On the Invertibility of EGARCH

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Abstract

Of the two most widely estimated univariate asymmetric conditional volatility models, the exponential GARCH (or EGARCH) specification can capture asymmetry, which refers to the different effects on conditional volatility of positive and negative effects of equal magnitude, and leverage, which refers to the negative correlation between the returns shocks and subsequent shocks to volatility. However, the statistical properties of the (quasi-) maximum likelihood estimator (QMLE) of the EGARCH parameters are not available under general conditions, but only for special cases under highly restrictive and unverifiable conditions. A limitation in the development of asymptotic properties of the QMLE for EGARCH is the lack of an invertibility condition for the returns shocks underlying the model. It is shown in this paper that the EGARCH model can be derived from a stochastic process, for which the invertibility conditions can be stated simply and explicitly. This will be useful in re-interpreting the existing properties of the QMLE of the EGARCH parameters.

Keywords: Leverage, asymmetry, existence, stochastic process, asymptotic properties, invertibility.

JEL classifications: C22, C52, C58, G32.

1. Introduction

In addition to modelling and forecasting volatility, and capturing clustering clustering, two key characteristics of univariate time-varying conditional volatility models in the GARCH class of Engle (1982) and Bollerslev (1986) are asymmetry and leverage. Asymmetry refers to the different impacts on volatility of positive and negative shocks of equal magnitude, whereas leverage, as a special case of asymmetry, captures the negative correlation between the returns shocks and subsequent shocks to volatility. Black (1976) defined leverage in terms of the debt-to-equity ratio, with increases in volatility arising from negative shocks to returns and decreases in volatility arising from positive shocks to returns.

The two most widely estimated asymmetric univariate models of conditional volatility are the exponential GARCH (or EGARCH) model of Nelson (1990, 1991), and the GJR (alternatively, asymmetric or threshold) model of Glosten, Jagannathan and Runkle (1992). As EGARCH is a discrete-time approximation to a continuous-time stochastic volatility process, and is expressed in logarithms, conditional volatility is guaranteed to be positive without any restrictions on the parameters. In order to capture leverage, the EGARCH model requires parametric restrictions to be satisfied. Leverage is not possible for GJR, unless the short run persistence parameter is negative, which is not consistent with the standard sufficient condition for conditional volatility to be positive.

As GARCH can be obtained from random coefficient autoregressive models (see Tsay (1987)), and similarly for GJR (see McAleer et al. (2007)), the statistical properties for the (quasi-) maximum likelihood estimator (QMLE) of the GARCH and GJR parameters are straightforward to establish. However, the statistical properties for the QMLE of the EGARCH parameters are not available under general conditions. A limitation in the development of asymptotic properties of the QMLE for EGARCH is the lack of an invertibility condition for the returns shocks underlying the model.

McAleer and Hafner (2014) showed that EGARCH could be derived from a random coefficient complex nonlinear moving average (RCCNMA) process. The reason for the lack of statistical properties of the QMLE of EGARCH under general conditions is that the stationarity and invertibility conditions for the RCCNMA process are not known, in part because the RCCNMA process is not in the class of random coefficient linear moving average models (for further details, see Marek (2005)).

The recent literature on the asymptotic properties of the QMLE of EGARCH shows that such properties are available only for some special cases, and under highly restrictive and unverifiable conditions. For example, Straumann and Mikosch (2006) derive some asymptotic results for the simple EGARCH(1,0) model, but their regularity conditions are difficult to interpret or verify. Wintenberger (2013) proves consistency and asymptotic normality for the quasi-maximum likelihood estimator of EGARCH(1,1) under the non-verifiable assumption of invertibility of the model. Demos and Kyriakopoulou (2014) present sufficient conditions for asymptotic normality under a highly restrictive conditions that are difficult to verify.

It is shown in this paper that the EGARCH model can, in fact, be derived from a stochastic process, for which the invertibility conditions can be stated simply and explicitly. This will be useful in re-interpreting the existing properties of the QMLE of the EGARCH parameters.

The remainder of the paper is organized as follows. In Section 2, the EGARCH model is discussed. Section 3 presents a stochastic process, from which EGARCH is derived. Some concluding comments are given in Section 4.

2. EGARCH

Consider the conditional mean of financial returns as in the following:

$$y_t = E(y_t \mid I_{t-1}) + \varepsilon_t \tag{1}$$

where the returns, $y_t = \Delta \log P_t$, represents the log-difference in stock prices (P_t), I_{t-1} is the information set at time *t-1*, and ε_t is conditionally heteroskedastic.

The EGARCH specification of Nelson (1990, 1991) is given as:

$$\log h_t = \omega + \alpha \, | \eta_{t-1} | + \gamma \eta_{t-1} + \beta \log h_{t-1} \ , \quad | \beta | < 1 \qquad (2)$$

where the standardized shocks, $\eta_t \sim$ *iid* $(0, \omega)$, and $| \beta | < 1$ is the stability condition when $\log h_{t-1}$ is included in the model. Asymmetry exists if $\gamma \neq 0$, with symmetry given by $\gamma = 0$, while leverage arises if the parametric conditions $\gamma < 0$ and $\gamma < \alpha < -\gamma$ are satisfied. The specification in equation (2) is EGARCH(1,1), with EARCH(1) = EGARCH(1,0) when $\beta = 0$, but the specification can easily be extended to EGARCH(*p*,*q*).

In the absence of a specific stochastic process for ε_t, it is not possible to state the specific conditions for invertibility of the process. For this reason, McAleer and Hafner (2014) proposed a random coefficient complex nonlinear moving average (RCCNMA) process for ε_t. However, it could not be shown that the RCCNMA process was invertible.

3. Invertibility of a Stochastic Process for Returns Shocks

In this section, a stochastic process for ε_t is proposed, for which there are simple and explicit invertibility conditions.

Consider the following stochastic process for returns shocks given as:

$$\varepsilon_t = \pi_t \exp\left(\alpha \, | \eta_{t-1} | / 2 + \gamma \eta_{t-1} / 2 + \eta_t / \sqrt{2} \right) \qquad (3)$$

where $\pi_t \sim$ *iid* (0,1), $\eta_t \sim$ *iid* N $(0, \omega)$, and $\alpha, \gamma \in \Re$.

The conditional expectation of ε_t is given as:

$$E[\varepsilon_t \mid I_{t-1}] = 0 \tag{4}$$

as the expectation of π_t is zero and the expectation of the exponential term in equation (3) is finite as the *iid* random variable is normal. It follows that both the unconditional and conditional means of ε_t in equation (3) are zero.

The conditional variance of ε_t is given as:

$$h_t \equiv E\left(\varepsilon_t^2 \mid I_{t-1}\right) = E\left\{\pi_t^2 \exp\left(\alpha \mid \eta_{t-1} \mid + \gamma\eta_{t-1} + \sqrt{2}\eta_t\right)\right\}$$

$$= \exp(\omega + \alpha \mid \eta_{t-1} \mid + \gamma\eta_{t-1})$$

which yields the EGARCH(1,0) = EARCH(1) model as:

$$\log h_t = \omega + \alpha \mid \eta_{t-1} \mid + \gamma\eta_{t-1} \,. \tag{5}$$

A distributed lag version of equation (3), with lags $\to \infty$, $\alpha_j = \alpha\beta^{j-1}$ and $\gamma_j = \gamma\beta^{j-1}$, would lead to the EGARCH(1,1) model.

From equation (3), $sign(\varepsilon_t) = sign(\pi_t)$. For invertibility, we need two conditions to hold, the first of which is given by:

Condition 1: $P(\pi_t = 0) = 0$ and $P(\mid \pi_t \mid < \infty) = 1$.

This condition is not restrictive for any variable with a distribution that is absolutely continuous under a Lebesque measure. Therefore, $\varepsilon_t / \pi_t > 0$ almost surely. It follows that:

$$\log(\varepsilon_t / \pi_t) = \eta_t / \sqrt{2} + \alpha \,|\, \eta_{t-1} \,|\, / 2 + \gamma \eta_{t-1} / 2$$

and

$$\eta_t = \sqrt{2} \log(\varepsilon_t / \pi_t) - \alpha \,|\, \eta_{t-1} \,|\, / \sqrt{2} - \gamma \eta_{t-1} / \sqrt{2} \,. \tag{6}$$

Equation (6) will be used to invert the stochastic process recursively.

In order to simplify the notation, consider the function given by:

$$f_{\alpha,\gamma}(x) \equiv -\alpha \,|\, x \,|\, / \sqrt{2} - \gamma x / \sqrt{2} \,.$$

This leads to the following proposition:

Proposition 1: For $x, y \in \Re$:

$$|\, f_{\alpha,\gamma}(x) - f_{\alpha,\gamma}(y) \,| \le \left((|\, \alpha \,| + |\, \gamma \,|) / \sqrt{2} \right) |\, x - y \,|$$

Proof: Consider the following four cases:

(i) $x \ge 0, y \ge 0$:

$$|\, f_{\alpha,\gamma}(x) - f_{\alpha,\gamma}(y) \,| = |\, (\alpha + \gamma) x / \sqrt{2} - (\alpha + \gamma) y / \sqrt{2} \,|$$

$$= |(\alpha + \gamma) / \sqrt{2} \,||\, x - y \,|$$

$$\le (|\, \alpha \,| + |\, \gamma \,| / \sqrt{2}) \,|\, x - y \,|$$

(ii) $x \ge 0, y < 0$:

$$|\, f_{\alpha,\gamma}(x) - f_{\alpha,\gamma}(y) \,| = |\, (\alpha + \gamma) x / \sqrt{2} - (-\alpha + \gamma) y / \sqrt{2} \,|$$

$$
\begin{aligned}
&= |\alpha(x+y)/\sqrt{2} + \gamma(x-y)/\sqrt{2}| \\
&\leq |\alpha||x+y|/\sqrt{2} + |\gamma||x-y|/\sqrt{2} \\
&\leq (|\alpha|+|\gamma|)|x-y|/\sqrt{2}
\end{aligned}
$$

as $|x+y| \leq |x-y|$.

(iii) $x<0, y<0$: as in case (i).

(iv) $x<0, y\geq 0$: as in case (ii). •

Therefore, it can be shown through recursive substitution that:

$$
\begin{aligned}
\eta_t &= \sqrt{2}\log(\varepsilon_t/\pi_t) + f_{\alpha,\gamma}(\eta_{t-1}) \\
&= \sqrt{2}\log(\varepsilon_t/\pi_t) + f_{\alpha,\gamma}(\sqrt{2}\log(\varepsilon_{t-1}/\pi_{t-1}) + f_{\alpha,\gamma}(\eta_{t-2})) \\
&= \sqrt{2}\log(\varepsilon_t/\pi_t) + f_{\alpha,\gamma}(\sqrt{2}\log(\varepsilon_{t-1}/\pi_{t-1}) + f_{\alpha,\gamma}(\sqrt{2}\log(\varepsilon_{t-2}/\pi_{t-2}) + f_{\alpha,\gamma}(\eta_{t-3})))
\end{aligned} \tag{7}
$$

and so on. Each expression depends on $\eta_{t'}, t'<t$, and we need to express η_t as a function of ε_t, $t'<t$, for invertibility.

In order to simplify notation, consider the two series that are defined recursively:

$$
\begin{aligned}
u_n^1 &= f_{\alpha,\gamma}(\eta_{t-n}) + \sqrt{2}\log(\varepsilon_{t+1-n}/\pi_{t+1-n}) \\
u_n^{k+1} &= \sqrt{2}\log(\varepsilon_{t+k-n}/\pi_{t+k-n}) + f_{\alpha,\gamma}(u_n^k),\ k\geq 1
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
v_n^1 &= \sqrt{2}\log(\varepsilon_{t+1-n}/\pi_{t+1-n}) \\
v_n^{k+1} &= \sqrt{2}\log(\varepsilon_{t+k-n}/\pi_{t+k-n}) + f_{\alpha,\gamma}(v_n^k),\ k\geq 1
\end{aligned} \tag{9}
$$

From these definitions, it follows from equations (4) and (5) that:

$\eta_t = u_n^n$, $\forall\, n \in N^*$

and v_n^n depends on the ε_t . Thus, it is necessary to prove that:

$v_n^n \to \eta_t$.

For invertibility, the second of two required conditions is given by:

Condition 2: $|\,\alpha\,| + |\,\gamma\,| < \sqrt{2}$.

The following Lemma will be useful in the derivation of the invertibility condition:

Lemma 1: $|\, v_n^n - \eta_t \,| \le \left| (|\,\alpha\,| + |\,\gamma\,|)/\sqrt{2} \right|^n |\,\eta_{t-n}\,|$, $\forall\, n \in N^*$.

Proof: It was shown from equations (4) and (5) that $\eta_t = u_n^n$, $\forall\, n \in N^*$, and it follows from equations (8) and (9), and for $n \ge 2$:

$$
\begin{aligned}
&|\, v_n^n - \eta_t \,| = |\, v_n^n - u_n^n \,| \\
&= |\, f_{\alpha,\gamma}(v_n^{n-1}) - f_{\alpha,\gamma}(u_n^{n-1}) \,| \\
&\le \left((|\,\alpha\,| + |\,\gamma\,|)/\sqrt{2} \right) |\, v_n^{n-1} - u_n^{n-1} \,|
\end{aligned}
$$

Hence, we can show recursively that:

$$
\begin{aligned}
&|\, v_n^n - \eta_t \,| \le \left| (|\,\alpha\,| + |\,\gamma\,|)/\sqrt{2} \right|^{n-1} |\, v_n^1 - u_n^1 \,| \\
&\le \left| (|\,\alpha\,| + |\,\gamma\,|)/\sqrt{2} \right|^{n-1} |\, f_{\alpha,\gamma}(\eta_{t-n}) \,|
\end{aligned}
$$

As $f_{\alpha,\gamma}(0)=0$, we have $|\, f_{\alpha,\gamma}(\eta_{t-n})\,|=|\, f_{\alpha,\gamma}(\eta_{t-n}) - f_{\alpha,\gamma}(0)\,|$, so the Lemma follows by Proposition 1. •

We now consider L^1 and P convergence in the following Proposition:

Proposition 2: Under Conditions 1 and 2, it follows that:

$$v_n^n \xrightarrow[n\to\infty]{L^1,P} \eta_t$$

which can be used to derive the invertibility conditions.

Proof: Under Lemma 1, it is straightforward to show that:

$$E\left(\,|v_n^n - \eta_t\,|\right) \le \left|(|\,\alpha\,|+|\,\gamma\,|)/\sqrt{2}\right|^n E\left(\,|\,\eta_{t-n}\,|\right) \xrightarrow[n\to\infty]{} 0.$$

As L^1 convergence implies P convergence, this proves the proposition. •

Remark: As all the moments of a normal distribution exist, it is straightforward to prove L^p convergence $\forall\ p > 0$.

Lemma 1 is not sufficient to prove almost sure convergence as we do not know how the series $\left(\eta_{t-n}(\omega)\right)_{n\in N^*}$ behaves for a fixed ω. Borel Cantelli's Lemma, which is given in Lemma 2, enables a demonstration of almost sure convergence:

Lemma 2: Define the probability space (Ω, A, P) and consider a series of sets, $(E_n)_{n\in N^8}$, where $E_n \in A$. If $\sum_n P(E_n)$ converges, then $P(\lim_{n\to\infty}\sup E_n)=0$.

Lemma 2 can be used to prove the following Proposition of almost sure convergence:

Proposition 3: Under Conditions 1 and 2, it follows that:

$v_n^n \xrightarrow[n\to\infty]{} \eta_t$ almost surely,

which proves invertibility.

Proof: By Lemma 1, for $\omega \in \Omega$ it holds that:

$$| v_n^n(\omega) - \eta_t(\omega) | \le \left|(|\alpha| + |\gamma|)/\sqrt{2}\right|^n |\eta_{t-n}(\omega)|, \ \forall\, n \in N^*.$$

By Condition 2:

$(|\alpha| + |\gamma|)/\sqrt{2} < 1$, so that $\exists\, a \in \Re$ as $(|\alpha| + |\gamma|)/\sqrt{2} < a < 1$.

The objective is to find a set A with probability one such that, $\forall\, \omega \in A$, $v_n^n(\omega) \xrightarrow[n\to\infty]{} \eta_t(\omega)$.

Define:

$$E_n = \left\{\omega : |\eta_{t-n}(\omega)| \ge (1/a)^n\right\} \text{ and } P(E_n) = 2\Phi\left(-(1/a)^n/\sqrt{\omega}\right)$$

where $\Phi(\bullet)$ is the cumulative density function of a standard $N(0,1)$ random variable:

$$\Phi\left(-(1/a)^n/\sqrt{\omega}\right) = \int_{-\infty}^{-(1/a)^n/\sqrt{\omega}} \left(1/\sqrt{2\pi}\right) e^{-x^2/2} dx\,.$$

Moreover, $\exists\, N \in N^*$ such that, $\forall\, n \ge N$, $(1/a)^n/\sqrt{\omega} \ge 1$. For $n \ge N$, it follows that:

$$
\begin{aligned}
&\int_{-\infty}^{-(1/a)^n/\sqrt{\omega}} \left(1/\sqrt{2\pi}\right) e^{-x^2/2} dx \le \int_{-\infty}^{-(1/a)^n/\sqrt{\omega}} \left(1/\sqrt{2\pi}\right) e^{-|x|/2} dx \\
&\le \int_{-\infty}^{-(1/a)^n/\sqrt{\omega}} \left(1/\sqrt{2\pi}\right) e^{x/2} dx \\
&\le (1/\sqrt{2\pi}) \left[2e^{x/2}\right]_{-\infty}^{-(1/a)^n/\sqrt{\omega}} \\
&\le (1/\sqrt{2\pi})\ 2e^{-(1/a)^n/(2\sqrt{\omega})} \\
&= o(a^n)
\end{aligned}
$$

Therefore, $P(E_n) = o(a^n)$, with $a < 1$, as convergence of $\sum_n a^n$ implies that $\sum_n P(E_n)$ also converges. By Lemma 2, it follows that $P(\lim_{n\to\infty} \sup E_n) = 0$. However, $\lim_{n\to\infty} \sup E_n \in A$ and

$$
\begin{aligned}
E &\equiv \lim_{n\to\infty} \sup E_n = \bigcap_n \bigcup_{k \ge n} E_k \\
&= \left\{ \omega : \forall\, n,\ \exists\, k \ge n \ni \omega \in E_k \right\} \\
&= \left\{ \omega : \forall\, n,\ \exists\, k \ge n \ni |\eta_{t-k}(\omega)| \ge (1/a)^k \right\}
\end{aligned}
$$

Therefore, ${}^cE = \left\{ \omega : \exists\, n,\ \forall\, k \ge n \ni |\eta_{t-k}(\omega)| < (1/a)^k \right\}$, and $P({}^cE) = 1$ as $P(E) = 0$.

Furthermore, ${}^cE \subset \left\{ \omega :\ |\eta_{t-n}(\omega)| = O\left((1/a)^n\right) \right\}$.

As $0 \le (|\alpha| + |\gamma|)/\sqrt{2} < a < 1$ implies $1/a < \sqrt{2}/(|\alpha| + |\gamma|)$, it follows that:

$|\eta_{t-n}(\omega)| = O\left((1/a)^n\right)$ implies $|\eta_{t-n}(\omega)| = o\left(\left(\sqrt{2}/(|\alpha| + |\gamma|)\right)^n\right)$ which implies, for $\omega \in {}^cE$:

$$\left(|\eta_{t-n}(\omega)|\right)\left((|\alpha| + |\gamma|)/\sqrt{2}\right)^n \xrightarrow[n\to\infty]{} 0.$$

By Lemma 1, $\omega \in {}^cE$ implies $|\eta_t(\omega) - v_n^n(\omega)| \xrightarrow[n\to\infty]{} 0$ and $P({}^cE) = 1$.

Therefore, $v_n^n \xrightarrow[n\to\infty]{} \eta_t$ almost surely. •

4. Conclusion

The two most widely estimated asymmetric univariate models of conditional volatility are the exponential GARCH (or EGARCH) model and the GJR model. Asymmetry refers to the different effects on conditional volatility of positive and negative effects of equal magnitude, As EGARCH is a discrete-time approximation to a continuous-time stochastic volatility process, and is expressed in logarithms, conditional volatility is guaranteed to be positive without any restrictions on the parameters. For leverage, which refers to the negative correlation between returns shocks and subsequent shocks to volatility, EGARCH requires parametric restrictions to be satisfied. Leverage is not possible for GJR, unless the short run persistence parameter is negative, which is unlikely in practice.

The statistical properties for the QMLE of the GJR parameters are straightforward to establish. However, the statistical properties for the QMLE of the EGARCH parameters are not available under general conditions, but rather only for special cases under highly restrictive and unverifiable conditions.

A limitation in the development of asymptotic properties of the QMLE for EGARCH is the lack of an invertibility condition for the returns shocks underlying the model. It was shown in the paper that the EGARCH model could be derived from a stochastic process, for which the invertibility conditions could be stated simply and explicitly (conditions 1 and 2). This should be useful in re-interpreting the existing properties of the QMLE of the EGARCH parameters.

References

Black, F. (1976), Studies of stock market volatility changes, 1976 Proceedings of the American Statistical Association, Business and Economic Statistics Section, pp. 177-181.

Bollerslev, T. (1986), Generalised autoregressive conditional heteroscedasticity, Journal of Econometrics, 31, 307-327.

Demos, A. and D. Kyriakopoulou (2014), Asymptotic normality of the QMLEs in the EGARCH(1,1) model.
Available online: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2236055 (accessed on 14 June 2014).

Engle, R.F. (1982), Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation, Econometrica, 50, 987-1007.

Glosten, L., R. Jagannathan and D. Runkle (1992), On the relation between the expected value and volatility of nominal excess return on stocks, Journal of Finance, 46, 1779-1801.

Marek, T. (2005), On invertibility of a random coefficient moving average model, Kybernetika, 41(6), 743-756.

McAleer, M., F. Chan and D. Marinova (2007), An econometric analysis of asymmetric volatility: Theory and application to patents, Journal of Econometrics, 139, 259-284.

McAleer, M. and C. Hafner (2014), A one line derivation of EGARCH, Econometrics, 2(2), 92-97.

Nelson, D.B. (1990), ARCH models as diffusion approximations, Journal of Econometrics, 45, 7-38.

Nelson, D.B. (1991), Conditional heteroskedasticity in asset returns: A new approach, Econometrica, 59, 347-370.

Tsay, R.S. (1987), Conditional heteroscedastic time series models, Journal of the American Statistical Association, 82, 590-604.

Straumann, D. and T. Mikosch (2006), Quasi-maximum-likelihood estimation in conditionally heteroscedastic time series: A stochastic recurrence equation approach, Annals of Statistics, 34, 2449–2495.

Wintenberger, O. (2013), Continuous invertibility and stable QML estimation of the EGARCH(1,1) model, Scandinavian Journal of Statistics, 40, 846–867.