set with $|V| \geq 2$, and let $z$ be a fixed vertex in $V$. We define the function $R$ by $R(u,v) = \{z\}$, for all $u, v$ in $V$. Clearly $R$ does not satisfy ($t1$). Also $R$ satisfies axioms ($t2$), ($b1$), ($b2$), ($U$) and ($c4$) trivially. □ □ □

Finally, we show the last independence of the axiom set in our main Theorem 4 on block graphs.

**Example 15** [$R$ does not satisfy ($b2$)]
Let $G$ be the graph consisting of the path $P = u_0 u_1 \ \ldots \ u_{2k}$ and an isolated vertex $z$, with $k \geq 1$. So $G$ is not connected, and $P$ is a path of even length at least 2. We define the transit function $R$ on $G$ as follows. It has $G$ as underlying graph, and on $P$ it is just the interval function $I_P$ of $P$. So far only the intervals between $z$ and any vertex on $P$ are yet undetermined. We define $R(u_i, z) = R(z, u_i) = \{u_i, u_{i+1}, \ \ldots \ , u_{i+k}, z\}$. Here we assume that the indices are taken modulo $n = 2k+1$, that is $u_{2k+1} = u_0$, and so forth. Loosely speaking, $R(u_i, z)$ consists of $z$, $u_i$ and the $k$ vertices of $P$ following $u_i$ (modulo $n$). We call these vertices the $k$ *followers* of $u_i$ on $P$.

Clearly, $R$ satisfies ($t1$) and ($t2$). On $P$ it satisfies ($b1$), ($b2$) and ($U$) as well. Now we check the cases where intervals of the type $R(u,z)$ or $R(z,v)$, with $u$ and $v$ on $P$, are involved. Take $x$ in $R(u,z)$ distinct from $z$. Then $R(u,x) = I_P(u,x)$, so that it does not contain $z$. Take $x$ in $R(z,u)$ distinct from $u$, say $u = u_i$ and $x = u_{i+\ell}$ with $\ell \leq k$. Then $u_i$ is not among the $k$ followers of $x$ on $P$. So $u$ is not in $R(z,x)$. So $R$ satisfies ($b1$) overall. Clearly, $u_{(i+1)+k} = u_{i+(k+1)}$ lies in $R(u_{i+1}, z)$ but not in $R(u_i, z)$. Hence $R(u_{i+1}, z)$ is not contained in $R(u_i, z)$. Therefore $R$ does not satisfy ($b2$).

Finally, we show that $R$ satisfies ($U^*$). Consider $R(u,x) \cap R(x,v)$. We only have to check the cases that $z$ is among $u, x, v$. First suppose that $z = x$. Then $R(u,z)$ and $R(z,v)$ both contain $k+1$ vertices on $P$. Hence they have at least one common vertex on $P$. So $|R(u,z) \cap R(z,v)| \geq 2$, and ($U^*$) is trivially satisfied. Now suppose that $z = u$, say. Let $x = u_i$ and $v = u_j$. In order that we have $R(z,x) \cap R(x,v) = \{x\}$, we need $v$ to be on the part of $P$ between $u_0$ and $u_i$ such that $v$ does not belong to the $k$ followers of $x$ on $P$. But now $R(z,x) \cup R(x,v)$ contains $z$ and the vertices of $P$ between $u_j$ and $u_{i+k}$. So it contains $R(z,v)$, and ($U^*$) is satisfied. □ □ □

Note that we have thus established independence of most of the axioms in our theorems. For one important case we do not have an answer yet. Is axiom ($b2$) independent from axioms $(t1)$, $(t2)$, $(b1)$ and $(U)$? We do not know whether these four axioms imply $(b2)$ or not. Similarly, what if we replace $(U)$ by $(U')$? This remains an open problem. Here we present some partial answers on this question of independence. Proposition 9 tells us that, if $(t1)$, $(t2)$ and $(U)$ are satisfied, then each component of $G_R$ is a tree. What we cannot prove is that $G_R$ is connected. For this we seem to need $(b1)$ and $(b2)$, or other axioms that would do the trick. So any example that would show that $(b2)$ is independent from the other four axioms in Theorem 6 must have a disconnected underlying graph. We present two examples that show some independencies.

**Example 16** [$R$ satisfies $(t1)$, $(t2)$ and $(U)$ but neither $(b1)$ nor $(b2)$]
Let $C$ be an odd cycle of length $2k+1$ with $k \geq 2$, and let $V$ be the vertex set of $C$. We define $R(u,u) = \{u\}$, for all $u$ in $V$. For distinct $u$ and $v$, we define $R(u,v)$ to be the set of all vertices on the longest $u,v$-path in $C$. Note that $G_R$ is the edgeless graph. Clearly, $R$ satisfies $(t1)$ and $(t2)$. We have $R(u,u) \cup R(u,v) = R(u,v)$. So in this case $(U)$ is trivially satisfied. Now take distinct $u, x, v$. Let $y$ be the neighbor of $x$ in $R(u,x)$. To avoid that $y$ is also in $R(x,v)$, we must have that $u$ is on the longest $x,v$-path. Therefore $R(u,x) \cap R(x,v) \neq \{x\}$, so that again $(U)$ is trivially satisfied. □ □ □

**Example 17** [$R$ satisfies $(t1)$, $(t2)$ and $(b1)$ but neither $(U)$ nor $(b2)$]
Let $S$ be the paw with vertex set $V = \{u, v, w, x\}$ with $u$ the vertex of degree 1 and $x$ the vertex of degree 3. Let $R$ be the transit function with $S$ as underlying graph defined as follows for the two non-adjacent pairs: $R(u,v) = \{u,v,w\}$ and $R(u,w) = \{u,w,x\}$. Now $w$ is in $R(u,v)$, but $R(u,w)$ is not a subset of $R(u,v)$. So $(b2)$ is not satisfied. Moreover, $R(u,v) \cap R(v,x) = \{v\}$, but $R(u,v) \cup R(v,x) = V \neq \{u,x\} = R(u,x)$. So $(U^*)$ does not hold, and in particular, $(U)$ does not hold. It is easy to verify that $(b1)$ holds. □ □ □

## 6 The interval function of special classes of trees

Two vertices that have maximum distance in a connected graph $G = (V,E)$ are called *diametrical*. Of course, diametrical pairs of vertices exist in any connected graph. In some graphs they play a special role. It might be that there is a pair $u, v$ in $G$ such that $I(u,v) = V$. Such a pair is necessarily diametrical. Graphs having such a pair are abundant. Some special instances are such different graphs as paths, hypercubes and even cycles. The hypercubes and the even cycles even have the property that any vertex is in such a pair. The following axiom catches the existence of such a pair.

**(D):** There exists $p$ and $q$ in $V$ such that $R(p,q) = V$.

If we combine this axiom with the ones that make $G_R$ a block graph, then obviously we get a path. So we have the following result.

**Theorem 18** *Let $R : V \times V \to 2^V$ be a function on $V$. Then $R$ satisfies axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$, $(U^*)$ and $(D)$ if and only if $G_R$ is a path and $I_{G_R} = R$.*

Here we would like to suggest the following question: is there a characterization of the interval function of a hypercube involving (almost) only elementary axioms, amongst which axiom $(D)$?

There is another subclass of the trees that admit a characterization involving an extra axiom, viz. the stars $K_{1,n}$ with $n > 1$. It turns out that with this extra axiom we can weaken axiom $(U)$. We prove a more general theorem. A *block star* is a block graph with at most one cut vertex. So it consists of complete graphs, all glued together along the same vertex.

**(St):** $|R(u_1,u_2)\cap R(v_1,v_2)|=1$, for distinct $u_1,u_2,v_1,v_2$ in $V$.

**(p2):** $R(u,x)=\{u,x\}$, $R(x,v)=\{x,v\}\Rightarrow R(u,v)\subseteq R(u,x)\cup R(x,v)$, for $u,v,x$ in $V$.

Note that axiom $(p2)$ in itself does not guarantee that $R(u,v)=V(P)$, for any shortest $u,v$-path $P$ in $G_R$. For instance, let $G$ be a graph without induced $C_4$ or $K_4-e$. Then the interval function of $G$ satisfies $(p2)$.

**Theorem 19** *Let $R: V\times V\to 2^V$ be a function on $V$. Then $R$ is a betweenness satisfying axioms $(p2)$ and $(St)$ if and only if $G_R$ is a block star and $R=I_{G_R}$.*

**Proof.** Assume that $R$ is a betweenness satisfying the extra axioms $(St)$ and $(p2)$. By Lemma 3, we know that $G_R$ is connected. Let $u$ and $v$ be non-adjacent vertices having a common neighbor $x$. Then, by $(p2)$, we have $R(u,v)=\{u,x,v\}$. Now it follows that $G_R$ does not contain a $C_4$ or $K_4-e$ as an induced subgraph. Assume the contrary, and let $u$ and $v$ be non-adjacent vertices in this subgraph and $x$ and $y$ be their common neighbors in the subgraph. Then we would have $R(u,v)=\{u,x,v\}=\{u,y,v\}$, a contradiction.

Also, $G_R$ does not contain a path on four vertices as induced subgraph. Assume to the contrary that $uxvy$ is such a path. Then we have $R(u,v)=\{u,x,v\}$ and $R(x,y)=\{x,v,y\}$. But this contradicts axiom $(St)$. Hence $G_R$ also does not contain an induced cycle of length at least 5. From all this we deduce that the blocks of $G_R$ are complete graphs, so that it is a block graph. If there were two distinct cut vertices, then we would get an induced path of length at least 3. Since this is impossible, $G_R$ contains at most one cut vertex, so that it is a block star.

The converse is obvious. □ □ □

As a corollary, we have a characterization of the interval function of a star.

**(p2*):** $R(u,x)=\{u,x\}$, $R(x,v)=\{x,v\}\Rightarrow R(u,v)=R(u,x)\cup R(x,v)$, for $u,v,x$ in $V$.

**Theorem 20** *Let $R: V\times V\to 2^V$ be a function on $V$. Then $R$ is a betweenness satisfying axioms $(p2^*)$ and $(St)$ if and only if $G_R$ is a star and $R=I_{G_R}$.*

**Proof.** Assume that $R$ is a betweenness satisfying the extra axioms $(St)$ and $(p2^*)$. By Theorem 19, we have that $G_R$ is a block star. Axiom $(p2^*)$ forbids triangles. So $G_R$ is a star. The converse is obvious. □ □ □

# 7 Concluding remarks

We obtained a characterization of the interval function of a block graph. As a consequence, we obtained a characterization of the interval function of a tree that used weaker and more elementary axioms than Sholander's classical result on "tree segments" of 1952. We also presented a characterization of the interval function of block stars and stars, in which the heavy duty axiom $(U)$ is replaced by two simpler axioms. Moreover we presented a number of examples that showed various independencies of axiom sets. But it is still open whether axiom $(b2)$ is independent from the other axioms in Theorem 6, viz. axioms $(t1)$, $(t2)$, $(b1)$ and $(U)$. So it might be a necessary axiom, or it might follow from these other axioms.

Along the way we mentioned a few interesting open problems. For instance, can we avoid axiom $(U)$ in the case of trees? Is there a characterization of the interval function of the hypercube that involves axiom $(D)$. What can we do when we replace axiom $(t2)$ with the dual $(t1')$ of $(t1)$ with $(t1')$ being the axiom: $v \in R(u, v)$, for $u, v$ in $V$? As Example 11 shows, we then move into the area of directed graphs. These are just a few of the many open problems that are still abundant in this area.

## References


[1] L. Burigana, Tree representation of betweenness relations defined by intersection and inclusion, *Mathematics and Social Sciences* **185** (2009) 5 – 36.

[2] M. Changat, S. Klavžar and H.M. Mulder, The all-paths transit function of a graph, *Czech. Math. J.* **51** (126) (2001) 439 – 448.

[3] M. Changat, A.K. Lakshmikuttyamma, J. Mathews, I. Peterin, P.G. Narasimha-Shenoi, G. Seethakuttyammae, S. Špacapan, A forbidden subgraph characterization of some graph classes using betweenness axioms, *Discrete. Math.* **313** (2013) 951 – 958.

[4] M. Changat and J. Mathew, On triangle path convexity in graphs, *Discrete Math.* **206** (1999) 91 – 95.

[5] M. Changat, J. Mathew and H.M. Mulder, Induced path transit function, betweenness and monotonicity, *Electron. Notes Discrete Math.* **15** (2003) 62 – 65.

[6] M. Changat, J. Mathew and H.M. Mulder, Induced path function, monotonicity and betweenness, *Discrete Appl. Math* **158** (2010) 426 – 433.

[7] M. Changat, H.M. Mulder and G. Sierksma, Convexities Related to Path Properties on Graphs, *Discrete Math.* **290** (2005) 117 – 131.

[8] M. Changat, G.N. Prasanth and J. Mathews, Triangle path transit functions, betweenness and pseudo-modular graphs, *Discrete Math.* **309** (2009), 1575 – 1583.

[9] V. Chvátal, D. Rautenbach and P.M. Schäfer, Finite Sholander trees, trees, and their betweenness, *Discrete Math.* **311** (2011) 2143 – 2147.

[10] W.D. Duthie, *Segments in ordered sets*, Ph.D.Thesis, Princeton University, 1940, 27 pp.

[11] W.D. Duthie, Segments of ordered sets, *Transactions AMS* **51** (1942) 1 – 14.

[12] A.K. Lakshmikuttyamma, *Geodesic and induced path transit functions, their generalizations, betweenness axioms and related graph classes*, PhD-Thesis, University of Kerala, Trivandrum, 2013.

[13] M.A. Morgana and H.M. Mulder, The induced path convexity, betweenness and svelte graphs, *Discrete Math.* **254** (2002) 349 – 370.

[14] H.M. Mulder, The structure of median graphs, *Discrete Math.* **24** (1978) 197–204.

[15] H.M. Mulder, *The Interval Function of a Graph*, MC Tract 132, Mathematisch Centrum, Amsterdam, 1980.

[16] H.M. Mulder, Transit functions on graphs (and posets), in: M. Changat, S. Klavžar, H.M. Mulder and A. Vijayakumar eds., *Convexity in Discrete Structures*, pp. 117–130, Ramanujan Math. Soc. Lect. Notes Ser. **5**, *Ramanujan Math. Soc.*, Mysore, 2008.

[17] H.M. Mulder and L. Nebeský, Axiomatic characterization of the interval function of a graph. *European J. Combin.* **30** (2009) 1172 – 1185.

[18] H.M. Mulder and A. Schrijver, Median graphs and Helly hypergraphs, *Discrete Math.* **25** (1979) 41–50.

[19] L. Nebeský, Graphic algebras, *Comment. Math. Univ. Carolinae* **11** (1970) 533–544.

[20] L. Nebeský, The induced paths in a connected graph and a ternary relation determined by them, *Math. Bohem.* 127 (2002) 397 – 408.

[21] L. Nebeský, Characterization of the set of all shortest paths in a connected graph, *Math. Boh.* **119** (1994) 15 – 20.

[22] L. Nebeský, Characterization of the interval function of a connected graph, *Czech. Math. J.* **44** (1994) 173 – 178.

[23] L. Nebeský, A characterization of the interval function of a (finite or infinite) connected graph, *Czech. Math. J.* **51** (2001) 635 – 642.

[24] E. Pitcher and M.F. Smiley, Transitivities of betweenness, *Transactions AMS* **52** (1942) 95 – 114.

[25] M. Sholander, Trees, lattices, order, and betweenness, *Proc. Amer. Math. Soc.* **3** (1952) 369 – 381.

[26] M. Sholander, Medians and betweenness, *Proc. Amer. Math. Soc.* **5** (1952) 801–807.

[27] M.L.J. van de Vel, *Theory of Convex Structures*, North Holland, Amsterdam, 1993.