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Optimal Stopping-Related Inequalities for I.I.D. Random Variables when the Future Is Discounted

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Comparisons are made between the maximal expected gain of a prophet and the maximal expected reward of an ordinary player observing a sequence of uniformly bounded i.i.d. random variables when the future is discounted. The player uses pure threshold stopping times which are asymptotically optimal. Both finite and infinite sequences of random variables are treated. Also, comparisons between optimal stopping values and expected rewards obtained by using asymptotically optimal pure threshold stopping time are given.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables on some probability space $(\Omega, \mathscr{F}, P)$ and taking values in the unit interval $[0, 1]$. Hill and Kertz [5] proved that

$$E(\max_{1 \leqslant i \leqslant n} X_i) - \sup_{\tau \in \mathscr{T}_n} EX_\tau \leqslant b_n, \tag{1}$$

where $\mathscr{T}_n$ is the collection of all stopping times which stop no later than n and the numbers b_n are fixed points of certain inductively defined functions (e.g., $b_2 = 0.0625$, $b_5 \approx 0.090$, $b_{100} \approx 0.110$, $b_{10,000} \approx 0.111$; for more details, see Hill and Kertz [5]). Moreover, the constants b_n are the best possible bounds for which (1) holds.

Recently, Samuel-Cahn [9] studied the same problem in a cost of observation setting. She showed that for i.i.d. random variables $X_1, X_2, \ldots$ taking values in $[0, 1]$ and for all $c > 0$ and $n \geqslant 1$

$$E(\max_{1 \leqslant i \leqslant n} (X_i - ic)) - \sup_{\tau \in \mathscr{T}_n} E(X_\tau - \tau c) < e^{-1}. \tag{2}$$

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The constant $e^{-1} \approx 0.3679$ is the best possible bound for which (2) holds. Also specific bounds b_n^* and $b^*(c)$ for n and c fixed, respectively, were obtained: $b_n^* = ((n-1)/n)^{n+1}$ and $b^*(c) = [1/c]\, c(1-c)^{[1/c]+1}$ ($[x]$ is the largest integer not exceeding x). Note that Samuel-Cahn's results do not imply inequality (1) (the cost model problem has a discontinuity at $c=0$).

This paper concerns another classical model in optimal stopping theory, namely the discount model. The main results are the following. For i.i.d. random variables $X_1, X_2, \ldots$ taking values in $[0, 1]$ we have

$$E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - \sup_{\tau \in \mathscr{T}_n} E(\beta^{\tau-1} X_\tau) \leqslant U_n(\beta), \tag{3}$$

where $U_n(\beta)$ is given in Definition 3.5 (see also Proposition 4.2(ii)), and

$$E(\sup_{i \geqslant 1} \beta^{i-1} X_i) - \sup_{\tau \in \mathscr{T}} E(\beta^{\tau-1} X_\tau) \leqslant U(\beta), \tag{4}$$

where $\mathscr{T}$ is the collection of all almost surely finite stopping times and $U(\beta)$ is given in Proposition 4.3(i). The constant $U_n(\beta)$ is the best possible bound for which (3) holds if $\beta \in D_n$, where D_n is defined in Definition 4.1. The same phenomenon appears as in the Samuel-Cahn problem: inequality (1) is not implied by inequality (3) since $1 \notin D_n$ for all $n \geqslant 2$. The constant $U(\beta)$ is the best possible bound for which (4) holds for all $\beta \in (0, 1)$.

Inequalities such as (1)–(4) have been called prophet inequalities due to the natural interpretation (for instance in (1)) of $E(\max_{1 \leqslant i \leqslant n} X_i)$ as the expected gain of a prophet (a player with complete foresight) and $\sup_{\tau \in \mathscr{T}_n} EX_\tau$ as the expected reward of an ordinary player using an optimal strategy.

Both in Samuel-Cahn [9] as in this paper pure threshold stopping times (of the form stop the first time a certain fixed level is exceeded), which are asymptotically optimal in some sense (see Definition 2.1), are used to obtain the desired inequalities (2)–(4). (See also Samuel-Cahn [8].)

The organization of this paper is as follows: Section 2 is a preliminary section where asymptotically optimal stopping times are defined. We also show what these stopping times can look like in the setting of i.i.d. random variables with a discount factor. In Section 3, inequality (3) is presented, and Section 4 gives an asymptotic analysis of the constant $U_n(\beta)$ appearing in (3). Also the prophet problem for infinite sequences of random variables (inequality (4)) is solved in Section 4. Finally, in Section 5, comparisons between the optimal stopping value $\sup_{\tau \in \mathscr{T}_n} EX_\tau$ and the expected reward $E(\beta^{\tau_n - 1} X_{\tau_n})$ are given. Here $\{\tau_n\}$ is a sequence of asymptotically optimal pure threshold stopping times as given in Proposition 2.2(ii).

2. Asymptotically Optimal Stopping Times

Throughout this paper all sequences $X_1, X_2, \ldots$ consist of i.i.d. random variables taking values in [0, 1]. Let us fix some notation: for real numbers x and y, $x \vee y$ and $x \wedge y$ denote the maximum and the minimum of x and y, respectively, $x^+ = x \vee 0$ denotes the positive part of x, and $[x]$ is the largest integer not exceeding x. For a sequence $X_1, X_2, \ldots,$ let $\mathscr{T}$ be the collection of all a.s. finite stopping times with respect to the filtration $\{\mathscr{F}_i\}_{i=1}^{\infty}$, where $\mathscr{F}_i = \sigma\{X_1, \ldots, X_i\}$ is the σ-algebra generated by $\{X_1, \ldots, X_i\}$, and let $\mathscr{T}_n = \{\tau \wedge n : \tau \in \mathscr{T}\}$. Also denote $V(X_1, X_2, \ldots) = \sup_{\tau \in \mathscr{T}} EX_\tau$ and $V(X_1, \ldots, X_n) = \sup_{\tau \in \mathscr{T}_n} EX_\tau$ the optimal stopping values for the infinite and the finite horizon case, respectively.

Definition 2.1. (i) A stopping time σ is called $\mathscr{S}$-optimal ($\mathscr{S}$ is a subcollection of $\mathscr{T}$) for a sequence $X_1, X_2, \ldots$ if $EX_\sigma = \sup_{\tau \in \mathscr{S}} EX_\tau$.

(ii) A sequence of stopping times $\{\tau_n\}$, such that $\tau_n \in \mathscr{T}_n$, is called asymptotically optimal (AO) if $\lim_{n \to \infty} EX_{\tau_n} = V(X_1, X_2, \ldots)$.

In the discount model it is very easy to find the optimal stopping time for an infinite sequence of random variables. The next proposition is due to Karlin [6]. Since we only deal with bounded random variables a very simple proof can be given and is included here for the sake of completeness. An optimal stopping time τ for an infinite sequence $X_1, \beta X_2, \beta^2 X_3, \ldots$ is of the pure threshold type: stop the first time a fixed level is exceeded. If the same threshold is used for the finite sequence $X_1, \beta X_2, \ldots, \beta^{n-1} X_n$, then the resulting sequence of stopping times $\{\tau_n\}$ evidently has the AO-property.

Note that stopping times having the AO-property are not necessarily $\mathscr{T}_n$-optimal for a sequence $X_1, \beta X_2, \ldots, \beta^{n-1} X_n$ (see also Section 5).

Proposition 2.2. *Let $X, X_1, X_2, \ldots$ be a sequence of* i.i.d. [0, 1]*-valued random variables and take any $\beta \in (0, 1)$. Let $V(\beta)$ be the unique solution of the equation $x = E(X \vee \beta x)$. Then*

(i) *$\tau = \inf\{i \geq 1 : X_i > \beta V(\beta)\}$ is an optimal stopping time for the sequence $X_1, \beta X_2, \beta^2 X_3, \ldots$ and $V(X_1, \beta X_2, \beta^2 X_3, \ldots) = V(\beta)$; and*

(ii)

$$E(\beta^{\tau_n - 1} X_{\tau_n}) = V(\beta) - (\beta u)^{n-1} E(\beta V(\beta) - X)^+,$$

where $u = P(X \leq \beta V(\beta))$ and $\tau_n = \tau \wedge n$.

Proof. (i) Since the $\{X_i\}$ are bounded, we have $\lim_{n\to\infty} V(X_1, \beta X_2, ..., \beta^{n-1}X_n) = V(X_1, \beta X_2, \beta^2 X_3, ...)$. The principle of backward induction (Chow *et al.* [2, Th. 3.2, p. 50]) yields

$$V(X_1, \beta X_2, ..., \beta^{n-1}X_n) = E(X_1 \vee \beta V(X_1, \beta X_2, ..., \beta^{n-2}X_{n-1})). \qquad (5)$$

Taking limits on both sides of the equality sign in (5) gives $V(X_1, \beta X_2, \beta^2 X_3, ...) = V(\beta)$. To see that τ is optimal, compute

$$\begin{aligned} E(\beta^{\tau-1}X_\tau) &= \sum_{j=1}^{\infty} E(\beta^{j-1}X_j I_{\{\tau=j\}}) \\ &= \sum_{j=1}^{\infty} E(X-\beta V(\beta))^+ (\beta u)^{j-1} \\ &\quad + \sum_{j=1}^{\infty} \beta V(\beta)(\beta u)^{j-1}(1-u) = V(\beta). \end{aligned}$$

(ii)

$$\begin{aligned} E(\beta^{\tau_n-1}X_{\tau_n}) &= \sum_{j=1}^{n-1} E(\beta^{j-1}X_j I_{\{\tau_n=j\}}) + E(\beta^{n-1}X_n)\, u^{n-1} \\ &= V(\beta) - \sum_{j=n}^{\infty} E(\beta^{j-1}X_j I_{\{X_j>\beta V(\beta)\}})\, u^{j-1} + (u\beta)^{n-1} EX \\ &= V(\beta) - (u\beta)^{n-1} V(\beta) + (u\beta)^{n-1} EX \\ &= V(\beta) - (\beta u)^{n-1} E(\beta V(\beta) - X)^+. \quad \blacksquare \end{aligned}$$

3. Prophet Inequalities

In this section the best possible upper bound $U_n(\beta)$ for the difference between $E(\max_{1\leqslant i\leqslant n} \beta^{i-1}X_i)$ and $E(\beta^{\tau_n-1}X_{\tau_n})$ is given, where $\{\tau_n\}$ are the stopping times as in Proposition 2.2(ii) (having the AO-property). Now, let $v \in [0, 1]$ and $\beta \in (0, 1)$ and let $X, X_1, X_2, ...$ be any sequence such that $V(X_1, \beta X_2, \beta^2 X_3, ...) = v$. It may be assumed, without loss of generality (as Lemma 3.3 shows), using a mass spreading technique called balayage (see Hill and Kertz [4]), that X takes values in the set $\{0, v\beta, 1\}$ if we are searching for a best possible upper bound for $E(\max_{1\leqslant i\leqslant n} \beta^{i-1}X_i) - E(\beta^{\tau_n-1}X_{\tau_n})$.

Definition 3.1. Given a $[0, 1]$-valued random variable X and constants $0 \leqslant x < y \leqslant 1$, let X_x^y denote a random variable with distribution

$$P(X_x^y \in B) = P(X \in B) \quad \text{if} \quad B \not\subset [x, y], \quad B \in \mathscr{B}$$

$$P(X_x^y = x) = (y-x)^{-1} \int_{[x, y]} (y-z)\, dP_X(z),$$

$$P(X_x^y = y) = (y-x)^{-1} \int_{[x, y]} (z-x)\, dP_X(z).$$

(P_X denotes the distribution of X and $\mathscr{B}$ is the family of Borel sets in $[0, 1]$.)

The next lemma states some basic properties of balayaged random variables. The proofs of parts (i)–(iii) can be found in [4] and part (iv) can be found in [3].

LEMMA 3.2. *Let X and Y be two $[0, 1]$-valued random variables and suppose that Y is independent of both X and X_x^y, $0 \leq x \leq y \leq 1$. Then*

(i) $E(X \vee Y) \leq E(X_x^y \vee Y)$;

(ii) $E(X_x^y) = EX$;

(iii) $E(X_x^y \vee x) = E(X \vee x)$; *and*

(iv) $E\phi(X) \leq E\phi(X_x^y)$ *for all convex functions* $\phi\colon [0, 1] \to [0, 1]$.

LEMMA 3.3. *Fix $\beta \in (0, 1)$, let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables taking values in $[0, 1]$ and let $v = V(X_1, \beta X_2, \beta^2 X_3, \ldots)$. Then there exists a sequence of i.i.d. random variables $\hat{X}_1, \hat{X}_2, \ldots$, all distributed as a random variable $\hat{X}$ of the type*

$$\hat{X} = \begin{cases} 1 & \\ \beta v & \text{with probability} \\ 0 & \end{cases} \quad \begin{matrix} v(1-\beta)/(1-\beta v) \\ (1-v)/(1-\beta v) - p \\ p \end{matrix}$$

for some $p \in [0, (1-v)/(1-\beta v)]$, such that

(i) $V(\hat{X}_1, \beta \hat{X}_2, \beta^2 \hat{X}_3, \ldots) = v$; *and*

(ii) $E(\max_{1 \leq i \leq n} \beta^{i-1} X_i) - E(\beta^{\tau_n - 1} X_{\tau_n}) \leq E(\max_{1 \leq i \leq n} \beta^{i-1} \hat{X}_i) - E(\beta^{\tau_n - 1} \hat{X}_{\tau_n})$, *where* $\tau_n = (\inf\{i \geq 1 : X_i > \beta v\}) \wedge n$.

Proof. (i) Define $\hat{X} = (X_0^{\beta v})_{\beta v}^1$. Lemma 3.2(iii) and Proposition 2.2(i) imply that $E(\hat{X} \vee \beta v) = v$ and thus $V(\hat{X}_1, \beta \hat{X}_2, \ldots) = v$. Note that $\hat{X}$ is of the type above.

(ii) Lemma 3.2(i) implies that $E(\max_{1 \leq i \leq n} \beta^{i-1} X_i) \leq E(\max_{1 \leq i \leq n} \beta^{i-1} \hat{X}_i)$. Further, Proposition 2.2(ii) yields

$$E(\beta^{\tau_n - 1} X_{\tau_n}) = v - (\beta P(X \leq \beta v))^{n-1} E(\beta v - X)^+.$$

By Lemma 3.2(ii) and (iii) we have

$$E(\beta v - X)^+ = E(\beta v - \hat{X})^+$$

(use the identity $(a-b)^+ = a \vee b - b$), and by the definition of $\hat{X}$ we have that $P(X \leqslant \beta v) \leqslant P(\hat{X} \leqslant \beta v)$. Hence

$$E(\beta^{\tau_n - 1} X_{\tau_n}) \geqslant E(\beta^{\tau_n - 1} \hat{X}_{\tau_n}),$$

which settles the proof of (ii). ∎

Remark. The reason that we work with a strict inequality in the definition of τ_n is that Lemma 3.3 does not hold for stopping times $\tilde{\tau}_n$ of the form $(\inf\{i \geqslant 1 : X_i \geqslant \beta v\}) \wedge n$, since

$$E(\beta^{\tilde{\tau}_n - 1} X_{\tilde{\tau}_n}) = v - (\beta P(X < \beta v))^{n-1} E(\beta v - X)^+$$

and $P(X < \beta v) \geqslant P(\hat{X} < \beta v)$ (so the inequality is in the wrong direction).

The next lemma gives an upper bound for the difference $E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - E(\beta^{\tau_n - 1} X_{\tau_n})$ for random variables $X_1, X_2, \ldots$ with $V(X_1, \beta X_2, \beta^2 X_3, \ldots) = v$, $v \in (0, 1)$ fixed. (In the sequel we exclude the trivial cases $v = 0$ and $v = 1$ corresponding to $X \equiv 0$ and $X \equiv 1$, respectively.)

LEMMA 3.4. *Fix $\beta \in (0, 1)$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $v = V(X_1, \beta X_2, \beta^2 X_3, \ldots) \in (0, 1)$ is fixed. Then*

$$E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - E(\beta^{\tau_n - 1} X_{\tau_n}) \leqslant d_n(p, v; \beta),$$

where

$$\begin{aligned} d_n(p, v; \beta) = {} & v\beta(u - p)((1 - \beta^{s-1} u^{s-1}((1 - (p\beta)^{n-s})/(1 - p\beta)) \\ & + (p\beta)^{n-s} u((u^{s-1} - (p\beta)^{s-1})/(u - p\beta)) + (p\beta)^{n-1}) \\ & + v\beta^n (p u^{n-1} - p^{n-s} u^s), \end{aligned}$$

with $u = u(v; \beta) = (1 - v)/(1 - \beta v)$ and $s = s_n(v; \beta) = ([\log v/\log \beta] + 2) \wedge n$.

Proof. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distributions as in the reduction Lemma 3.3. For $n = 1, 2, \ldots$ define $m_n = E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i)$, $e_n = E(\beta^{\tau_n - 1} X_{\tau_n})$ and $d_n = m_n - e_n$. The idea is to express d_n in terms of d_{n-1} and then to obtain the expression for d_n by induction. Now,

$$\begin{aligned} d_n &= E(\max_{2 \leqslant i \leqslant n} \beta^{i-1} X_i - X_1)^+ + EX_1 - e_n \\ &= p\beta m_{n-1} + (u - p)\, \beta E(\max_{1 \leqslant i \leqslant n-1} \beta^{i-1} X_i - v)^+ \\ &\quad + (1 - u) + \beta v(u - p) - e_n \end{aligned}$$

$$= p\beta d_{n-1} + (u-p)\,\beta E(\max_{1\leqslant i\leqslant n-1} \beta^{i-1}X_i - v)^+$$

$$+ (1-p\beta)\,v + p\beta e_{n-1} - e_n,$$

where $u = u(v;\beta) = (1-v)/(1-\beta v)$. By Proposition 2.2(ii) we have

$$p\beta e_{n-1} - e_n = pv\beta^n u^{n-2}(u-p) - v(1-p\beta).$$

Also note that

$$E(\max_{1\leqslant i\leqslant n-1} \beta^{i-1}X_i - v)^+ = \sum_{i=0}^{j_n} (\beta^i - v)\,u^i(1-u) = vu^{j_n+1}(1-\beta^{j_n+1}),$$

where $j_n = j_n(v;\beta) = \max\{j\colon \beta^j \geqslant v\} = [\log v/\log \beta] \wedge (n-2)$. Hence,

$$d_n = p\beta d_{n-1} + (u-p)\,v(\beta(1-\beta^{j_n+1})\,u^{j_n+1} + p\beta^n u^{n-2}).$$

Now, put $C_n = \beta(1-\beta^{s_n-1})\,u^{s_n-1} + p\beta^n u^{n-2}$ with $s_n = s_n(v;\beta) = j_n(v;\beta) + 2$. By induction on n it follows that

$$d_n = (u-p)\,v\sum_{i=1}^{n-1} (p\beta)^{i-1}\,C_{n-i+1}. \tag{6}$$

(We use the convention that an empty sum is equal to zero, so $d_1 = 0$.) Next, we calculate the sum in (6). Define a partition $\{B_s\}_{s=n,\ldots,2}$ of $(0, 1]$ by $B_s = (\beta^{s-1}, \beta^{s-2}]$, $s = 2, \ldots, n-1$, and $B_n = (0, \beta^{n-2}]$. For $v \in B_s$ we have $s_n = s \wedge n$ and thus $s_{n-i+1} = s$ for $1 \leqslant i \leqslant n-s$ and $s_{n-i+1} = n-i+1$ for $n-s+1 \leqslant i \leqslant n-1$. Hence for $v \in B_s$

$$\begin{aligned}
d_n &= (u-p)\,v\sum_{i=1}^{n-1} (p\beta)^{i-1}\,C_{n-i+1}\\
&= (u-p)\,v\left\{\sum_{i=1}^{n-s} (p\beta)^{i-1}\,\beta(1-\beta^{s-1})\,u^{s-1}\right.\\
&\quad + \sum_{i=n-s+1}^{n-1} (p\beta)^{i-1}\,\beta(1-\beta^{n-i})\,u^{n-i}\\
&\quad \left. + \sum_{i=1}^{n-1} (p\beta)^{i-1}\,p\beta^{n-i+1}u^{n-i-1}\right\}\\
&= v\beta(u-p)\{(1-\beta^{s-1})\,u^{s-1}(1-(p\beta)^{n-s})/(1-p\beta)\\
&\quad + (p\beta)^{n-s}\,u(u^{s-1} - (p\beta)^{s-1})/(u-p\beta) + (p\beta)^{n-1}\}\\
&\quad + v\beta^n(pu^{n-1} - p^{n-s}u^s),
\end{aligned}$$

which completes the proof of the lemma. ∎

The next theorem gives the best possible upper bound for $E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - E(\beta^{\tau_n - 1} X_{\tau_n})$ for arbitrary sequences of i.i.d. $[0, 1]$-valued random variables $X_1, X_2, \ldots$. The bound $U_n(\beta)$ is given as the maximum of the function d_n (computed in Lemma 3.4) on an appropriate domain.

DEFINITION 3.5. Let d_n be the function as given in Lemma 3.4. For $\beta \in (0, 1)$, define $U_n(\beta) = \max\{d_n(p, v; \beta) : 0 \leqslant v \leqslant 1,\ 0 \leqslant p \leqslant (1-v)/(1-\beta v)\}$.

THEOREM 3.6. *Let $X_1, X_2, \ldots$ be a sequence of $[0, 1]$-valued i.i.d. random variables and let $\beta \in (0, 1)$. Then for $n > 1$*

$$E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - E(\beta^{\tau_n - 1} X_{\tau_n}) \leqslant U_n(\beta),$$

where $\tau_n = (\inf\{i \geqslant 1 : X_i > \beta V(X_1, \beta X_2, \beta^2 X_3, \ldots)\}) \wedge n$. The constant $U_n(\beta)$ is the best possible.

Proof. Follows immediately from Lemma 3.4. ∎

It is obvious, since $E(\beta^{\tau_n - 1} X_{\tau_n}) \leqslant V(X_1, \beta X_2, \ldots, \beta^{n-1} X_n)$, that $U_n(\beta)$ is also an upper bound for $E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - V(X_1, \beta X_2, \ldots, \beta^{n-1} X_n)$. The next example shows that for $n = 2$, $U_2(\beta)$ is the best possible upper bound for $E(X_1 \vee \beta X_2) - V(X_1, \beta X_2)$ for $\beta \leqslant 1/2$. Also we compute the best possible upper bound $\tilde{U}_2(\beta)$ for the difference above for $1/2 < \beta \leqslant 1$.

In Example 3.8 we given some numerical approximations for the bounds $U_n(\beta)$ for $n = 3, 4, 5, 10, 100$ and several values of β.

EXAMPLE 3.7. Let $n = 2$. Note that $s_2(v; \beta) = 2$ for all v and β. Thus

$$d_2(p, v; \beta) = v\beta(u^2(1-\beta) + p(2\beta - 1)\,u - \beta p^2).$$

If $\beta \geqslant 1/2$, then $\max\{d_2(p, v; \beta) : 0 \leqslant v \leqslant 1,\ 0 \leqslant p \leqslant (1-v)/(1-\beta v)\} = \max\{1/4v((1-v)/(1-\beta v))^2 : 0 \leqslant v \leqslant 1\} = 1/4v_2^*((1-v_2^*)/(1-v_2^*\beta))^2$ with $v_2^* = (3 - \beta - \sqrt{(9-\beta)(1-\beta)})/(2\beta)$. On the other hand, if $\beta \leqslant 1/2$, then $\max\{d_2(p, v; \beta) : 0 \leqslant v \leqslant 1,\ 0 \leqslant p \leqslant (1-v)/(1-\beta v)\} = \max\{d_2(0, v; \beta) : 0 \leqslant v \leqslant 1\} = \beta(1-\beta)\, v_2^*((1-v_2^*)/(1-v_2^*\beta))^2$.

Now, we determine the constant

$$\tilde{U}_2(\beta) = \sup\{E(X_1 \vee \beta X_2) - V(X_1, \beta X_2) : X \text{ is a } [0, 1]\text{-valued r.v.}\}.$$

An extremal random variable for $E(X_1 \vee \beta X_2) - V(X_1, \beta X_2)$ has the form

$$X = \begin{cases} 1 & \\ \beta\mu & \text{with probability} \\ 0 & \end{cases} \quad \begin{matrix} (1 - q\beta)\,\mu \\ q \\ 1 - q - (1 - q\beta)\,\mu. \end{matrix}$$

Here, $\mu = EX$ and $0 \leqslant q \leqslant (1-\mu)/(1-\mu\beta)$. (Again, use a balayage argument as in Lemma 3.3.) For such a random variable X (X_1, X_2 distributed as X) we have

$$E(X_1 \vee \beta X_2) - V(X_1, \beta X_2) = q\beta E(X-\mu)^+ = \beta\mu(1-\mu)\, q(1-q\beta).$$

Call this function $\tilde{d}_2(q, \mu; \beta)$. Then $\tilde{U}_2(\beta) = \max\{\tilde{d}_2(q, \mu; \beta): 0 \leqslant \mu \leqslant 1, 0 \leqslant q \leqslant (1-\mu)/(1-\mu\beta)\} = \tilde{d}_2(1/2, 1/(2\beta); \beta) = 1/16$ if $\beta \geqslant 2/3$, and $\tilde{U}_2(\beta) = \max\{\tilde{d}_2((1-\mu)/(1-\mu\beta), \mu; \beta): 0 \leqslant \mu \leqslant 1\} = \beta(1-\beta)\, v_2^*((1-v_2^*)/(1-v_2^*\beta))^2$ if $\beta \leqslant 2/3$. Hence $U_2(\beta)$ and $\tilde{U}_2(\beta)$ coincide on the interval $(0, 1/2]$. Note that $(1-v_2^*)/(1-v_2^*\beta) = (3+\beta-\sqrt{(9-\beta)(1-\beta)})/(4\beta)$.

EXAMPLE 3.8. For $n = 3$, $U_3(1/2) \approx d_3(0, 0.3542) \approx 0.0642$; $U_3(\beta_3^*) \approx d_3(0, 0.3970) \approx 0.0774$, where $\beta_3^* = (\sqrt{5}-1)/2 \approx 0.6180$; and $U_3(3/4) \approx d_3(0.2748, 0.4648) \approx 0.0967$.

For $n = 4$, $U_4(1/2) \approx d_4(0, 0.3542) \approx 0.0642$; $U_4(\beta_4^*) \approx d_4(0, 0.3703) \approx 0.0869$, where $\beta_4^* \approx 0.6823$; and $U_4(3/4) \approx d_4(0.2359, 0.4093) \approx 0.0982$.

For $n = 5$, $U_5(1/2) \approx d_5(0, 0.3542) \approx 0.0642$; $U_5(\beta_5^*) \approx d_5(0, 0.3512) \approx 0.0920$, where $\beta_5^* \approx 0.7245$; and $U_5(3/4) \approx d_5(0.1285, 0.3670) \approx 0.0964$.

The discount factors β_3^*, β_4^*, and β_5^* have the following interpretation: The numerical approximations for $n = 3$, 4, and 5 suggest that the largest discount factor β for which the maximum (p_n^*, v_n^*) lies on the axis $p = 0$ is equal to the unique solution (in $(0, 1)$) of the equation $\beta^{n-1} + \beta = 1$. It is not hard to show that if $s_n(v; \beta) = n$ (in the expression for d_n in Lemma 3.4), then the partial derivative of d_n (with $\beta = \beta_n^*$) in $p = 0$ is equal to zero, and v_n^* so that $\max\{d_n(0, v; \beta_n^*): 0 \leqslant v \leqslant 1\} = d_n(0, v_n; \beta_n^*)$ satisfies $s_n(v_n^*, \beta_n^*) = n$ for $n = 2, 3, 4, 5$. For $n > 5$ it is much harder to come up with a clean expression for β_n^* with the latter property, although for fixed n approximations can be made. (See also the remark before Proposition 4.3.)

For $n = 10$, $U_{10}(3/4) \approx d_{10}(0, 0.3670) \approx 0.0956$; and $U_{10}(0.95) \approx d_{10}(0.6610, 0.5082) \approx 0.2017$.

For $n = 100$, $U_{100}(0.95) \approx d_{100}(0, 0.3910) \approx 0.1230$.

Hill and Kertz [5] conjectured that the constants $\{b_n\}$, the best possible upper bounds for the differences $E(\max_{1 \leqslant i \leqslant n} X_i) - V(X_1, ..., X_n)$ form a monotonically increasing sequence. Note that in the discount model $\{U_n(\beta)\}$ is not necessarily an increasing sequence for fixed β: $U_3(3/4) < U_4(3/4) > U_5(3/4)$.

4. ASYMPTOTIC ANALYSIS AND THE INFINITE HORIZON PROPHET PROBLEM

In this section the bound $U_n(\beta)$ is analyzed. It is clear that $E(\beta^{\tau_n - 1} X_{\tau_n}) = V(X_1, \beta X_2, ..., \beta^{n-1} X_n)$ for i.i.d. random variables $X_1, ..., X_n$ taking at most two values. So if the extremal random variable (the random variable for which $U_n(\beta)$ is attained) takes at most two values, then $U_n(\beta)$ is also the best possible upper bound for $E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - V(X_1, \beta X_2, ..., \beta^{n-1} X_n)$. We will say something about those β for which this is the case (see Proposition 4.2). Remember that we showed in Example 3.7 that for $n=2$ the extremal random variable takes two values if $\beta \leqslant 1/2$.

Furthermore, we prove that $U(\beta) = \lim_{n \to \infty} U_n(\beta)$ is the best possible upper bound for $E(\sup_{i \geqslant 1} \beta^{i-1} X_i) - V(X_1, \beta X_2, \beta^2 X_3, ...)$ in the infinite horizon prophet problem (see Corollary 4.5).

DEFINITION 4.1. Let $D_n \subset (0, 1)$ be the set of discount factors β for which $d_n(p, v; \beta)$ is maximized by a point $(p_n^*(\beta), v_n^*(\beta))$ somewhere on the axis $p=0$, i.e., $D_n = \{\beta \in (0, 1) : p_n^*(\beta) = 0\}$.

PROPOSITION 4.2. *Fix $n > 1$.*

(i) *Let $X_1, X_2, ...$ be a sequence of $[0, 1]$-valued i.i.d. random variables and let $\beta \in (0, 1)$. Then*

$$E(\max_{1 \leqslant i \leqslant n} \beta^{i-1} X_i) - V(X_1, \beta X_2, ..., \beta^{n-1} X_n) \leqslant U_n(\beta).$$

(ii) *If $\beta \in D_n$, with D_n as in Definition 4.1, then the constant $U_n(\beta)$ is the best possible for the inequality in (i) and $U_n(\beta)$ can be expressed as*

$$U_n(\beta) = \max_{2 \leqslant s \leqslant n} \beta(1-\beta)^{s-1} v_s(u_s)^s.$$

Here $v_s = (\beta^{s-1} \vee v_s^*) \wedge \beta^{s-2}$, $s = 2, ..., n-1$ and $v_n = v_n^* \wedge \beta^{n-2}$, where

$$v_s^* = \frac{(s+1) - (s-1)\beta - \sqrt{((s+1)^2 - (s-1)^2 \beta)(1-\beta)}}{2\beta}, \qquad s = 2, ..., n,$$

and $u_s = (1 - v_s)/(1 - v_s \beta) = (w_{s-2} \vee u_s^*) \wedge w_{s-1}$, $s = 2, ..., n$, where $w_i = (1-\beta^i)/(1-\beta^{i+1})$, $i = 0, ..., n-2$ and $w_{n-1} = 1$, and

$$u_s^* = \frac{(s+1) + (s-1)\beta - \sqrt{((s+1)^2 - (s-1)^2 \beta)(1-\beta)}}{2s\beta}, \qquad s = 2, ..., n.$$

Proof. Part (i) follows immediately from Theorem 3.6, since $V(X_1, \beta X_2, ..., \beta^{n-1} X_n) \geqslant E(\beta^{\tau_n - 1} X_{\tau_n})$. For part (ii) we have to show that $\max\{\tilde{d}_n(v; \beta) : 0 \leqslant v \leqslant 1\} = U_n(\beta)$, where $\tilde{d}_n(v; \beta) = d_n(p=0, v; \beta) = \beta(1-\beta^{s_n - 1})\, v((1-v)/(1-\beta v))^{s_n}$; $s_n = ([\log v / \log \beta] + 2) \wedge n$. Recall the partition (of $[0, 1]$) $\{B_s\}_{s=n, ..., 2}$ as defined in the proof of Lemma 3.4. If $v \in B_s$, some $2 \leqslant s \leqslant n$, then $s_n = s$. Thus $\tilde{d}_n$ restricted to B_s is equal to

$\beta(1-\beta^{s-1}) f_s(v)$, where $f_s(v)=v((1-v)/(1-\beta v))^s$. Straightforward calculus shows that $\max\{f_s(v): 0 \leqslant v \leqslant 1\} = f_s(v_s^*)$, where

$$v_s^* = \frac{(s+1)-(s-1)\,\beta - \sqrt{((s+1)^2-(s-1)^2\,\beta)(1-\beta)}}{2\beta}, \qquad s=2, ..., n.$$

In order to maximize $\tilde{d}_n$ on B_s we have to check whether $v_s^* \in B_s$, $v_s^* \leqslant \beta^{s-1}$ or $v_s^* > \beta^{s-2}$. In the first case we have

$$\max\{\tilde{d}_n(v;\beta): v \in B_s\} = \beta(1-\beta^{s-1})\, v_s^*(u_s^*)^s,$$

and in the latter two cases we have

$$\max\{\tilde{d}_n(v;\beta): v \in B_s\} = \tilde{d}_n(\beta^{s-1};\beta) = \beta^s(1-\beta^{s-1})^{s+1}\,(1-\beta^s)^{-s}$$

and

$$\max\{\tilde{d}_n(v;\beta): v \in B_s\} = \tilde{d}_n(\beta^{s-2};\beta) = \beta^{s-1}(1-\beta^{s-2})^s\,(1-\beta^{s-1})^{-(s-1)},$$

respectively. Hence, if $\beta \in D_n$, then

$$U_n(\beta) = \max_{2 \leqslant s \leqslant n} \max\{\tilde{d}_n(v;\beta): v \in B_s\} = \max_{2 \leqslant s \leqslant n} \beta(1-\beta^{s-1})\, v_s(u_s)^s.$$

Since the extremal random variable takes only the values βv^*, some $v^* \in (0, 1)$, and 1 we have that $\max_{2 \leqslant s \leqslant n} \beta(1-\beta^{s-1})\, v_s(u_s)^s$ is also the best possible upper bound for $E(\max_{1 \leqslant i \leqslant n} \beta^{i-1}X_i) - V(X_1, \beta X_2, ..., \beta^{n-1}X_n)$. This observation completes the proof of the theorem. ∎

Remark. It is not directly obvious that the sets D_n appearing in Definition 4.1 and Proposition 4.2(ii) are non-empty. However, Example 3.7 shows that $D_2=(0, 1/2]$. Also we prove in Proposition 4.3(ii) that $\lim_{n\to\infty} U_n(\beta) = \lim_{n\to\infty} d_n(0, v_n^*; \beta)$ is asymptotitcally the best upper bound for $E(\max_{1 \leqslant i \leqslant n} \beta^{i-1}X_i) - V(X_1, \beta X_2, ..., \beta^{n-1}X_n)$. This implies that $\bigcup_{n=2}^{\infty} D_n = (0, 1)$. Numerical examples, for instance, Example 3.8, suggest the following conjectures:

(i) $D_n \subset D_{n+1}$, $n=2, 3, ...$; and

(ii) D_n is an interval of the form $(0, \beta_n]$ for some $\beta_n \in (0, 1)$.

The author was not able to prove either (i) or (ii) since it is extremely hard to analyse the partial derivatives of d_n.

Note that the Hill and Kertz prophet inequality (1) implies that $1 \notin D_n$ since $b_n \neq U_n(1)$, for all $n \geqslant 2$.

In the next proposition we give an expression for $\lim_{n\to\infty} U_n(\beta)$ and we show that this limit is asymptotically the best possible upper bound for $E(\max_{1 \leqslant i \leqslant n} \beta^{i-1}X_i) - V(X_1, \beta X_2, ..., \beta^{n-1}X_n)$.

PROPOSITION 4.3. *Fix* $\beta \in (0, 1)$.

(i) $U(\beta) = \lim_{n\to\infty} U_n(\beta) = \sup_{s\geq 2} \beta(1-\beta^{s-1})\, v_s(u_s)^s$, *where* v_s *and* u_s *were defined in Proposition* 4.2(ii).

(ii) $\lim_{n\to\infty} \sup\{E(\max_{1\leq i\leq n} \beta^{i-1}X_i) - V(X_1, \beta X_2, ..., \beta^{n-1}X_n)$: $X_1, ..., X_n$ i.i.d. $[0, 1]$-*valued*$\} = U(\beta)$.

Proof. (i) It follows from the definition of $U_n(\beta)$ (Definition 3.5) that

$$\max_{0\leq v\leq 1} d_n(0, v; \beta) \leq U_n(\beta) \leq \max_{0\leq v\leq 1} d_n(0, v; \beta) + \beta^{n-s_n-1} + 2\beta^n. \tag{7}$$

To see the second inequality in (7) write d_n as $d_n^{(1)} + d_n^{(2)}$, where

$$d_n^{(1)}(p, v; \beta) = v\beta(1-\beta^{s_n-1})(u-p)\, u^{s_n-1}(1-(\beta p)^{n-s_n})/(1-\beta p),$$

and

$$d_n^{(2)}(p, v; \beta) = v\beta^{n-s_n-1}p^{n-s_n}u(u^{s_n-1} - (\beta p)^{s_n-1})(u-p)/(u-\beta p)$$
$$+ v\beta^n((u-p)\, p^n + pu^{n-1} - p^{n-s_n}u^{s_n}).$$

Since $d_n^{(1)}$ is decreasing in p, we have $\max\{d_n^{(1)}(p, v; \beta): 0\leq v\leq 1, 0\leq p\leq u\} = \max\{d_n^{(1)}(0, v; \beta): 0\leq v\leq 1\} = \max\{d_n(0, v; \beta): 0\leq v\leq 1\}$. Recall that $u = (1-v)/(1-\beta v)$. It is obvious that $d_n^{(2)} \leq \beta^{n-s_n-1} + 2\beta^n$, since all variables involved are in $[0, 1]$. Now, fix $2\leq s\leq n$ and let $v\in B_s$ (the $\{B_s\}$ were defined in the proof of Lemma 3.4). Then $s_n = s$ and

$$\max_{0\leq v\leq 1} d_n(0, v; \beta) = \max_{2\leq s\leq n} \beta(1-\beta^{s-1}) \max_{v\in B_s} vu^s$$
$$= \max_{2\leq s\leq n} \beta(1-\beta^{s-1})\, v_s((1-v_s)/1-v_s\beta))^s.$$

Letting n tend to infinite etablishes part (i) of the proposition, since $n - s_n = (n - [\log v/\log \beta] - 2)^+ \to \infty$ if $n\to\infty$.

(ii) Since $V(X_1, \beta X_2, ..., \beta^{n-1}X_n) \geq E(\beta^{\tau_n-1}X_{\tau_n})$,

$$E(\max_{1\leq i\leq n} \beta^{i-1}X_i) - V(X_1, \beta X_2, ..., \beta^{n-1}X_n) \leq U_n(\beta) \to U(\beta),$$

and hence

$$\limsup_{n\to\infty} \sup\{E(\max_{1\leq i\leq n} \beta^{i-1}X_i) - V(X_1, \beta X_2, ..., \beta^{n-1}X_n):$$
$$X_1, ..., X_n \text{ i.i.d. } [0, 1]\text{-valued}\} \leq U(\beta).$$

To establish the reverse inequality let $\tilde{v}$ be the number such that $\max_{0\leq v\leq 1} d_n(0, v; \beta) = \beta(1-\beta^{s_n-1})\, \tilde{v}((1-\tilde{v})/(1-\tilde{v}\beta))^{s_n}$. Let $\tilde{X}$ be a random variable with distribution $P(\tilde{X} = \beta\tilde{v}) = (1 - \tilde{v})/(1 - \tilde{v}\beta) = 1 - P(\tilde{X} = 1)$ and let $\tilde{X}_1, ..., \tilde{X}_n$ be i.i.d, distributed as $\tilde{X}$. Then

$$\begin{aligned}
&\sup\{E(\max_{1\leqslant i\leqslant n} \beta^{i-1}X_i) - V(X_1, \beta X_2, ..., \beta^{n-1}X_n): \\
&\quad X_1, ..., X_n \text{ i.i.d. } [0, 1]\text{-valued}\} \\
&\quad \geqslant E(\max_{1\leqslant i\leqslant n} \beta^{i-1}\tilde{X}_i) - V(\tilde{X}_1, \beta\tilde{X}_2, ..., \beta^{n-1}\tilde{X}_n) \\
&\quad = E(\max_{1\leqslant i\leqslant n} \beta^{i-1}\tilde{X}_i) - E(\beta^{\tau_n-1}\tilde{X}_{\tau_n}) \\
&\quad = \beta(1-\beta^{s_n-1})\,\tilde{v}((1-\tilde{v})/(1-\tilde{v}\beta))^{s_n} \\
&\quad = \max_{0\leqslant v\leqslant 1} d_n(0, v; \beta) \to U(\beta) \qquad \text{if} \quad n\to\infty,
\end{aligned}$$

where the first equality follows since

$$E(\beta^{\tau_n-1}\tilde{X}_{\tau_n}) = \tilde{v} = V(\tilde{X}_1, \beta\tilde{X}_2, ..., \beta^{n-1}\tilde{X}_n),$$

and the second equality follows from Lemma 3.4 since $P(\tilde{X}=0)=0$. ∎

The next theorem is the solution to the infinite horizon prophet problem in a general sense, namely, we describe all ordered pairs $(V(X_1, \beta X_2, ...), E(\sup_{i\geqslant 1} \beta^{i-1}X_i))$ as a subset of $[0, 1]^2$, where $X_1, X_2, ...$ runs through the class of sequences of $[0, 1]$-valued i.i.d. random variables. The set containing these pairs is usually called a prophet region. See Kertz [7] for the case $\beta = 1$ (which is not included here) and Boshuizen [1] for the case that the random variables involved are not necessarily identically distributed. An immediate consequence of the next theorem is that the constant $U(\beta)$ defined in Proposition 4.3(i) is the best upper bound for $E(\sup_{i\geqslant 1} \beta^{i-1}X_i) - V(X_1, \beta X_2, ...)$.

THEOREM 4.4. *Let R be the set $\{(x, y): x = V(X_1, \beta X_2, ...), y = E(\sup_{i\geqslant 1} \beta^{i-1}X_i), X_i, X_2, ...$ i.i.d. $[0, 1]$-valued$\}$. Then*

$$R = \bigcup_{s=2}^{\infty} \{(x, y): \beta^{s-1} \leqslant x \leqslant \beta^{s-2}, x \leqslant y \leqslant \Psi_s(x)\} \cup \{(0, 0)\}, \tag{8}$$

where $\Psi_s(x) = x + \beta(1-\beta^{s-1})\,x((1-x)/(1-\beta x))^s$.

Proof. Let S be the set at the right hand side of (8). First, we prove that $R \subset S$. Obviously, $E(\sup_{i\geqslant 1} \beta^{i-1}X_i) = 0$ whenever $V(X_1, \beta X_2, ...) = 0$, so it is enough to prove that $E(\sup_{i\geqslant 1} \beta^{i-1}X_i) \leqslant \Psi_s(v)$, for a sequence $X_1, X_2, ...$ of i.i.d. $[0, 1]$-valued random variables such that $V(X_1, \beta X_2, ...) = v \in (\beta^{s-1}, \beta^{s-2}]$, for some $s = 2, 3, ...$ Again we may assume, without loss of generality, that the X_j take values in the set $\{0, \beta v, 1\}$. Letting $n \to \infty$ in Lemma 3.4, we get

$$E(\sup_{i \geq 1} \beta^{i-1} X_i) = v + \frac{\beta(1-\beta^{s-1})\, v((1-v)/(1-\beta v))^{s-1}\, ((1-v)/(1-\beta v) - p)}{1 - p\beta},$$

where $p = P(X_1 = 0)$. The function $g(p) = ((1-v)/(1-\beta v) - p)/(1 - p\beta)$ is decreasing on $[0, (1-v)/(1-\beta v)]$. Hence

$$E(\sup_{i \geq 1} \beta^{i-1} X_i) \leqslant v + \beta(1-\beta^{s-1})\, v((1-v)/(1-\beta v))^s = \Psi_s(v).$$

To prove $S \subset R$ we have to show that for any $(x, y) \in S$ there exists a sequence $X_1, X_2, ...$ such that $V(X_1, \beta X_2, ...) = x$ and $E(\sup_{i \geq 1} \beta^{i-1} X_i) = y$. If $(x, y) = (0, 0)$, then take $X_j \equiv 0$, $j = 1, 2, ...$. Now, take any point $S \ni (v, y) \neq (0, 0)$. Let $s(v; \beta) = [\log v / \log \beta] + 2$ and let X be a random variable with distribution $P(X = 1) = v(1-\beta)/(1 - v\beta)$, $P(X = v\beta) = (1-v)/(1-\beta v)$. Then $E(\sup_{i \geq 1} \beta^{i-1} X_i) = \Psi_{s(v;\beta)}(v)$ and $V(X_1, \beta X_2, ...) = v$, where $X_1, X_2, ...$ are i.i.d. and distributed as X. Let $a = (y - v)/(\Psi_{s(v;\beta)}(v) - v)$, and define $\tilde{X}_1, \tilde{X}_2, ...$ by $\tilde{X}_j = aX_j + (1-a)$. Then $V(\tilde{X}_1, \beta \tilde{X}_2, ...) = v$ and $E(\sup_{i \geq 1} \beta^{i-1} \tilde{X}_i) = y$. ∎

COROLLARY 4.5. (i) *Let* $X_1, X_2, ...$ *be a sequence of* i.i.d. $[0, 1]$-*valued random variables and let* $U(\beta)$ *be the constant defined in Proposition* 4.3(i). *Then*

$$E(\sup_{i \geq 1} \beta^{i-1} X_i) - V(X_1, \beta X_2, ...) \leqslant U(\beta).$$

The constant $U(\beta)$ *is the best possible.*

(ii) *Let* $X_1, X_2, ...$ *be a sequence of* i.i.d *nonnegative random variables. Then*

$$E(\sup_{i \geq 1} \beta^{i-1} X_i) \leqslant (1 + \beta)\, V(X_1, \beta X_2, ...).$$

The inequality is sharp.

Proof. We only prove part (ii). If $EX_1 = \infty$, then there is nothing to prove. So assume that $EX_1 < \infty$. By appropriate scaling (and then passing to limits) it is enough to prove (ii) for $[0, 1]$-valued random variables. From Theorem 4.4 it is easy to deduct

$$\frac{E(\sup_{i \geq 1} \beta^{i-1} X_i)}{V(X_1, \beta X_2, ...)} \leqslant 1 + \sup_{s \geq 2} \max_{v \in B_s} \beta(1-\beta^{s-1}) \left(\frac{1-v}{1-v\beta}\right)^s. \tag{9}$$

The maximum on the right hand side of (9) is equal to $\beta(1-\beta^{s-1})((1-\beta^{s-1})/(1-\beta^s))^s$, since $(1-v)/(1-\beta v)$ is decreasing in v. The inequality now follows from $\sup_{s\geq 2}(1-\beta^{s-1})((1-\beta^{s-1})/(1-\beta^s))^s=1$. ∎

5. AO versus Optimal Stopping Times

In Section 2 we remarked that the stopping times $\{\tau_n\}$ with the AO-property are not necessarily optimal in the finite horizon stopping problems in that it is possible that $V(X_1, \beta X_2, ..., \beta^{n-1}X_n) > E(\beta^{\tau_n-1}X_{\tau_n})$ for some sequence $X_1, ..., X_n$. What we do know is that the difference between the latter two quantities tends to zero if the horizon tends to infinity. We can raise the following question: What is the maximal difference between $V(X_1, \beta X_2, ..., \beta^{n-1}X_n)$ and $E(\beta^{\tau_n-1}X_{\tau_n})$ if $X_1, ..., X_n$ is a sequence of i.i.d. $[0, 1]$-valued random variables, and $\beta \in (0, 1)$ and $n > 1$ are fixed? (In practice, it is far more easy to work with a stopping time τ_n then with a $\mathcal{T}_n$-optimal stopping time since only one stopping threshold has to be computed. Theorem 5.1 basically says that you will lose at most $W_n(\beta)$ in doing so.) Again, by using a similar result as Lemma 3.3, we may assume, without loss of generality, that X_1 is of the type defined in that lemma (here we basically apply parts (iii) and (iv) of Lemma 3.2). So in the following theorem all random variables are assumed to be i.i.d. and $\{0, \beta v, 1\}$-valued, where $v = V(X_1, \beta X_2, \beta^2 X_3, ...)$.

Theorem 5.1. *Let $X_1, X_2, ...$ be a sequence if i.i.d. $[0, 1]$-valued random variables. Then*

$$V(X_1, \beta X_2, ..., \beta^{n-1}X_n) - E(\beta^{\tau_n-1}X_{\tau_n}) \leq W_n(\beta),$$

where

$$W_n(\beta) = \beta^n (n-1)\, n^{-n/(n-1)}\, v_n^*(u_n^*)^n,$$

with

$$v_n^* = \frac{(n+1)-(n-1)\beta - \sqrt{((n+1)^2-(n-1)^2\beta)(1-\beta)}}{2\beta},$$

$$u_n^* = \frac{(n+1)+(n-1)\beta - \sqrt{((n+1)^2-(n-1)^2\beta)(1-\beta)}}{2n\beta},$$

and τ_n is the stopping time defined in Proposition 2.2(ii). The constant $W_n(\beta)$ is the best possible.

Remark. Obviously, $\lim_{n \to \infty} W_n(\beta) = 0$, since the sequence of stopping times $\{\tau_n\}$ has the AO-property. However, if $\beta = \beta_n$ depends on n, then it is possible that $\lim_{n \to \infty} W_n(\beta) = \eta$, for some $\eta > 0$, as the next example shows. Also we calculate some values of $W_n(\beta)$ explicitly for $\beta = 0.95$ and several n.

EXAMPLE 5.2. (i) $W_2(0.95) \approx 0.1280$, $W_3(0.95) \approx 0.1659$, $W_5(0.95) \approx 0.1734$, $W_{10}(0.95) \approx 0.1281$ and $W_{100}(0.95) \approx 0.605$.

(ii) Let $\{\beta_n\}$ be a sequence of numbers such that $\beta_n = n/(n+1) + o(1)$. Then $\lim_{n \to \infty} W_n(\beta_n) = (1/2)(3 - \sqrt{5})\, e^{-(\sqrt{5}+1)/2} \approx 0.757$.

Proof of Theorem 5.1. For i.i.d. random variables of the same type as the extremal random variable in Lemma 3.3 (so $v = V(X_1, \beta X_2, \beta^2 X_3, ...)$ is fixed) we have by Proposition 2.2(ii)

$$E(\beta^{\tau_n - 1} X_{\tau_n}) = v(1 - p\beta^n u^{n-1}), \tag{10}$$

where $u = (1-v)/(1-\beta v)$. On the other hand, it follows by backward induction [2, p. 50] that

$$V(\beta^{i-1} X_i, ..., \beta^{n-1} X_n) = \beta^{i-1} v(1 - (p\beta)^{n-i+1}), \qquad i = n, ..., 1. \tag{11}$$

Together (8) and (9) imply

$$V(X_1, \beta X_2, ..., \beta^{n-1} X_n) - E(\beta^{\tau_n - 1} X_{\tau_n}) = v\beta^n (pu^{n-1} - p^n) =: H_n(p, v).$$

It is easy to see that

$$\max_{0 \leqslant p \leqslant u} H_n(p, v) = H_n(n^{-1/(n-1)} u, v)$$
$$= \beta^n (n-1)\, n^{-n/(n-1)}\, v((1-v)/(1-\beta v))^n =: \tilde{H}_n(v).$$

By direct calculations (compare the proof of Theorem 4.2) it follows that $\max_{0 \leqslant v \leqslant 1} \tilde{H}_n(v) = W_n(\beta)$. ∎

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