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On the Asymptotic Worst Case Behavior of Harmonic Fit

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In the parametric bin packing problem we must pack a list of items with size smaller than or equal to $1/r$ in a minimal number of unit-capacity bins. Among the approximation algorithms, the class of Harmonic Fit algorithms (HF_M) plays an important role. Lee and Lee (*J. Assoc. Comput. Mach.* **32** (1985), 562–572) and Galambos (*Ann. Univ. Sci. Budapest Sect. Comput.* **9** (1988), 121–126) provide upper bounds for the asymptotic worst case ratio of HF_M and show tightness for certain values of the parameter M. In this paper we provide worst case examples that meet the known upper bound for additional values of M, and we show that for remaining values of M the known upper bound is not tight. For the classical bin packing problem ($r = 1$), we prove an asymptotic worst case ratio of $\frac{12}{7}$ for the case $M = 4$ and 1.7 for the case $M = 5$. We give improved lower bounds for some interesting cases that are left open.

1. INTRODUCTION

One of the famous problems in combinatorial optimization is the so-called *bin packing problem*. We are given a list of items $L = (a_1, \dots, a_n)$. An item a_i has size $s(a_i)$, which is greater than 0 and smaller than or equal to 1. We are also given an infinite number of unit-capacity bins. The problem is to assign the items to a minimal number of bins, such that the items that any bin receives have total size smaller than or equal to 1. Throughout, we will use the size-operator $s(\cdot)$ also to denote the total size of items in a bin B or a list L (resp. $s(B)$ and $s(L)$).

Since this problem is NP-hard [3], we can not expect to find an algorithm that gives an optimal solution in reasonable (polynomial) time. Therefore, research has focused on finding fast algorithms that give near-optimal solutions. The most commonly used performance measure for these kind of algorithms is the *asymptotic worst case ratio* (a.w.c.r.). Let us denote by $A(L)$ the number of bins that an algorithm A uses to pack list L and let

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$OPT(L)$ be the minimal number of bins for list L. Then the asymptotic worst case ratio of algorithm A, denoted by R_A^∞, is defined by

$$R_A^\infty = \limsup_{k\to\infty}\Big(\max_L\{A(L)/OPT(L) | OPT(L) = k\}\Big).$$

In other words, the asymptotic worst case ratio of an algorithm A is the minimal number α, such that

$$A(L) \leq \alpha OPT(L) + o(OPT(L))$$

holds for every list L. It may also be interesting to investigate the asymptotic worst case behavior when we restrict our attention to a special class of lists L. Let r be an integer greater than or equal to 1. Then we denote by $£(r)$ the family of lists that only contain items of size less than or equal to $1/r$. When we restrict ourselves to lists of $£(r)$, we speak of the *parametric* bin packing problem with parameter r, and we use

$$R_A^\infty(r) = \limsup_{k\to\infty}\Big(\max_{L\in £(r)}\{A(L)/OPT(L) | OPT(L) = k\}\Big)$$

to denote the a.w.c.r. for these cases.

Probably the oldest and simplest algorithm for bin packing is the *Next Fit* algorithm. Next Fit begins with opening the first bin which becomes the active bin. Then items are considered on a one-by-one basis: as long as the current item fits in the active bin, it is added; when the active bin can no longer accommodate this item, the active bin is closed and the item is placed in the next bin which has then become the active bin. It can easily be established that the a.w.c.r. of NF is equal to 2 for $r = 1$ and $r/(r-1)$ for $r \geq 2$.

In this paper we will study another basic algorithm for the bin packing problem, which is the *Harmonic Fit* (HF_M) algorithm that was introduced by Lee and Lee [4]. Given a parameter M, we divide the interval $(0,1]$ into M disjoint intervals:

$$I_j = \left(\frac{1}{j+1}, \frac{1}{j}\right] \quad \forall 1 \leq j \leq M-1$$

$$I_M = \left(0, \frac{1}{M}\right].$$

All items are classified according to their size: an item a_i is called an I_j-item if $s(a_i) \in I_j$. I_j-items are packed together in so-called I_j-bins. Exactly j items of I_j, $1 \leq j \leq M-1$, can be packed together in a bin. Items of I_M are packed in I_M-bins by Next Fit.

If we consider lists from $\mathcal{L}(r)$, Harmonic Fit reduces to Next Fit if $M \leq r$. Since Next Fit has been studied extensively, we will only be interested in the cases where $M \geq r + 1$. In order to implement HF_M it suffices to use $M - r + 1$ active bins, one for each interval I_j, $r \leq j \leq M$. Items can be packed in an on-line manner, which means that they can be packed in the order that they are given, without considering the sizes of subsequent items. The time complexity of the algorithm is $O(n)$ for any fixed M.

In order to investigate the asymptotic worst case behavior of Harmonic Fit, we need to define the series $t_j(r)$ as

$$t_1(r) = r + 1, \qquad t_2(r) = r + 2,$$

$$t_j(r) = t_{j-1}(r)\big(t_{j-1}(r) - 1\big) + 1 \quad \forall j \geq 3.$$

This doubly exponential series has the following important property that we will use throughout:

$$\frac{r}{t_1(r)} + \sum_{j=2}^{i} \frac{1}{t_j(r)} = 1 - \frac{1}{t_{i+1}(r) - 1} < 1 \quad \forall i \geq 1.$$

This means that if we have a bin that contains r items of size $1/t_1(r) + \varepsilon$ and one item of size $1/t_j(r) + \varepsilon$ for all $2 \leq j \leq i$ (assume $\varepsilon > 0$ is arbitrarily small), then $t_{i+1}(r)$ is the smallest integer s such that we can add an item of size $1/s + \varepsilon$ to this bin. Furthermore, it has the property that $t_k(r) - 1$ is a multiple of both $t_j(r)$ and $t_j(r) - 1$ for all pairs of j and k with $j < k$. Table 1 displays some values of $t_j(r)$ for small j and r.

Lee and Lee [4] gave an upper bound for the asymptotic worst case ratio of HF_M for the nonparametric case ($r = 1$) and $M \geq 3$. They showed that their bound is tight for $M = t_{i+1}(1) - 1$, $i \geq 2$. By taking M sufficiently large, they got an a.w.c.r. of $1.691\ldots$ for HF_M.

After HF_M, a number of modifications of HF_M were suggested to improve upon its a.w.c.r. In the same paper, Lee and Lee introduced the

TABLE 1
Values of $t_j(r)$ for Small j and r

	$r = 1$	$r = 2$	$r = 3$
$t_1(r)$	2	3	4
$t_2(r)$	3	4	5
$t_3(r)$	7	13	21
$t_4(r)$	43	157	421
$t_5(r)$	1807	24493	176821

Refined Harmonic Fit algorithm which has an a.w.c.r. of 1.636. Ramanan *et al.* [5] gave birth to the Modified Harmonic Fit algorithm with a bound of 1.612, and finally Richey [6] designed an algorithm with an a.w.c.r. of 1.589. All these modifications of HF_M are also on-line algorithms and must therefore exceed the 1.540 lower bound for on-line algorithms that was proven by Van Vliet [7].

Galambos [2] generalized the upper bound for $R^{\infty}_{HF_M}$ to the parametric case:

THEOREM 1 (GALAMBOS). *Let $i \geq 2$ and $t_i(r) \leq M < t_{i+1}(r)$. Then*

$$R^{\infty}_{HF_M}(r) \leq 1 + \sum_{j=2}^{i} \frac{1}{t_j(r) - 1} + \frac{M}{(M-1)\left(t_{i+1}(r) - 1\right)}.$$

Throughout, we will denote this upper bound by $Q_M(r)$. Galambos showed that this bound is tight for $M = t_{i+1}(r) - 1$, $i \geq 2$.

In summary, Lee and Lee [4] and Galambos [2] investigate the asymptotic worst case behavior of Harmonic Fit for $M \geq r + 2$. They give an upper bound for the asymptotic worst case ratio, but show tightness for only a limited number of values of M. In this paper we will first discuss the case $M = r + 1$. Next, we show that the upper bound $Q_M(r)$ is also tight for $M = t_{i+1}(r)$, $i \geq 1$, and that it is not tight for other values of M. The remainder of the paper is devoted to some special cases. In two subsequent sections we deal with the cases $M = 4$ and $M = 5$ for the nonparametric case and prove asymptotic worst case ratios of respectively $\frac{12}{7}$ and $\frac{17}{10}$. Finally, we give some improved lower bounds for small values of M and r. We have summarized our results in Table 2.

In a recent study of on-line algorithms for the bin packing problem, Csirik and Johnson [1] also realized that the upper bound $Q_M(r)$ is not tight for all values of M. They constructed similar worst case examples for the nonparametric case of $M = 3, 4, 7, 8, 9$, and 10 that yield the same lower bounds as in this paper.

2. THE CASE $M = r + 1$

As Lee and Lee [4] and Galambos [2] did not cover the case $M = r + 1$, we fill in this blank spot. We do this as follows:

THEOREM 2.

$$R^{\infty}_{HF_{r+1}}(r) = \frac{r+1}{r}.$$

TABLE 2
Values of $R^{\infty}_{HF_M}(r)$ and the Upper Bound $Q_M(r)$ (Rounded Off to Four Decimals)

	$r = 1$		$r = 2$		$r = 3$	
M	$R^{\infty}_{HF_M}$	Q_M	$R^{\infty}_{HF_M}$	Q_M	$R^{\infty}_{HF_M}$	Q_M
2	2.0000	—	—	—	—	—
3	1.7500	1.7500*	1.5000	—	—	—
4	1.7143	1.7222	1.4444	1.4444*	1.3333	—
5	1.7000	1.7083	$\geq$ 1.4343	1.4375	1.3125	1.3125*
6	1.7000	1.7000*	$\geq$ 1.4314	1.4333	$\geq$ 1.3092	1.3100
7	1.6944	1.6944*	$\geq$ 1.4282	1.4306	$\geq$ 1.3071	1.3083
8	$\geq$ 1.6938	1.6939	$\geq$ 1.4259	1.4286	$\geq$ 1.3065	1.3071
9	$\geq$ 1.6933	1.6935	$\geq$ 1.4253	1.4271	$\geq$ 1.3055	1.3063
10	$\geq$ 1.6929	1.6931	$\geq$ 1.4242	1.4259	$\geq$ 1.3047	1.3056
11	$\geq$ 1.6926	1.6929	$\geq$ 1.4242	1.4250	$\geq$ 1.3041	1.3050
12	$\geq$ 1.6925	1.6926	1.4242	1.4242*	$\geq$ 1.3039	1.3045
∞	1.6910	1.6910*	1.4231	1.4231*	1.3024	1.3024*

Note. Values of $Q_M(r)$ that are tight are marked by *.

Proof. Since all items have size smaller than or equal to $1/r$, we only have items with size in the intervals

$$I_r = \left(\frac{1}{r+1}, \frac{1}{r}\right] \quad \text{and} \quad I_{r+1} = \left(0, \frac{1}{r+1}\right].$$

Items with size in $(1/(r+1), 1/r]$ are packed with r in a bin in bins of type I_r. Therefore, all I_r-bins (except possibly the last) are at least $r/(r+1)$ full. Items from $(0, 1/(r+1)]$ are packed by Next Fit in bins of type I_{r+1}. Since the biggest item in a I_{r+1}-bin has size smaller than or equal to $1/(r+1)$, all I_{r+1}-bins (except possibly the last) are at least $r/(r+1)$ full. So,

$$HF_{r+1}(L) \leq \frac{r+1}{r} s(L) + 2 \leq \frac{r+1}{r} OPT(L) + 2.$$

Hereby it follows that $R^{\infty}_{HF_{r+1}}(r) \leq (r+1)/r$.

Let n be a multiple of r and take $\varepsilon = r/(r^2 + r + 1)n$. Let the list L be a concatenation of $1 + n/r$ sublists: $L = L_1 L_{21} L_{22} \cdots L_{2n/r}$. L_1 contains nr items of size $1/(r+1) + \varepsilon$ and L_{2i} contains r items of size $1/(r+1) - r\varepsilon$ followed by one item of size $(r^2 + r + 1)\varepsilon$. HF_{r+1} packs list L_1 in n I_r-bins and it packs every list L_{2i} in a separate I_{r+1}-bin. So, $HF_{r+1}(L) = n + n/r$. On the other hand, we can pack n bins with r items of L_1 and one item of size $1/(r+1) - r\varepsilon$ each, and one bin with all the items of size $(r^2 + r + 1)\varepsilon$. Since all $n+1$ bins are then completely full, this is an optimal solution. So,

$$\frac{HF_{r+1}(L)}{OPT(L)} = \frac{n + n/r}{n+1} = \frac{1 + 1/r}{1 + 1/n}$$

We can take n arbitrarily large, so $R^{\infty}_{HF_{r+1}}(r) \geq (r+1)/r$. ∎

Among others, this gives us an asymptotic worst case ratio equal to 2 for the nonparametric case of $M = 2$.

3. TIGHTNESS OF THE UPPER BOUND FOR $M \geq r + 2$

In this section we will discuss in what cases the upper bound given in Theorem 1 is tight or not. As we already mentioned, Galambos proved the following:

THEOREM 3 (GALAMBOS). *Let $i \geq 2$ and let $M = t_{i+1}(r) - 1$, then*

$$R^{\infty}_{HF_M}(r) \geq 1 + \sum_{j=2}^{i} \frac{1}{t_j(r) - 1} + \frac{1}{M-1}.$$

Although this result only applies to certain values of M, it has an important consequence:

$$\lim_{M \to \infty} R^{\infty}_{HF_M}(r) = 1 + \sum_{i=2}^{\infty} \frac{1}{t_i(r) - 1}.$$

We can also show tightness of the upper bound for other values of M:

THEOREM 4. *Let $i \geq 1$ and let $M = t_{i+1}(r)$, then*

$$R^{\infty}_{HF_M}(r) \geq 1 + \sum_{j=2}^{i} \frac{1}{t_j(r) - 1} + \frac{M}{(M-1)\big(t_{i+1}(r) - 1\big)}.$$

Proof. Let n be a suitable large number. We take

$$\varepsilon = \frac{1}{(t_{i+2}(r) - 1)n}$$

$$n_x = \left(\frac{1}{t_{i+1}(r) - 1} - \frac{1}{(t_{i+1}(r) - 2)\big(t_{i+1}(r) - 1\big)^2} \right) n$$

$$n_y = \frac{1}{(t_{i+1}(r) - 2)\big(t_{i+1}(r) - 1\big)} n.$$

We assume that we have chosen n such that n_x and n_y have integer values. Let L be a concatenation of $i + n_x + n_y$ sublists: $L = L_1 \cdots L_i L_{x1} \cdots L_{xn_x} L_{y1} \cdots L_{yn_y}$. L_1 contains nr items of size $1/(r+1) + \varepsilon/ri$; L_j, $2 \le j \le i$, contains n elements of size $1/t_j(r) + \varepsilon/i$, L_{xj}, $1 \le j \le n_x$, contains $t_{i+1}(r) - 1$ items of size $1/t_{i+1}(r)$ followed by one item of size ε; and L_{yj}, $1 \le j \le n_y$, contains one item of size $1/t_{i+1}(r)$ followed by $(t_{i+1}(r) - 1)(t_{i+1}(r) - 2)n + 1$ items of size ε. This construction is such that

$$s(L_{xj}) = s(L_{yj}) = 1 - \frac{1}{t_{i+1}(r)} + \varepsilon$$

$$(t_{i+1}(r) - 1)n_x + n_y = n.$$

The last equation implies that list L contains exactly n items of size $1/t_{i+1}(r)$.

HF_M packs the items of L_1 in n bins; it packs the items of L_j, $2 \le j \le i$, in $n/(t_j(r) - 1)$ bins; and it packs every list L_{xj} and L_{yj} in a separate bin. So, with

$$\begin{aligned} n_x + n_y &= \left(\frac{1}{t_{i+1}(r) - 1} - \frac{1}{(t_{i+1}(r) - 2)(t_{i+1}(r) - 1)^2} \right. \\ &\qquad \left. + \frac{1}{(t_{i+1}(r) - 2)(t_{i+1}(r) - 1)} \right) n \\ &= \left(\frac{1}{t_{i+1}(r) - 1} + \frac{1}{(t_{i+1}(r) - 1)^2} \right) n = \frac{t_{i+1}(r)}{(t_{i+1}(r) - 1)^2} n, \end{aligned}$$

we get

$$\begin{aligned} HF_M(L) &= n + \sum_{j=2}^{i} \frac{n}{t_j(r) - 1} + n_x + n_y \\ &= \left(1 + \sum_{j=2}^{i} \frac{1}{t_j(r) - 1} + \frac{t_{i+1}(r)}{(t_{i+1}(r) - 1)^2} \right) n. \end{aligned}$$

One can easily verify that r items of L_1 together with one item of L_j for all $2 \le j \le i$, one item of size $1/t_{i+1}(r)$ and $n - 1$ items of size ε fit exactly in one bin. If we pack n bins like this, $(1 + t_{i+1}(r)/(t_{i+1}(r) - 1)^2)n$

items of size ε remain unpacked. They fit easily together in one bin, so $OPT(L) = n + 1$. As we can take n arbitrarily large, the desired result follows. ∎

Next we will show that for other values of M the upper bound given in Theorem 1 is not tight.

THEOREM 5. *Let* $i \geq 2$ *and* $t_i(r) + 1 \leq M < t_{i+1}(r) - 1$. *Then there exists an* $\varepsilon > 0$ *and a positive integer K such that*

$$\frac{HF_M(L)}{OPT(L)} \leq Q_M(r) - \varepsilon \quad \forall L \in \pounds(r) : OPT(L) \geq K.$$

Proof. We define the constants θ_1 and θ_2 by

$$\theta_1 = \frac{1}{M(M-1)(t_{i+1}(r) - 1)} \quad \text{and} \quad \theta_2 = \frac{1}{t_{i+2}(r) - 1}.$$

Further, we define the constant δ by $\delta = 2\theta_1$ and we choose $\varepsilon = \theta_1 \theta_2$ and $K = \lceil (M - r + 1)/\varepsilon \rceil$.

Consider a given list L with $OPT(L) = k \geq K$. Let $n_r, \ldots, n_{M-1}$ denote the number of items of this list that have size in respectively $I_r, \ldots, I_{M-1}$, and let S_M denote the total size of items from I_M. Further, we count the number of items in the interval $(1/(M+1), 1/M]$ by n_M.

The Next Fit packing of the items from $I_M = (0, 1/M]$ uses at most $\lceil [M/(M-1)] S_M \rceil$ bins, because every I_M-bin (except possibly the last) is at least $(M-1)/M$ full. Therefore, we can bound the number of bins that HF_M uses to pack list L by

$$\begin{aligned} HF_M(L) &\leq \sum_{j=r}^{M-1} \left\lceil \frac{n_j}{j} \right\rceil + \left\lceil \frac{M}{M-1} S_M \right\rceil \\ &\leq \sum_{j=r}^{M-1} \frac{n_j}{j} + \frac{M}{M-1} S_M + (M - r + 1). \end{aligned}$$

If $n_M = 0$ then every I_M bin (except possibly the last) is at least $M/(M+1)$ full and thus there can be at most $\lceil [(M+1)/M] S_M \rceil$ I_M-bins. In principle every item from $(1/(M+1), 1/M]$ can cause one I_M-bin to be less than $M/(M+1)$ full, but never less than $(M-1)/M$ full. This leads us to the following upper bound for $HF_M(L)$ (note that we allow x to be negative in $\lceil x \rceil$):

$$HF_M(L) \le \sum_{j=r}^{M-1} \left\lceil \frac{n_j}{j} \right\rceil + n_M + \left\lceil \frac{M+1}{M}\left(S_M - \frac{M-1}{M} n_M\right) \right\rceil$$

$$\le \sum_{j=r}^{M-1} \frac{n_j}{j} + \frac{1}{M^2} n_M + \frac{M+1}{M} s_M + (M - r + 1).$$

Let $\mathscr{B} = \{B_1, \ldots, B_k\}$ denote the set of bins in the optimal packing of list L. Let $\mathscr{B}_1 \subseteq \mathscr{B}$ denote the subset of bins that contain r items from I_r and one item from $I_{t_j(r)-1}$ for every $2 \le j \le i$, and let $\mathscr{B}_2 = \mathscr{B} \setminus \mathscr{B}_1$.

We now split our analysis into two cases:

Case 1. $n_M \ge \delta Mk$. Analogously to [2] we define the weighting function $W(x)$ as

$$W(x) = \begin{cases} \dfrac{1}{j} & \text{if } x \in I_j \text{ and } r \le j \le M-1 \\ \dfrac{M}{M-1} x & \text{if } x \in I_M. \end{cases}$$

Then,

$$HF_M(L) \le \sum_{j=r}^{M-1} \frac{n_j}{j} + \frac{M}{M-1} S_M + (M - r + 1)$$

$$= \sum_{j=1}^{k} \sum_{a \in B_j} W(s(a)) + (M - r + 1).$$

In the proof of Theorem 1 (see [2]), it is shown that for every bin B in a feasible packing

$$W(B) = \sum_{a \in B} W(s(a)) \le Q_M(r)$$

holds. If B is an element of $\mathscr{B}_1$, this inequality can hold with equality. We will show that this inequality is a strict inequality whenever B is an element of $\mathscr{B}_2$.

Consider a given bin $B \in \mathscr{B}_2$. Let us first suppose that this bin contains less than r items from I_r. Denote the number of I_r-items by q, $0 \le q < r$. Since $M \ge r + 2$, we have that

$$\frac{W(x)}{x} \le \frac{r+2}{r+1} \quad \text{for } x \le \frac{1}{r+1}.$$

This gives us

$$\begin{aligned} W(B) &\le q\frac{1}{r} + \frac{r+2}{r+1}\left(1 - \frac{q}{r+1}\right) \\ &= 1 + \frac{1}{r+1} + q\frac{1}{r(r+1)^2} \\ &\le 1 + \frac{1}{r+1} + \frac{r-1}{r(r+1)^2} \\ &= 1 + \frac{1}{r+1} + \frac{1}{(r+1)(r+2)} - \frac{2}{r(r+1)^2(r+2)} \\ &= 1 + \sum_{j=2}^{3} \frac{1}{t_j(r) - 1} - \frac{2}{r(r+1)^2(r+2)} \\ &< Q_M(r) - \frac{2}{r(r+1)^2(r+2)} < Q_M(r) - \frac{1}{t_{i+2}(r) - 1}. \end{aligned}$$

Second, let us suppose that $B \in \mathscr{B}_2$ contains r items from I_r. Let l, $2 \le l \le i$, be the minimal index for which bin B does not contain an element of $I_{t_l(r)-1}$. As a consequence, B contains exactly one item from every interval $I_{t_j(r)-1}$, $2 \le j \le l-1$. In this case, such a bin B has at most

$$1 - \frac{r}{r+1} - \sum_{j=2}^{l-1} \frac{1}{t_j(r)} = \frac{1}{t_l(r) - 1}$$

space left to accommodate items smaller than or equal to $1/t_l(r)$. For these small items we have that $W(s(a))/s(a) \le (t_l(r) + 1)/t_l(r)$, so

$$\begin{aligned} W(B) &\le r\frac{1}{r} + \sum_{j=2}^{l-1} \frac{1}{t_j(r) - 1} + \frac{1}{t_l(r) - 1}\,\frac{t_l(r) + 1}{t_l(r)} \\ &= 1 + \sum_{j=2}^{l+1} \frac{1}{t_j(r) - 1} \\ &< Q_M(r) - \frac{1}{t_{i+2}(r) - 1}. \end{aligned}$$

We conclude that $W(B) < Q_M(r) - \theta_2$ for every $B \in \mathscr{B}_2$.

Whenever B is an element of $\mathscr{B}_1$, it cannot contain an item from $(1/(M+1), 1/M]$. First, each item from I_r and $I_{t_j(r)-1}$, $2 \le j \le i$, is not in

$(1/(M+1), 1/M]$ because of $M \geq t_i(r) + 1$ and $i \geq 2$. Second, any additional item has size strictly less than

$$1 - \frac{r}{r+1} - \sum_{j=2}^{i} \frac{1}{t_j(r)} = \frac{1}{t_{i+1}(r) - 1} \leq \frac{1}{M+1}.$$

So, from $n_M \geq \delta Mk$ we may conclude that $|\mathscr{B}_2| \geq \delta k$ and thus

$$\begin{aligned} HF_M(L) &\leq \sum_{B \in \mathscr{B}_1} W(B) + \sum_{B \in \mathscr{B}_2} W(B) + (M - r + 1) \\ &\leq |\mathscr{B}_1| Q_M(r) + |\mathscr{B}_2| \big(Q_M(r) - \theta_2\big) + (M - r + 1) \\ &\leq k Q_M(r) - \delta\theta_2 k + (M - r + 1). \end{aligned}$$

Since $\delta\theta_2 = 2\varepsilon$, dividing both sides by $OPT(L) = k$ gives us that $HF_M(r)/OPT(L) \leq Q_M(r) - \varepsilon$.

Case 2. $n_M < \delta Mk$. We define the weighting function $V(x)$ as

$$V(x) = \begin{cases} \dfrac{1}{j} & \text{if } x \in I_j \text{ and } r \leq j \leq M - 1 \\ \dfrac{M+1}{M} x & \text{if } x \in I_M. \end{cases}$$

Then

$$\begin{aligned} HF_M(L) &\leq \sum_{j=r}^{M-1} \frac{n_j}{j} + \frac{n_M}{M^2} + \frac{M+1}{M} S_M + (M - r + 1) \\ &= \frac{n_M}{M^2} + \sum_{j=1}^{k} \sum_{a \in B_j} V(s(a)) + (M - r + 1). \end{aligned}$$

For $B \in \mathscr{B}_2$ we have that $V(B) \leq W(B) \leq Q_M(r) - \theta_2$. For $B \in \mathscr{B}_1$ we get that

$$\begin{aligned} V(B) &\leq 1 + \sum_{j=2}^{i} \frac{1}{t_j(r) - 1} + \frac{M+1}{M\big(t_{i+1}(r) - 1\big)} \\ &= Q_M(r) - \frac{1}{M(M-1)\big(t_{i+1}(r) - 1\big)} = Q_M(r) - \theta_1. \end{aligned}$$

Using the fact that $\theta_1 < \theta_2$, we get

$$\begin{aligned} HF_M(L) &\le \frac{n_M}{M^2} + \sum_{B \in \mathscr{B}_1} V(B) + \sum_{B \in \mathscr{B}_2} V(B) + (M - r + 1) \\ &\le \frac{\delta k}{M} + |\mathscr{B}_1|(Q_M(r) - \theta_1) + |\mathscr{B}_2|(Q_M(r) - \theta_2) + (M - r + 1) \\ &\le kQ_M(r) - k\theta_1 + \frac{\delta k}{M} + (M - r + 1). \end{aligned}$$

Since $\theta_1 - \delta/M \ge 2\varepsilon$, dividing both sides by $OPT(L) = k$ gives us that $HF_M(r)/OPT(L) \le Q_M(r) - \varepsilon$.

4. THE CASE $M = 4$ AND $r = 1$

In the previous section we have shown that the upper bound $Q_M(r)$ is not tight for many values of M and r. In this section we will come up with the asymptotic worst case ratio for an interesting case that remains: $M = 4$ and $r = 1$. Before we start with the worst case analysis of HF_4, we will first discuss the Next Fit packings of items from $(0, 1/M]$.

Let $L = (a_1, \dots, a_p)$ be a list of items with size smaller than or equal to $\frac{1}{2}$. Let $a_{(i)}$ denote the ith biggest item of L. So, we have $a_{(1)} \ge a_{(2)} \ge \cdots \ge a_{(p)}$. We say that list L satisfies the *Next Fit Maximality* condition, if and only if

$$NF(L) \ge s(L) + \sum_{i=1}^{NF(L)-1} s(a_{(i)}).$$

If a list L does not satisfy this condition, then one can construct a new list L' from L by reordering and splitting up of items such that $NF(L') > NF(L)$. Note that the optimal number of bins can never increase by applying these operations to the list.

As an example, consider the list $L = (0.41, 0.35, 0.30, 0.25, 0.25, 0.25, 0.25, 0.21)$. This list does not satisfy the Next Fit Maximality condition, since $NF(L) = 3$, $s(L) = 2.27$, and $s(a_{(1)}) + s(a_{(2)}) = 0.76$. If we split the item of size 0.21 into one item of size 0.11 and one item of size 0.10, then we can reorder the items such that we get the list $L' = (0.25, 0.25, 0.10, 0.41, 0.25, 0.35, 0.25, 0.11, 0.30)$ with $NF(L') = 4$.

For a given list L, let k be defined by

$$k = \min\left\{ l \mid l \ge s(L) + \sum_{i=1}^{l-1} s(a_{(i)}) \right\}.$$

We will show that we can always construct a new list L' from L such that $NF(L') = k$. We will construct L' by specifying the Next Fit packing of L'.

Since k is the minimal number, for which the condition holds, we have

$$k - 1 < s(L) + \sum_{i=1}^{k-2} s(a_{(i)}).$$

We take ε and δ as

$$\varepsilon = \min\left\{\left(s(L) + \sum_{i=1}^{k-2} s(a_{(i)}) - k + 1\right)\Big/(k-1), s(a_{(k-1)})\right\}$$

$$\delta = s(L) + \sum_{i=1}^{k-2} s(a_{(i)}) - k + 1 - (k-1)\varepsilon.$$

The Next Fit packing of the modified list is constructed as follows. First, item $a_{(i)}$ is placed as the first item in bin $i+1$, $1 \le i \le k-1$. Then the remaining items are split up and distributed over the bins such that bin i, $1 \le i \le k-1$, receives a total size of $1 - s(a_{(i)}) + \varepsilon$, and that bin k receives $s(a_{(k-1)}) + \delta$. This gives us a valid Next Fit packing and thus a construction of L'. From the definition of k it immediately follows that L' satisfies the Next Fit Maximality condition.

In our worst case analysis of HF_4 we will make use of the fact that if

$$k - 1 < s(L) + \sum_{i=1}^{k-2} s(a_{(i)})$$

holds for a list L, we can construct a list L' from L with $NF(L') \ge k$.

Now we will return to the worst case analysis of HF_4. In order to give an upper bound for $R^{\infty}_{HF_4}(1)$, we will first prove some lemmas that help us to exclude lists from our analysis.

LEMMA 6. *From every list L we can construct a list L' that satisfies the conditions*

(a) $OPT(L') = OPT(L)$,

(b) $HF_4(L') = HF_4(L)$,

(c) *no bin in the optimal packing of L' contains* 2 *items of size* $(\frac{1}{3}, \frac{1}{2}]$.

Proof. We will give such a construction. Suppose that there is a bin in the optimal packing of list L that contains two items of size $(\frac{1}{3}, \frac{1}{2}]$. Then we can replace these two items by one item of their combined size. Since the number of bins that HF_4 uses to pack items of size $(\frac{1}{4}, 1]$ only depends on the number of items in every subinterval, it fills one bin less of type I_2, and one extra bin of type I_1. Of course, our adjustment of the list does not change the optimal number of bins. ∎

LEMMA 7. *From every list L we can construct a list L′ that satisfies the conditions*

(a) $OPT(L') \leq OPT(L)$,

(b) $HF_4(L') \geq HF_4(L)$,

(c) *L' contains at most two items of size* $(\frac{1}{4}, \frac{1}{3}]$.

Proof. Suppose that there are three items of, respectively, size $\frac{1}{4} + \varepsilon_1$, $\frac{1}{4} + \varepsilon_2$, and $\frac{1}{4} + \varepsilon_3$ $(0 < \varepsilon_1, \varepsilon_2, \varepsilon_3 \leq \frac{1}{12})$. Let us denote the items of L that are smaller than or equal to $\frac{1}{4}$ by $L_4 = (b_1, \dots, b_q)$, and let us denote the number of I_4-bins that HF_4 uses by $k = NF(L_4)$. Clearly, k satisfies

$$k - 1 < s(L_4) + \sum_{i=1}^{k-2} s(b_{(i)})$$

We remove the three items with size in $(\frac{1}{4}, \frac{1}{3}]$ from the list and add items of, respectively, size $\frac{1}{4}, \varepsilon_1, \frac{1}{4}, \varepsilon_2, \frac{1}{4}, \varepsilon_3$ to the list. Let us denote the new list by L' and let us denote by $L'_4 = (b'_1, \dots, b'_{q+6})$ the items from $(0, \frac{1}{4}]$ on the new list. One immediately sees that

$$s(L'_4) = s(L_4) + \frac{3}{4} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\sum_{i=1}^{k-1} s(b'_{(i)}) \geq \sum_{i=1}^{k-2} s(b_{(i)}) + \frac{1}{4}.$$

This gives us

$$s(L'_4) + \sum_{i=1}^{k-1} s(b'_{(i)}) \geq s(L_4) + \sum_{i=1}^{k-2} s(b_{(i)}) + 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 > k$$

and so L'_4 can be further modified such that Next Fit will need at least $k + 1$ bins. Since we need one bin less of type I_3 and at least one bin more of type I_4, we conclude that $HF_4(L') \geq HF_4(L)$. Since division of items into smaller items cannot increase the optimal number of bins, we get that $OPT(L') \leq OPT(L)$. Repeating this procedure as long as there are at least three items from $(\frac{1}{4}, \frac{1}{3}]$, we end up with a list L' that contains at most two items from $(\frac{1}{4}, \frac{1}{3}]$. ∎

LEMMA 8. *From every list L we can construct a list L′ that satisfies the conditions*

(a) $OPT(L') \leq OPT(L)$,

(b) $HF_4(L') \geq HF_4(L)$,

(c) *In the optimal packing of list L' there is at most one bin that contains one item of* $(\frac{1}{3}, \frac{1}{2}]$ *together with items of* $(0, \frac{1}{4}]$ *only.*

Proof. As long as there are two bins in the optimal packing of L that contain both one item from $(\frac{1}{3}, \frac{1}{2}]$ together with items that are smaller than or equal to $\frac{1}{4}$, we apply the following procedure. Let us denote that two items from $(\frac{1}{3}, \frac{1}{2}]$ by a_1 and a_2. We replace these two items by one *large* item of size $s(a_1) + s(a_2) - \frac{1}{6}$ and a *small* item of size $\frac{1}{6}$. It is clear that HF_4 will need an extra bin to pack the items of $(\frac{1}{2}, 1]$ and one bin less to pack the items of $(\frac{1}{3}, \frac{1}{2}]$. In order to maintain a feasible packing with $OPT(L)$ bins, we may have to split one of the items, say a_i, from $(0, \frac{1}{4}]$ into two smaller items: a_{i1} and a_{i2}. If we replace a_i on list L by $\{\frac{1}{6}, a_{i1}, a_{i2}\}$ (assume $s(a_{i1}) \geq s(a_{i2})$), it follows from $\frac{1}{6} + s(a_{i1}) \geq s(a_i)$ that HF_4 will need at least as many I_4-bins as before. So, $HF_4(L')$ is greater than or equal to $HF_4(L)$. ∎

LEMMA 9. *From every list L we can construct a list L' that satisfies the conditions*

(a) $OPT(L') \leq OPT(L)$,

(b) $HF_4(L') \geq HF_4(L)$,

(c) *In the optimal packing of L', there is no bin that contains items from* $(0, \frac{1}{4}]$ *only.*

Proof. To every bin in the optimal packing of L that contains items from $(0, \frac{1}{4}]$ only, we apply the following procedure. Let us denote the items of such a bin by $c_1, \ldots, c_p$. Let $L_4 = (b_1, \ldots, b_q)$ denote the items of L that have size in $(0, \frac{1}{4}]$, and let $k = NF(L_4)$. Because we have a feasible packing of L_4 in k bins, the inequality

$$k - 1 < s(L_4) + \sum_{i=1}^{k-2} s\big(b_{(i)}\big)$$

holds. We will denote the gap in this inequality by δ. We make the following adjustments to L.

We choose ε such that $0 < \varepsilon < \delta/2$. The items $c_1, \ldots, c_p$ are removed from L, while we add one item of size $\frac{1}{2} + \varepsilon$ and two items of size $(1/4) - (\varepsilon/2)$ to list L. This gives us the list L', and we use $L'_4 = (b'_1, \ldots, b'_{q-p+2})$ to denote the items smaller than or equal to $\frac{1}{4}$ in the new list. Clearly, HF_4 will need an extra bin of type I_1 to pack the extra item of size $\frac{1}{2} + \varepsilon$. So, we need to show that we can modify L'_4 such that Next Fit will need at least $k - 1$ bins. One can easily verify that

$$s(L'_4) \geq s(L_4) - \left(\frac{1}{2} + \varepsilon\right)$$
$$\sum_{i=1}^{k-3} s\left(b'_{(i)}\right) \geq \sum_{i=1}^{k-2} s\left(b_{(i)}\right) - \left(\frac{1}{2} + \varepsilon\right).$$

This gives us that

$$\begin{aligned}\sum_{i=1}^{k-3} s\left(b'_{(i)}\right) + s(L'_4) &\geq \sum_{i=1}^{k-2} s\left(b_{(i)}\right) + s(L_4) - 1 - 2\varepsilon \\ &= k - 2 - 2\varepsilon + \delta \\ &> k - 2.\end{aligned}$$

From this we conclude that L'_4 can be modified to yield $NF(L'_4) \geq k - 1$.

It is clear that list L' allows a feasible packing in $OPT(L)$ bins. Therefore, we get that $OPT(L') \leq OPT(L)$. ∎

If we apply the reconstructions from the above proofs consecutively to an arbitrary list L, then we get a list L' with

$$\frac{HF_4(L')}{OPT(L')} \geq \frac{HF_4(L)}{OPT(L)}$$

and that satisfies the conditions

1. No bin in the optimal packing of L' contains two items with size in $(\frac{1}{3}, \frac{1}{2}]$.
2. L' contains at most two items of size $(\frac{1}{4}, \frac{1}{3}]$.
3. In the optimal packing of list L' there is at most one bin that contains one item with size in $(\frac{1}{3}, \frac{1}{2}]$ together with items from $(0, \frac{1}{4}]$ only.
4. In the optimal packing of list L', there is no bin that contains items from $(0, \frac{1}{4}]$ only.

Because we are interested in the *asymptotic* worst case behavior of HF_4, we may disregard the one or two items from $(\frac{1}{4}, \frac{1}{3}]$ and the bin in the optimal packing that contains one item of $(\frac{1}{3}, \frac{1}{2}]$ together with items from $(0, \frac{1}{4}]$ only. This gives us the following corollary:

COROLLARY 10. *In order to investigate the asymptotic worst case behavior of HF_4 we only need to consider lists L that satisfy*

(a) *L contains no items from* $(\frac{1}{4}, \frac{1}{3}]$,

(b) *every bin in the optimal packing of L contains* 1 *item from* $(\frac{1}{2}, 1]$.

This enables us to prove the following upper bound for HF_4:

THEOREM 11.

$$R^{\infty}_{HF_4}(1) \leq \tfrac{12}{7}.$$

Proof. Consider a list L with $OPT(L) = n$. Let n_j denote the number of I_j-items in list L. We split the interval $I_4 = (0, \frac{1}{4}]$ in the following two intervals: $I_{4a} = (\frac{1}{6}, \frac{1}{4}]$ and $I_{4b} = (0, \frac{1}{6}]$ and use n_{4a} to denote the number of items in I_{4a}. Due to Corollary 10 we have $n_1 = n$ and $n_3 = 0$. Every bin in the optimal packing of L can contain at most two items from I_{4a}, given the fact that $n_1 = n$. However, if a bin contains an item from I_2, then it cannot contain any item from I_{4a}. From this we get that $n_{4a} \leq 2(n - n_2)$.

Let m_j denote the number of I_j-bins in the packing produced by HF_4. Clearly, $m_1 = n_1$ and $m_2 = \lceil n_2/2 \rceil$. Further, we will denote the total size of I_4-items by S_4. S_4 satisfies

$$S_4 < n - \frac{n_1}{2} - \frac{n_2}{3} = \frac{n}{2} - \frac{n_2}{3}.$$

We consider two cases in order to bound $HF_4(L)$:

Case 1. $n_2 \leq \frac{6}{7}n$. Since every I_4-bin (except possibly the last) is at least $\frac{3}{4}$ full, we get

$$\begin{aligned} HF_4(L) &= m_1 + m_2 + m_4 \leq n + \left\lceil \frac{n_2}{2} \right\rceil + \left\lceil \frac{4}{3} S_4 \right\rceil \\ &\leq n + \frac{n_2}{2} + \frac{4}{3}\left(\frac{n}{2} - \frac{n_2}{3} \right) + 2 = \frac{5}{3} n + \frac{1}{18} n_2 + 2 \\ &\leq \frac{12}{7} n + 2. \end{aligned}$$

Case 2. $n_2 > \frac{6}{7}n$. If $n_{4a} = 0$ then every I_4-bin (except possibly the last) is at least $\frac{5}{6}$ full and thus $m_4 \leq \lceil \frac{6}{5} S_4 \rceil$. Every item in I_{4a} can cause one I_4-bin to be less than $\frac{5}{6}$ full, but still it holds that this I_4-bin is more than $\frac{3}{4}$

full. Therefore, we can bound the number of bins as

$$\begin{aligned} HF_4(L) &= m_1 + m_2 + m_4 \le n + \left\lceil \frac{n_2}{2} \right\rceil + n_{4a} + \left\lceil \frac{6}{5}\left(S_4 - \frac{3}{4} n_{4a} \right) \right\rceil \\ &\le n + \frac{n_2}{2} + n_{4a} + \frac{6}{5}\left(S_4 - \frac{3}{4} n_{4a} \right) + 2 \\ &= n + \frac{n_2}{2} + \frac{1}{10} n_{4a} + \frac{6}{5} S_4 + 2 \\ &\le n + \frac{n_2}{2} + \frac{1}{5}(n - n_2) + \frac{6}{5}\left(\frac{1}{2} n - \frac{1}{3} n_2 \right) + 2 \\ &= \frac{9}{5} n - \frac{1}{10} n_2 + 2 \\ &\le \frac{12}{7} n + 2. \end{aligned}$$

So, in both cases we have that $HF_4(L) \le \frac{12}{7} OPT(L) + 2$. ▮

In order to prove the tightness of this upper bound for $R^{\infty}_{HF_4}(1)$, we will provide lower bounds for the a.w.c.r. of Harmonic Fit for several values of M and r that include the case $M = 4$ and $r = 1$.

THEOREM 12. *If $i \ge 2$, $2 \le m \le t_i(r) - 1$, and $M = m(t_i(r) - 1)$, then*

$$\begin{aligned} R^{\infty}_{HF_M}(r) \ge 1 &+ \sum_{k=2}^{i-1} \frac{1}{t_k(r) - 1} \\ &+ \frac{m(t_i(r))^2 - 2t_i(r) - m}{m(t_{i+1}(r) - 1) - 2t_i(r) + 1} \frac{1}{t_i(r) - 1}. \end{aligned}$$

Proof. Let n be a multiple of $m(t_{i+1}(r) - 1) - 2t_i(r) + 1$, and let $\varepsilon > 0$ and $\delta > 0$ be some suitable small numbers. We take

$$n_x = \frac{m}{m(t_{i+1}(r) - 1) - 2t_i(r) + 1} n.$$

Let L be a concatenation of $i + n_x$ sublists: $L = L_1 \cdots L_i L_{x1} \cdots L_{xn_x}$. L_1 contains nr items of size $1/(r+1) + \varepsilon/ri$; L_k, $2 \le k \le i - 1$, contains n items of size $1/t_k(r) + \varepsilon/i$; L_i contains $n - n_x/m$ items of size $1/t_i(r) + \varepsilon/i$, and every list L_{xj} contains one item of size $1/m(t_i(r) - 1) - \varepsilon/m$ followed by $m(t_{i+1}(r) - 1) - 2t_i(r)$ items of size $1/m(t_{i+1}(r) - 1) - \varepsilon/m$ and one item of size δ.

We can always choose ε and δ such that $\delta > (\varepsilon/m)(2 + m(t_{i+1}(r) - 1) - 2t_i(r))$. Under this condition we have that

$$s(L_{xj}) + \frac{1}{m(t_i(r) - 1)} - \frac{\varepsilon}{m} > 1.$$

HF_M packs list L_1 in n bins; it packs list L_k, $2 \le k \le i - 1$, in $n/(t_k(r) - 1)$ bins; it packs list L_i in $(n - (n_x/m)/(t_i(r) - 1)$ bins; and it packs every list L_{xj} in a separate bin. So,

$$HF_M(L) = n + \sum_{k=2}^{i-1} \frac{n}{t_k(r) - 1} + \frac{n - n_x/m}{t_i(r) - 1} + n_x$$

$$= \left(1 + \sum_{k=2}^{i-1} \frac{1}{t_k(r) - 1} + \frac{m(t_i(r))^2 - 2t_i(r) - m}{m(t_{i+1}(r) - 1) - 2t_i(r) + 1} \times \frac{1}{t_i(r) - 1}\right) n.$$

The optimal solution is constructed as follows. We pack $n - n_x/m$ bins each with r items of L_1 and one item of L_k for all $2 \le k \le i$ and m items of size $(1/m(t_{i+1}(r) - 1)) - (\varepsilon/m)$, and we pack n_x/m bins each with r items of L_1 and one item of L_k for all $2 \le k \le i - 1$ and m items of size $(1/m(t_i(r) - 1)) - (\varepsilon/m)$. After packing these n bins, only n_x items of size δ remain. These items can be packed in a single bin (take $\delta \le 1/n_x$), so the optimal packing uses at most $n + 1$ bins.

We can take n arbitrarily large, so the desired result follows. ∎

In order to apply this theorem on our case of $M = 4$ and $r = 1$, we must take $i = 2$ and $m = 2$. Evaluation of the formula gives us that

$$R^{\infty}_{HF_4}(1) \ge 1 + \frac{2(t_2(1))^2 - 2t_2(1) - 2}{2(t_3(1) - 1) - 2t_2(1) + 1} \frac{1}{t_2(1) - 1} = \frac{12}{7}.$$

Applying this theorem to other cases of M and r yields $R^{\infty}_{HF_{12}}(1) \ge 721/426$, $R^{\infty}_{HF_6}(2) \ge 73/51$, $R^{\infty}_{HF_9}(2) \ge 124/87$, $R^{\infty}_{HF_8}(3) \ge 81/62$, and $R^{\infty}_{HF_{12}}(3) \ge 133/102$ among others.

5. THE CASE $M = 5$ AND $r = 1$

In this section we will prove an asymptotic worst case ratio of 1.7 for the case $M = 5$ and $r = 1$. This implies that $R^{\infty}_{HF_5}(1) = R^{\infty}_{HF_6}(1)$, which means that the worst case ratio of HF_M is not strictly decreasing with M. Since the worst case example for $M = 6$ in [4] does not contain items in the interval I_5, it is also valid for $M = 5$. Therefore we only need to prove that

$R^{\infty}_{HF_5}(1) \le 1.7$. Similar to the upper bound proof of $R^{\infty}_{HF_4}(1)$, we will first prove some lemmas that help us to exclude lists from our analysis.

LEMMA 13. *From every list L we can construct a list L′ that satisfies the conditions*

(a) $OPT(L') = OPT(L)$,

(b) $HF_5(L') \ge HF_5(L)$,

(c) *At most one bin in the optimal packing of L′ contains two items of size* $(\frac{1}{4}, \frac{1}{2}]$.

Proof. Suppose that there is a bin in the optimal packing of list L that contains two items of size $(\frac{1}{3}, \frac{1}{2}]$. Then we can replace these two items by one item of their combined size. This does not change the optimal number of bins and the number of bins used by HF_5.

Suppose that there is a bin in the optimal packing of list L that contains two or three items from $(\frac{1}{4}, \frac{1}{3}]$. If we replace these items by one item of their combined size, this does not change the optimal number of bins and the number of bins used by HF_5 will not decrease.

Let there be two bins in the optimal packing of list L that contain one item from $(\frac{1}{3}, \frac{1}{2}]$ and one item from $(\frac{1}{4}, \frac{1}{3}]$. In both bins we replace those two items by one item of their combined size. This does not change the optimal number of bins and the number of bins used by HF_5 will not decrease.

In this way we may leave at most one bin with one item of $(\frac{1}{3}, \frac{1}{2}]$ and one item from $(\frac{1}{4}, \frac{1}{3}]$ unchanged. ∎

LEMMA 14. *From every list L we can construct a list L′ that satisfies the conditions*

(a) $OPT(L') \le OPT(L)$,

(b) $HF_5(L') \ge HF_5(L)$,

(c) *L′ contains at most three items of size* $(\frac{1}{5}, \frac{1}{4}]$.

Proof. Analogously to the proof of Lemma 7 we replace every four items of, respectively, size $\frac{1}{5} + \varepsilon_1$, $\frac{1}{5} + \varepsilon_2$, $\frac{1}{5} + \varepsilon_3$, and $\frac{1}{5} + \varepsilon_4$ $(0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \le \frac{1}{20})$, by eight items of respectively size $\frac{1}{5}$, ε_1, $\frac{1}{5}$, ε_2, $\frac{1}{5}$, ε_3, $\frac{1}{5}$, ε_4. ∎

LEMMA 15. *From every list L we can construct a list L′ that satisfies the conditions*

(a) $OPT(L') \le OPT(L)$,

(b) $HF_5(L') \ge HF_5(L)$,

(c) *In the optimal packing of list L′ there is at most one bin that contains one item of* $(\frac{1}{3}, \frac{1}{2}]$ *together with items of* $(0, \frac{1}{5}]$ *only.*

(d) *In the optimal packing of list L' there are at most two bins that contain one item of* $(\frac{1}{4}, \frac{1}{3}]$ *together with items of* $(0, \frac{1}{5}]$ *only*.

Proof. As long as there are two bins in the optimal packing of L' that contain one item from $(\frac{1}{3}, \frac{1}{2}]$ together with items from $(0, \frac{1}{5}]$ only, we will adjust the items just like we did in the Proof of Lemma 8.

As long as there are three bins that contain one item from $(\frac{1}{4}, \frac{1}{3}]$ together with items from $(0, \frac{1}{5}]$ only, we apply the following procedure.

Let us denote that three items from $(\frac{1}{4}, \frac{1}{3}]$ by a_1, a_2, and a_3. We replace these three items by one *large* item of size $s(a_1) + s(a_2) + s(a_3) - \frac{1}{4}$ and two *small* items of size $\frac{1}{8}$. It is clear that HF_5 will need an extra bin to pack the items of $(\frac{1}{2}, 1]$ and one bin less to pack the items of $(\frac{1}{4}, \frac{1}{3}]$. In order to maintain a feasible packing with $OPT(L)$ bins, we may have to split two items, say a_i and a_j, from $(0, \frac{1}{5}]$ into two smaller items: a_{i1} and a_{i2}, and a_{j1} and a_{j2}. We replace a_i on list L by $\{\frac{1}{8}, a_{i1}, a_{i2}\}$ (assume $s(a_{i1}) \geq s(a_{i2})$), and a_j by $\{\frac{1}{8}, a_{j1}, a_{j2}\}$ (assume $s(a_{j1}) \geq s(a_{j2})$). Then it follows from $\frac{1}{8} + s(a_{i1}) \geq s(a_i)$ and $\frac{1}{8} + s(a_{j1}) \geq s(a_j)$ that HF_5 will need at least as many bins as before to pack the items of $(0, \frac{1}{5}]$. So, $HF_5(L')$ is greater than or equal to $HF_5(L)$.

Our construction gives us a feasible packing of list L' in $OPT(L)$ bins. So, $OPT(L') \leq OPT(L)$.

LEMMA 16. *From every list L we can construct a list L' that satisfies the conditions*

(a) $OPT(L') \leq OPT(L)$,

(b) $HF_5(L') \geq HF_5(L)$,

(c) *In the optimal packing of $OPT(L')$, there is no bin that contains items of* $(0, \frac{1}{5}]$ *only*.

Proof. Similar to Lemma 9. ∎

These lemmas allow us to conclude that:

COROLLARY 17. *In order to investigate the asymptotic worst case behavior of HF_5 we only need to consider lists L that satisfy*

(a) *L contains no items from* $(\frac{1}{5}, \frac{1}{4}]$,

(b) *every bin in the optimal packing of L contains* 1 *item from* $(\frac{1}{2}, 1]$.

This corollary helps us to prove the following result:

THEOREM 18.

$$R^{\infty}_{HF_5}(1) \leq \tfrac{17}{10}.$$

Proof. Consider a list L with $OPT(L) = n$. Let n_j denote the number of I_j-items in list L. We split the interval $I_5 = (0, \frac{1}{5}]$ into the two intervals

$I_{5a} = (\frac{1}{6}, \frac{1}{5}]$ and $I_{5b} = (0, \frac{1}{6}]$ and use n_{5a} to denote the number of items in I_{5a}. Due to Corollary 17 we have $n_1 = n$ and $n_4 = 0$.

Every bin in the optimal packing contains one item from I_1. Therefore, every bin can contain at most one job from I_2 or at most two jobs from $I_3 \cup I_{5a}$. This gives us the following inequality that bounds the number of items in I_2, I_3, and I_{5a}:

$$n_2 + \tfrac{1}{2}n_3 + \tfrac{1}{2}n_{5a} \le n.$$

The total size of I_5-items is denoted by S_5 and satisfies

$$S_5 \le \frac{n}{2} - \frac{n_2}{3} - \frac{n_3}{4}.$$

Let m_j denote the number of I_j-bins in the packing produced by HF_5. The straightforward way to bound m_5 is to use $m_5 \le \lceil \frac{5}{4}S_5 \rceil$. Taking into account the number of I_{5a}-items, however, we get

$$\begin{aligned} m_5 &\le n_{5a} + \left\lceil \tfrac{6}{5}\left(S_5 - \tfrac{4}{5}n_{5a}\right)\right\rceil \\ &\le \tfrac{1}{25}n_{5a} + \tfrac{6}{5}S_5 + 1 \end{aligned}$$

using a similar argument as was used in the Proof of Theorem 11.

Putting things together, we get

$$\begin{aligned} HF_5(L) &= m_1 + m_2 + m_3 + m_5 \\ &\le n + \frac{n_2}{2} + \frac{n_3}{3} + \frac{1}{25}n_{5a} + \frac{6}{5}S_5 + 3 \\ &\le n + \frac{n_2}{2} + \frac{n_3}{3} + \frac{n_{5a}}{25} + \frac{6}{5}\left(\frac{n}{2} - \frac{n_2}{3} - \frac{n_3}{4}\right) + 3 \\ &= \frac{16}{10}n + \frac{1}{10}n_2 + \frac{1}{30}n_3 + \frac{1}{25}n_{5a} + 3 \\ &\le \frac{16}{10}n + \frac{1}{10}\left(n_2 + \frac{1}{2}n_3 + \frac{1}{2}n_{5a}\right) + 3 \\ &\le \frac{17}{10}n + 3. \end{aligned}$$

So, we conclude that $R^{\infty}_{HF_5}(1) \le \frac{17}{10}$. ∎

6. CONCLUSION

We have shown for which values of M and r the upper bound $Q_M(r)$ is tight. The most interesting cases for which this upper bound is not tight are $M = 4$ and $M = 5$ for the nonparametric case ($r = 1$). For these cases we proved an asymptotic worst case ratio of respectively $\frac{12}{7}$ and 1.7.

In Theorem 12 we provided lower bounds for some special values of M and r. It is not too difficult to construct bad lists for other cases as well. Some routine work gives us

$$R^{\infty}_{HF_M}(1) \geq \frac{365M - 420}{216M - 252} \quad \text{for } M = 8, 9, 10, 11;$$

$$R^{\infty}_{HF_5}(2) \geq \frac{142}{99}; \; R^{\infty}_{HF_M}(2) \geq \frac{89M - 96}{63M - 72} \quad \text{for } M = 7, 8;$$

$$R^{\infty}_{HF_M}(3) \geq \frac{83M - 100}{64M - 80} \quad \text{for } M = 6, 7;$$

and

$$R^{\infty}_{HF_M}(3) \geq \frac{187M - 200}{144M - 160} \quad \text{for } M = 9, 10, 11.$$

In order to provide better upper bounds than $Q_M(r)$, a very detailed analysis may be needed. Since the remaining gaps between our lower bounds and $Q_M(r)$ are not too large, we leave them as they are.

We have summarized the values (or lower bounds) for $R^{\infty}_{HF_M}(r)$ and the values for the upper bound $Q_M(r)$ in Table 2 for the most interesting cases of M and r. Indeed, we were not able to construct examples for $M = 10, 11$ and $r = 2$ that gave us a lower bound better than $1.4242\ldots$.

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