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Large Quantile Estimation in a Multivariate Setting

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An asymptotic theory is developed for the estimation of high quantile curves, i.e., sets of points in higher dimensional space for which the exeedance probability is p_n, with $np_n \to 0$ $(n \to \infty)$. Here n is the number of available observations. This is the situation of interest if one wants to protect against a calamity that has not yet occurred. Asymptotic normality of the estimated quantile curve is proved under appropriate conditions, including the domain of the attraction condition for multivariate extremes.

1. INTRODUCTION

In de Haan and Rootzén [6] we constructed confidence intervals for extreme quantiles, i.e., quantiles outside the scope of the sample under extreme-value conditions. Estimation of extreme quantiles is necessary, e.g., when determining the height of a projected sea-dike: on the basis of high tide water levels observed during 100 years one has to design the height of the dike in such a way that the return period of a flood is 10,000 years.

In this paper we consider the multi-dimensional problem. The requirement is now, e.g., that the return period of a flood at either one of two places along the coast is 10,000 years. This leads to the problem of estimation of extreme quantile curves in two-dimensional distributions. We formulate the problem and its solution in two dimensions in order to keep the notation relatively simple. Generalization to higher dimensions is straightforward.

Our aim is to estimate the curve of all values $(x(p), y(p))$ for which $p = 1 - F(x(p), y(p))$, where F is some unknown distribution function from which a sample has been taken.

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The main requirement is that the probability distribution is in the domain of attraction of a two-dimensional extreme-value distribution. That is, if F is the underlying distribution, for some positive functions a_1 and a_2 and functions b_1 and b_2,

$$\lim_{n \to \infty} F^n(a_1(n)\, x + b_1(n), a_2(n)\, y + b_2(n)) = G(x, y) \tag{1.1}$$

weakly with G a non-degenerate distribution function. We choose b_i ($i = 1, 2$) such that $1 - F(b_1(t), \infty) \sim 1 - F(\infty, b_2(t)) \sim t^{-1}$ ($t \to \infty$) and a_i such that the marginal distribution of G are standard, that is, $G(x, \infty) = \exp - (1 + \gamma_1 x)^{-1/\gamma_1}$, and $G(\infty, y) = \exp - (1 + \gamma_2 y)^{-1/\gamma_2}$, where γ_1 and γ_2 are the extreme-value indices (cf. [3]) of the two marginal distributions, both of which are in the domain of attraction of an extreme-value distribution by (1.1). We also assume that the upper endpoints of the marginal distributions of F are positive. Recall that the limit distriution G in (1.1) satisfies [4, 8] the relation

$$-\log G\left(\frac{(ax)^{\gamma_1} - 1}{\gamma_1}, \frac{(ay)^{\gamma_2} - 1}{\gamma_2}\right) = a^{-1}\left\{-\log G\left(\frac{x^{\gamma_1} - 1}{\gamma_1}, \frac{y^{\gamma_2} - 1}{2}\right)\right\} \tag{1.2}$$

for $a, x, y > 0$.

We have estimators for γ_1, γ_2, and G: Let $k = k(n) \to \infty$, $k(n)/n \to 0$ ($n \to \infty$); then if (X_1, Y_1), (X_2, Y_2), ... are i.i.d. F and $\{X_{(i,n)}\}_{i=1}^n$, $\{Y_{(j,n)}\}_{j=1}^n$ are their nth order statistics, we define

$$\hat{\gamma}_1 := M_n^{(1)} + 1 - \tfrac{1}{2}\{1 - (M_n^{(1)})^2/M_n^{(2)}\}^{-1} \tag{1.3}$$

with $M_n^{(r)} := (1/k) \sum_{i=0}^{k-1} \{\log X_{n-i,n} - \log X_{n-k,n}\}^r$ ($r = 1, 2$),

$$\hat{\gamma}_2 := P_n^{(1)} + 1 - \tfrac{1}{2}\{1 - (P_n^{(1)})^2/P_n^{(2)}\}^{-1} \tag{1.4}$$

with $P_n^{(r)} := (1/k) \sum_{i=0}^{k-1} \{\log Y_{n-i,n} - \log Y_{n-k,n}\}^r$ ($r = 1, 2$),

$$-\log \hat{G}(x, y) := \frac{1}{k} \sum_{i=1}^{n} 1_{\{X_i > \hat{b}_1(n/k) + x\hat{a}_1(n/k) \text{ or } Y_i > \hat{b}_2(n/k) + y\hat{a}_2(n/k)\}} \tag{1.5}$$

with

$$\begin{aligned} \hat{b}_1\left(\frac{n}{k}\right) &:= X_{n-k,n} \\ \hat{a}_1\left(\frac{n}{k}\right) &:= X_{n-k,n}\, M_n^{(1)} \cdot \max(1, 1 - \hat{\gamma}_1) \end{aligned} \tag{1.6}$$

and

$$\hat{b}_2\left(\frac{n}{k}\right) := Y_{n-k,n}$$
$$\hat{a}_2\left(\frac{n}{k}\right) := Y_{n-k,n} P_n^{(1)} \cdot \max(1, 1-\hat{\gamma}_2). \tag{1.7}$$

One has the asymptotic normality result under some extra conditions on F, G, and the sequence $k(n)$ (see [2, 5]):

$$\sqrt{k}\left(\frac{\hat{a}_i(n/k)}{a_i(n/k)} - 1\right) \to A_i \tag{1.8}$$

$$\sqrt{k}\left(\frac{\hat{b}_i(n/k) - b_i(n/k)}{a_i(n/k)} - 1\right) \to B_i$$

$$\sqrt{k}(\hat{\gamma}_i - \gamma_i) \to \Gamma_i \qquad \text{in distribution } (i = 1, 2),$$

$$\begin{aligned}\sqrt{k}(-\log \hat{G}(x, y) + \log G(x, y)) \to V(x, y) &:= W(x, y)\\ &+ (B_1 + xA_1)(-\log G)_1 (x, y)\\ &+ (B_2 + yA_2)(-\log G)_2 (x, y)\end{aligned} \tag{1.9}$$

in D-space, where $W(x, y)$ is a zero-mean Gaussian random field $(-\{\max(0, \gamma_1)\}^{-1} < x \leqslant \infty,\ -\{\max(0, \gamma_2)\}^{-1} < y \leqslant \infty)$ with covariance function

$$\begin{aligned}\operatorname{Cov}(W(x, y), W(s, t)) = \nu(&([x_0, x] \times [y_0, y))^c\\ &\cap ([x_0, s] \times [y_0, t])^c)\end{aligned} \tag{1.10}$$

and ν is a σ-finite measure such that [4]

$$-\log G(x, y) = \nu([x_0, x] \times [y_0, y])^c;$$

$(-\log G)_1$ and $(-\log G)_2$ are the first-order derivatives of $-\log G$.

The joint distribution of A_i, B_i, Γ_i $(i = 1, 2)$ and W is given as

$$\begin{aligned}\Gamma_i &= \{\gamma_i - \bar{\gamma}_i + 2(1 + \bar{\gamma}_i)^2 (1 - 2\bar{\gamma}_i)\} P_i + \frac{(1 - \bar{\gamma}_i)^2 (1 - 2\bar{\gamma}_i)^2}{2} Q_i,\\ A_i &= \gamma_i W_i(0) + \frac{(1 - \bar{\gamma}_i)^2 (1 - 2\bar{\gamma}_i)}{(1 - 4\bar{\gamma}_i)} \left(\frac{P_i}{1 - \bar{\gamma}_i} + \frac{Q_i}{2}\right)\\ &\quad - \frac{1}{2} \frac{6(1 - 2\bar{\gamma}_i)^3 + 2(1 - \bar{\gamma}_i)(1 - 2\bar{\gamma}_i)^2 - 8(1 - \bar{\gamma}_i)^3}{(1 - 4\bar{\gamma}_i)^2 (1 - \bar{\gamma}_i)(1 - 2\bar{\gamma}_i)} \Gamma_i,\\ B_i &= W_i(0),\end{aligned} \tag{1.11}$$

where

$$P_i := \int_1^\infty W_i(s)\,\frac{ds}{s} - W_i(0)$$
$$Q_i := 2\int_1^\infty W_i(s)\,(\log s)\,\frac{ds}{s} - 2W_i(0) \qquad (i=1,2). \tag{1.12}$$

Converge in D-space means that (according to Skorohod's a.s. construction) one can find a sequence of processes $\hat{G}_*$ and a process W_* (with its functionals A_i^* and B_i^* as in (1.11)) defined on one sample space and such that a.s.

$$\lim_{n\to\infty} \sqrt{k}(-\log \hat{G}_*(x,y) + \log G(x,y))$$
$$= W^*(x,y) + (B_1^* + xA_1^*)(-\log G)_1\,(x,y)$$
$$+ (B_2^* + xA_2^*)(-\log G)_2\,(x,y)$$

locally uniformly for $(x,y) \in (-\{\max(0,\gamma_1)\}^{-1}, \infty] \times (-\{\max(0,\gamma_2)\}^{-1}, \infty]$.

Note that here and in the rest of the paper we do not indicate explicitly the dependence of the estimators on n, the number of observations.

We now proceed to give a heuristic introduction of our estimators for extreme quantiles and begin with a review of the one-dimensional case [3, 6]. One has to estimate a value x_p such that

$$p = 1 - F_1(x_p) = \frac{k}{n}\left[\frac{n}{k}\left\{1 - F_1\left(b_1\left(\frac{n}{k}\right) + a_1\left(\frac{n}{k}\right)\frac{x_p - b_1(n/k)}{a_1(n/k)}\right)\right\}\right]$$
$$\approx \frac{k}{n}\left\{-\log G_1\left(\frac{x_p - b_1(n/k)}{a_1(n/k)}\right)\right\} = \frac{k}{n}\left\{\left(1 + \gamma_1\,\frac{x_p - b_1(n/k)}{a_1(n/k)}\right)^{-1/\gamma_1}\right\}$$

by (1.1), where F_1 and G_1 are the distribution functions of the first marginal distributions in (1.1).

It follows that

$$x_p \approx b_1\left(\frac{n}{k}\right) + a_1\left(\frac{n}{k}\right)\cdot\frac{(k/np)^{\gamma_1} - 1}{\gamma_1}.$$

Hence we introduce the estimator

$$\hat{x}_p := \hat{b}_1\left(\frac{n}{k}\right) + \hat{a}_1\left(\frac{n}{k}\right)\cdot\frac{(k/np)^{\hat{\gamma}_1} - 1}{\hat{\gamma}_1}, \tag{1.13}$$

where we can take $\hat{\gamma}_1$ as in (1.3) and $\hat{b}_1$ and $\hat{a}_1$ as in (1.6).

In de Haan and Rootzén [6] we gave an asymptotic confidence interval for x_p based on the estimator $\hat{x}_p$ in the situation $p = p_n \to 0$, $np_n \to 0$ $(n \to \infty)$, which is the situation of interest. In the proof of the main result in [6] we made extensive use of the inverse function of the distribution function which is not availabe in the multidimensional situation.

Moving now to the two-dimensional case it will prove useful to introduce a parameterization of the quantile curve $(x(p, \theta), y(p, \theta))$ for $0 < \theta < \pi/2$, i.e., the curve for which

$$p = 1 - F(x(p, \theta), y(p, \theta))$$

and a sequence of positive functions $\rho_n(\theta)$ by means of the equalities

$$x(p) = x(p, \theta) = b_1\left(\frac{n}{k}\right) + a_1\left(\frac{n}{k}\right) \cdot \frac{(k\rho_n(\theta)\cos\theta/np)^{\gamma_1} - 1}{\gamma_1} \tag{1.14}$$

$$y(p) = y(p, 0) = b_2\left(\frac{n}{k}\right) + a_2\left(\frac{n}{k}\right) \cdot \frac{(k\rho_n(\theta)\sin\theta/np)^{\gamma_2} - 1}{\gamma_2}. \tag{1.15}$$

That is, the function $\rho_n(\theta)$ is the solution of Eqs. (1.14) and (1.15).

Remark 1.1. By considering the cases $\gamma_i > 0$, $\gamma_i < 0$, $\gamma_i = 0$ separately and by using $a_i(t) \sim \gamma_i b_i(t)$ for $\gamma_i > 0$ and $a_i(t) \sim -\gamma_i(b_i(\infty) - b_i(t))$ for $\gamma_i < 0$ $(t \to \infty)$, one sees that Eqs. (1.14) and (1.15), indeed, admit a positive solution $\rho_n(\theta)$ for all choices of γ_i $(i = 1, 2)$. To see this, e.g., for the case $\gamma_1 > 0$, replace $a_1(t)$ by $\gamma_1 b_1(t)$ in (1.14) so that the equation becomes

$$x(p) = \gamma_1^{-1} b_1\left(\frac{n}{k}\right)\left(\frac{k\rho_n(\theta)\cos\theta}{np}\right)^{\gamma_1}.$$

Since $x(p)$ and $b_1(n/k)$ are positive in this case,

$$\rho_n(\theta) = np\left\{\gamma_1 x(p)/b_1\left(\frac{n}{k}\right)\right\}^{1/\gamma_1} / (k\cos\theta)$$

is a positive solution of (1.14).

Remark 1.2. Note that the function $\rho_n(\theta)$ characterizes the quantile curve.

We now proceed heuristically. Note that (1.1) implies that

$$\lim_{n\to\infty} \frac{n}{k}\left\{1 - F\left(b_1\left(\frac{n}{k}\right) + xa_1\left(\frac{n}{k}\right), b_2\left(\frac{n}{k}\right) + ya_2\left(\frac{n}{k}\right)\right)\right\} = -\log G(x, y).$$

Hence,

$$p = 1 - F(x(p), y(p)) \approx \frac{k}{n}\left\{-\log G\left(\frac{x(p)-b_1(n/k)}{a_1(n/k)}, \frac{y(p)-b_2(n/k)}{a_2(n/k)}\right)\right\}$$

$$= \frac{k}{n}\left\{-\log G\left(\frac{(k\rho_n(\theta)\cos\theta/np)^{\gamma_1}-1}{\gamma_1}, \frac{(k\rho_n(\theta)\sin\theta/np)^{\gamma_2}-1}{\gamma_2}\right)\right\}$$

$$= -\frac{p}{\rho_n(\theta)}\log G\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right).$$

The later equality follows by (1.2). Hence,

$$\rho_n(\theta) \approx -\log G\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right).$$

This leads us to define a parameter $\rho(\theta)$ by

$$\rho(\theta) := -\log G\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right). \tag{1.16}$$

Note that the curve $(\{(\rho(\theta)\cos\theta)^{\gamma_1}-1\}/\gamma_1, \{(\rho(\theta)\cos\theta)^{\gamma_2}-1\}/\gamma_2)$ for $0<\theta<\pi/2$ is the e^{-1}-quantile curve of G. A little reflection shows that all quantile curves of G have the same shape.

A natural estimator of $\rho(\theta)$ is $\hat{\rho}(\theta)$ defined by

$$\hat{\rho}(\theta) := -\log \hat{G}\left(\frac{(\cos\theta)^{\hat{\gamma}_1}-1}{\hat{\gamma}_1}, \frac{(\sin\theta)^{\hat{\gamma}_2}-1}{\hat{\gamma}_2}\right). \tag{1.17}$$

We shall later show that $\rho_n(\theta)$ is close to $\rho(\theta)$ and that $\hat{\rho}(\theta)$ is a good estimator for $\rho(\theta)$. This leads us to introduce the following estimators for $x(p,\theta)$ and $y(p,\theta)$ (cf. (1.14) and (1.15)):

$$\hat{x}(p,\theta) := \hat{b}_1\left(\frac{n}{k}\right) + \hat{a}_1\left(\frac{n}{k}\right)\frac{(k\hat{\rho}(\theta)\cos\theta/np)^{\hat{\gamma}_1}-1}{\hat{\gamma}_1} \tag{1.18}$$

$$\hat{y}(p,\theta) := \hat{b}_2\left(\frac{n}{k}\right) + \hat{a}_2\left(\frac{n}{k}\right)\frac{(k\hat{\rho}(\theta)\sin\theta/np)^{\hat{\gamma}_2}-1}{\hat{\gamma}_2}. \tag{1.19}$$

It will further be useful to introduce the functions $x_n(p,\theta)$ and $y_n(p,\theta)$ defined as (cf. (1.14) and (1.15))

$$x_n(p,\theta) := b_1\left(\frac{n}{k}\right) + a_1\left(\frac{n}{k}\right)\cdot\frac{(k\rho(\theta)\cos\theta/np)^{\gamma_1}-1}{\gamma_1}$$

$$y_n(p,\theta) := b_2\left(\frac{n}{k}\right) + a_2\left(\frac{n}{k}\right)\frac{(k\rho(\theta)\sin\theta/np)^{\gamma_2}-1}{\hat{\gamma}_2}.$$

We shall prove (Theorem 2.1) the asymptotic normality of $(\hat{x}(p,\theta)-x(p,\theta), \hat{y}(p,\theta)-y(p,\theta))$, normalized, and we sketch now the course of the proof.

First it is proved (Lemma 2.1) that $\hat{p}(\theta)-p(\theta)$, normalized, is asymptotically normal. This will imply (Lemma 2.2) that $(\hat{x}(p,\theta)-x_n(p,\theta), \hat{y}(p,\theta)-y_n(p,\theta))$, normalized, is asymptotically normal. Next it is proved (Lemma 2.3) that $p_n(\theta)-p(\theta)$, normalized, converges to zero. This will imply (Lemma 2.4) that $(x_n(p,\theta)-x(p,\theta), y_n(p,\theta)-y(p,\theta))$, normalized, also converges to zero.

The result is then proved by combining the statements of Lemmas 2.2 and 2.4. We shall need a strong second-order condition on F in the spirit of conditions (1.5) and (1.6) of the Haan and Rootzén [6] resulting in (2.7) of the same paper. The condition is needed in two forms. The equivalence of the two forms is proved in an appendix.

The problem of estimating quantiles in the tail of a multidimensional distribution has been considered before in Joe, Smith, and Weissman [7]. See also Coles and Tawn [1]. The setup there is somewhat different: it is assumed that the marginal distributions and the dependence structure coincide exactly with the limiting situation from some point on; further a restricted parameteric model is considered for the dependence structure, whereas the present paper is non-parameteric. Joe, Smith, and Weissman confront their models and methods with real data. Also they have a method for testing the model. This has not yet been done with the methods in the present paper.

2. The Result

Note that (1.1) is equivalent to

$$\lim_{t\to\infty} t\{1-F(b_1(t)+xa_1(t), b_2(t)+ya_2(t))\} = -\log G(x,y). \tag{2.1}$$

A second-order (rate of convergence) condition for (2.1) is (cf. [5])

$$\lim_{t\to\infty} \frac{t\{1-F(b_1(t)+xa_1(t), b_2(t)+ya_2(t)\} + \log G(x,y)}{c(b_1(t), b_2(t))} = \psi(x,y) \tag{2.2}$$

locally uniformly for $x, y \in (x_0, \infty]$ for some non-constant function ψ and a positive function c, where $c(b_1(t), b_2(t))$ is regularly varying, $c(b_1(t), b_2(t)) \to 0$ $(t\to\infty)$.

THEOREM 2.1. *Let $a_n := k(n)/np_n$. Suppose that* (2.2) *holds and, moreover, the following strong second order condition*:

$$\lim_{n\to\infty}\left[\frac{n}{k}\left\{1-F\left(b_1\left(\frac{n}{k}\right)+\frac{(a_nx)^{\gamma_1}-1}{\gamma_1}a_1\left(\frac{n}{k}\right), b_2\left(\frac{n}{k}\right)\right.\right.\right.$$
$$\left.\left.+\frac{(a_ny)^{\gamma_2}-1}{\gamma_2}a_2\left(\frac{n}{k}\right)\right)\right\}$$
$$\left.+\log G\left(\frac{(a_nx)^{\gamma_1}-1}{\gamma_1},\frac{(a_ny)^{\gamma_2}-1}{\gamma_2}\right)\right]\bigg/\left[c\left(b_1\left(\frac{n}{k}\right), b_2\left(\frac{n}{k}\right)\right)\right.$$
$$\left.\times\psi\left(\frac{(a_nx)^{\gamma_1}-1}{\gamma_1},\frac{(a_ny)^{\gamma_2}-1}{\gamma_2}\right)\right]=1 \tag{2.3}$$

locally uniformly for $0<x, y\leqslant\infty$, where $c(b_1(t), b_2(t))$ is a regularly varying function tending to zero ($t\to\infty$) and ψ is a non-constant function. Write

$$\psi_0(x, y) := \psi\left(\frac{x^{\gamma_1}-1}{\gamma_1},\frac{y^{\gamma_2}-1}{\gamma_2}\right).$$

One can show that ψ_0 is a homogeneous function of degree $\alpha\leqslant 0$. Suppose $\alpha=0$. Suppose further that $-\log G$ has continuous first-order derivatives $(-\log G)_1$ and $(-\log G)_2$. Set

$$\psi_1(x) := \psi\left(x, \lim_{y\to\infty}\frac{y^{\gamma_2}-1}{\gamma_2}\right)$$

and

$$\psi_2(y) := \psi\left(\lim_{x\to\infty}\frac{x^{\gamma_1}-1}{\gamma_1}, y\right).$$

Suppose, finally, that

$$\lim_{n\to\infty}(\log a_n)/\sqrt{k}=0 \tag{2.4}$$

and

$$\lim_{n\to\infty}\sqrt{k}\cdot c\left(b_1\left(\frac{n}{k}\right), b_2\left(\frac{n}{k}\right)\right)a_n\psi\left(\frac{a_n^{\gamma_1}-1}{\gamma_1},\frac{a_n^{\gamma_2}-1}{\gamma_2}\right)=0 \tag{2.5}$$

(see also the Appendix about these conditions). Then

$$\sqrt{k}\left(\frac{\hat{x}(p,\theta)-x(p,\theta)}{\hat{a}_1(n/k)\int_1^{a_n}s^{\hat{\gamma}_1-1}(\log s)\,ds},\frac{\hat{y}(p,\theta)-y(p,\theta)}{\hat{a}_2(n/k)\int_1^{a_n}s^{\hat{\gamma}_2-1}(\log s)\,ds}\right) \tag{2.6}$$

converges in distribution to

$$(\{\rho(\theta)\cos\theta\}^{\gamma_1}\Gamma_1-(0\wedge\gamma_1)A_1+(0\wedge\gamma_1)^2B_1,$$
$$\{\rho(\theta)\sin\theta\}^{\gamma_2}\Gamma_2-(0\wedge\gamma_2)A_2+(0\wedge\gamma_2)^2B_2). \tag{2.7}$$

Remark. It is surprising that the limit process does not depend explicitly on the process V from (1.10): the limit process (2.7) depends on just six random variables.

Remark. Relations (2.4) and (2.5) put limits on the growth of the sequence $\{k(n)\}$ related to the sequence $\{a_n\}$; (2.4) gives a lower bound and (2.5) gives an upper bound. Relation (2.3) is essentially a smoothness condition on F related to the sequences $\{k(n)\}$ and $\{a_n\}$. Cf. the conditions in the one-dimensional case [6].

EXAMPLE. The Cauchy distribution in $\mathbb{R}_2$ has density

$$\frac{1}{2\pi}\frac{\alpha}{(1+x^2+y^2)^{\alpha/2+1}}$$

and the distribution function can be written for x, $y>0$

$$1-F(x,y)=\frac{1}{2\pi}\int_{-\pi/2}^{\pi}\{1+R^2(x,y,\theta)\}^{-\alpha/2}\,d\theta,$$

where

$$R(x,y,\theta)=\begin{cases}\dfrac{x}{\cos\theta}, & -\dfrac{\pi}{2}\leqslant\theta\leqslant\operatorname{arc\,tg}\dfrac{y}{x}\\[2mm] \dfrac{y}{\sin\theta}, & \operatorname{arc\,tg}\dfrac{y}{x}\leqslant\theta\leqslant\pi.\end{cases}$$

Note that $R(tx,ty,\theta)=tR(x,y,\theta)$ for t, x, $y>0$.

For $x>0$ introduce

$$P_\alpha(x)=x^{2+\alpha}\left\{(1+x^2)^{-\alpha/2}-x^{-\alpha}+\frac{\alpha}{2}x^{-2-\alpha}\right\}.$$

it is easily checked that $P_\alpha(x)\to 0$ $(x\to\infty)$. Now

$$\frac{t^\alpha\{1-F(tx,ty)\}-(1/2\pi)\int_{-\pi/2}^{\pi}R^{-\alpha}(x,y,\theta)\,d\theta}{-(\alpha/4\pi)\,t^{-2}\int_{-\pi/2}^{\pi}R^{-\alpha-2}(x,y,\theta)\,d\theta}$$
$$=1-\frac{4\pi}{\alpha}\frac{\int_{-\pi/2}^{\pi}R^{-\alpha-2}(x,y,\theta)\,P_\alpha(tR(x,y,\theta))\,d\theta}{\int_{-\pi/2}^{\pi}R^{-\alpha-2}(x,y,\theta)\,d\theta}.$$

This expression tends to 1 uniformly for $x \wedge y > \varepsilon$ ($t \to \infty$). Hence conditions (2.2) and (2.3) hold. It is easily checked that condition (2.5) is fulfilled for sequences $k(n)$ and p_n satisfying

$$\lim \sqrt{k}\, p_n^{2/\alpha} = 0.$$

The proof of Theorem 2.1 is broken up into four lemmas.

LEMMA 2.1. *If* $-\log G$ *has continuous first-order derivatives* $(-\log G)_1$ *and* $(-\log G)_2$, *then*

$$\sqrt{k}(\hat{\rho}(\theta) - \rho(\theta)) \tag{2.8}$$

converges in D-space to

$$V\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right) + (-\log G)_1\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right)\left\{\int_1^{\cos\theta} (\log s)\, s^{\gamma_1-1}\, ds\right\} \Gamma_1 + (-\log G)_2\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right)\left\{\int_1^{\sin\theta} (\log s)\, s^{\gamma_2-1}\, ds\right\} \Gamma_2. \tag{2.9}$$

Proof.

$$\begin{aligned}\sqrt{k}\{\hat{\rho}(\theta) - \rho(\theta)\} = {} & \sqrt{k}\left\{-\log \hat{G}\left(\frac{(\cos\theta)^{\hat{\gamma}_1}-1}{\hat{\gamma}_1}, \frac{(\sin\theta)^{\hat{\gamma}_2}-1}{\hat{\gamma}_2}\right)\right. \\ & \left. + \log G\left(\frac{(\cos\theta)^{\hat{\gamma}_1}-1}{\hat{\gamma}_1}, \frac{(\sin\theta)^{\hat{\gamma}_2}-1}{\hat{\gamma}_2}\right)\right\} \\ & + \sqrt{k}\left\{-\log G\left(\frac{(\cos\theta)^{\hat{\gamma}_1}-1}{\hat{\gamma}_1}, \frac{(\sin\theta)^{\hat{\gamma}_2}-1}{\hat{\gamma}_2}\right)\right. \\ & \left. + \log G\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right)\right\}.\end{aligned}$$

The first part converges to $V([(\cos\theta)^{\gamma_1} - 1]/\gamma_1, [(\sin\theta)^{\gamma_2} - 1]/\gamma_2)$ by the local uniformity in (1.9) and the second part clearly converges to the second part of (2.9) (cf. 1.8)).

LEMMA 2.2. *Under the conditions of the theorem,*

$$\sqrt{k}\left(\frac{\{\hat{x}(p,\theta) - x_n(p,\theta)\}}{\hat{a}_1(n/k)\int_1^{a_n} s^{\hat{\gamma}_1-1}(\log s)\, ds}\right) \tag{2.10}$$

converges in distribution ($n \to \infty$) *to*

$$(\rho(\theta)\cos\theta)^{\gamma_1}\Gamma_1 - (0 \wedge \gamma_1)A_1 + (0 \wedge \gamma_1)^2 B_1, \tag{2.11}$$

and similarly for the second component $\hat{y}(p, \theta)$.

Proof. The proof follows the line of the proof of the theorem in [6]. Note that (from Proposition A.1)

$$\lim_{n\to\infty} \frac{[U_1(a_n xn/k) - b_1(n/k)]/a_1(n/k) - [(a_n x)^{\gamma_1} - 1]/\gamma_1}{c(b_1(n/k), b_2(n/k))(a_n x)^{\gamma_1+1}\,\psi_1([(a_n x)^{\gamma_1} - 1]/\gamma_1)} = 1 \tag{2.12}$$

locally uniformly, $0 < x < \infty$. This, together with (2.4), suffices for the proof of the theorem in [6] (since conditions (1.5) and (1.6) of that paper just serve to derive (2.7) of the same paper and since our condition (2.5) implies that $\sqrt{k}\, c(b_1(n/k), b_2(n/k)) \to 0$ by (A.12)). Now

$$\hat{x}(p,\theta) - x_n(p,\theta)$$

$$= \left\{\frac{(a_n\hat{\rho}(\theta)\cos\theta)^{\hat{\gamma}_1} - 1}{\hat{\gamma}_1} - \frac{(a_n\rho(\theta)\cos\theta)^{\gamma_1} - 1}{\gamma_1}\right\}\hat{a}_1\left(\frac{n}{k}\right) \tag{2.13}$$

$$+ \frac{(a_n\rho(\theta)\cos\theta)^{\gamma_1} - 1}{\gamma_1}\left\{\hat{a}_1\left(\frac{n}{k}\right) - a_1\left(\frac{n}{k}\right)\right\} \tag{2.14}$$

$$+ \hat{b}_1\left(\frac{n}{k}\right) - b_1\left(\frac{n}{k}\right) \tag{2.15}$$

$$- \left\{U_1\left(\frac{\rho(\theta)\cos\theta}{p_n}\right) - U_1\left(\frac{n}{k}\right) - \frac{(a_n\rho(\theta)\cos\theta)^{\gamma_1} - 1}{\gamma_1}\right.$$

$$\left.\times a_1\left(\frac{n}{k}\right)\right\}. \tag{2.16}$$

The parts (2.14), (2.15), and (2.16) can be treated as in [6].

Note that the present conditions are somewhat more general than the conditions in [6]. The part (2.13) equals

$$\hat{a}_1\left(\frac{n}{k}\right)\int_1^{a_n\rho(\theta)\cos\theta} (s^{\hat{\gamma}_1 - 1} - s^{\gamma_1 - 1})\,ds + \hat{a}_1\left(\frac{n}{k}\right)$$

$$\times\left\{(a_n\cos\theta)^{\hat{\gamma}_1}\frac{(\hat{\rho}(\theta))^{\hat{\gamma}_1} - (\rho(\theta))^{\hat{\gamma}_1}}{\hat{\gamma}_1}\right\}. \tag{2.17}$$

The first term of (2.17) can be treated as the corresponding term in the proof of the theorem in [6]. By Lemma 2.1 and since $(\log a_n)/\sqrt{k} \to 0$ $(n \to \infty)$ by (2.4), the second term is asymptotically equal to

$$\begin{aligned} &\hat{a}_1\left(\frac{n}{k}\right)(a_n \cos\theta)^{\gamma_1} \frac{\hat{\gamma}_1 \log \hat{\rho}(\theta) - \hat{\gamma}_1 \log \rho(\theta)}{\hat{\gamma}_1} (\rho(\theta))^{\gamma_1} \\ &\quad \sim \hat{a}_1\left(\frac{n}{k}\right)(a_n \rho(\theta)\cos\theta)^{\gamma_1} \frac{\hat{\rho}(\theta) - \rho(\theta)}{\rho(\theta)} \\ &\quad = \hat{a}_1\left(\frac{n}{k}\right)\frac{a_n^{\gamma_1}}{\sqrt{k}} O_p(1) \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Hence the second term of (2.17) does not contribute to the asymptotic normal distribution of $\hat{x}(p, \theta)$.

LEMMA 2.3. *Under the conditions of the theorem*

$$\lim_{n\to\infty} \sqrt{k}\{\rho_n(\theta) - \rho(\theta)\} = 0 \qquad \textit{locally uniformly}. \tag{2.18}$$

Proof. A little reflection shows that $\rho_n(\theta)$ avoids a neighbourhood of zero for $0 < \theta < \pi/2$ (cf. Remark 1.1). The conditions of the theorem imply (and we use the homogeneity relation (1.2) for G) that

$$\begin{aligned} \lim_{n\to\infty} \sqrt{k}\, a_n \Bigg[\frac{n}{k} \Bigg\{ 1 - F\Bigg(& b_1\left(\frac{n}{k}\right) + a_1\left(\frac{n}{k}\right) \frac{(a_n \rho_n(\theta)\cos\theta)^{\gamma_1} - 1}{\gamma_1}, \\ & b_2\left(\frac{n}{k}\right) + a_2\left(\frac{n}{k}\right) \frac{(a_n \rho_n(\theta)\sin\theta)^{\gamma_2} - 1}{\gamma_2} \Bigg) \Bigg\} \\ & + \log G\left(\frac{(a_n \rho_n(\theta)\cos\theta)^{\gamma_1} - 1}{\gamma_1}, \frac{(a_n \rho_n(\theta)\sin\theta)^{\gamma_2} - 1}{\gamma_2} \right) \Bigg] = 0. \end{aligned} \tag{2.19}$$

Now by the definitions of ρ_n and ρ

$$\begin{aligned} &\frac{n}{k}\left\{ 1 - F\left(b_1\left(\frac{n}{k}\right) + a_1\left(\frac{n}{k}\right) \frac{(a_n \rho_n(\theta)\cos\theta)^{\gamma_1} - 1}{\gamma_1}, b_2\left(\frac{n}{k}\right) \right.\right. \\ &\qquad \left.\left. + a_2\left(\frac{n}{k}\right) \frac{(a_n \rho_n(\theta)\sin\theta)^{\gamma_2} - 1}{\gamma_2} \right) \right\} \\ &\quad = \frac{np}{k} = a_n^{-1} = -(a_n \rho(\theta))^{-1} \\ &\qquad \times \log G\left(\frac{(\cos\theta)^{\gamma_1} - 1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2} - 1}{\gamma_2} \right). \end{aligned}$$

Hence the left-hand side of (2.19) becomes

$$\left\{-\log G\left(\frac{(\cos\theta)^{\gamma_1}-1}{\gamma_1}, \frac{(\sin\theta)^{\gamma_2}-1}{\gamma_2}\right)\right\}\sqrt{k}\left\{\frac{1}{\rho(\theta)}-\frac{1}{\rho_n(\theta)}\right\}.$$

This completes the proof.

LEMMA 2.4. *Under the conditions of theorem,*

$$\lim_{n\to\infty}\frac{\sqrt{k}\{x_n(p,\theta)-x(p,\theta)\}}{\hat{a}_1(n/k)\,a_n^{\hat{\gamma}_1+1}\psi_1((a_n^{\hat{\gamma}_1}-1)/\hat{\gamma}_1)}=0 \qquad \textit{locally uniformly.}$$

Proof.

$$x(p,\theta)-x_n(p,\theta)=a_1\left(\frac{n}{k}\right)\left\{\frac{(a_n\rho_n(\theta)\cos\theta)^{\gamma_1}-1}{\gamma_1}-\frac{(a_n\rho(\theta)\cos\theta)^{\gamma_1}-1}{\gamma_1}\right\}\sim a_1\left(\frac{n}{k}\right)(a_n\cos\theta)^{\gamma_1}$$
$$\times\{\rho(\theta)\}^{\gamma_1-1}(\rho_n(\theta)-\rho(\theta)) \qquad (n\to\infty).$$

The result now follows from Lemma 2.3 (use Corollary A3 for $\alpha=0$).

APPENDIX: THE STRONG SECOND-ORDER RELATION

The second-order condition (4.1) on the distribution function F from de Haan and Resnick [5] reads

$$\lim_{t\to\infty}\frac{t\{1-F(b_1(t)+xa_1(t), b_2(t)+ya_2(t)\}+\log G(x,y)}{c(b_1(t),b_2(t))}=\psi(x,y) \quad \text{(A.1)}$$

locally uniformly for $x, y\in(x_0,\infty]$ for some non-constant function ψ and a positive function c, where $c(b_1(t),b_2(t))$ is regularly varying, $c(b_1(t),b_2(t))\to 0$ $(t\to\infty)$, and

$$b_i(t)=\left(\frac{1}{1-F_i}\right)^{\leftarrow}(t).$$

Recall that F_i is the ith marginal distribution function of F.

We require here a stronger form of (A.1), namely,

$$\lim_{n\to\infty}\left[\frac{n}{k}\left\{1-F\left(b_1\left(\frac{n}{k}\right)+\frac{(a_n x)^{\gamma_1}-1}{\gamma_1}a_1\left(\frac{n}{k}\right),\right.\right.\right.$$
$$\left.b_2\left(\frac{n}{k}\right)+a_2\left(\frac{n}{k}\right)\frac{(a_n y)^{\gamma_2}-1}{\gamma_2}\right)\Bigg\}$$
$$+\log G\left(\frac{(a_n x)^{\gamma_1}-1}{\gamma_1},\frac{(a_n y)^{\gamma_2}-1}{\gamma_2}\right)\Bigg]\Bigg/\Bigg[c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)$$
$$\times\psi\left(\frac{(a_n x)^{\gamma_1}-1}{\gamma_1},\frac{(a_n y)^{\gamma_2}-1}{\gamma_2}\right)\Bigg]=1 \tag{A.2}$$

locally uniformly for $x, y\in(0,\infty]$, where $k=k(n)\to\infty$, $k(n)/n\to 0$ $(n\to\infty)$, and $a_n=k/np_n$ as before.

Relation (A.2) can be specialized to (e.g.) the first marginal as

$$\lim_{n\to\infty}\frac{n/k\{1-F(b_1(n/k)+a_n(n/k)((a_n x)^{\gamma_1}-1)/\gamma_1)\}-1/a_n x}{c(b_1(n/k),b_2(n/k))\,\psi_1(((a_n x)^{\gamma_1}-1)/\gamma_1)}=1 \tag{A.3}$$

locally uniformly, $0<x\leqslant\infty$, where $\psi_1(x):=\psi(x,\lim_{y\to\infty}((y^{\gamma_2}-1)/\gamma_2)$. Our aim is to translate (A.3) into a strong second-order condition for

$$U_1:=\left(\frac{1}{1-F_1}\right)^{\leftarrow},$$

which is needed in Lemma 2.3. The inversion is as folows.

PROPOSITION A.1. *Suppose* (A.3) *holds locally uniformly on* $(0,\infty)$ *with* $c(b_1(t),b_2(t))$ *regularly varying and tending to zero* $(t\to\infty)$. *Suppose, further, that* ψ_1 *is non-constant and*

$$\lim_{n\to\infty}c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)a_n\psi_1\left(\frac{a_n^{\gamma_1}-1}{\gamma_1}\right)=0; \tag{A.4}$$

then

$$\lim_{n\to\infty}\frac{\left[U_1\left(a_n x\frac{n}{k}\right)-b_1\left(\frac{n}{k}\right)\right]\Big/a_1\left(\frac{n}{k}\right)-\dfrac{(a_n x)^{\gamma_1}-1}{\gamma_1}}{c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)(a_n)^{\gamma_1+1}\psi_1\left(\dfrac{(a_n x)^{\gamma_1}-1}{\gamma_1}\right)}=1 \tag{A.5}$$

locally uniformly, $x\in(0,\infty)$, *where*

$$U_1:=\left(\frac{1}{1-F_1}\right)^{\leftarrow}.$$

Before giving the proof we first identify the function ψ_1. The following lemma is useful.

LEMMA A.2. *If*

$$\lim_{t\to\infty} \frac{t\{1-F_1(b_1(t)+xa_1(t))\}-(1+\gamma_1 x)^{-1/\gamma_1}}{c(b_1(t), b_2(t))} = \psi_1(x), \tag{A.6}$$

locally uniformly on $(0, \infty)$, *with* $c(b_1(t), b_2(t)) \to 0$ $(t \to \infty)$, *then*

$$\lim_{t\to\infty} \frac{\dfrac{U_1(tx)-b_1(t)}{a_1(t)} - \dfrac{x^{\gamma_1}-1}{\gamma_1}}{c(b_1(t), b_2(t))} = x^{\gamma_1+1}\psi_1\left(\frac{x^{\gamma_1}-1}{\gamma_1}\right) \tag{A.7}$$

locally uniformly on $(0, \infty)$.

Proof. Relation (A.6) implies that

$$\lim_{t\to\infty} t\{1-F_1(b_1(t)+xa_1(t))\} = (1+\gamma_1 x)^{-1/\gamma_1}; \tag{A.8}$$

hence (A.6) is equivalent to

$$\lim_{t\to\infty} \frac{(1/t\{1-F_1(b_1(t)+a_1(t)(x^{\gamma_1}-1)/\gamma_1)\})-x}{c(b_1(t), b_2(t))} = -x^2\psi_1\left(\frac{x^{\gamma_1}-1}{\gamma_1}\right),$$

which by Vervaat's lemma [9] implies that

$$\lim_{t\to\infty} \frac{\{1+\gamma_1[(U_1(tx)-b_1(t))/a_1(t)]\}^{1/\gamma_1}-x}{c(b_1(t), b_2(t))} = x^2\psi_1\left(\frac{x^{\gamma_1}-1}{\gamma_1}\right). \tag{A.9}$$

Writing the second term of the numerator on the left-hand side as $(1+\gamma_1((x^{\gamma_1}-1)/\gamma_1))^{1/\gamma_1}$ and using

$$\lim_{t\to\infty} \frac{U_1(tx)-b_1(t)}{a_1(t)} = \frac{x^{\gamma_1}-1}{\gamma_1}, \tag{A.10}$$

we get (A.7) from (A.9).

COROLLARY A.3. *For some constants* $\alpha \leqslant 0$, c_1, *and* c_2,

$$\psi_1\left(\frac{x^{\gamma_1}-1}{\gamma_1}\right) = x^{-\gamma_1-1}\left\{c_1\int_1^x s^{\gamma_1-1}\frac{s^\alpha-1}{\alpha}\,ds + c_2\frac{x^{\gamma_1+\alpha}-1}{\gamma_1+\alpha}\right\}. \tag{A.11}$$

It follows, e.g., that, if $\alpha = 0$, as $x \to \infty$,

$$\psi_1\left(\frac{x^{\gamma_1}-1}{\gamma_1}\right) \sim \begin{cases} c_1\gamma^{-1}x^{-1}\log x, & \gamma_1 > 0 \\ c_1 2^{-1}x^{-1}(\log x)^2, & \gamma_1 = 0 \\ c_1 x^{-\gamma_1-1}\gamma_1^{-2}, & \gamma_1 < 0. \end{cases} \tag{A.12}$$

Proof. $x^{\gamma_1+1}\psi_1((x^{\gamma_1}-1)/\gamma_1)$ is the same as the function $H(x)$ from Appendix A in [2].

Proof of the Proposition. We give the proof for the case $\alpha=0$ in (A.11). The proof for $\alpha<0$ is similar. Note that (A.3), (A.4), and (A.7) imply that

$$\lim_{n\to\infty} a_n\frac{n}{k}\left\{1-F_1\left(b_1\left(\frac{n}{k}\right)+\frac{(a_n x)^{\gamma_1}-1}{\gamma_1}a_1\left(\frac{n}{k}\right)\right)=x^{-1}\right. \tag{A.13}$$

locally uniformly. Hence (A.3) implies that

$$\frac{1\Big/\left(a_n\frac{n}{k}\left\{1-F_1(b_1\left(\frac{n}{k}\right)+\frac{(a_n x)^{\gamma_1}-1}{\gamma_1}a_1\left(\frac{n}{k}\right)\right\}\right)-x}{c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)a_n x^2\psi_1\left(\frac{(a_n x)^{\gamma_1}-1}{\gamma_1}\right)}=-1 \tag{A.14}$$

locally uniformly. By (A.12) the denominator is asymptotic to c_1 times

$$\begin{cases} c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)\gamma_1^{-1}x\log a_n & \gamma_1>0\\ c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)2^{-1}x(\log a_n)^2 & \gamma_1=0\\ c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)\gamma_1^{-2}x^{1-\gamma_1}a_n^{-\gamma_1} & \gamma_1<0.\end{cases} \tag{A.15}$$

So by (A.4) we can apply Vervaat's lemma and get from (A.14)

$$\lim_{n\to\infty}\frac{a_n^{-1}\left\{1+\gamma_1\left[U_1\left(a_n x\frac{n}{k}\right)-b_1\left(\frac{n}{k}\right)\right]\Big/a_1\left(\frac{n}{k}\right)\right\}^{1/\gamma_1}-x}{c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)a_n x^2\psi_1\left(\frac{(a_n x)^{\gamma_1}-1}{\gamma_1}\right)}=1. \tag{A.16}$$

Using again the fact that the denominator of (A.16) converges to zero ($n\to\infty$), we find from (A.16) that

$$\lim_{n\to\infty}\frac{\left\{1+\gamma_1\left[U_1\left(\frac{n}{k}a_n x\right)-b_1\left(\frac{n}{k}\right)\right]\Big/a_1\left(\frac{n}{k}\right)\right\}\Big/(a_n x)^{\gamma_1}-1}{\gamma_1 c\left(b_1\left(\frac{n}{k}\right),b_2\left(\frac{n}{k}\right)\right)a_n x\,\psi_1\left(\frac{(a_n x)^{\gamma_1}-1}{\gamma_1}\right)}=1;$$

i.e., (A.5) holds.

ACKNOWLEDGMENTS

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