Discrete vs continuous time for large extremes of Gaussian processes.

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Abstract

We prove a limit theorem for joint distribution of maximums of a Gaussian process in continuous and discrete time.

Key words: *Gaussian processes, extremes, tail dependence.*

1 Introduction. Main results.

The problem how continuous time modeling relates to discrete time modelling is intensively discussed in financial literature, [2], [3], [6]. Recent availability of high-frequency data offers empirical researches the opportunity to apply strong mathematical results for continuous time, which was difficult with daily data or data collected at lower frequencies. On the other hand, modelling and analyzing near-continuous records of data pose new challenges. The extent to which either the empirical findings or the theoretical concepts developed in daily or weekly data are applicable in the high-frequency domain is not apparent and needs to be explored. In particular it concerns high extremes modelling, when continuous records for values at risk estimates may never be realized in practice but strongly depends of the records frequency. The purpose of the present paper is to elaborate, for a simple mathematical model, relation between high extremes distributions in continuous and discrete time. Well-developed techniques for asymptotic studying of Gaussian processes, [1], [5] allow us to get exact results.

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Let X_t be a Gaussian stationary process on the real line with zero mean, unit variance and covariance function $r(t)$ such that for some $\alpha \in (0,2)$,

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha) \text{ as } t \to 0 \text{ and } r(t) < 1 \text{ for all } t > 0. \tag{1}$$

We also assume that

$$r(t) = o(1/\log t), \quad \text{as } t \to \infty. \tag{2}$$

It is well known, see for example [4], that from (1) and (2) it follows that

$$\mathbf{P}(\max_{t \in [0,T]} X_t \le b_T + x/a_T) \to e^{-e^{-x}} \tag{3}$$

as $T \to \infty$ uniformly in x from any closed interval, where

$$a_T = \sqrt{2\log T}, \quad b_T = \sqrt{2\log T} + \frac{\log\left((2\pi)^{-1/2} H_\alpha (2\log T)^{-1/2+1/\alpha}\right)}{\sqrt{2\log T}}. \tag{4}$$

Here H_α denotes the Pickands constant, which is defined as the limit $H_\alpha = \lim_{T\to\infty} H_\alpha(T)/T$, with

$$H_\alpha(T) = \mathbf{E}\exp\left(\max_{t\in[0,T]} \sqrt{2} B_{\alpha/2}(t) - t^\alpha\right)$$

and B_H is the Fractional Brownian Motion, that is a Gaussian zero mean process with stationary increments such that $\mathbf{E}B_H^2(t) = |t|^{2H}$. It is also well known that $0 < H_\alpha < \infty$. We may also consider maximum X_t in discrete time. We shall consider three types of grids on the real line. Let $\delta > 0$, denote

$$\mathcal{R} = \mathcal{R}(\delta) = \{k\delta,\ k \in \mathbb{Z} \cap [0,T]\},$$

where $\mathbb{Z}$ is set of all integers. We will call the grid $\mathcal{R}$ *dense* if $\delta = \delta(T) = o((\log T)^{-1/\alpha})$ as $T \to \infty$. Further, the grid with δ such that $\delta(T)(2\log T)^{1/\alpha} \to \infty$ as $T \to \infty$ but for some $\delta_0 > 0$ $\delta(T) \le \delta_0$, will be called *sparse grid.* Finally, the grid $\mathcal{R}$ such that for all T, $\delta(T)(2\log T)^{1/\alpha} = a$, where $0 < a < \infty$ will be called *Pickands' grid.*

We study limit behavior of the distribution of the vector (M_T, M_T^δ) as $T \to \infty$, where

$$(M_T, M_T^\delta) := \left(\max_{t\in[0,T]} X_t,\ \max_{t\in[0,T]\cap\mathcal{R}} X_t\right).$$

Let for some fixed $\delta > 0$, the relation (2) be fulfilled for the sequence $r(k\delta)$, $k = 1, 2, \dots$, that is

$$r(k\delta) = o(1/\log k) \text{ as } k \to \infty. \tag{5}$$

Then, [1],

$$\mathbf{P}(\max_{t\in[0,T]\cap\mathcal{R}} X_t \le b_T^\delta + x/a_T) \to e^{-e^{-x}} \tag{6}$$

as $T \to \infty$ uniformly in x from any closed interval, where

$$b_T^\delta = \sqrt{2\log T} + \frac{\log\left((2\pi)^{-1/2}\delta^{-1}(2\log T)^{-1/2}\right)}{\sqrt{2\log T}}. \tag{7}$$

We shall see that (4) and (7) give normalizing constants in limit theorems for the vector (M_T, M_T^δ) in case of any sparse grid. It is convenient to denote $u = \sqrt{2\log T}$, so that Pickands' grid can be written as $\mathcal{R} = \mathcal{R}(au^{-2/\alpha})$. For this grid we have, [5], Chapter 4,

$$\mathbf{P}(\max_{t\in[0,q]\cap\mathcal{R}} X_t > u) \sim \mathbf{P}(\max_{t\in[0,q]} X_t > u) \sim qH_{a,\alpha}u^{2/\alpha}\Psi(u)$$

as $u \to \infty$, where for any positive a, $H_{a,\alpha} = \lim_{T\to\infty} H_{a,\alpha}(T)/T$, $0 < H_{a,\alpha} < \infty$, with

$$H_{a,\alpha}(T) = \mathbf{E}\exp\left(\max_{ka\in[0,T]} \sqrt{2}B_{\alpha/2}(ka) - (ka)^\alpha\right).$$

We use the standard notation

$$\Psi(u) = \frac{1}{\sqrt{2\pi}u}e^{-\frac{1}{2}u^2} \sim \mathbf{P}(X_0 > u) \text{ as } u \to \infty.$$

For such grid $\mathcal{R} = \mathcal{R}(au^{-2/\alpha})$, it will be proven that ($\delta = au^{-2/\alpha} = (2\log T)^{-1/\alpha}$)

$$\mathbf{P}(\max_{t\in[0,T]\cap\mathcal{R}} X_t \le b_T^\delta + x/a_T) \to e^{-e^{-x}} \tag{8}$$

as $T \to \infty$ uniformly in x from any closed interval, where now

$$b_T^\delta = \sqrt{2\log T} + \frac{\log\left((2\pi)^{-1/2}H_{a,\alpha}(2\log T)^{-1/2+1/\alpha}\right)}{\sqrt{2\log T}}. \tag{9}$$

Now we formulate main results of the present paper.

Theorem 1 *Let X_t be a stationary centered Gaussian process with covariance function $r(t)$ satisfying (1) and (2). Let $\delta = \delta(T)$ be such that $\delta(T) \le \delta_0$ for some $\delta_0 > 0$ and $\delta(T)^\alpha \log T \to \infty$ as $T \to \infty$ (sparse grid), in particular $\delta(T) \equiv \delta_0$. Then*

$$F_T^{\text{sparse}}(x,y) := \mathbf{P}\left(a_T(M_T - b_T) < x, a_T(M_T^\delta - b_T^\delta) < y\right) \to \exp\left(-e^{-x} - e^{-y}\right)$$

as $T \to \infty$ so that extreme values of the Gaussian process on the interval $[0,T]$ and on the sparse uniform grid $[0,T]\cap\mathcal{R}_\delta$ are tail independent.

Define,

$$H^{x,y}_{a,\alpha}(T)$$
$$:= \int_{-\infty}^{\infty} e^s \mathbf{P}\left(\max_{k:ka\in[0,T]} \sqrt{2}B_{\alpha/2}(ka) - (ka)^\alpha > s + x, \max_{t\in[0,T]} \sqrt{2}B_{\alpha/2}(t) - t^\alpha > s + y \right) ds.$$

Theorem 2 *Let X_t be a stationary centered Gaussian process with covariance function $r(t)$ satisfying (1) and (2). Let $\delta = a(2\log T)^{-1/\alpha}$, so that $\mathcal{R} = \mathcal{R}(a(2\log T)^{-1/\alpha})$, $a > 0$, the Pickands' grid. Then for any x, y there exists the limit*

$$H^{x,y}_{a,\alpha} := \lim_{T\to\infty} \frac{H^{x,y}_{a,\alpha}(T)}{T}, \quad 0 < H^{x,y}_{a,\alpha} < \infty$$

and

$$\begin{aligned} F^{\text{Pickands}}_T(x,y) &= \mathbf{P}\left(a_T(M_T - b_T) < x, a_T(M^\delta_T - b^\delta_T) < y\right) \\ &\to \exp\left(-e^{-x} - e^{-y} + H^{\log H_{a,\alpha}+x, \log H_\alpha + y}_{a,\alpha}\right) \end{aligned}$$

as $T \to \infty$, where b^δ_T is given by (9), so that the extremes are tail dependent in the case of continuous time and Pickands' grid.

Theorem 3 *Let X_t be a stationary centered Gaussian process with covariance function $r(t)$ satisfying (1) and (2). Then for any dense grid $\mathcal{R}(\delta)$, $\delta(\log T)^{1/\alpha} \to 0$ as $T \to \infty$, we have*

$$F^{\text{dense}}_T(x,y) = \mathbf{P}\left(a_T(M_T - b_T) < x, a_T(M^\delta_T - b_T) < y\right) \to \exp\left(-e^{-x\wedge y}\right)$$

as $T \to \infty$ so that extreme values of the Gaussian process on the interval $[0,T]$ and on dense grid are completely tail dependent.

2 Sparse grid.

We will use the notation $u = \sqrt{2\log T}$ so that $\delta = \delta(u) = l(u)u^{-2/\alpha}$, where $l(u) \to \infty$ as $u \to \infty$, recall that $\delta(u) \le \delta_0$ for some positive δ_0, in particular, $\delta(u) \equiv \delta_0$. Consider first the probability

$$P(u,x) = \mathbf{P}\left(X_0 > u, \max_{t\in[-\delta,\delta]} X_t > u + \frac{\frac{2}{\alpha}\log u + \log\delta + x}{u} \right),$$

where x varies in a closed interval, say, $x \in [-A, A]$, $A < \infty$. To shorten formulas we denote

$$v := \sqrt{\frac{2}{\alpha}\log u + \log\delta} \quad (= \sqrt{l(u)}).$$

An obvious estimation $P(u,x)$ from below is

$$P(u,x) \geq \mathbf{P}\left(X_0 > u + \frac{v^2 + x}{u}\right) \geq 0.9(2\pi)^{-1/2} e^{-x}\delta^{-1}u^{-1-2/\alpha}e^{-u^2/2}, \qquad (10)$$

for all sufficiently large u. Note that the marginal probabilities have up to a positive multiplier the same asymptotic behavior. Indeed, by the Pickands' Theorem (see Theorem D.2 [5] or [4])

$$\begin{aligned}
\mathbf{P}\left(\max_{t\in[-\delta,\delta]} X_t > u + \frac{v^2+x}{u}\right) &\sim 2\delta H_\alpha \left(u + \frac{v^2+x}{u}\right)^{2/\alpha} \Psi\left(u + \frac{v^2+x}{u}\right) \\
&\sim 2\delta H_\alpha u^{2/\alpha} \exp\left(-\frac{2}{\alpha}\log u - \log\delta\right)\Psi(u) \\
&\sim 2H_\alpha \mathbf{P}(X_0 > u) \qquad (11)
\end{aligned}$$

as $u \to \infty$. Let now $L(u)$ be a positive function such that $L(u) \to \infty$, $L(u)u^{-2/\alpha} \to 0$ and $L(u) = o(l(u))$ as $u \to \infty$. Then, by Pickands' Theorem, similarly to (11),

$$\begin{aligned}
\mathbf{P}\left(\max_{t\in[-L(u)u^{-2/\alpha}, L(u)u^{-2/\alpha}]} X_t > u + \frac{v^2+x}{u}\right) &\sim 2H_\alpha L(u)u^{2/\alpha} l(u)^{-1}u^{-2/\alpha}\Psi(u) \\
&= o(\mathbf{P}(X_0 > u)) \qquad (12)
\end{aligned}$$

as $u \to \infty$. From here it obviously follows that

$$P(u,x) = \mathbf{P}\left(X_0 > u, \max_{\delta \geq |t| \geq L(u)u^{-2/\alpha}} X_t > u + \frac{v^2+x}{u}\right) + O\left(L(u)l(u)^{-1}\mathbf{P}(X_0 > u)\right) \qquad (13)$$

as $u \to \infty$. Denote by $P_1(u,x)$ the probability in (13) and by $\varphi(z)$ the standard normal density. Changing variables $z = u + \frac{vy}{u}$, we have,

$$\begin{aligned}
P_1(u,x) &= \int_u^\infty \mathbf{P}\left(\max_{\delta \geq |t| \geq L(u)u^{-2/\alpha}} X_t > u + \frac{v^2+x}{u} \,\middle|\, X_0 = z\right)\varphi(z)dz \\
&= \int_u^{u+(v^2+x)/u} \mathbf{P}\left(\max_{\delta \geq |t| \geq L(u)u^{-2/\alpha}} X_t > u + \frac{v^2+x}{u} \,\middle|\, X_0 = z\right)\varphi(z)dz
\end{aligned}$$

$$
\begin{aligned}
& +\int_{u+(v^2+x)/u}^{\infty} \varphi(z)dz \\
= & \frac{v}{u}\int_0^{v+x/v} \mathbf{P}\left(\max_{\delta\geq|t|\geq L(u)u^{-2/\alpha}} X_t > u+\frac{v^2+x}{u} \middle| X_0 = u+\frac{vy}{u}\right)\varphi(u+vy/u)dy \\
& +\Psi(u+(v^2+x)/u) \\
= & \frac{v}{\sqrt{2\pi}u}e^{-u^2/2}\int_0^{v+x/v} e^{-vy-(vy)^2/(2u^2)} \times \\
& \qquad \times \mathbf{P}\left(\max_{\delta\geq|t|\geq L(u)u^{-2/\alpha}} X_t > u+\frac{v^2+x}{u} \middle| X_0 = u+\frac{vy}{u}\right) dy \\
& +\Psi(u+(v^2+x)/u)
\end{aligned}
$$

Now we study the conditional probability under the integral, denote it by $P(u,x,y)$. We have,

$$
\mathbf{E}(X_t|X_0) = u+vy/u) = (u+vy/u)r(t); \quad \mathbf{Var}(X_t|X_0) = 1-r(t)^2; \tag{14}
$$

$$
\mathbf{Var}(X_t - X_s|X_0) = 2(1-r(t-s)) - (r(t)-r(s))^2; \tag{15}
$$

Consider the Gaussian zero mean process $Y(t)$, $t\in R$, $Y(0)=0$ a.s., with covariance function satisfying (15), that is

$$
\begin{aligned}
\mathbf{E}Y(t)Y(s) &= \frac{1}{2}(\mathbf{E}Y(t)^2+\mathbf{E}Y(s)^2-\mathbf{E}(Y(t)-Y(s))^2) \\
&= \frac{1}{2}(1-r(t)^2+1-r(s)^2-2(1-r(t-s))+(r(t)-r(s))^2) \\
&= r(t-s)-r(t)r(s). \qquad (16)
\end{aligned}
$$

We have

$$
\begin{aligned}
P(u,x,y) &= \mathbf{P}\left(\max_{\delta\geq|t|\geq L(u)u^{-2/\alpha}} Y(t) + \left(u+\frac{vy}{u}\right)r(t) > u+\frac{v^2+x}{u}\right) \\
&= \mathbf{P}\left(\max_{\delta\geq|t|\geq L(u)u^{-2/\alpha}} Y(t) + \frac{vy}{u}r(t) - (1-r(t))u > \frac{v^2+x}{u}\right) \\
&= \mathbf{P}\left(\exists t,\ \delta\geq|t|\geq L(u)u^{-2/\alpha} : Y(t) > -\frac{vy}{u}r(t)+(1-r(t))u+\frac{v^2+x}{u}\right) \\
&= \mathbf{P}\left(\exists t,\ \delta\geq|t|\geq L(u)u^{-2/\alpha} : \frac{Y(t)}{\sqrt{1-r(t)^2}} > -\frac{vy}{u}\frac{r(t)}{\sqrt{1-r(t)^2}} + \right. \\
&\qquad \left. +u\sqrt{\frac{1-r(t)}{1+r(t)}}+\frac{v^2+x}{u\sqrt{1-r(t)^2}}\right).
\end{aligned}
$$

Consider the behavior of the function

$$F(t) = -\frac{vy}{u}\frac{r(t)}{\sqrt{1-r(t)^2}} + u\sqrt{\frac{1-r(t)}{1+r(t)}} + \frac{v^2+x}{u\sqrt{1-r(t)^2}}$$

near its point of minimum. By differentiation we get that the minimum of the function reaches at the point

$$r(t) = \frac{u^2+vy}{u^2+v^2+x} = 1 - \frac{v^2}{u^2} + \frac{vy}{u^2} + O\left(\frac{1}{u^3}\right),$$

that is, for fixed y, by (1) at the point

$$t \sim w = w(u) := (v/u)^{2/\alpha}$$

as $u \to \infty$. The value of the minimum is

$$\sigma = \frac{-\frac{vy}{u}\frac{u^2+vy}{u^2+v^2+x} + u\frac{v^2+x-vy}{u^2+v^2+x} + \frac{v^2+x}{u}}{\frac{\sqrt{v^2+x-vy}\sqrt{2u^2+v^2+x+vy}}{u^2+v^2+x}} = \frac{\sqrt{v^2+x-vy}\sqrt{2u^2+v^2+x+vy}}{u},$$

so that for fixed y,

$$\sigma \sim \sqrt{2}v$$

as $u \to \infty$. Consider the following family of random Gaussian processes

$$\begin{aligned} Z_u(t) &= \frac{vY(wt)}{u(1-r(wt)-(vy/u)r(wt)+(v^2+x)/u}, \\ \delta/w &\geq |t| \geq L(u)u^{-2/\alpha}/w = L(u)v^{-2/\alpha},\ u > 0, \end{aligned}$$

so that

$$P(u,x,y) = \mathbf{P}\left(\max_{\delta/w\geq|t|\geq L(u)v^{-2/\alpha}} Z_u(t) > v\right).$$

For the standardized process

$$\widehat{Y}(t) = \frac{Y(t)}{\sqrt{\mathbf{E}Y(t)^2}} = \frac{Y(t)}{\sqrt{1-r(t)^2}}$$

we have

$$Z_u(t) = D(u,t)\widehat{Y}(wt),$$

where

$$D(u,t)=\frac{v\sqrt{1-r(wt)^2}}{u(1-r(wt))-(vy/u)r(wt)+(v^2+x)/u}. \tag{17}$$

Now we are in a position to study the behavior of $D(u,t)$ as u tends to infinity and $|t|$ varies between $L(u)v^{-2/\alpha}$ and δ/w. For the nominator of (17) using (1) we have

$$\begin{aligned} v\sqrt{1-r(wt)^2} &= v\sqrt{(1-r(wt))(1+r(wt))} \\ &= v\sqrt{((v/u)^2|t|^\alpha+O(|t|^\beta(v/u)^{2\beta/\alpha})(2+O(|t|^\alpha(v/u)^2)} \\ &= \sqrt{2}v^2u^{-1}|t|^{\alpha/2}(1+O(|t|^\alpha(v/u)^2))+O(|t|^{\beta-\alpha}(v/u)^{2(\beta-\alpha)/\alpha})), \end{aligned}$$

as $u\to\infty$. For the denominator of (17) we have

$$\begin{aligned} & u(1-r(wt))-(yv/u)r(wt)+(v^2+x)/u \\ = \;& u((v/u)^2|t|^\alpha+O(|t|^\beta(v/u)^{2\beta/\alpha})-(yv/u)(1-|t|^\alpha(v/u)^2+O(|t|^\beta(v/u)^{2\beta/\alpha})) \\ & \qquad +v^2/u+x/u \\ = \;& v^2u^{-1}(|t|^\alpha+1-v^{-1}y+xv^{-2}+yO(|t|^\alpha u^{-2})+yO(|t|^\beta v^{\frac{2\beta}{\alpha}-1}u^{-2\beta/\alpha}), \end{aligned}$$

as $u\to\infty$. Thus we get

$$D(u,t)=\frac{\sqrt{2}|t|^{\alpha/2}(1+O(|t|^\alpha(v/u)^2))+O(|t|^{\beta-\alpha}(v/u)^{2(\beta-\alpha)/\alpha}))}{|t|^\alpha+1-v^{-1}y+xv^{-2}+yO(|t|^\alpha u^{-2})+yO(|t|^\beta v^{\frac{2\beta}{\alpha}-1}u^{-2\beta/\alpha})}, \tag{18}$$

as $u\to\infty$. Now, using that $y\le v+x/v\le v+A/v$, we have for all sufficiently large u and some positive C depending of δ,

$$D(u,t)\le\frac{C|t|^{\alpha/2}}{|t|^\alpha+1/2},$$

so that for any $d>0$ one can find T such that for all $|t|\ge T$,

$$D(u,t)\le d. \tag{19}$$

Chose $L(u)$ as large as $L(u)v^{-2/\alpha}\ge T$. We can do that because

$$l(u)v^{-2/\alpha}=l(u)(\log l(u))^{-1/\alpha}\to\infty \text{ as } u\to\infty.$$

Our aim is now to estimate from above the probability

$$\mathbf{P}(\max_{\delta/w\ge|t|\ge L(u)v^{-2/\alpha}} Z_u(t)>v).$$

To this end we consider the correlation function $r_Z(t,s)$ of $Z_u(t)$. First observe that

$$r_Z(t,s) = \mathbf{Corr}(Z_u(t), Z_u(s)) = \mathbf{E}\widehat{Y}(wt)\widehat{Y}(ws) = \frac{\mathbf{E}Y(wt)Y(ws)}{\sqrt{\mathbf{E}Y(wt)^2\mathbf{E}Y(ws)^2}}.$$

Using (16), we obtain

$$\begin{aligned}
& 1-\mathbf{Corr}(Y(t),Y(s))^2 = 1-\frac{(\mathbf{E}Y(t)Y(s))^2}{\mathbf{E}Y(t)^2\mathbf{E}Y(s)^2} \\
= & \ 1-\frac{\frac{1}{4}(\mathbf{E}Y(t)^2+\mathbf{E}Y(s)^2-\mathbf{E}(Y(t)-Y(s))^2)^2}{(1-r(t)^2)(1-r(s)^2)} \\
= & \ 1-\frac{\frac{1}{4}(1-r(t)^2+1-r(s)^2-2(1-r(t-s))+(r(t)-r(s))^2)^2}{(1-r(t)^2)(1-r(s)^2)} \\
= & \ 1-\frac{(r(t-s)-r(t)r(s))^2}{(1-r(t)^2)(1-r(s)^2)} \\
= & \ \frac{1-r(t)^2-r(s)^2+r(t)^2r(s)^2-r(t-s)^2-r(t)^2r(s)^2+2r(t-s)r(t)r(s)}{(1-r(t)^2)(1-r(s)^2)} \\
= & \ \frac{(1-r(t-s))(1+r(t-s)-2r(t)r(s))-(r(t)-r(s))^2}{(1-r(t)^2)(1-r(s)^2)}.
\end{aligned}$$

Then we have,

$$\begin{aligned}
1-r_Z^2(t,s) &= \frac{(1-r(w(t-s)))(1+r(w(t-s))-2r(wt)r(ws))-(r(wt)-r(ws))^2}{(1-r(wt)^2)(1-r(ws)^2)} \\
&\leq \frac{(1-r(w(t-s)))(1+r(w(t-s))-2r(wt)r(ws))}{(1-r(wt)^2)(1-r(ws)^2)}. \qquad (20)
\end{aligned}$$

For the nominator we have for some constant C, all t, s and all sufficiently large u,

$$1-r(w(t-s)) \leq C(v/u)^2|t-s|^\alpha;$$

$$|1+r(w(t-s))-2r(wt)r(ws)| \leq |1-r(w(t-s))| + 2|1-r(wt)r(ws)| \leq C(v/u)^2.$$

For the denominator of (20) we have, using that $|t|,|s| \geq T$,

$$(1-r(wt)^2)(1-r(ws)^2) = 4(v/u)^4|t|^\alpha|s|^\alpha(1+o(1)) \geq C(v/u)^4. \qquad (21)$$

Thus we have for all $t, s \in [-\delta/w, -L(u)v^{-2/\alpha}] \cup [L(u)v^{-2/\alpha}, \delta/w]$,

$$1-r_Z(t,s) \leq C|t-s|^\alpha. \qquad (22)$$

Thus since (19) and (22), applying Theorem 8.1, [5], we get for a sufficiently large T,

$$\mathbf{P}(\max_{\delta/w \geq |t| \geq T} Z_u(t) > v) \leq C(\delta/w)e^{-v^2/3d^2} = Cl(u)v^{-2/\alpha}l(u)^{-1/(3\alpha d^2)}, \tag{23}$$

where d can be made arbitrarily small by choosing T sufficiently large. Collecting above estimations (11, 12, 13, 23), we have for some C, T and $L(u)$ such that $L(u)v^{-2/\alpha} \to \infty$, $L(u)/l(u) \to 0$ and all positive u,

$$P(u,x) \leq CL(u)l(u)^{-1}\mathbf{P}\left(X_0 > u\right). \tag{24}$$

Now we consider the probability

$$P_S(u,x) = \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t > u, \max_{t\in[0,S]} X_t > u + \frac{v^2+x}{u}\right)$$

when we will allow S tend to infinity with u but not too fast. Define

$$\rho(\varepsilon) = \inf_{t\geq\varepsilon}(1 - r(t),$$

$\rho(\varepsilon)$ is positive for all positive ε.

Proposition 1 *Let X_t and δ be as described above and let $S = S(u) \geq 2\delta$ for all u and $S(u) = o\left(\exp\left(u^2\rho(\varepsilon)/8\right)\right)$ as $u \to \infty$. Then there exists $\varepsilon > 0$ such that*

$$\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t > u\right) \sim S\delta^{-1}\Psi(u), \tag{25}$$

$$\mathbf{P}\left(\max_{t\in[0,S]} X_t > u + \frac{v^2+x}{u}\right) \sim S\delta^{-1}e^{-x}H_\alpha\Psi(u) \tag{26}$$

as $u \to \infty$ and

$$P_S(u,x) = o\left(\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t > u\right) + \mathbf{P}\left(\max_{t\in[0,S]} X_t > u + \frac{v^2+x}{u}\right)\right)$$

as $u \to \infty$, so that

$$\begin{aligned} 1 - \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t \leq u, \max_{t\in[0,S]} X_t \leq u + \frac{v^2+x}{u}\right) \\ \sim \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t > u\right) + \mathbf{P}\left(\max_{t\in[0,S]} X_t > u + \frac{v^2+x}{u}\right) \\ \sim S\delta^{-1}\Psi(u)(1 + e^{-x}H_\alpha) \end{aligned} \tag{27}$$

as $u \to \infty$.

Proof. The relation (26) follows from Theorem D.2, [5]. Now we prove that for the sparse grid the double probability $P_S(u,x)$ tends to zero faster then right-hand part of (26). We have,

$$\begin{aligned} P_S(u,x) &\le \sum_{k,l=0}^{[S/\delta]} \mathbf{P}\left(X_{k\delta} > u, \max_{t\in[(l-1)\delta,(l+1)\delta]} X_t > u + \frac{v^2+x}{u}\right) =: \sum_{k,l=0}^{[S/\delta]} p_{k,l} \\ &= \sum_{k,l=0,|k-l|\le 1}^{[S/\delta]} p_{k,l} + \sum_{k,l=0,|k-l|>1}^{[S/\delta]} p_{k,l}. \end{aligned} \tag{28}$$

The members of the first sum can be estimated by (24) so that

$$\sum_{k,l=0,|k-l|\le 1}^{[S/\delta]} p_{k,l} \le CSl(u)^{-1}L(u)\Psi(u). \tag{29}$$

Consider now the probability $p_{k,k+\nu} = p_{0,\nu}$. Obviously we may consider only the case $\nu > 1$. We have, using Theorem 8.1, [5],

$$\begin{aligned} p_{0,\nu} &\le \mathbf{P}\left(\max_{t\in[(\nu-1)\delta,(\nu+1)\delta]} (X_0 + X_t) > 2u + \frac{v^2+x}{u}\right) \\ &\le C\delta u^{2/\alpha-1} \exp\left(-\frac{(2u+(v^2+x)/u)^2}{2\max_{t\ge(\nu-1)\delta}(2+2r(t))}\right) \\ &\le Cl(u)u^{-1} \exp\left(-\frac{4u^2+4(v^2+x)}{8\max_{t\ge(\nu-1)\delta}(1-\frac{1}{2}(1-r(t))}\right) \\ &\le Cl(u)u^{-1} \exp\left(-\frac{u^2+v^2}{2(1-\frac{1}{2}\min_{t\ge(\nu-1)\delta}(1-r(t))}\right) \\ &\le Cl(u)u^{-1} \exp\left(-\frac{1}{2}(u^2+v^2)(1+\frac{1}{2}\min_{t\ge(\nu-1)\delta}(1-r(t))\right) \\ &\le Cl(u)^{1/2}\Psi(u) \exp\left(-\frac{1}{4}u^2 \min_{t\ge(\nu-1)\delta}(1-r(t))\right) \\ &\le \begin{cases} Cl(u)^{1/2}\Psi(u)\exp\left(-u^2(\nu-1)^\alpha\delta^\alpha/8\right) \text{ if } (\nu-1)\delta \le \varepsilon, \\ Cl(u)^{1/2}\Psi(u)\exp\left(-u^2\rho(\varepsilon)/8\right) \text{ if } (\nu-1)\delta > \varepsilon, \end{cases} \end{aligned}$$

where ε is such that $1-r(t) \ge |t|^\alpha/2$ for all $t \in (0,\varepsilon]$. Using these inequalities separately in the cases $(\nu-1)\delta \le \varepsilon$, when we get the summable series $\Sigma \exp(-(|\nu|-1)l(u)^\alpha)$, and in the case $(\nu-1)\delta > \varepsilon$, when we uniformly bound all corresponding summands, we get that for some C and all sufficiently large u,

$$\sum_{k,l=0,|k-l|>1}^{[S/\delta]} p_{k,l} \leq CS\Psi(u)\left\{\exp(-l(u)^{\alpha}/16)+S\exp\left(-u^2\rho(\varepsilon)/8\right)\right\}. \qquad (30)$$

Thus from (29) and (30) it follows that $P_S(u,x)$ is infinitely smaller than probability in (26).

Now prove the relation (25). We have for all k and $l>k$,

$$\mathbf{P}\left(X_{k\delta}>u, X_{l\delta}>u\right) \leq p_{k,l+1},$$

hence

$$\sum_{k,l=0,k\neq l}^{[S/\delta]} \mathbf{P}\left(X_{k\delta}>u, X_{l\delta}>u\right) \leq \sum_{k,l=0,|k-l|>1}^{[S/\delta]} p_{k,l},$$

which follows that the double sum in the above left-hand side tends to zero faster that $S\delta^{-1}\Psi(u)$ as $u\to\infty$. Thus the relation (25) and therefore Proposition 1 are proven.

3 Pickands' and dense grids.

Let $a>0$, in this section $\mathcal{R}$ is the Pickands' grid, that is $\mathcal{R}=\{aku^{-2/\alpha},\ k\in\mathbb{Z}\}$. We will evaluate the asymptotic behavior of the probability

$$P'_S(u,x)=\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t>u, \max_{t\in[0,S]} X_t>u+\frac{x}{u}\right).$$

We begin with a short interval. Let $T>a$. Then it can be proved quite similar to the proof of Lemma D.1, [5], that

$$P'_{Tu^{-2/\alpha}}(u,x)\sim H^x_{a,\alpha}(T)\Psi(u) \text{ as } u\to\infty,$$

where

$$H^{0,x}_{a,\alpha}(T)=\int_{-\infty}^{\infty} e^y\mathbf{P}\left(\max_{k:ka\in[0,T]}\sqrt{2}B_{\alpha/2}(ka)-(ka)^{\alpha}>y, \max_{t\in[0,T]}\sqrt{2}B_{\alpha/2}(t)-t^{\alpha}>y+x\right)dy.$$

It also can be proved in the similar to the proof of Theorem D.2 [5] way that

$$0<H^{0,x}_{a,\alpha}:=\lim_{T\to\infty}\frac{H^{0,x}_{a,\alpha}(T)}{T}<\infty$$

and there exists $\kappa > 0$ such that for any $S = S(u)$ with $S(u)u^{-2/\alpha} \to \infty$ and $S(u) = O(\exp(\kappa u^2))$ as $u \to \infty$ we have,

$$P'_S(u,x) \sim SH^{0,x}_{a.\alpha} u^{2/\alpha}\Psi(u) \text{ as } u \to \infty. \tag{31}$$

From here we have for the Pickands' grid.

Proposition 2 *For any a and $\mathcal{R} = \{kau^{-2/\alpha},\ k \in \mathbb{Z}\}$,*

$$\begin{aligned}
1 - \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t \le u, \max_{t\in[0,S]} X_t \le u + \frac{v^2+x}{u}\right) \\
\sim \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t > u\right) + \mathbf{P}\left(\max_{t\in[0,S]} X_t > u + \frac{v^2+x}{u}\right) \\
-\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t > u, \max_{t\in[0,S]} X_t > u + \frac{v^2+x}{u}\right) \\
\sim SH_{a,\alpha}u^{2/\alpha}\Psi(u) + SH_\alpha(u+(v^2+x)/u)^{2/\alpha}\Psi(u+(v^2+x)/u) \\
-SH^{0,x}_{a,\alpha}u^{2/\alpha}\Psi(u)
\end{aligned} \tag{32}$$

as $u \to \infty$.

Now consider another functional similar to the Pickands' constant,

$$H^-_{a,\alpha}(T) = \int_0^\infty e^y \mathbf{P}\left(\max_{k:ka\in[0,T]} \sqrt{2}B_{\alpha/2}(ka) - (ka)^\alpha \le y, \max_{t\in[0,T]} \sqrt{2}B_{\alpha/2}(t) - t^\alpha > y\right) dy.$$

Again, similarly to the proof of Lemma D.1 and Lemma 15.3 , [5], one can show that for all sufficiently large u,

$$\begin{aligned}
\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t \le u\right) - \mathbf{P}\left(\max_{t\in[0,S]} X_t \le u\right) &= \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t \le u,\ \max_{t\in[0,S]} X_t > u\right) \\
&\le 2SH^-_{a,\alpha}(1)u^{2/\alpha}\Psi(u).
\end{aligned} \tag{33}$$

Following further the proof of Lemmas D.1 and 15.3, [5], we get that $H^-_{a,\alpha}(1) \to 0$ as $a \to 0$, hence we have,

Proposition 3 *For any $\varepsilon > 0$ one can find $b > 0$ such that for the Pickands' grid $\mathcal{R} = \mathcal{R}(bu^{-2/\alpha})$ and all sufficiently large u and all S,*

$$\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t \le u\right) - \mathbf{P}\left(\max_{t\in[0,S]} X_t \le u\right) \le \varepsilon S u^{2/\alpha}\Psi(u).$$

As immediate consequence of the Proposition 3 we get for any dense grid:

Proposition 4 *Let $\mathcal{R} = \mathcal{R}(\delta)$ be a dense grid, that is $\delta = \delta(u) = o(u^{-2/\alpha}$ as $u \to \infty$. Then for any $S = S(u)$ with $S(u)u^{-2/\alpha} \to \infty$,*

$$\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}} X_t \le u\right) - \mathbf{P}\left(\max_{t\in[0,S]} X_t \le u\right) = o\left(Su^{2/\alpha}\Psi(u)\right)$$

as $u \to \infty$.

4 Proof of main results.

We divide interval $[0, T]$ with intervals of lengths S intermitting shorter intervals of lengths R. We let $S = T^a$, $R = T^b$, $0 < b < a < 1$, further restrictions on a and b we will give later on. We denote long intervals by S_k, $k = 1, 2, ..., n = [T/(S + R)]$, and short intervals by R_k, $k = 1, ..., n$. We will easily see that possible rest interval with different of S or R length plays no role in our considerations. Denote $\mathbf{S} = \cup S_k$, $\mathbf{R} = \cup R_k$, so that $[0, T] = \mathbf{S} \cup \mathbf{R}$ (up to the rest interval). Let $X_t^1, ..., X_t^n, ...$ be independent copies of X_t, $t \in \mathbb{R}$. Introduce a Gaussian process, defined on $\mathbf{S}$,

$$Y_t = X_t^k \text{ for } \quad t \in S_k,\ k = 1, ..., n.$$

Denote

$$M_{\mathbf{S},T} = \max_{t\in\mathbf{S}} X_t,\ M_{\mathbf{S},T}^{\delta} = \max_{t\in\mathbf{S}\cap\mathcal{R}(\delta)} X_t.$$

We use the following simple estimation.

Proposition 5 *For any $B > 0$ one can find a constant C such that for all $x, y \in [-B, B]$,*

$$\begin{aligned}&|\mathbf{P}\left(a_T(M_T - b_T) < x, a_T(M_T^{\delta} - b_T^{\delta}) < y\right)\\ &\quad -\mathbf{P}\left(a_T(M_{\mathbf{S},T} - b_T) < x, a_T(M_{\mathbf{S},T}^{\delta} - b_T^{\delta}) < y\right)| \le C(\log T)^{1/\alpha-1/2}T^{b-1}.\end{aligned}$$

Proof. We use obvious inequality

$$\begin{aligned}&|\mathbf{P}\left(a_T(M_T - b_T) < x, a_T(M_T^{\delta} - b_T^{\delta}) < y\right)\\ &\quad -\mathbf{P}\left(a_T(M_{\mathbf{S},T} - b_T) < x, a_T(M_{\mathbf{S},T}^{\delta} - b_T^{\delta}) < y\right)|\\ \le\ &\mathbf{P}(\max_{t\in\mathbf{R}} X_t > b_T + x/a_T) + \mathbf{P}(\max_{t\in\mathbf{R}} X_t > b_T' + y/a_T).\end{aligned}$$

Now by the Pickands' theorem,

$$\mathbf{P}(\max_{t\in\mathbf{R}} X_t > b_T + x/a_T) = \mathrm{mes}(\mathbf{R})(b_T + x/a_T)^{2/\alpha}\Psi(b_T + x/a_T)(1 + o(1))$$

as $T \to \infty$, with following simple calculations. The Proposition is proven.

From Proposition 5 it will follow that since $b < 1$ one can reduce the proofs of the main results to the parameter set $\mathbf{S}$.

Next key step to prove the main results is a comparison inequality of the probabilities for processes X_t and Y_t, respectively. The following proposition is a reformulation of Theorem 1.2, [5], which generalizes Berman's inequality on mixing properties of Gaussian stationary sequences. Denote $r^h(t,s) = hr(t-s) + (1-h)r_Y(t,s)$, where r_Y is the covariance function of the process Y_t.

Proposition 6 *Let $\mathcal{T}_1 = \{0 = t_{10} < t_{11} < ... < t_{1n} = T\}$ and $\mathcal{T}_2 = \{0 = t_{20} < t_{21} < ... < t_{2n} = T\}$ be partitions of interval $[0,T]$ and u_1, u_2 be positive numbers. Then*

$$\left|\mathbf{P}\left(\max_{t\in\mathcal{T}_1\cap\mathbf{S}} X_t \le u_1, \max_{t\in\mathcal{T}_2\cap\mathbf{S}} X_t \le u_2\right) - \mathbf{P}\left(\max_{t\in\mathcal{T}_1\cap\mathbf{S}} Y_t \le u_1, \max_{t\in\mathcal{T}_2\cap\mathbf{S}} Y_t \le u_2\right)\right|$$
$$\le \sum_{t,s\in(\mathcal{T}_1\cup\mathcal{T}_2)\cap\mathbf{S}, t\ne s} |r(t-s) - r_Y(t,s)| \sum_{i,j=1,2} \int_0^1 \varphi(u_i, u_j; r^h(t,s))dh,$$

where $\varphi(x,y;r)$ is Gaussian two-dimensional density with zero mean, unit variances and covariance r.

Note that $r(t-s) - r_Y(t,s) = 0$ for $t,s \in S_k$, $k = 1,...,n$, and $r_Y(t,s) = 0$ for $t \in S_k$, $s \in S_l$, $k \ne l$, $k,l = 1,2,...,n$. So the right-hand part of the latter inequality can be re-written as

$$\sum_{k,l=1,k\ne l}^{n} \sum_{t\in(\mathcal{T}_1\cup\mathcal{T}_2)\cap S_k, s\in(\mathcal{T}_1\cup\mathcal{T}_2)\cap S_l} |r(t-s)| \sum_{i,j=1,2} \int_0^1 \varphi(u_i, u_j; hr(t-s))dh.$$

Now, since distance between S_k and S_l is not less then T^b, by (2), $|r(t-s)| = o(1/\log T)$ as $t \to \infty$ uniformly in k, l, $k \ne l$. Hence,

$$\left|\mathbf{P}\left(\max_{t\in\mathcal{T}_1\cap\mathbf{S}} X_t \le u_1, \max_{t\in\mathcal{T}_2\cap\mathbf{S}} X_t \le u_2\right) - \mathbf{P}\left(\max_{t\in\mathcal{T}_1\cap\mathbf{S}} Y_t \le u_1, \max_{t\in\mathcal{T}_2\cap\mathbf{S}} Y_t \le u_2\right)\right| \quad (34)$$
$$= \frac{o(1)}{\log T} \sum_{k,l=1,k\ne l}^{n} \sum_{t\in(\mathcal{T}_1\cup\mathcal{T}_2)\cap S_k, s\in(\mathcal{T}_1\cup\mathcal{T}_2)\cap S_l} \sum_{i,j=1,2} \int_0^1 \varphi(u_i, u_j; hr(t-s))dh.$$

Now we treat the integrals of Gaussian densities. We consider two types of levels which are defined in (9) and (7) for Pickands' and sparse grids. We have from (9) and (7) that

$$u_i = \sqrt{2\log T} - \frac{1}{2}\frac{\log\log T}{\sqrt{2\log T}} + d_i \frac{\log\log T}{\sqrt{2\log T}} + O\left(\frac{1}{\sqrt{\log T}}\right)$$

as $T \to \infty$, where $d_i = 1/\alpha$ in case of Pickands' grid (9), and $d_i = \frac{\log \delta^{-1}}{\log\log T}$ in case of sparse grid (7). In particular, when δ is a constant, one can let $d_i = 0$. From here we have

$$\begin{aligned} u_i^2 &= 2\log T - \log\log T + 2d_i \log\log T + O(1), \\ u_i u_j &= 2\log T - \log\log T + (d_i + d_j)\log\log T + O(1), \end{aligned} \tag{35}$$

as $T \to \infty$. So

$$\begin{aligned} \frac{1}{2}\frac{u_i^2 - 2u_iu_j + u_j^2}{1-r^2} &= \frac{(1-r)(2\log T - \log\log T + (d_i+d_j)\log\log T) + O(1)}{1-r^2} \\ &= \frac{2\log T - \log\log T + (d_i+d_j)\log\log T + O(1)}{1+r} \end{aligned}$$

(To shorten the above formula we have denoted $r := hr(t-s)$). Now using (2) we have for any t, s, $t \in S_k$, $s \in S_l$, $k \neq l$, (so that $t - s \geq T^b$),

$$\varphi(u_i, u_j; hr(t-s)) = \frac{\exp\left(-\frac{2\log T - \log\log T + (d_i+d_j)\log\log T}{1+hr(t-s)} + O(1)\right)}{2\pi\sqrt{1-h^2r(t-s)^2}} \tag{36}$$

$$\begin{aligned} &= O(1)T^{-2}\log T(\log T)^{-(d_i+d_j)}\exp\left(O(r(t-s)\log T)\right) \\ &= O(1)T^{-2}\log T(\log T)^{-(d_i+d_j)}. \end{aligned}$$

Continuing (34), we get

$$\left|\mathbf{P}\left(\max_{t\in\mathcal{T}_1\cap\mathbf{S}} X_t \leq u_1, \max_{t\in\mathcal{T}_2\cap\mathbf{S}} X_t \leq u_2\right) - \mathbf{P}\left(\max_{t\in\mathcal{T}_1\cap\mathbf{S}} Y_t \leq u_1, \max_{t\in\mathcal{T}_2\cap\mathbf{S}} Y_t \leq u_2\right)\right| \tag{37}$$

$$\begin{aligned} &= \frac{o(1)}{\log T}\sum_{k,l=1,k\neq l}^{n}\ \sum_{t\in(\mathcal{T}_1\cup\mathcal{T}_2)\cap S_k, s\in(\mathcal{T}_1\cup\mathcal{T}_2)\cap S_l}\ \sum_{i,j=1,2} T^{-2}\log T(\log T)^{-(d_i+d_j)} \\ &= \frac{o(1)}{\log T}T^{-2}\log T(\log T)^{-(d_i+d_j)}n^2\#((\mathcal{T}_1\cup\mathcal{T}_2)\cap S_k)\#((\mathcal{T}_1\cup\mathcal{T}_2)\cap S_l) \\ &= o(1)T^{-2a}(\log T)^{-(d_i+d_j)}\#((\mathcal{T}_1\cup\mathcal{T}_2)\cap S_k)\#((\mathcal{T}_1\cup\mathcal{T}_2)\cap S_l). \end{aligned}$$

Now let assumptions of Theorem 1 be fulfilled. We are to choose in (37) $\mathcal{T}_1 = \mathcal{R}(\delta)$ and $\mathcal{T}_2 = \mathcal{R}(\varepsilon(\log T)^{-1/\alpha})$, $\varepsilon > 0$ and get

$$\#((\mathcal{T}_1 \cup \mathcal{T}_2) \cap S_k) = O(T^a (\log T)^{1/\alpha})$$

as $T \to \infty$, hence

$$\left| \mathbf{P}\left(\max_{t \in \mathcal{T}_1 \cap \mathbf{S}} X_t \le u_1, \max_{t \in \mathcal{T}_2 \cap \mathbf{S}} X_t \le u_2 \right) - \mathbf{P}\left(\max_{t \in \mathcal{T}_1 \cap \mathbf{S}} Y_t \le u_1, \max_{t \in \mathcal{T}_2 \cap \mathbf{S}} Y_t \le u_2 \right) \right| = o(1) \quad (38)$$

as $T \to \infty$.

Let now the assumptions of Theorem 2 be fulfilled, we have $\mathcal{T}_1 = \mathcal{R}(a(\log T)^{-1/\alpha})$ and $\mathcal{T}_2 = \mathcal{R}(\varepsilon(\log T)^{-1/\alpha})$, $\varepsilon > 0$, and again easily get (4).

Next, letting $u_1 = a_T + x/b_T^{\delta_1}$, $u_2 = a_T + y/b_T^{\delta_2}$ where $\delta_2 = \varepsilon(\log T)^{-1/\alpha})$ and δ_1 takes values for the above described grids, we have

$$\begin{aligned}
&\mathbf{P}\left(\max_{t \in \mathcal{T}_1 \cap \mathbf{S}} Y_t \le u_1, \max_{t \in \mathcal{T}_2 \cap \mathbf{S}} Y_t \le u_2 \right) \\
&\quad = \left(\mathbf{P}\left(\max_{t \in [0,S] \cap \mathcal{R}(\delta)} X_t \le u_1, \max_{t \in [0,S] \cap \mathcal{R}(\varepsilon(\log T)^{-1/\alpha})} X_t \le u_2 \right) \right)^n \\
&\quad = \exp\left(n \log \left(\mathbf{P}\left(\max_{t \in [0,S] \cap \mathcal{R}(\delta)} X_t \le u_1, \max_{t \in [0,S] \cap \mathcal{R}(\varepsilon(\log T)^{-1/\alpha})} X_t \le u_2 \right) \right) \right) \\
&\quad = \exp\left(-n \left(1 - \mathbf{P}\left(\max_{t \in [0,S] \cap \mathcal{R}(\delta)} X_t \le u_1, \max_{t \in [0,S] \cap \mathcal{R}(\varepsilon(\log T)^{-1/\alpha})} X_t \le u_2 \right) \right) + R \right),
\end{aligned}$$

where since

$$P_u := \mathbf{P}\left(\max_{t \in [0,S] \cap \mathcal{R}(\delta)} X_t \le u_1, \max_{t \in [0,S] \cap \mathcal{R}(\varepsilon(\log T)^{-1/\alpha})} X_t \le u_2 \right) \to 1$$

as $u \to \infty$, the reminder R can be estimated as $|R| \le n(1 - P_u)^2$.

In assumptions of Theorem 1, it is immediately follows from the Proposition 1 and definitions of the norming constants a_T, b_T and b_T^δ that

$$\begin{aligned}
&n\left[1 - \mathbf{P}\left(\max_{t \in [0,S] \cap \mathcal{R}(\delta)} X_t \le b_T^\delta + x/a_T, \max_{t \in [0,S]} X_t \le b_T + y/a_T \right) \right] \\
&\sim nST^{-1}(e^{-x} + e^{-y}) \sim (e^{-x} + e^{-y})
\end{aligned}$$

as $T \to \infty$. Hence, by Proposition 3, we get that for any $\varepsilon > 0$ one can find b such that

$$\limsup_{u \to \infty} |n(1 - P_u) - (e^{-x} + e^{-y})| \le \epsilon.$$

Thus using (4), then again Proposition 3 and then Proposition 5, we get the assertion of Theorem 1.

Theorem 3 easily follows from Proposition 4 and the same arguments.

Proof of Theorem 2 word by word repeats the above arguments, making use Proposition 2 instead of Proposition 1. We consecutively pass from uniform time in Proposition 3 to the grid $\mathcal{R}(bu^{-2/\alpha})$ working in the interval $(0,S]$ and come back from the grid to continuous time using (4) and Proposition 3 to conclude the proof of Theorem 2. To get the final representation for the limit distribution we rewrite the assertion of Proposition 2 as follows,

$$\begin{aligned}
1 &- \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}(\delta)} X_t \le b_T^\delta + x/a_T, \max_{t\in[0,S]} X_t \le b_T + y/a_T\right) \\
&= \mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}(\delta)} X_t > b_T^\delta + x/a_T\right) + \mathbf{P}\left(\max_{t\in[0,S]} X_t > b_T + y/a_T\right) \\
&\qquad -\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}(\delta)} X_t > b_T^\delta + x/a_T, \max_{t\in[0,S]} X_t > b_T + y/a_T\right) \\
&\sim ST^{-1}e^{-x} + ST^{-1}e^{-y} \\
&\qquad -\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}(\delta)} X_t > b_T^\delta + x/a_T, \max_{t\in[0,S]} X_t > b_T + y/a_T\right)
\end{aligned}$$

as $u\to\infty$. To transform the last term, we shorten notation $U = b_T^\delta + x/a_T$ and get

$$\begin{aligned}
b_T + y/a_T &= U + b_T - b_T'' + (y-x)/a_T \\
&= U + \frac{\log H_\alpha - \log H_{a,\alpha} + y - x}{\sqrt{2\log T}} \\
&= U + \frac{\log H_\alpha - \log H_{a,\alpha} + y - x}{U} + O((\log\log T)^2(\log T)^{-3/2}). \quad (39)
\end{aligned}$$

Observing that $U \sim (\log T)^{1/2}$, we see that the reminder $O(\cdot)$ in (39) plays a negligible role. Therefore, by (31),

$$\begin{aligned}
\mathbf{P}\left(\max_{t\in[0,S]\cap\mathcal{R}(\delta)} X_t > b_T^\delta + x/a_T, \max_{t\in[0,S]} X_t > b_T + y/a_T\right) &= SH_{a,\alpha}^{0,x}U^{2/\alpha}\Psi(U)(1+o(1)) \\
&= ST^{-1}H_{a.\alpha}^{0,Z}(H_{a.\alpha})^{-1}e^{-x}(1+o(1)),
\end{aligned}$$

where $Z = \log H_\alpha - \log H_{a,\alpha} + y - x$. Now changing variables in the definition of the definition of $H_{a.\alpha}^{x,y}$ we get that $H_{a.\alpha}^{0,Z}(H_{a.\alpha})^{-1}e^{-x} = H_{a,\alpha}^{\log H_{a,\alpha}+x, \log H_\alpha + y}$.

This concludes the proof of Theorem 2.

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