TI 2003-035/1
Tinbergen Institute Discussion Paper

Strategic Targeted Advertising

Andrea Galeotti[1,3]
José Luis Moraga-Gonzalez[2,3]

[1] *Econometric Institute,,*
[2] *Department of Economics, ,*
[3] *Faculty of Economics, Erasmus University Rotterdam, and Tinbergen Institute.*

Tinbergen Institute
The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam, and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Roetersstraat 31
1018 WB Amsterdam
The Netherlands
Tel.: +31(0)20 551 3500
Fax: +31(0)20 551 3555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900
Fax: +31(0)10 408 9031

Please send questions and/or remarks of non-scientific nature to driessen@tinbergen.nl.
Most TI discussion papers can be downloaded at http://www.tinbergen.nl.

Strategic Targeted Advertising*

Andrea Galeotti† José Luis Moraga-González‡

First version: May 10, 2003

Abstract

We present a strategic game of pricing and targeted-advertising. Firms can simultaneously target price advertisements to different groups of customers, or to the entire market. Pure strategy equilibria do not exist and thus market segmentation cannot occur surely. Equilibria exhibit random advertising –to induce an unequal distribution of information in the market– and random pricing –to obtain profits from badly informed buyers–. We characterize a positive profits equilibrium where firms advertise low prices to a segment of consumers, high prices to a distinct segment of consumers, and intermediate prices to the entire market. As a result the market is segmented only from time to time and presents substantial price dispersion across segments.

JEL Classification: D43, D83

Keywords: targeted advertising, oligopoly, price dispersion, segmented markets.

*We thank Stefan Buehler, Sanjeev Goyal, Luis Ubeda and the seminar participants at Alicante, CORE (Belgium) and Erasmus for their comments.

†Tinbergen Institute and Econometric Institute, Erasmus University Rotterdam. E-mail: galeotti@few.eur.nl

‡Department of Economics, Erasmus University Rotterdam, Tinbergen Institute and CESifo. E-mail: moraga@few.eur.nl

1 Introduction

Advertising technologies have evolved a great deal in recent years. Sellers currently have at their disposal a vast amount of distinct advertising media that enable them to target their ads to specific sets of customers with accuracy.[1] By targeting price advertisements to consumers who are not in the target set of the rivals, firms can safely appropriate a great deal of the consumers surplus. In this paper we investigate how firms will use targeted advertising in a strategic context and to what extent targeted advertising can lead to the segmentation of the product market.

We present a strategic game of pricing and targeted-advertising. The population of consumers is segmented in regard to the advertising media through which they can be reached by the sellers. Firms sell a homogeneous product and simultaneously decide which price to charge and to which consumer segment to target their costly price-advertisements.[2] We are interested in the Nash equilibria of this game in which firms obtain strictly positive profits. We show that pure-strategy equilibria do not exist and thus an equilibrium where firms surely send their ads to distinct groups of buyers cannot be sustained. Interestingly, equilibria exhibit both random pricing and random advertising. Firms employ random advertising to induce an unequal distribution of information in the market, which gives firms potential market power. Random pricing enables a firm to realize market power by overcoming detrimental undercutting behavior by rival firms.

In the simplest version of our model, two groups of consumers A and B of possibly different sizes each reads a distinct newspaper. A firm can target its price ads to one of the groups, say A, by placing ads in newspaper A; alternatively, it can target its ads to the other group of consumers by placing ads in the other newspaper; finally, a firm may decide to announce its price to the entire market by placing ads in both newspapers. The main results we obtain are summarized as follows. First, we find a unique equilibrium in which firms obtain positive profits by using targeted price advertising. We note that if firms were unable to target ads to different groups of buyers, firms profits would be zero in the unique equilibrium. Thus, our result is important since it shows that just the minimum amount of segmentation enables firms to make profits in highly competitive environments. In equilibrium firms randomize between advertising low prices in one of the consumer segments, advertising high prices in the other consumer segment, and advertising intermediate

[1] For example, in recent years sellers operating in countries with several official languages such as Belgium or Spain –or in the US where the spanish speaking community is sizeable– can target their advertising campaigns to a group of buyers who speak a particular language by inserting commercials in TV channels which only broadcast in that language, or by inserting ads in newspapers and magazines written in that language.

[2] Since we deal with a homogeneous good market, this implies that buyers can be segmented according to some attribute –e.g., mother language, gender, location, marital status, age, education level, etc. – which does not have a bearing on their willingness to pay for the products. In other words, consumers are segmented in terms of their exposure to the different media and not in the way they value distinct firms' offerings.

prices in the entire market. As a result, genuine market segmentation –in the sense that one of the consumer groups is always served by one firm and the other group is always served by the other firm– never arises. Instead the market is segmented only from time to time. Remarkably, our model does not generate 'sponsored' sales because low prices are not advertised heavily but intermediate prices. It is finally worth noting that an unequal distribution of information arises in equilibrium as a result of strategic considerations. This differs from previous models of price advertising where informational segmentation derives from imperfections of the advertising technology in the sense that it is prohibitively costly that a given consumer is reached by a firm with probability one (see e.g. Butters, 1977; Grossman and Shapiro, 1984; and Stahl, 1994).

Second, we discuss the comparative statics results of parameter changes. One, we find that an increase in advertising fees makes market segmentation more likely: firms increase the probability with which they target ads to the distinct consumer segments and decrease the probability with which they send ads to the entire market. Interestingly, this results in a profits increase when the advertising cost is low to begin with, and in a profits fall otherwise. The intuition behind the fact that profits are non-monotonic in advertising costs is as follows. An increase in advertising costs makes competition for the entire market a relatively unattractive strategy compared to competition for the segments. Since a firm must be indifferent between the three advertising strategies in equilibrium, firms decrease the frequency with which they compete for the entire market. At the same time, in an attempt to overcome rival's undercutting behavior, firms increase the dispersion of advertised prices in the segments. These effects result in a weakening of competitiveness and in an increase in revenues. When fees are low to begin with, this revenue effect outweighs the cost effect and firms' profits rise. Finally, we also see that as advertising fees go to zero, the extent to which segmentation arises converges to zero, the distribution of advertised prices converges to a price distribution that is degenerate at the marginal cost and firms profits converge to zero.

Two, an increase in the difference between the two newspapers' advertising fees may be beneficial for the firms. When advertising fees difference rises, firms increase the probability with which they advertise in the cheapest newspaper, and decrease the probability with which they advertise in the entire market, while the probability of advertising in the other segment is unaltered. At the same time, in an attempt to weaken competition, firms increase price dispersion in the segment that is most competitive. Profits are found to be non-monotonic in the difference between advertising fees. Essentially, what happens is that if cost asymmetries are quite large to start with, an increase in fees difference makes it very unattractive to advertise in the most expensive newspaper and firms end up competing very aggressively for just one of the consumer segments, which results in lower profits. If fees asymmetries are small initially, firms benefit from an increase in fees differences because competition for the entire market occurs less frequently.

Three, we discuss how a change in the customer distribution across newspapers affects our results. We note that this parameter change does not impact the cost side of the problem. An increase in the number of consumers reading one of the newspapers makes advertising in that newspaper more attractive. Equilibrium requires firms to increase the probability with which they advertise in the newspaper with the largest readership, which makes it less attractive in turn and firms increase the dispersion of advertised prices in such newspaper. Again firms profits are non-monotonic in customer distribution asymmetries across newspapers.

In a first extension of the basic model we examine the implications of allowing for a fraction of consumers who read both newspapers. These consumers are always fully informed, in the sense that given any firms' advertising-strategy profile, they always observe the prices charged by the two firms. We show that the equilibrium of the benchmark model discussed above survives this change in the distribution of consumers across newspapers. Interestingly we find that an increase in the number of consumers who read both newspapers may be beneficial for the firms. The reason is that the frequency with which firms advertise their prices in the entire market declines as the fraction of these new consumers rises. As a result market competitiveness weakens and firms' profits may increase. In a second extension of the basic model we allow for entry of firms. We show that the random advertising and pricing equilibrium described above exists also in a N-firms oligopoly. Moreover, we find that equilibrium profits decline as the number of firms the market hosts rises. Finally, we show that if firms did operate in a setting where they could announce distinct prices to the different consumer segments when they advertise in the entire market, then firms profits would be zero in equilibrium. This suggests that firms benefit from restrictions on price discrimination when they compete in segmented markets.

The economics literature about informative *price* advertising has in general ignored the possibility that firms can target their advertisements to distinct consumer groups.[3] The only papers we know where firms can use targeted advertising in a strategic context are Bester and Petrakis (1995), Iyer *et al.* (2002), and Roy (2000).[4] The first two papers model price-advertising as a short-run variable, while Roy (2000) models product-advertising as a long-run variable. Bester and Petrakis (1995) study a model where two firms located in different neighborhoods compete to attract consumers

[3]See e.g. Bester (1995), Butters (1977), Caminal (1996), Grossman and Shapiro (1984), Robert and Stahl (1993), Shapiro (1980), Stahl (1994), and Stegeman (1991).

[4]In monopolized markets, Esteban *et al.* (2001) show that a firm may have incentives to use specialized magazines as a vehicle to target price advertisements to the consumers who value the good more. This results in higher advertised prices and is thus potentially detrimental from a welfare viewpoint. Esteban *et al.* (2001), also in a non-strategic context, study the influence between advertising-strategy and product quality. They find that customer directed advertising has a bearing on the price and the design of new products.

from rival's location. Consumers initially know the price charged by the neighboring store but ignore the price charged at the distant location; moreover, for similar prices buyers prefer to buy locally. In this situation advertising locally is wasteful and thus firms target their ads to the rival's region. If advertising costs are high, a pure-strategy equilibrium exists where firms charge monopoly prices. If advertising costs are low, equilibrium entails randomization over two prices, a low price that is advertised at the rival's location and a high price that is not advertised. Iyer *et al.* (2002) also examine a product differentiated duopoly market but, in contrast to Bester and Petrakis (1995), consumers have no information of their own. Some consumers are loyal to one firm, some other consumers are loyal to the other firm and the rest of the consumers are price-sensitive.[5] Firms can target ads to distinct consumer segments and the authors compare price-dispersed equilibria with regular advertising and with targeted advertising. They find that regular advertising leads to a zero-profit equilibrium while targeted advertising leads to an equilibrium with positive profits which accrue from the loyal segment. Our paper differs from these two papers in that consumers see all products as identical and thus they are all equally price-sensitive. In addition, in our model consumers hold no price information at all *ex-ante*. These two aspects enable us to understand the role of targeted advertising in generating positive firm profits in highly competitive environments. Further, our paper offers an explanation for temporary market segmentation based on pure strategic considerations, rather than on consumer loyalty. Furthermore, in our paper some prices are more heavily advertised than others while in Bester and Petrakis (1995) and Iyer *et al.* (2002) all prices are equally 'sponsored.'[6]

The nature of advertising is different in Roy (2000). He studies a two-stage model where two firms first send product-advertisements to the consumers and then choose their prices. In his model advertising has a long-run nature and a commitment to not invade the 'natural' market of the rival enables firms to segment the market and appropriate consumer surplus. His model applies to markets where advertising provides product information, perhaps intended to create brand image and consumer awareness, and later buyers discover prices costlessly. Our paper, by contrast, examines targeted advertising in environments where advertising has a short-run nature and conveys price information. A great bulk of the advertising we observe in real world markets has this type of nature, for instance, advertising that announces sales, price reductions and promotions.

Our model can also be seen as a strategic game of entry in multiple markets with advertising fees playing the role of entry costs. In this connection, it is related to the work on entry and competition by, e.g., Elberfeld and Wolfstetter (1999) and Sharkey and Sibley (1993). The former paper differs

[5] See Rosenthal (1980) for a similar model with perfect price information.

[6] See also Stahl (1994) for a price-advertising game where random prices are advertised with the same intensity.

from ours in that entry is a long-run strategic decision. In the latter paper decisions on entry and pricing are simultaneously taken, like in our paper. The main difference between the work of Sharkey and Sibley (1993) and ours is that they examine an entry game in a single market while we investigate entry in multiple markets. Remarkably, while they find that firms profits are zero in the unique equilibrium of the game, we characterize an equilibrium with positive profits. Thus, the mere existence of some kind of market segmentation can be exploited by firms to make money in situations of extreme competitiveness.

The remainder of the paper is organized as follows. Section 2 describes the basic model. Section 3.1 discusses some preliminary results. Section 3.2 presents equilibria with random pricing and random advertising and comparative statics results. Section 4 is devoted to discuss some extensions of the basic model. Section 5 closes the paper with a review of the main conclusions. Some proofs are relegated to the Appendix.

2 The model

We study a market for a homogeneous good. On the demand side of the market, there is a unitary mass of identical consumers. *A priori*, consumers ignore the existence and the price of the good. Thus, a potential consumer cannot be an actual buyer unless sellers invest in advertising.[7] A consumer buys at most a single unit of the product, and does so at the minimum price known to her; the maximum price a consumer is willing to pay for the good is $v > 0$.[8] On the supply side of the market there are two firms indexed $i = 1, 2$.[9] These firms produce the good at constant returns to scale and their identical unit production cost is normalized to zero, without loss of generality.

To examine the scope of strategic targeted advertising, we shall assume that consumers are segmented in regard to the advertising channel through which firms can reach them. The simplest example is that in which there are two newspapers in the economy which are read by different consumers.[10] Letting $j = A, B$ be the newspaper index, we assume that a fraction μ_A of the consumers reads newspaper A, while the rest of the consumers μ_B reads newspaper B, with $\mu_A + \mu_B = 1$. An individual firm must necessarily advertise its price in at least a newspaper to be able to sell. Advertising is costly; assume that placing an advertisement in a newspaper j costs $\phi_j > 0$ to the firms, $j = A, B$. A firm can target its advertising campaign to the readers of newspaper A, or

[7] Advertising also conveys product and price information in Butters (1977), Grossman and Shapiro (1984) and Stahl (1994). The natural interpretation is that firms sell newly introduced items.

[8] The results of this paper would not change if we considered instead that all consumers hold a downward sloping demand function.

[9] In Section 4 we consider entry of firms an examine the robustness of our results in an N-firm game.

[10] In Section 4 we discuss how our results change when some of the consumers read both newspapers.

to the readers of newspaper B, or else can advertise its price in both newspapers A and B. We assume that if a firm advertises a price in newspaper j, all consumers in segment j are reached with probability one.[11] A firm's advertising-strategy is thus a probability function over the set $\{A, B, M\}$, where M indicates that a firm advertises its price in both newspapers thus bearing a total advertising cost equal to $\phi_A + \phi_B$. Let λ_j^i denote the probability with which a firm $i = 1, 2$ advertises its price in newspaper $j = A, B, M$.[12]

A firm's pricing-strategy is denoted by a distribution of prices $F_j^i(p)$ accompanying advertising decisions. Let σ_j^i denote the support of $F_j^i(p)$ and let $\overline{p}_j^i$ and $\underline{p}_j^i$ denote the maximum and the minimum price in σ_j^i, respectively. Since j can take on value M, this means that a firm cannot advertise distinct prices in different newspapers.[13] A firm's strategy is thus denoted by a collection or pairs $\{\lambda_j^i, F_j^i(p)\}_{j=A,B,M}$, $i = 1, 2$.

Firms play a simultaneous move game. An individual firm decides which price to charge and in which newspaper to advertise it taking as given the pricing and advertising strategies of the other firm. Our interest lies on the existence and characterization of Nash equilibria in which firms obtain positive profits.[14]

3 Analysis

3.1 Preliminary results

An individual seller must choose a price and decide whether to advertise it in newspaper A, or in newspaper B, or in both newspapers taking as given the advertising fees and the price and advertising decision of the rival firm. These actions lead to a number of different strategy profiles, pure as well as mixed. The pure advertising-strategy profiles are summarized in Table 1.

[11] We note that this assumption captures an extreme case of competitiveness; this implies that, if the market were not segmented, then the unique equilibrium would be such that firms make zero profits. We shall show that, even in such case of extreme competition, segmentation may lead to a positive profits equilibrium. Alternatively, one could assume that advertising technology is imperfect and that ads do not reach consumers surely. In that case, market segmentation should also lead to an equilibrium with even higher profits.

[12] We note that a firm may also decide to get out of the market altogether. Since our focus is on whether segmentation leads to positive profits equilibria, we shall ignore strategy profiles where firms put positive probability on staying out of the market (see footnote 14).

[13] We assume that price discrimination is unfeasible –e.g. because of legal restraints or due to consumer arbitrage possibilities–. In Section 4 we discuss the role that price discrimination would have if it were feasible.

[14] We have found that there may potentially be two types of equilibria with zero profits. In the first type of equilibrium, firms randomize between staying out of the market, advertising a random price in newspaper A, and advertising a random price in newspaper B. If there is an equilibrium of this type, we have seen that it is unique and that it requires advertising cost to be large enough. In the second type of equilibrium, firms randomize between staying out of the market, advertising a price in segment A, advertising a price in segment B and advertising a price in the entire market, M. We have seen that there may be a continuum of equilibria of this second type.

Symmetric	Asymmetric I	Asymmetric II
(i) $\lambda_A^i = 1, i = 1, 2$	(iv) $\lambda_A^1 = 1, \lambda_B^2 = 1$	(vii) $\lambda_B^1 = 1, \lambda_M^2 = 1$
(ii) $\lambda_B^i = 1, i = 1, 2$	(v) $\lambda_A^1 = 1, \lambda_M^2 = 1$	(viii) $\lambda_M^1 = 1, \lambda_A^2 = 1$
(iii) $\lambda_M^i = 1, i = 1, 2$	(vi) $\lambda_B^1 = 1, \lambda_A^2 = 1$	(ix) $\lambda_M^1 = 1, \lambda_B^2 = 1$

Table 1: Pure advertising-strategy profiles

Our first remark is that an equilibrium where firms play a pure advertising-strategy does not exist.

Lemma 1 *A pure advertising-strategy cannot be part of an equilibrium.*

Proof: We first rule out symmetric advertising-strategy profiles. Suppose that both firms advertise their price in newspaper j with probability one in equilibrium, $j = A, B, M$. Then, it is obvious that both firms should accompany their advertising decisions with prices equal to marginal cost; but if this is so, firms would not cover their advertising costs. Thus, this cannot be part of an equilibrium. We now rule out asymmetric advertising-strategy profiles. Suppose firms advertise in different newspapers in equilibrium, e.g., firm 1 advertises in newspaper A and firm 2 does so in B. Then, it is obvious that they should accompany their advertising strategies with monopoly pricing, which would yield profits of $\pi_1 = \mu_A v - \phi_A$ and $\pi_2 = \mu_B v - \phi_B$ to the firms. But if this were so in equilibrium, a firm would gain by slightly undercutting the monopoly price and advertising it in both newspapers, obtaining a profit of $v - \phi_A - \phi_B$.

Consider finally that a firm, say firm 1, advertises in just a newspaper, say A, and the other firm in both newspapers. Let $s^1 = \{\lambda_A = 1, F_A(p)\}$ be firm 1's strategy and let $s^2 = \{\lambda_M = 1, F_M(p)\}$ denote firm 2's strategy. If an equilibrium exists firms' profits would be given by:

$$\begin{aligned} E\pi_1(\lambda_A &= 1, p_1; s^2) = p_1\mu_A(1 - F_M(p_1)) - \phi_A \\ E\pi_2(\lambda_M &= 1, p_2; s^1) = p_2[\mu_A(1 - F_A(p_2)) + \mu_B] - \phi_A - \phi_B. \end{aligned}$$

We note that it must be the case that $\overline{p}_A < \overline{p}_M$ because otherwise firm 1 charging the upper bound $\overline{p}_A$ in A would make negative profits. From this, it follows that $\overline{p}_M = v$ because a firm advertising a different upper bound in the entire market would gain by increasing its price. Since firm 2 must be indifferent between advertising any price in the support σ_M, it follows that firm 2's expected profits must be $E\pi_2 = \mu_B v - \phi_A - \phi_B$. Now it is obvious that firm 2 would gain by deviating and advertising v only in B since the firm would save on advertising costs. The other pure strategy profiles (see Table 1) are ruled out similarly. The proof is now complete. ■

The interesting implication of Lemma 1 is that equilibria in this game of price and targeted-advertising must involve random advertising strategies. This is important because this means that targeted advertising cannot be a permanent phenomenon in homogeneous product markets. In what follows we examine the different symmetric mixed advertising-strategy profiles. These profiles are presented in the next Table.[15]

(i) $\lambda_A^i + \lambda_B^i = 1, i = 1, 2$
(ii) $\lambda_A^i + \lambda_M^i = 1, i = 1, 2$
(iii) $\lambda_B^i + \lambda_M^i = 1, i = 1, 2$
(iv) $\lambda_A^i + \lambda_B^i + \lambda_M^i = 1, i = 1, 2$

Table 2: Symmetric mixed advertising-strategy profiles

Our second observation is that firms must advertise in both newspapers from time to time.

Lemma 2 *A symmetric equilibrium where $\lambda_M = 0$ does not exist.*

Proof: Suppose, on the contrary, that firms randomize between advertising in newspaper A and advertising in newspaper B, i.e., $\lambda_A + \lambda_B = 1$, $\lambda_A, \lambda_B > 0$. Let us denote firms strategies as $s^i = \{(\lambda_A, F_A(p)), (\lambda_B, F_B(p))\}$, $i = 1, 2$. Given the strategy of firm 2, firm 1's payoff from advertising a price p in newspaper A is

$$E\pi_1\left(\lambda_A = 1, p; s^2\right) = p\mu_A[\lambda_A(1 - F_A(p)) + \lambda_B] - \phi_A.$$

Likewise, firm 1's payoff from advertising a price p in B is

$$E\pi_1\left(\lambda_B = 1, p; s^2\right) = p\mu_B[\lambda_A + \lambda_B(1 - F_B(p))] - \phi_B.$$

We first note that the upper bound of the price distributions cannot be lower than v because a firm charging the upper bound would gain by slightly raising its price, i.e., $\overline{p}_j = v$, $j = A, B$. We secondly note that firm 1 must be indifferent between advertising a price $p \in \sigma_A \cap \sigma_B$ in either of the segments in a mixed strategy equilibrium, i.e., $\forall p \in \sigma_A \cap \sigma_B$ it must be the case that $E\pi_1\left(\lambda_A = 1, p; s^2\right) = E\pi_1\left(\lambda_B = 1, p; s^2\right)$. Since $v \in \sigma_j$, $j = A, B$ it follows that

$$E\pi_1\left(\lambda_A = 1, v; s^2\right) = \lambda_B\mu_A v - \phi_A = E\pi_1\left(\lambda_B = 1, v\right) = \lambda_A\mu_B v - \phi_B > 0.$$

Then, a firm, say firm 1, can profitably deviate by advertising the monopoly price in the entire market. Indeed, profits to firm 1 from such deviation are

$$E\pi_1^d\left(\lambda_M = 1, v; s^2\right) = \lambda_B\mu_A v + \lambda_A\mu_B v - \phi_A - \phi_B$$

[15] We note that asymmetric mixed strategy profiles are very difficult to deal with in our setting; thus we focus on symmetric ones.

which are clearly greater than $\lambda_A \mu_B v - \phi_B$. This contradicts Nash notion. ■

We now note that firms introduce quite a bit of noise in the market to maximize profits.

Lemma 3 *If a symmetric equilibrium exists, then it must be the case that* $\lambda_j > 0$, $j = A, B, M$.

Proof: On the basis of Lemma 2, we only need to show that $\lambda_A + \lambda_M = 1$, $\lambda_j > 0, j = A, M$ and $\lambda_B + \lambda_M = 1$, $\lambda_j > 0, j = B, M$ cannot be part of an equilibrium. Suppose first that $\lambda_A + \lambda_M = 1$. Let's denote firm i's strategy as $s^i = \{(\lambda_A, F_A(p)), (\lambda_M, F_M(p))\}, i = 1, 2$. Taking as given s^2, the profit to firm 1 from advertising p in newspaper A is

$$E\pi_1\left(\lambda_A = 1, p; s^2\right) = p\mu_A[\lambda_A(1 - F_A(p)) + \lambda_M(1 - F_M(p))] - \phi_A.$$

Likewise, the profit to firm 1 from advertising a price p in both newspapers is

$$E\pi_1\left(\lambda_M = 1, p; s^2\right) = \lambda_A p \mu_B + \lambda_A \mu_A p(1 - F_A(p)) + \lambda_M p(1 - F_M(p)) - \phi_A - \phi_B.$$

As above we observe that $F_A(p)$ and $F_M(p)$ cannot have atoms. We now note that $\overline{p}_A < \overline{p}_M$; indeed if $\overline{p}_A \geq \overline{p}_M$ a firm advertising the upper bound $\overline{p}_A$ in newspaper A would always obtain negative profits. Observe next that it must be the case that $\overline{p}_M = v$; this is because there is a strictly positive probability that a firm advertising in both newspapers is the only one advertising in newspaper B; then, a firm advertising a different upper bound would gain by raising its price. Firms must be indifferent between the distinct price and advertising strategies; therefore equilibrium profits would be:

$$E\pi_1\left(\lambda_M = 1, v; s^2\right) = \lambda_A v \mu_B - \phi_A - \phi_B$$

We now note that a firm, say firm 1, can gain by advertising the monopoly price only in newspaper B. Indeed, profits to firm 1 from such a deviation are:

$$E\pi_1^d\left(\lambda_B = 1, v; s^2\right) = \lambda_A v \mu_B - \phi_B$$

which are clearly greater than equilibrium profits for all $\phi_A > 0$. This contradicts Nash notion. The proof that $\lambda_B + \lambda_M = 1$ cannot be part of an equilibrium is analogous. ■

3.2 Equilibria

Lemmas 1-3 have shown that if an equilibrium with positive profits exists, firms must put positive probability on all pure advertising strategies. In what follows, we shall examine the existence and characterization of equilibria in this game of pricing and targeted advertising.

Let $s^i = \{(\lambda_A, F_A(p)), (\lambda_B, F_B(p)), (\lambda_M, F_M(p))\}$, $i = 1, 2$ denote firms' strategies. The payoffs to a firm from the different advertising strategies are:

$$E\pi_1(\lambda_A = 1, p; s^2) = p\mu_A[\lambda_A(1 - F_A(p)) + \lambda_B + \lambda_M(1 - F_M(p))] - \phi_A \tag{1}$$
$$E\pi_1(\lambda_B = 1, p; s^2) = p\mu_B[\lambda_A + \lambda_B(1 - F_B(p)) + \lambda_M(1 - F_M(p))] - \phi_B \tag{2}$$
$$E\pi_1(\lambda_M = 1, p; s^2) = E\pi_1(\lambda_A = 1, p; s^2) + E\pi_1(\lambda_B = 1, p; s^2) \tag{3}$$

The next result shows that prices must be dispersed in equilibrium.

Lemma 4 *If a symmetric equilibrium exists, the price distributions $F_A(p)$, $F_B(p)$ and $F_M(p)$ must be atomless.*

Proof. The proof follows from the fact that atoms can be profitably undercut as discussed above in the proof of Lemma 2. ■

Our next result establishes an equilibrium property of the supports of the price distributions.

Lemma 5 *If a symmetric equilibrium exists, $\underline{p}_j < \underline{p}_M = \overline{p}_j < \underline{p}_k < \overline{p}_M = \overline{p}_k = v$, $j, k = A, B$, $j \neq k$.*

The proof, which is in the Appendix, proceeds as follows. We first prove that there cannot be a price common to the three equilibrium supports σ_A, σ_B and σ_M. This follows from inspection of payoff functions (1)-(3). Then a series of claims proves that the support configuration must satisfy the inequality above. These claims exploit two facts: first, that a firm advertising in a single segment cannot increase its profits by deviating and advertising v in the entire market; second, that a firm advertising in the entire market cannot increase its profits by deviating and advertising in a single segment.

Lemma 5 implies that the only support configuration (up to a permutation of segment labels) that can be part of an equilibrium is as represented in the following graph.

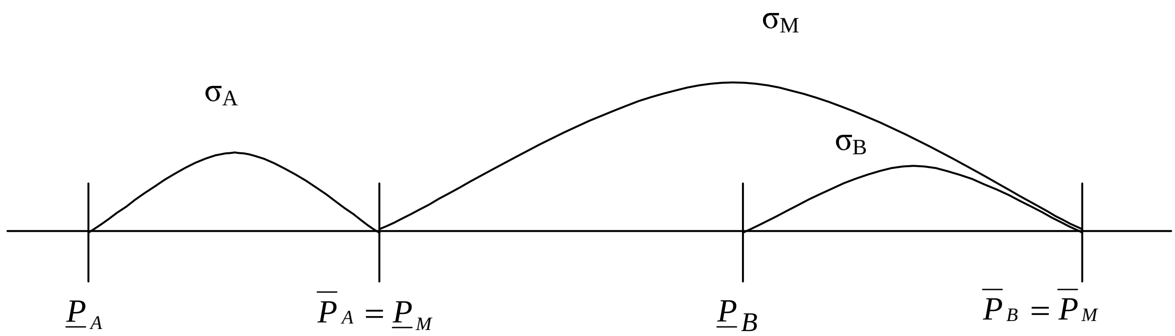

Figure 1: Equilibrium support configuration.

In what follows we shall see that a positive profits equilibrium indeed exists for this support configuration. Moreover, we shall show in Section 4 that this equilibrium is robust to the introduction of a fraction of consumers who read both newspapers, as well as to an increase in the number of firms.

To characterize this equilibrium, we exploit the fact that a firm must be indifferent between the distinct pricing and advertising strategies. A firm must be indifferent between advertising v to consumer segment B and in the entire market, that is, $E\pi_1(\lambda_B = 1, v; s^2) = E\pi_1(\lambda_M = 1, v; s^2)$. Using equations (1)-(3), this implies that $\mu_B v\lambda_A - \phi_B = \mu_B v\lambda_A - \phi_B + \mu_A v\lambda_B - \phi_A$. This expression yields the frequency with which firms advertise prices in newspaper B :

$$\lambda_B = \frac{\phi_A}{\mu_A v}. \tag{4}$$

A firm must be indifferent between advertising a price from σ_B in newspaper B and advertising it in the entire market. That is, $E\pi_1(\lambda_M = 1, p; s^2) = E\pi_1(\lambda_B = 1, p; s^2)$ for all $p \in \sigma_B \cap \sigma_M$. This implies that $\mu_A p[\lambda_B + \lambda_M(1 - F_M(p))] - \phi_A = 0$. Solving for $F_M(p)$ yields

$$F_M(p) = 1 - \frac{1}{\lambda_M}\left(\frac{\phi_A}{\mu_A p} - \lambda_B\right) \text{ for all } p \in \sigma_B \cap \sigma_M. \tag{5}$$

A firm must gain the same by advertising $\overline{p}_A$ in newspaper A and by doing so in the entire market, i.e., $E\pi_1(\lambda_A = 1, \overline{p}_A; s^2) = E\pi_1(\lambda_M = 1, \overline{p}_A; s^2)$. This implies that $\overline{p}_A = \phi_B/\mu_B (= \underline{p}_M)$.

If a firm charges a price $p \in [\overline{p}_A, \underline{p}_B]$ in the entire market, this firm obtains a profit:

$$\mu_A p[\lambda_B + \lambda_M(1 - F_M(p))] - \phi_A + \mu_B p[\lambda_A + \lambda_B + \lambda_M(1 - F_M(p))] - \phi_B. \tag{6}$$

This benefit must be equal to the profit a firm gains by advertising v in both newspapers:

$$\mu_A v\lambda_B - \phi_A + \mu_B v\lambda_A - \phi_B \tag{7}$$

Equating (6) and (7) yields an expression for $F_M(p)$ in $[\overline{p}_A, \underline{p}_B]$:

$$F_M(p) = 1 - \frac{\lambda_A \mu_B(v - p) - \lambda_B p + \phi_A}{\lambda_M p} \text{ for all } p \in [\overline{p}_A, \underline{p}_B] \tag{8}$$

Since $F_M(p)$ must be atomless, we note that at $p = \underline{p}_B$, the two equilibrium price distributions (5) and (8) must be equal; using (4), we find $\underline{p}_B = \frac{(\lambda_A - \lambda_B)v}{\lambda_A}$.

Consider now the profits from advertising a price $p \in \sigma_B \cap \sigma_M$ in newspaper B. This profit must equal the profit from advertising v in B. Therefore:

$$\mu_B p[\lambda_A + \lambda_B(1 - F_B(p) + \lambda_M(1 - F_M(p))] - \phi_B = v\mu_B\lambda_A - \phi_B.$$

Substituting (5) into this expression, we can solve for the distribution of advertised prices in newspaper B:

$$F_B(p) = 1 - \frac{\lambda_A - \lambda_B}{\lambda_B}\left[\frac{v-p}{p}\right]$$

Equating the profits from advertising $\underline{p}_M$ and v in the entire market yields an expression for the frequency with which firms advertise prices in newspaper A:

$$\lambda_A = \frac{\phi_B - \phi_A\mu_B}{\mu_B^2 v + \phi_B\mu_A}$$

It remains to find the distribution of advertised prices in newspaper A. A firm advertising in A must be indifferent between advertising any price in the support of σ_A. Solving $E\pi_1(\lambda_A = 1, p; s^2) = E\pi_1(\lambda_A = 1, \overline{p}_A; s^2)$ yields

$$F_A(p) = 1 - \frac{\lambda_B + \lambda_M}{\lambda_A}\frac{(\overline{p}_A - p)}{p}$$

The lower bound of σ_A is obtained by setting $F_A(\underline{p}_A) = 1$; thus, $\underline{p}_A = (1-\lambda_A)\overline{p}_A$.

The next result summarizes these findings. Let $\underline{p}_B = \frac{v(\lambda_A - \lambda_B)}{\lambda_A}$ and $\overline{p}_A = \underline{p}_M = \frac{\phi_B}{\mu_B}$.

Proposition 1 *There exists a non-empty set of parameters μ_A, μ_B, ϕ_A and ϕ_B for which the following positive-profit symmetric equilibrium of the price and targeted-advertising game exists: With probability $\lambda_A = \frac{\phi_B - \phi_A\mu_B}{\mu_B^2 v + \phi_B\mu_A}$ firms advertise in segment A a price p randomly chosen from the set $[(1-\lambda_A)\overline{p}_A, \overline{p}_A]$ according to the price distribution $F_A(p) = 1 - \frac{\lambda_B+\lambda_M}{\lambda_A}\frac{(\overline{p}_A - p)}{p}$; with probability $\lambda_B = \phi_A/\mu_A v$ firms advertise in segment B a price p randomly chosen from the set $[\underline{p}_B, v]$ according to the price distribution $F_B(p) = 1 - \frac{\lambda_A - \lambda_B}{\lambda_B}\left[\frac{v-p}{p}\right]$; and with the remaining probability firms advertise in the entire market a price p randomly chosen from the set $[\underline{p}_M, v]$ according to the price distribution*

$$F_M(p) = \begin{cases} 1 - \frac{\lambda_A\mu_B(v-p) - \lambda_B p + \phi_A}{\lambda_M p} & \text{for all } p \in [\underline{p}_M, \underline{p}_B] \\ 1 - \frac{1}{\lambda_M}\left(\frac{\phi_A}{\mu_A p} - \lambda_B\right) & \text{for all } p \in [\underline{p}_B, v] \end{cases}$$

This is the only symmetric equilibrium in which firms obtain strictly positive profits $E\pi = \lambda_A\mu_B v - \phi_B$ (up to a permutation of segment labels).

The existence part of this result is proved in the Appendix. We show that firms cannot profitably deviate from the strategy profile described in the Proposition and that the set of parameters for which an equilibrium exists is non-empty. In equilibrium, firms obtain positive profits by employing random pricing and random targeted-advertising strategies. This twofold randomization enables firms to induce an unequal distribution of information in the market and to overcome detrimental undercutting behavior by the rival firm. We note that market segmentation can only occur from time to time.

The equilibrium described in Proposition 1 is illustrated in Figure 2 below for a situation where the two newspapers have the same readership. In this graph we have represented the equilibrium distribution of advertised prices in newspaper A, B and in the entire market.

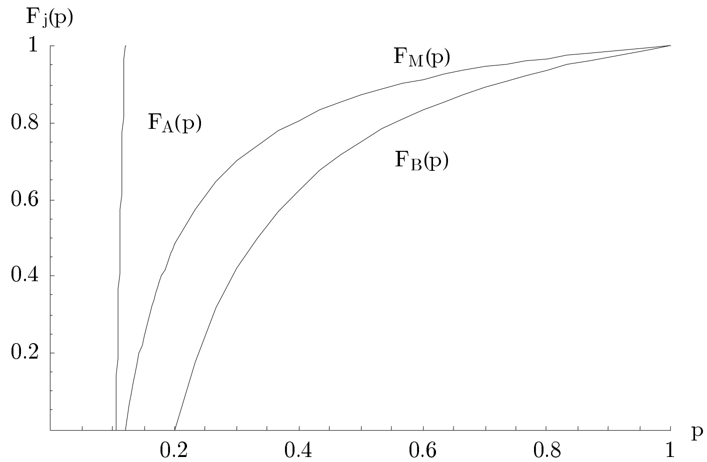

Figure 2: Equilibrium price distributions. ($\phi_A = 1/20$, $\phi_B = 3/20, v = 1$).

Our next result establishes that the equilibrium distributions of advertised prices can be ranked using the criterion of first-order stochastic dominance.

Proposition 2 *The distribution of advertised prices $F_B(p)$ dominates $F_M(p)$ in a first-order stochastic sense; moreover, $F_M(p)$ dominates $F_A(p)$ under the same dominance criterion.*

The proof is in the appendix. The implication of this result is that, in equilibrium, firms randomize between announcing high prices in a single newspaper, announcing low prices in the other newspaper, and advertising intermediate prices to the entire market. Thus, when market segmentation arises, expected prices may be more or less dispersed. Another observation worth pointing out is that our model does not generate 'sponsored' sales because low prices are not heavily advertised but intermediate prices; this contrasts with previous research on price-advertising.

We now take a closer look at the existence conditions derived in the Appendix. Let us assume, without loss of generality, that $\phi_A = \gamma v$, with $\gamma \in (0, 1)$ and $\phi_B = \beta\phi_A = \beta\gamma v$, with $\beta \in (0, 1/\gamma)$. It can be seen that, for the case in which both newspapers have the same readership, an equilibrium exists if and only if $\beta \in (1, 1/\gamma)$ and $\gamma \in (0, (\beta - 1)/2\beta^2$. We have represented these two conditions in Figure 3.

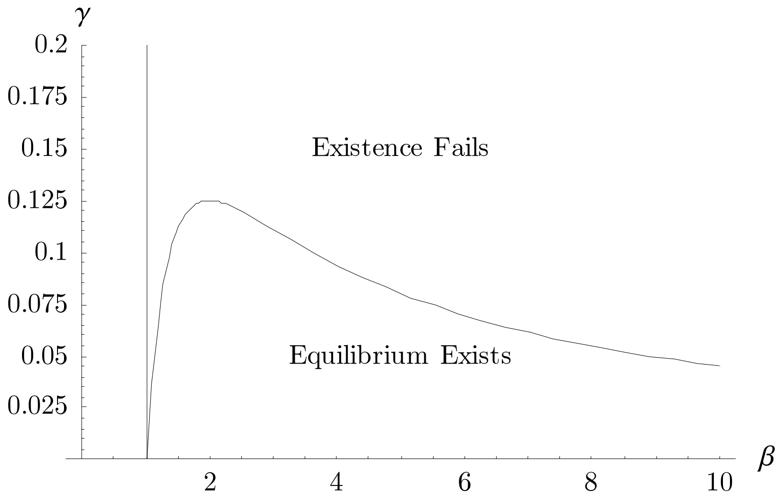

Figure 3: Existence when newspapers have the same readership.

We observe that an equilibrium exists only if newspapers' advertising fees are neither too dissimilar, nor too similar. To see this, fix γ in Figure 3. If advertising fees are very similar (low β), the only way firms can be indifferent between advertising low prices in A and high prices in B is when competition for B's readers is very vigorous, which implies that λ_A must be very small and this results in firms making negative profits. If advertising fees are very dissimilar instead (β high), a similar argument applies.

This discussion motivates a careful examination of how equilibrium changes when we alter the parameters of the model, in particular, when advertising fees increase, when advertising fees differences fall and when the fraction of consumers in one of the segments increases. We note that parameter changes affect both advertising probabilities and price distributions. The following result, proved in the appendix, is useful for this purpose:

Proposition 3 *In equilibrium the following relations hold.*

(1) $\frac{\partial \lambda_A}{\partial \gamma} > 0$, $\frac{\partial \lambda_B}{\partial \gamma} > 0$, *and* $\frac{\partial \lambda_M}{\partial \gamma} < 0$. *Moreover, an increase in* γ *widens* σ_A *and* σ_B*, and narrows* σ_M*. Moreover, as* $\gamma \to 0$, $\lambda_M \to 1$ *and* $F_M(p)$ *converges to a price distribution that is degenerate at the marginal cost.*

(2) $\frac{\partial \lambda_A}{\partial \beta} > 0$, $\frac{\partial \lambda_B}{\partial \beta} = 0$, *and* $\frac{\partial \lambda_M}{\partial \beta} < 0$. *Moreover, an increase in* β *widens* σ_A*, and narrows both* σ_B *and* σ_M*.*

(3) $\frac{\partial \lambda_A}{\partial \mu_B} < 0$, $\frac{\partial \lambda_B}{\partial \mu_B} > 0$, *and* $\frac{\partial \lambda_M}{\partial \mu_B}$ *is indeterminate. Moreover, an increase in* μ_B *narrows* σ_A*, and widens both* σ_B *and* σ_M*.*

Building on this Proposition, we now elaborate on the comparative statics results of parameter changes. Consider first a decrease in advertising fees, which is captured in our model by a decrease in γ. A fall in advertising fees leads to less likely market segmentation and to less extreme pricing; that is, firms decrease the probability with which they advertise high prices in newspaper B, and also decrease the probability with which they advertise low prices in newspaper A –by implication, firms increase the probability with which they advertise intermediate prices in both newspapers. In addition, firms narrow the set of prices advertised in A and in B and widen the set of prices advertised to the entire market. Keeping everything else constant, what happens is that a decrease in both fees makes advertising in both newspapers relatively inexpensive compared to advertising in just a newspaper. For firms to remain indifferent between the different advertising strategies, they must decrease competition for the distinct segments of consumers and increase competition for the whole market. This results in lower advertising probabilities and lower price dispersion at the segment level, and greater price dispersion at the market level. Interestingly, this has implications for firms' profits. We can show that profits are non-monotonic in advertising fees, first increasing and then decreasing. We observe that making advertising in the two newspapers a less attractive strategy, weakens competitiveness and boosts firms revenues. When γ is low to begin with, gains from weaker competitiveness offset the cost increase; by contrast, when advertising fees are high enough, the increase in revenues is too small to outweigh the cost increase. The influence of γ on the equilibrium can be seen in Figure 4. The left graph shows the equilibrium price distributions when advertising fees are low while the right graph represents a market with high advertising fees. We finally notice that as γ converges to zero, λ_M converges to 1 and the distribution of advertised prices converges to a degenerate price distribution at the marginal cost.

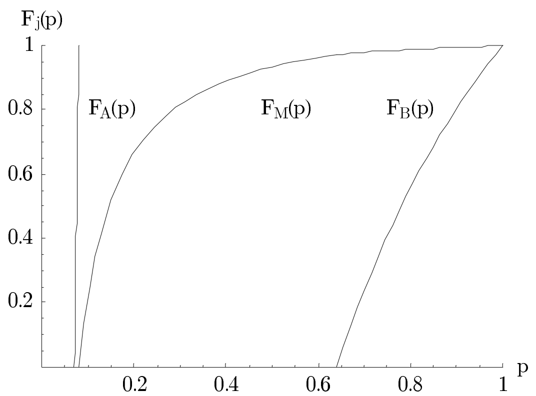

Figure 4a. Low γ ($\gamma = 1/50; \beta = 2$).

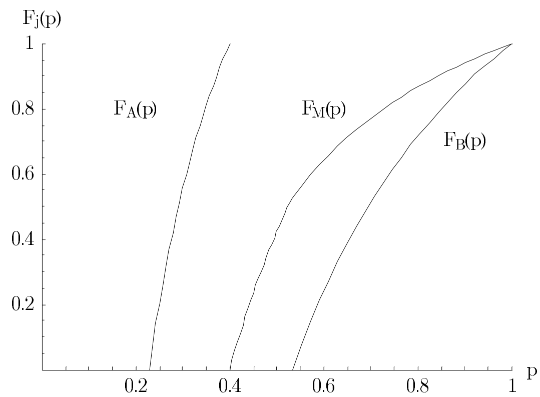

Figure 4b. High γ ($\gamma = 1/10$; $\beta = 2$).

We now discuss how changes in newspapers' advertising cost differences influence our results. An increase in β captures an increase in newspaper B's advertising fee keeping constant newspaper A's advertising fee. Thus, β increasing raises newspapers' cost asymmetries. This results in an increase in the probability λ_A with which firms advertise in newspaper A, while the probability λ_B with which firms advertise in newspaper B is unaltered; as a consequence firms advertise in the entire market less frequently, i.e., λ_M decreases. The reason is that in equilibrium the three advertising strategies must be equally attractive, so firms must increase competition for newspaper A's readers and decrease competition for newspaper B's readers. Interestingly, competition for consumer segment B is weakened due to a fall in the frequency with which firms advertise in the entire market. The distributions of advertised prices also change accordingly (see Figure 5 below). When firms compete more frequently for a segment in the market, they increase price dispersion to overcome rival's undercutting behavior. These observations translate into profits being non-monotonic in β. The reason why profits can increase is again related to the weakening of competitiveness that originates from the fact that firms meet at competing for the entire market less frequently.

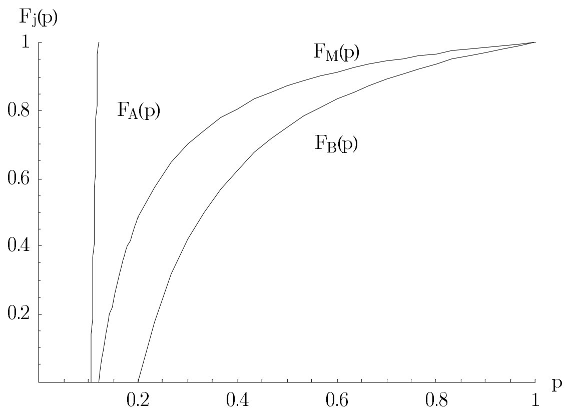

Figure 5a. Low β ($\gamma = 1/20; \beta = 1.2$).

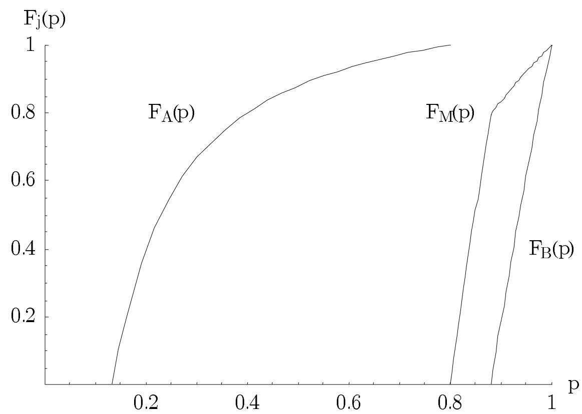

Figure 5b. High β ($\gamma = 1/20; \beta = 8$).

Finally, we discuss the implications of a change in the distribution of consumers across segments. An increase in μ_A leads to an increase in the probability λ_A with which firms advertise in newspaper A and a to decrease in the probability with which they advertise in newspaper B. This results in effects similar to those when β goes up. Profits again exhibit a non-monotonic pattern with respect to the parameter of interest, the readership of newspaper A in this case. The reason is as follows. Note that equilibrium profits are $\lambda_A \mu_B v - \phi_B$. An increase in μ_A tends to decrease profits due to a fall-in-demand effect. However, there is also a competition effect by which firms advertise more frequently in A, which tends to increase profits. We see that, for similar readerships, the competition effect is dominant and profits increase as μ_A rises; when the two newspapers are already quite different, the demand effect is stronger and profits decrease. See Figure 6 for an illustration of part 3 of Proposition 3.

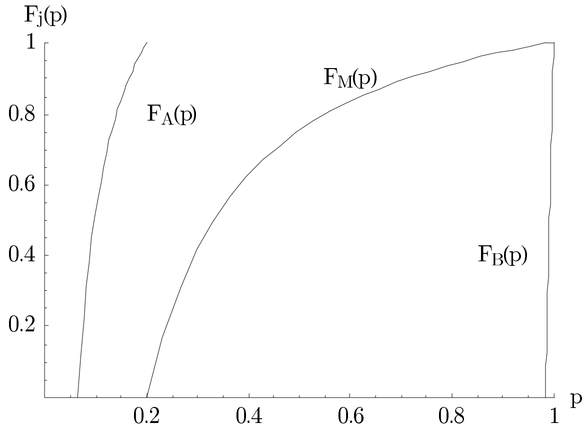

Figure 6a. High μ_A ($\mu_A = 0.9; \gamma = 1/100; \beta = 2$).

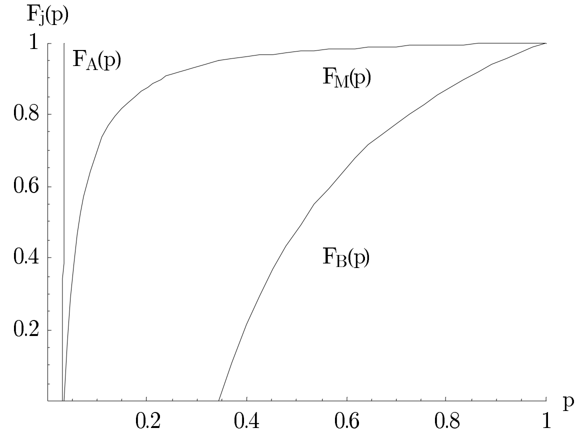

Figure 6b. Low μ_A ($\mu_A = 0.4; \gamma = 1/100; \beta = 2$).

4 Extension

In the previous section we have characterized Nash equilibria for a situation in which every consumer reads a single newspaper and there are just two firms operating in the industry. In this Section, we investigate the implications of buyer multiple readership and entry of firms. In addition, we discuss the implications of allowing firms to price discriminate across segments.

4.1 Multiple newspaper readership

In the market described above, consider that some consumers read both newspapers; we will refer to the multi-readership consumers segment as μ_M and therefore we assume consumer readerships to satisfy $\mu_A + \mu_B + \mu_M = 1$, with $\mu_i > 0$, $i = A, B, M$. We note that the fraction of consumers μ_M is always fully informed in the sense that, given any firms' advertising-strategy profile, these consumers always observe both firms' advertised prices.

We start investigating the robustness of the equilibrium in Proposition 1. The next result provides a natural extension of Proposition 1 to a market setting in which some buyers read both newspapers. Let $\underline{p}_M = \overline{p}_A = \frac{\lambda_A \mu_B v + \phi_A - \phi_B}{(1-\lambda_A)(1-\mu_B)}$ and $\underline{p}_B = v\left(\frac{\lambda_A \mu_B - \lambda_B(1-\mu_A)}{\lambda_A \mu_B}\right)$.

Proposition 4 *There exists a non-empty set of parameters $\phi_A, \phi_B, \mu_A, \mu_B$ and μ_M for which the following positive-profit symmetric equilibrium of the price and targeted-advertising game exists:*

With probability $\lambda_A = \frac{\phi_B - \phi_A \mu_B}{\mu_B^2 v + \phi_B(1-\mu_B)}$ *firms advertise in segment* A *a price* p *randomly chosen from the set* $[(1-\lambda_A)\overline{p}_A, \overline{p}_A]$ *according to the price distribution* $F_A(p) = 1 - \frac{1-\lambda_A}{\lambda_A}\left(\frac{\overline{p}_A - p}{p}\right)$; *with probability* $\lambda_B = \frac{\phi_A}{\mu_A v}$ *firms advertise in segment* B *a price* p *randomly chosen from the set* $[\underline{p}_B, v]$ *according to the price distribution* $F_B(p) = \frac{v}{p} - \frac{\lambda_A}{\lambda_B}\frac{\mu_B}{1-\mu_A}\frac{(v-p)}{p}$; *and with the remaining probability firms advertise in the entire market a price* p *randomly chosen from the set* $[\underline{p}_M, v]$ *according to the price distribution*

$$F_M(p) = \begin{cases} 1 - \frac{\lambda_A \mu_B (v-p) - \lambda_B p + \phi_A}{\lambda_M p} & \text{for all } p \in [\underline{p}_M, \underline{p}_B] \\ 1 - \frac{\lambda_B}{\lambda_M}\frac{(v-p)}{p} & \text{for all } p \in [\underline{p}_B, v]. \end{cases}$$

In equilibrium firms obtain strictly positive profits $E\pi = \lambda_A \mu_B v - \phi_B$.

The proof of this proposition is similar to the equilibrium characterization behind Proposition 1 above and is thus relegated to the Appendix. We note that setting $\mu_M = 0$ in Proposition 4 yields Proposition 1. Moreover, we observe that the introduction of a fraction of consumers who read both newspapers does not undermine the comparative statics results provided above in the context of Proposition 1. In particular, an increase in the advertising fee (γ) or an increase in the advertising cost asymmetry (β) raises the probability that firms advertise in newspaper A. Taking into consideration the cost effect caused by either an increase in γ or in β, we can see that equilibrium expected profit is non-monotonic with respect to γ and β. We now investigate the implications of an increase in the fraction of consumers who read both newspapers on the market equilibrium. We distinguish between the case in which an increase in μ_M is accompanied by a decrease in μ_A, and the case in which an increase in μ_M comes with a decrease in μ_B.

Proposition 5 *In equilibrium the following relations hold:*

(1) *Holding* μ_B *constant,* $\frac{\partial \lambda_A}{\partial \mu_M} = 0$, $\frac{\partial \lambda_B}{\partial \mu_M} > 0$, *and* $\frac{\partial \lambda_M}{\partial \mu_M} < 0$; *moreover, an increase in* μ_M *widens* σ_B, *while* σ_A, σ_M *and equilibrium profit are not altered.*

(2) *Let* $\overline{\gamma} = \frac{\mu_B^2}{\beta - \mu_B(2-\mu_B)}$. *Then, holding* μ_A *constant,* $\frac{\partial \lambda_A}{\partial \mu_M} > 0$, $\frac{\partial \lambda_B}{\partial \mu_M} = 0$, *and* $\frac{\partial \lambda_M}{\partial \mu_M} < 0$; *moreover, an increase in* μ_M *widens* σ_A *and narrows* σ_M, *while if* $\gamma < (>)\overline{\gamma}$ *then it narrows (widens)* σ_B *and increases (decreases) equilibrium profit.*

The proof is in the Appendix.

We now elaborate on the intuition behind this result. When μ_M increases in such a way that $\mu_A + \mu_M$ is constant (part 1), it turns out that advertising in newspaper A remains equally attractive. This can be seen by looking at the payoff function (13). As a result λ_A does not change. However, advertising in newspaper B becomes more attractive; the reason is that as μ_M increases, the number of customers a firm advertising in B may sell to rises (see equation (14)). As a result λ_B increases and σ_B widens. Finally, an increase in μ_M makes advertising in the entire market a less attractive strategy (see equation (15)). As a result, competition for the entire market must occur less frequently so λ_M must fall to restore equilibrium. Interestingly, profits are insensitive to changes in μ_M in this case.

When μ_M increases in such a way that $\mu_B+\mu_M$ is constant (part 2 of Proposition), we note that advertising in newspaper A becomes a more attractive strategy. The reason is that the number of customers a firm may sell to when advertising in newspaper A rises (see payoff function (13)). As a result firms must increase competition for segment A and σ_A correspondingly widens. By contrast, an increase in μ_M makes advertising in the entire market a less attractive strategy (see (15)) and λ_M falls to restore the equilibrium. It turns out that equilibrium profits may rise with increasing fraction of fully informed consumers; the reason is that firms compete for the entire market less frequently and this reduces overall market competitiveness.

4.2 N-firm oligopoly

Proposition 1 can be extended to consider an oligopolistic market with N firms. If all firms in the market randomize between advertising prices from σ_A in A, prices from σ_B in B and prices from σ_M in the entire market, where σ_A, σ_B and σ_M satisfy Lemma 5, the payoff to a firm from advertising a price from σ_A in A is:

$$E\pi_i(\lambda_A = 1, p \in \sigma_A; s^{-i}) = \sum_{j=0}^{N-1} \binom{N-1}{j} \lambda_A^j (1-F_A(p))^j \left[\sum_{i=0}^{N-1-j} \binom{N-1-j}{i} \lambda_B^i \lambda_M^{N-1-j-i} \right] - \phi_A$$

If a firm advertises a price p from σ_B in newspaper B, this firm obtains a profit:

$$\begin{aligned} E\pi_i(\lambda_B &= 1, p \in \sigma_B; s^{-i}) \\ &= \sum_{j=0}^{N-1} \binom{N-1}{j} \lambda_B^j (1-F_B(p))^j \left[\sum_{i=0}^{N-1-j} \binom{N-1-j}{i} \lambda_A^i \lambda_M^{N-1-j-i} (1-F_M(p))^{N-1-j-i} \right] - \phi_B \end{aligned}$$

If a firm advertises a price in the entire market, it may be the case that this price satisfies $p \in \sigma_M \cap \sigma_B$ or else $p \in \sigma_M \backslash \sigma_B$. The profits a firm obtains in those cases are:

$$\begin{aligned} E\pi_i(\lambda_M &= 1, p \in \sigma_M \cap \sigma_B; s^{-i}) = E\pi(\lambda_B = 1, p \in \sigma_B) + \\ & p\mu_A \left[\sum_{j=0}^{N-1} \binom{N-1}{j} \lambda_B^j \lambda_M^{N-1-j} (1-F_M(p))^{N-1-j} \right] - \phi_A - \phi_B \end{aligned}$$

$$
\begin{aligned}
E\pi_i(\lambda_M &= 1, p \in \sigma_M \backslash \sigma_B; s^{-1}) = p\mu_A \left[\sum_{j=0}^{N-1} \binom{N-1}{j} \lambda_B^j \lambda_M^{N-1-j} (1 - F_M(p))^{N-1-j} \right] \\
&+ p\mu_B \sum_{j=0}^{N-1} \binom{N-1}{j} \lambda_A^j \left[\sum_{i=0}^{N-1-j} \binom{N-1-j}{i} \lambda_B^i \lambda_M^{N-1-j-i} (1 - F_M(p))^{N-1-j-i} \right] - \phi_A - \phi_B
\end{aligned}
$$

Using the binomial theorem, these profit expressions can be simplified:

$$
\begin{aligned}
E\pi_i(\lambda_A &= 1, p \in \sigma_A; s^{-i}) = p\mu_A [1 - \lambda_A + \lambda_A (1 - F_A(p)]^{N-1} - \phi_A \\
E\pi_i(\lambda_B &= 1, p \in \sigma_B; s^{-i}) = p\mu_B [\lambda_A + \lambda_M (1 - F_M(p)) + \lambda_B (1 - F_B(p))]^{N-1} - \phi_B \\
E\pi_i(\lambda_M &= 1, p \in \sigma_M \cap \sigma_B; s^{-i}) = E\pi(\lambda_B = 1, p \in \sigma_B) + p\mu_A [\lambda_B + \lambda_M (1 - F_M(p))]^{N-1} - \phi_A - \phi_B \\
E\pi_i(\lambda_M &= 1, p \in \sigma_M \backslash \sigma_B; s^{-i}) = p\mu_A [\lambda_B + \lambda_M (1 - F_M(p))]^{N-1} \\
&+ p\mu_B [\lambda_A + \lambda_B \lambda_M (1 - F_M(p))]^{N-1} - \phi_A - \phi_B.
\end{aligned}
$$

A similar procedure as that outlined before Proposition 1, enables us to find a characterization of equilibrium in this N-firm market. Our interest is on how entry affects firms' profits in equilibrium. The profits from advertising $\underline{p}_M$ and v in the entire market must be equal. This yields an expression for the equilibrium frequency with which firms advertise their prices in newspaper A :

$$
v\mu_B \lambda_A^{N-1} - \frac{\mu_A}{\mu_B} \phi_B (1 - \lambda_A)^{N-1} - \phi_B + \phi_A = 0 \tag{9}
$$

Unfortunately this expression cannot be solved for λ_A explicitly. However we note that equilibrium profits are given by

$$
E\pi = v\mu_B \lambda_A^{N-1} - \phi_B
$$

so, using (9), we can examine how profits change with the number of firms:

$$
\begin{aligned}
\frac{d\lambda_A^{N-1}}{dN} &= (N-1) \frac{\partial \lambda_A}{\partial N} + \lambda_A \ln[\lambda_A] \\
&= (N-1) \frac{-v\mu_B \lambda_A^{N-1} \ln[\lambda_A] + \frac{\mu_A}{\mu_B} \phi_B (1 - \lambda_A)^{N-1} \ln[1 - \lambda_A]}{v\mu_B (N-1) \lambda_A^{N-2} + \frac{\mu_A}{\mu_B} \phi_B (N-1)(1 - \lambda_A)^{N-2}} + \lambda_A \ln[\lambda_A] \\
&= \frac{\frac{\mu_A}{\mu_B} \phi_B (1 - \lambda_A)^{N-1} \ln[1 - \lambda_A] + \frac{\mu_A}{\mu_B} \phi_B (1 - \lambda_A)^{N-2} \lambda_A \ln[\lambda_A]}{v\mu_B \lambda_A^{N-2} + \frac{\mu_A}{\mu_B} \phi_B (1 - \lambda_A)^{N-2}} < 0.
\end{aligned}
$$

As a result:

Proposition 6 *Let $\mu_A + \mu_B = 1, \mu_j > 0, j = A, B$. Let N firms randomize between advertising prices from σ_A in segment A, prices from σ_B in segment B, and prices from σ_M in the entire* *market, where* σ_j, $j = A, B, M$ *satisfy Lemma 5. Then, entry of firms causes equilibrium profits to fall.*

4.3 Price discrimination

We now discuss the implications of allowing firms to announce distinct prices to the different consumer segments when they advertise in the entire market. Our next result shows that firms benefit from restrictions on price discrimination when they compete in segmented markets.

Proposition 7 *Let* $\mu_A + \mu_B = 1, \mu_j > 0, j = A, B$. *If firms can practise price discrimination when they advertise in the entire market, then an equilibrium with positive profits does not exist.*

The proof is in the Appendix. The intuition behind this result is that price discrimination enables firms to see the different consumer segments as completely separate markets. This implies that a firm's profit from a segment is independent of the competitiveness of the other segment.

5 Concluding remarks

We have examined a strategic game of pricing and targeted-advertising. Firms sell a homogeneous good and must place ads in different newspapers to attract distinct groups of consumers. In the basic model each buyer reads only a single newspaper. We have shown that pure-strategy equilibria do not exist, which implies that an equilibrium where firms surely send their ads to distinct groups of buyers cannot be sustained. We have then focused on symmetric mixed strategy equilibria with positive profits. We have found that equilibrium must exhibit both random pricing and random advertising. In the unique equilibrium with positive profits, firms target ads with low prices to one segment of consumers, ads with high prices to the other segment of consumers and ads with intermediate prices to the entire market. This strategy enables firms to induce an unequal distribution of information in the market and the role of random pricing is to weaken competition. Interestingly, neither low prices nor high prices are heavily advertised, but intermediate ones.

The features of the equilibrium we have characterized prove to be robust to buyer multiple readership and entry of firms. We have shown that firms may obtain higher profits when the number of buyers who read both newspapers becomes larger; by contrast, we have seen that firms' profits fall as the market hosts more firms.

The model we have examined also applies to advertising in geographically segmented markets. In this connection, our model can naturally be seen as a model of strategic entry in multiple markets. The most remarkable result we derive is that firms are able to obtain positive profits in equilibrium by using targeted advertising.

6 Appendix

6.1 Appendix A

Proof of Lemma 5: The proof follows from a series of nine claims.

Claim 1: If an equilibrium exists, $\nexists p \in \sigma_A \cap \sigma_B \cap \sigma_M$.
Proof. We show this by contradiction. Suppose a price common to the supports of the price distributions existed. Then a firm would only be indifferent between advertising such a price in A and in the entire market if the profits from advertising it in B were negative, which yields a contradiction. ■

Claim 2: If an equilibrium exists, the set of prices which is exclusively advertised in newspaper j must be a continuum set, $j = A, B, M$.
Proof. We prove this by contradiction. Suppose there was an open interval of prices not chosen with positive probability in the set of prices that a firm exclusively advertises in, say, newspaper A. Consider a firm charging the infimum of this open interval; this firm would gain by increasing this price. ■

Claim 3: If an equilibrium exists, $\underline{p}_M \leq \overline{p}_B$ and $\underline{p}_M \leq \overline{p}_A$.
Proof. Suppose on the contrary that $\underline{p}_M > \overline{p}_B$. Then a firm advertising $\overline{p}_B$ in B would gain by increasing the price till $\underline{p}_M$. Proving that $\underline{p}_M \leq \overline{p}_A$ is analogous. ■

Claim 4: If an equilibrium exists, $\overline{p}_B \neq \overline{p}_A$.
Proof. To see this, suppose on the contrary that $\overline{p}_B = \overline{p}_A$. The claims above imply that it must be the case that $\overline{p}_M < \overline{p}_B = \overline{p}_A$. From this, it follows that $\overline{p}_B = \overline{p}_A = v$. Thus, a firm advertising v in newspaper A would make a profit of $v\mu_A\lambda_B - \phi_A$. A firm advertising v in B would obtain a benefit of $v\mu_B\lambda_A - \phi_B$. Since these two profits must be equal and strictly positive in equilibrium, this implies that a firm would gain by advertising v in both newspapers, which constitutes a contradiction. ■

We shall assume without loss of generality that $\overline{p}_A < \overline{p}_B$ in what follows. Claims 1-4 imply that if an equilibrium exists $\underline{p}_M \leq \overline{p}_A < \overline{p}_B$.

Claim 5: If an equilibrium exists $\overline{p}_M > \underline{p}_B$.
Proof. Suppose, on the contrary, that $\overline{p}_M < \underline{p}_B$; then we have two possibilities. First, $\overline{p}_M > \overline{p}_A$; in this case a firm advertising $\overline{p}_M$ in M, would strictly gain by increasing the price until $\underline{p}_B$. Second, $\overline{p}_M \leq \overline{p}_A$; if this is the case then a firm advertising $\overline{p}_A$ in A would gain by raising it until v, which contradicts Claim 4. It remains to prove that $\overline{p}_M = \underline{p}_B$ cannot be part of an equilibrium. We start noting that it must be the case that $\overline{p}_M = \underline{p}_B > \overline{p}_A$. To see this note that if

$\overline{p}_M = \underline{p}_B \leq \overline{p}_A$ then a firm advertising $\overline{p}_A$ in A would gain by raising this price. Further, $\overline{p}_B = v$ and therefore firms' expected profit in equilibrium is $\lambda_A \mu_B v - \phi_B > 0$. Furthermore, in equilibrium $E\pi_1\left(\lambda_M = 1, \overline{p}_M; s^2\right) = E\pi_1\left(\lambda_B = 1, \overline{p}_M; s^2\right)$, which is satisfied if and only if $\lambda_B \mu_A \overline{p}_M - \phi_A = 0$. However we note that if firm 1 deviates and advertises $p = v > \overline{p}_M$ in the whole market will obtain $E\pi_1^d\left(\lambda_M = 1, v; s^2\right) = \lambda_A \mu_B v - \phi_B + \lambda_B \mu_A v - \phi_A > \lambda_A \mu_B v - \phi_B$. Hence, the proof follows. ■

Claim 6: If an equilibrium exists then $\underline{p}_B > \overline{p}_A$.
Proof. Suppose not, i.e., $\underline{p}_B \leq \overline{p}_A$. The case $\underline{p}_B = \overline{p}_A$ is simply ruled out by Claim 1 above. Consider now $\underline{p}_B < \overline{p}_A$; the claims above imply that it must be the case that $\underline{p}_B < \overline{p}_M < \underline{p}_A < \overline{p}_A < \overline{p}_B = v$. If this were so, a firm advertising $\overline{p}_A$ in A would gain by raising its price till v, which contradicts Claim 4. ■

Claim 7: If an equilibrium exists then $\overline{p}_M = \overline{p}_B = v$.
Proof. We prove this by contradiction. First, suppose that $\overline{p}_B < \overline{p}_M$. The support configuration would then be: $\underline{p}_M \leq \overline{p}_A < \underline{p}_B < \overline{p}_B < \overline{p}_M = v$, where $\overline{p}_M = v$ because otherwise a firm advertising $\overline{p}_M$ in the entire market would gain by increasing its price. A firm must be indifferent between advertising any price $p \in [\overline{p}_B, v]$ in both newspapers. Then, it must be the case that $E\pi_1\left(\lambda_M = 1, p; s^2\right) = E\pi_1\left(\lambda_M = 1, v; s^2\right)$, which, using (3), yields:

$$F_M(p) = 1 - \frac{\lambda_A \mu_B + \lambda_B \mu_A}{\lambda_M}\left(\frac{v - p}{p}\right), \text{ for all } p \in [\overline{p}_B, v]. \tag{10}$$

Moreover, a firm must be indifferent between advertising any price $p \in \sigma_B(\subset \sigma_M) = [\underline{p}_B, \overline{p}_B]$ in newspaper B or in both newspapers. Thus, it must hold that $E\pi_1\left(\lambda_M = 1, p; s^2\right) = E\pi_1\left(\lambda_B = 1, p; s^2\right)$, which, using (2) and (3), yields:

$$F_M(p) = 1 - \frac{1}{\lambda_M}\left(\frac{\phi_A}{\mu_A p} - \lambda_B\right) \text{ for all } p \in [\underline{p}_B, \overline{p}_B] \tag{11}$$

The price distributions (10) and (11) must be equal at $p = \overline{p}_B$. Imposing this condition we obtain:

$$\overline{p}_B = \frac{\left(\lambda_A \mu_B + \lambda_B \mu_A\right) v - \frac{\phi_A}{\mu_A}}{\left(\lambda_A - \lambda_B\right) \mu_B}$$

A firm must be indifferent between advertising a price $p \in \sigma_B$ in B and advertising v in both newspapers, i.e., $E\pi_1\left(\lambda_B = 1, p; s^2\right) = E\pi_1\left(\lambda_M = 1, v; s^2\right)$. Using (11), this yields an expression for the distribution of advertised prices in B :

$$F_B(p) = \frac{\phi_A}{\mu_A \mu_B \lambda_B p} + \frac{\lambda_A}{\lambda_B} - \frac{\lambda_A \mu_B + \lambda_B \mu_A}{\lambda_B \mu_B} \frac{v}{p} \tag{12}$$

We can now determine the lowerbound of σ_B by solving $F_B(\underline{p}_B) = 0$ in (12):

$$\underline{p}_B = \frac{(\lambda_A\mu_B + \lambda_B\mu_A)\, v - \frac{\phi_A}{\mu_A}}{\lambda_A\mu_B}$$

Since $\underline{p}_B$ must be positive in equilibrium, it must be the case that $(\lambda_A\mu_B + \lambda_B\mu_A)\, v - \frac{\phi_A}{\mu_A} > 0$. Since $\overline{p}_B$ must also be positive, this implies that $\lambda_A - \lambda_B > 0$. Now we can compare $\overline{p}_B$ and v. For $\overline{p}_B < v$ to hold, it must be the case that $\lambda_B\mu_A v - \phi_A < 0$; but this implies that a firm advertising v in the entire market, which makes a profit of $\lambda_A\mu_B v - \lambda_B\mu_A v - \phi_A - \phi_B$, would gain by advertising it only in newspaper B. This contradicts Nash notion and proves that $\overline{p}_B \geq \overline{p}_M$.

It remains to prove that $\overline{p}_B > \overline{p}_M$ cannot be part of an equilibrium. If this were so, then $\overline{p}_B = v$, and therefore the support configuration would be $\underline{p}_M \leq \overline{p}_A < \underline{p}_B < \overline{p}_M < \overline{p}_B = v$. Moreover, in equilibrium $E\pi_1\left(\lambda_B = 1, v; s^2\right) = \lambda_A\mu_B v - \phi_B$. We know that a firm which deviates and advertises v in the entire market would obtain a profit $E^d\pi_1\left(\lambda_M = 1, v; s^2\right) = E\pi_1\left(\lambda_B = 1, v; s^2\right) + \lambda_B\mu_A v - \phi_A$. For this deviation not to be profitable, it must be the case that $\lambda_B\mu_A v - \phi_A \leq 0$. Further, in equilibrium a firm should be indifferent between advertising any price $p \in [\overline{p}_M, v]$ in B. This implies that $F_B(p) = 1 - \frac{\lambda_A}{\lambda_B}\left(\frac{v-p}{p}\right)$. Furthermore, for any $p \in [\underline{p}_B, \overline{p}_M]$, $E\pi_1\left(\lambda_M = 1, p; s^2\right) = E\pi_1\left(\lambda_B = 1, p; s^2\right)$, which yields $F_M(p) = 1 - \frac{\phi_A}{\lambda_M\mu_A p} + \frac{\lambda_B}{\lambda_M}$. The condition $F_M(\overline{p}_M) = 1$ yields $\lambda_B\mu_A\overline{p}_M - \phi_A = 0$, which is in contradiction with the condition above that $\lambda_B\mu_A v - \phi_A \leq 0$.

Therefore, if an equilibrium exists, then $\overline{p}_B = \overline{p}_M$, and by the usual arguments $\overline{p}_B = \overline{p}_M = v$. Hence, the claim follows. ■

Claim 8: An equilibrium exists only if $\underline{p}_A < \underline{p}_M$.
Proof. Let us assume that $\underline{p}_A \geq \underline{p}_M$, i.e., $\underline{p}_M \leq \underline{p}_A < \overline{p}_A < \underline{p}_B < \overline{p}_B = \overline{p}_M = v$. In equilibrium, for any $p \in \sigma_A$, $E\pi_1\left(\lambda_M = 1, p; s^2\right) = E\pi_1\left(\lambda_A = 1, p; s^2\right)$. This holds if and only if

$$p\mu_B[\lambda_A + \lambda_B + \lambda_M(1 - F_M(p))] - \phi_B = 0.$$

This yields an expression for the advertised prices in the entire market $F_M(p) = 1 - \frac{\phi_B}{\lambda_M\mu_B p} + \frac{\lambda_A + \lambda_B}{\lambda_M}$. Further, in equilibrium $E\pi_1\left(\lambda_A = 1, p; s^2\right) = \lambda_A\mu_B v - \phi_B$. Using $F_M(p)$, this equality leads to an expression for the advertised prices in A: $F_A(p) = \frac{\phi_B - \lambda_A\mu_B^2 v - \phi_A\mu_B}{\lambda_A\mu_A\mu_B p}$. Since $F_A(p) > 0$ for all $p \in \sigma_A$, it must be the case that $\phi_B - \lambda_A\mu_B^2 v - \phi_A\mu_B > 0$. But then $F_A(p)$ would be strictly decreasing in p, which cannot happen in equilibrium. ■

Claim 9: An equilibrium exists only if $\underline{p}_M = \overline{p}_A$.
Proof. The proof of this result is analogous to the proof of Claim 8 and therefore omitted. ■

This completes the proof the Lemma 5. ■

Proof of Proposition 1: To prove that the characterization summarized in Proposition 1 is indeed an equilibrium we need to show that: (i) firms do not have an incentive to deviate and (ii) there exists a non-empty set of parameters μ_A, μ_B, ϕ_A and ϕ_B for which $\lambda_A, \lambda_B, \lambda_M \in (0,1)$, $\lambda_A + \lambda_B + \lambda_M = 1$, the lower and upper bounds of the supports of the price distributions satisfy inequality given in Lemma 5, price distributions are well-behaved and expected profits are strictly positive. We start showing firms cannot profitably deviate. We note that there are various ways in which a firm may deviate. A firm may do so by advertising a price $p^d \notin \sigma_j$ in newspaper j. We now prove that this cannot constitute a profitable deviation. Take as given firm 2' strategy and let firm 1 deviate by advertising a price $p^d \notin \sigma_A$ in A. We have two possibilities. One, let $p^d \in (\overline{p}_A, \underline{p}_B]$, then using (1), the expected profit to firm 1 is $E\pi_1\left(\lambda_A = 1, p^d; s^2\right) = \lambda_B\mu_A p^d + \lambda_M\mu_A p^d(1 - F_M\left(p^d\right)) - \phi_A$. Using (8), we note that $E\pi_1\left(\lambda_A = 1, p^d; s^2\right) = \lambda_A\mu_A\mu_B\left(v - p^d\right) - \phi_A\left(1 - \mu_A\right)$, which is strictly decreasing in p^d. Therefore, this deviation is not profitable. Two, let $p^d \in [\underline{p}_B, v]$; then using (1) and (5) leads to $E\pi_1\left(\lambda_A = 1, p^d; s^2\right) = 0$. Thus, firm 1 cannot increase its profits by advertising a price $p^d \notin \sigma_A$ in segment A. Second, let firm 1 deviate by advertising a price $p^d \notin \sigma_B$ in B; again, we have two possibilities. One, let $p^d \in [\underline{p}_A, \overline{p}_A]$; then, using (2), we observe that $E\pi_1\left(\lambda_B = 1, p^d; s^2\right) = \mu_B p^d - \phi_B$. Since this expression is strictly increasing in p^d, if firm 1 deviates in such a way it will set $p^d = \overline{p}_A = \frac{\phi_B}{\mu_B}$; however this yields zero profits. Thus, this deviation is not profitable. Two, let $p^d \in [\overline{p}_A, \underline{p}_B)$, then using (2) and (8), the expected profit to firm 1 is $E\pi_1\left(\lambda_B = 1, p^d; s^2\right) = \lambda_A\mu_B\mu_A p^d + \lambda_A\mu_B v + \phi_A\mu_B - \phi_B$. Since this expression is strictly increasing in p^d, firm 1 does not find deviating profitable. Third, suppose firm 1 deviates by advertising a price $p^d \notin \sigma_M$ in the entire market. Then $p^d \in [\underline{p}_A, \overline{p}_A)$. Using (3), the profit to firm 1 is $E\pi_1\left(\lambda_M = 1, p^d; s^2\right) = (1 - \lambda_A)\, p^d + \lambda_A\mu_B p^d + \lambda_A\mu_A p^d(1 - F_A\left(p^d\right)) - \phi_A - \phi_B$. Using the expression for $F_A(p)$ obtained above this profit can be rewritten as $E\pi_1\left(\lambda_M = 1, p^d; s^2\right) = \mu_B p^d + (1 - \lambda_A)\,\mu_A\overline{p}_A - \phi_A - \phi_B$, which is strictly increasing in p^d. Hence, firm 1 has no incentive to deviate. We now remark that a firm may also deviate by advertising a price $p \in \sigma_j$ in newspaper $j' \neq j$. This type of deviation is however ruled out by the cases above where a firm advertises a price $p \notin \sigma_{j'}$ in j'. Finally, a firm may also deviate by announcing a price $p \notin \sigma_j$ in $j' \neq j$, but these deviations are also ruled out by the previous arguments. This completes the proof of (i).

We now show (ii). Let us assume without loss of generality, that $\phi_A = \gamma v$, with $\gamma \in (0,1)$ and $\phi_B = \beta\phi_A = \beta\gamma v$, with $\beta \in \left(0, \frac{1}{\gamma}\right)$. Then the above conditions are satisfied if and only if the following inequalities hold:

$\lambda_A \in (0,1) \Leftrightarrow \beta > \max\{\mu_B; 1\}$ and $\gamma < \frac{\mu_B}{\beta - 1}$;
$\lambda_B \in (0,1) \Leftrightarrow \gamma < \mu_A$;

$\lambda_M > 0 \Leftrightarrow \lambda_A + \lambda_B < 1 \Leftrightarrow \mu_A \gamma (\beta - \mu_B) - \left(\mu_B^2 + \beta\gamma\mu_A\right)(\mu_A - \gamma) < 0;$
$E\pi_1 > 0 \Leftrightarrow \gamma < \frac{\mu_B(\mu_A\beta - \mu_B)}{\beta^2\mu_A}$ and $\beta > \frac{\mu_B}{\mu_A}$;
$\underline{p}_B > 0 \Leftrightarrow \gamma < \frac{\mu_A\beta - \mu_B}{\beta\mu_A}$;
$\underline{p}_M < \underline{p}_B \Leftrightarrow \gamma < \frac{\mu_B(\mu_A\beta - \mu_B)}{\beta^2\mu_A}$;

Therefore an equilibrium exists if and only if:
Condition 1.1: $\beta > \max\left\{\mu_B, 1, \frac{\mu_B}{\mu_A}\right\}$
Condition 1.2: $\gamma < \mu_A$
Condition 1.3: $\gamma < \frac{\mu_B}{\beta - 1}$
Condition 1.4: $\gamma < \frac{\mu_B(\mu_A\beta - \mu_B)}{\beta^2\mu_A}$
Condition 1.5: $\mu_A\gamma(\beta - \mu_B) < \left(\mu_B^2 + \beta\gamma\mu_A\right)(\mu_A - \gamma)$.
Condition 1.6: $\gamma < \frac{\mu_A\beta - \mu_B}{\beta\mu_A}$.

First, we note that if 1.1 and 1.4 hold, then 1.3 holds. Also, note that if 1.4 and 1.1 hold then 1.6 is satisfied. Thus, conditions $1.1, 1.2, 1.4$ and 1.5 suffice for an equilibrium. To prove that the set of parameters satisfying these conditions is not empty, we consider the case in which the two newspapers have the same consumer base, *i.e.*, $\mu_A = \mu_B = \frac{1}{2}$. Given this, conditions $1.1, 1.2, 1.4$ and 1.5 can be rewritten as follows: $\beta > 1$, $\gamma < \min\left\{\frac{1}{2}, \frac{\beta-1}{2\beta^2}\right\}$and $\frac{1}{4} - \frac{1}{2}\beta\gamma - \gamma^2\beta > 0$. Given $\beta > 1$, $\min\left\{\frac{1}{2}, \frac{\beta-1}{2\beta^2}\right\} = \frac{\beta-1}{2\beta^2}$. Further, given $\beta > 1$, then $\frac{1}{4} - \frac{1}{2}\beta\gamma - \gamma^2\beta > 0$, for any $\gamma \in \left(0, \frac{\beta-1}{2\beta^2}\right)$. This is because $\frac{1}{4} - \frac{1}{2}\beta\gamma - \gamma^2\beta$ is decreasing in γ and because at $\gamma = \frac{\beta-1}{2\beta^2}$, the expression $\frac{1}{4} - \frac{1}{2}\beta\gamma - \gamma^2\beta = \frac{2\beta-1}{4\beta^3} > 0$. Hence, given $\mu_A = \mu_B$, an equilibrium exists if $\beta > 1$ and $\gamma < \frac{\beta-1}{2\beta^2}$. This condition is represented above in Figure 2. ■

Proof of Proposition 2. Given that the σ_A, σ_B and σ_M must satisfy Lemma 5, we only need to show that $F_M(p) > F_B(p)$ for all $p \in \sigma_B$. Using the expressions above, it follows that this amounts to show that $\lambda_A(1 - \lambda_A) > \lambda_B$. For convenience we assume again that $\phi_A = \gamma v$, with $\gamma \in (0, 1)$ and $\phi_B = \beta\phi_A = \beta\gamma v$, with $\beta \in \left(0, \frac{1}{\gamma}\right)$. Using the expressions for λ_A and λ_B given in Proposition 1, one gets that $\lambda_A(1 - \lambda_A) > \lambda_B$ if and only if

$$\frac{\gamma(\beta - \mu_B)}{\mu_B^2 + \gamma\beta\mu_A}\left(1 - \frac{\gamma(\beta - \mu_B)}{\mu_B^2 + \gamma\beta\mu_A}\right) > \frac{\gamma}{\mu_A},$$

or

$$-\gamma\left(\gamma\mu_A + \mu_B\right)\frac{\mu_B^2(1 + \beta) + \beta^2\gamma\mu_A - \beta\mu_B}{\left(\mu_B^2 + \beta\gamma - \beta\gamma\mu_B\right)^2\mu_A} > 0,$$

or

$$\gamma < \frac{\beta\mu_B - \mu_B^2(1+\beta)}{\beta^2\mu_A} = \frac{\mu_B(\beta - \mu_B(1+\beta))}{\beta^2\mu_A} = \frac{\mu_B(\beta\mu_A + \mu_B)}{\beta^2\mu_A}$$

which is always satisfied in equilibrium (condition 1.4). ■

Proof of Proposition 3. (1) First, $\frac{\partial\lambda_A}{\partial\gamma} = \frac{\mu_B^2(\beta-\mu_B)}{\left(\mu_B^2+\mu_A\beta\gamma\right)^2}$ which is strictly positive given the condition 1.1. Second, $\frac{\partial\lambda_B}{\partial\gamma} = \frac{\gamma}{\mu_A} > 0$; as a consequence $\frac{\partial\lambda_M}{\partial\gamma} < 0$. Third, we claim that an increase in γ widens σ_B. To see this note that $\underline{p}_B = v\left(1 - \frac{\lambda_B}{\lambda_A}\right)$ and $\frac{\lambda_B}{\lambda_A} = \frac{\mu_B^2+\mu_A\beta\gamma}{\mu_A(\beta-1)}$. Since an increase in γ raises $\frac{\lambda_B}{\lambda_A}$, it follows that $\frac{\partial\underline{p}_B}{\partial\gamma} < 0$; thus the claim follows. Fourth, an increase in γ widens σ_A if and only if $\frac{\partial\left(\frac{\overline{p}_A}{\underline{p}_A}\right)}{\partial\gamma} = \frac{\partial\left(\frac{1}{1-\lambda_A}\right)}{\partial\gamma} > 0$; since $\frac{\partial\lambda_A}{\partial\gamma} > 0$ this is always satisfied. Fifth, an increase in γ narrows σ_M because $\frac{\partial\underline{p}_M}{\partial\gamma} = \frac{\beta v}{\mu_B} > 0$. Finally, we note that as $\gamma \to 0$, λ_A and λ_B converge to zero which implies that $\lambda_M \to 1$. In addition $\underline{p}_M$ goes to zero and $F_M(p) \to 1$. We now prove (2). First, $\frac{\partial\lambda_A}{\partial\beta} = \frac{\gamma\mu_B(\mu_B+\mu_A\gamma)}{\left(\mu_B^2+\mu_A\beta\gamma\right)^2}$, which is always strictly positive. Second, $\frac{\partial\lambda_B}{\partial\beta} = 0$. These two points imply that $\frac{\partial\lambda_M}{\partial\beta} < 0$. Third, since $\underline{p}_B = v\left(1 - \frac{\lambda_B}{\lambda_A}\right)$, $\frac{\partial\lambda_A}{\partial\beta} > 0$ and $\frac{\partial\lambda_B}{\partial\beta} = 0$, it follows that an increase in β narrows σ_B, *i.e.* $\frac{\partial\underline{p}_B}{\partial\beta} > 0$. Fourth, similar to (1), since $\frac{\partial\lambda_A}{\partial\beta} > 0$, it follows that an increase in β widens σ_A. Finally, an increase in β narrows σ_M because $\frac{\partial\underline{p}_M}{\partial\beta} = \frac{\gamma v}{\mu_B} > 0$. We conclude with (3). First, we claim that $\frac{\partial\lambda_A}{\partial\mu_B} < 0$. We note that $\frac{\partial\lambda_A}{\partial\mu_B} = \frac{\gamma[\beta\gamma(\beta-1)-\mu_B(2\beta-\mu_B)]}{\left(\mu_B^2+\mu_A\beta\gamma\right)^2} < 0$ if and only if $\beta\gamma(\beta-1) - \mu_B(2\beta-\mu_B) < 0$; this condition is satisfied if and only if $\gamma < \frac{\mu_B(2\beta-\mu_B)}{\beta(\beta-1)}$, which holds given the parameter restrictions above; hence, the claim follows. Second, $\frac{\partial\lambda_B}{\partial\mu_B} = \frac{\gamma}{\mu_A} > 0$. Third, since $\underline{p}_B = v\left(1 - \frac{\lambda_B}{\lambda_A}\right)$, $\frac{\partial\lambda_A}{\partial\mu_B} < 0$ and $\frac{\partial\lambda_B}{\partial\mu_B} > 0$, it follows that an increase in μ_B widens σ_B, *i.e.* $\frac{\partial\underline{p}_B}{\partial\mu_B} < 0$. Fourth, since $\frac{\partial\lambda_A}{\partial\mu_B} < 0$, an increase in μ_B narrows σ_A. Finally, since $\underline{p}_M = \frac{\beta\gamma v}{\mu_B}$ an increase in μ_B widens σ_M. The proof is now complete. ■

Proof of Proposition 4: Firms' payoffs from the different advertising strategies are given by:

$$\begin{aligned}
E\pi_1(\lambda_A &= 1, p \in \sigma_A; s^2) = p\left(\mu_A + \mu_M\right)\left[\lambda_A(1 - F_A(p)) + \lambda_B + \lambda_M\right] - \phi_A && (13)\\
E\pi_1(\lambda_B &= 1, p \in \sigma_B; s^2) = p[\mu_B\lambda_A + \lambda_B(\mu_B + \mu_M)(1 - F_B(p)) && (14)\\
&\quad + \lambda_M(\mu_B + \mu_M)(1 - F_M(p))] - \phi_B \\
E\pi_1(\lambda_M &= 1, p \in \sigma_M \cap \sigma_B; s^2) = p[\mu_B\lambda_A + \lambda_B[\mu_A + (\mu_B + \mu_M)(1 - F_B(p))] && (15)\\
&\quad + \lambda_M(1 - F_M(p))] - \phi_A - \phi_B
\end{aligned}$$

Since $p = v \in \sigma_B \cap \sigma_M$ it must be the case that $E\pi_1\left(\lambda_B = 1, v; s^2\right) = E\pi_1\left(\lambda_M = 1, v; s^2\right)$. This implies that $\lambda_B\mu_A v - \phi_A = 0$ and therefore $\lambda_B = \frac{\phi_A}{\mu_A v}$. Moreover, for all $p \in [\underline{p}_B, v]$ it must be the case that $E\pi_1\left(\lambda_B = 1, p; s^2\right) = E\pi_1\left(\lambda_M = 1, p; s^2\right)$, which is satisfied if and only if $\lambda_B\mu_A p + \lambda_M\mu_A p(1 - F_M(p)) - \phi_A = 0$. Solving this condition we obtain:

$$F_M(p) = 1 - \frac{\lambda_B}{\lambda_M}\frac{(v-p)}{p}$$

Using this last expression and solving the equilibrium condition $E\pi_1(\lambda_B = 1, p) = E\pi_1(\lambda_B = 1, v)$, yields to:

$$F_B(p) = \frac{v}{p} - \frac{\lambda_A}{\lambda_B}\frac{\mu_B}{1-\mu_A}\frac{(v-p)}{p}$$

For an equilibrium it must be the case that $F_B(\underline{p}_B) = 0$, which yields $\underline{p}_B = v\left(\frac{\lambda_A\mu_B - (1-\mu_A)\lambda_B}{\lambda_A\mu_B}\right)$

Next, for any $p \in (\underline{p}_M, \underline{p}_B)$ we impose that $E\pi_1\left(\lambda_M = 1, p; s^2\right) = E\pi_1\left(\lambda_B = 1, v; s^2\right)$. This condition is satisfied if and only if:

$$F_M(p) = \frac{1-\lambda_A}{\lambda_M} - \frac{\lambda_A\mu_B}{\lambda_M}\frac{(v-p)}{p} - \frac{\phi_A}{\lambda_M p}$$

Using this last expression we can derive $\underline{p}_M$ by solving $F_M(\underline{p}_M) = 0$, which yields $\underline{p}_M = \frac{(\lambda_A\mu_B + \lambda_B\mu_A)v}{\mu_B + (1-\lambda_A)(1-\mu_B)}$. Finally, for any $p \in [\underline{p}_A, \underline{p}_M]$, $E\pi_1\left(\lambda_A, p; s^2\right)$ and $E\pi_1\left(\lambda_B, v; s^2\right)$ must be equal, which holds if and only if:

$$F_A(p) = 1 - \frac{\mu_B}{1-\mu_B}\frac{v}{p} + \frac{(1-\lambda_A)}{\lambda_A} - \frac{\phi_A - \phi_B}{\lambda_A(1-\mu_B)p}$$

We can now determine $\underline{p}_A$ and $\overline{p}_A$ by solving $F_A(\underline{p}_A) = 0$ and $F_A(\overline{p}_A) = 1$, respectively. This yields $\underline{p}_A = \frac{\mu_B\lambda_A v + \phi_A - \phi_B}{(1-\mu_B)}$ and $\overline{p}_A = \frac{\mu_B\lambda_A v + \phi_A - \phi_B}{(1-\lambda_A)(1-\mu_B)}$. Finally, since $\overline{p}_A$ and $\underline{p}_M$ must be equal, we obtain $\lambda_A = \frac{\phi_B - \phi_A\mu_B}{\mu_B^2 v + \phi_B(1-\mu_B)}$.

We now turn to the existence of this equilibrium. We start analyzing possible deviations; we assume, without loss of generality, that firm 2 follows the strategy prescribed by Proposition 3. Let firm 1 advertise a price $p^d \notin \sigma_A$ in A. Then we have two possibilities. One, assume $p^d \in (\overline{p}_A, \underline{p}_B]$, then $E\pi_1\left(\lambda_A = 1, p^d; s^2\right) = \lambda_B(1-\mu_B)p^d + \lambda_M(1-\mu_B)p^d(1 - F_M(p^d)) - \phi_A$. Using the expression for $F_M(p)$ in Proposition 3 yields $E\pi_1\left(\lambda_A = 1, p^d; s^2\right) = (1-\mu_B)\left(\lambda_A\mu_B\left(v - p^d\right) + \phi_A\right) - \phi_A$, which is strictly decreasing in p^d. Thus, firm 1 will not deviate. Two, suppose $p^d \in [\underline{p}_B, v]$, then $E\pi_1\left(\lambda_A = 1, p^d; s^2\right) = \lambda_B\mu_A p^d + \lambda_B\mu_M p^d(1 - F_B(p^d)) + \lambda_M(1-\mu_B)p^d(1 - F_M(p^d)) - \phi_A$. Using the expressions for $F_B(p)$ and $F_M(p)$ derived above we obtain: $E\pi_1\left(\lambda_A = 1, p^d; s^2\right) =$

$\frac{\mu_M}{(1-\mu_A)}\lambda_A\mu_B\left(v-p^d\right)$. This profit is decreasing in p^d and therefore the firm will set $p^d = \underline{p}_B = v\left(\frac{\lambda_A\mu_B-\lambda_B(1-\mu_A)}{\lambda_A\mu_B}\right)$, which yields $E\pi_1\left(\lambda_A=1,p^d;s^2\right)=v\lambda_B\mu_M$. For this deviation not to be profitable, it must be the case that $E\pi_1\left(\lambda_A=1,p^d;s^2\right)=v\lambda_B\mu_M<v\lambda_A\mu_B-\phi_B$. Second, let firm 1 advertise a price $p^d\notin\sigma_B$ in B. Again, we have two possibilities. One, assume $p^d\in[\underline{p}_A,\overline{p}_A)$, then $E\pi_1\left(\lambda_B=1,p^d;s^2\right)=\lambda_A\mu_Bp^d+\lambda_A\mu_Mp^d(1-F_A\left(p^d\right))+(1-\lambda_A)\left(1-\mu_A\right)p^d-\phi_B$. Using the expression for $F_A\left(p\right)$ derived above we obtain $E\pi_1\left(\lambda_B=1,p^d;s^2\right)=\mu_M\left(1-\lambda_A\right)\overline{p}_A+\mu_Bp^d-\phi_B$. Since this expression is strictly increasing in p^d the most profitable deviation consists of setting $p^d=\overline{p}_A$ and therefore: $E\pi_1\left(\lambda_B=1,p^d;s^2\right)=\mu_M\left(1-\lambda_A\right)\overline{p}_A+\mu_B\overline{p}_A-\phi_B$. Since in equilibrium $E\left(\lambda_A=1,\overline{p}_A;s^2\right)=\left(1-\lambda_A\right)\left(1-\mu_B\right)\overline{p}_A-\phi_A$, firm 1 will not deviate if and only if $\mu_B\overline{p}_A-\phi_B<\left(1-\lambda_A\right)\mu_A\overline{p}_A-\phi_A$, which holds if and only if: $\overline{p}_A\left(\mu_B-\mu_A+\lambda_A\mu_A\right)<\phi_B-\phi_A$. Two, let $p^d\in[\overline{p}_A,\underline{p}_B)$, then $E\pi_1\left(\lambda_B=1,p^d;s^2\right)=\lambda_A\mu_Bp^d+\lambda_B\left(1-\mu_A\right)p^d+\lambda_M\left(1-\mu_A\right)p^d(1-F_M\left(p^d\right))-\phi_B$. Using the expression for $F_M\left(p\right)$ we obtain $E\pi_1\left(\lambda_B=1,p^d;s^2\right)=\lambda_A\mu_B\left(1-\mu_A\right)v+p^d\lambda_A\mu_B\mu_A+\phi_A\left(1-\mu_A\right)-\phi_B$, which is strictly increasing in p^d. Thus, firm 1 will not deviate. Third, let firm 1 advertise a price $p^d\notin\sigma_M$ in the entire market, then firm 1 will set a price $p^d\in[\underline{p}_A;\overline{p}_A)$. The expected profit will then be $E\pi_1\left(\lambda_M=1,p^d;s^2\right)=\lambda_A\mu_Bp^d+\lambda_A\left(1-\mu_B\right)p^d(1-F_A\left(p^d\right))+\left(1-\lambda_A\right)p^d-\phi_A-\phi_B$. Using the expression for $F_A\left(p\right)$ we obtain $E\pi_1\left(\lambda_M=1,p^d;s^2\right)=\left(1-\mu_B\right)\left(1-\lambda_A\right)\overline{p}_A+p^d\mu_B-\phi_A-\phi_B$, which is strictly increasing in p^d. Thus firm 1 will not deviate. Summarizing, for a firm not to have incentives to deviate, the following two conditions must hold: (i) $v\lambda_B\mu_M<v\lambda_A\mu_B-\phi_B$ and $(ii)\overline{p}_A\left(\mu_B-\mu_A+\lambda_A\mu_A\right)<\phi_B-\phi_A$.

We now show that there exists a non-empty set of μ_A,μ_B,ϕ_A and ϕ_B such that the characterization provided in Proposition 3 is well defined and the no-deviation conditions are satisfied. In particular, for this equilibrium to exist we need to verify that λ_A,$\lambda_B,\lambda_M\in(0,1)$, that $\lambda_A+\lambda_B+\lambda_M=1$, that lower and upper bounds of the supports of the price distributions satisfy the above inequality, and that expected profits are strictly positive. Let us assume, without loss of generality, that $\phi_A=\gamma v$, with $\gamma\in(0,1)$ and $\phi_B=\beta\phi_A=\beta\gamma v$, with $\beta\in\left(0,\frac{1}{\gamma}\right)$. Assume also that $\mu_A\geq\mu_B$. Then the above conditions are satisfied if and only if the following inequalities hold:

$\lambda_A\in(0,1)\Leftrightarrow\beta>\max\{\mu_B;1\}$ and $\gamma<\frac{\mu_B}{\beta-1}$;

$\lambda_B\in(0,1)\Leftrightarrow\gamma<\mu_A$;

$\lambda_A+\lambda_B<1\Leftrightarrow\mu_A\gamma\left(\beta-\mu_B\right)-\left(\mu_B^2+\beta\gamma\left(1-\mu_B\right)\right)\left(\mu_A-\gamma\right)<0$;

$E\pi_1\left(\lambda_B=1,v\right)>0\Leftrightarrow\gamma<\frac{\mu_B}{\beta(1-\mu_B)}\left(\frac{\beta(1-\mu_B)-\mu_B}{\beta}\right)$ and $\beta>\frac{1-\mu_B}{\mu_B}$;

$\underline{p}_B>0\Leftrightarrow\gamma<\frac{\mu_B(\beta\mu_A-\mu_B)}{\beta(1-\mu_A)(1-\mu_B)}$ and $\beta>\frac{\mu_B}{\mu_A}$;

$\underline{p}_M<\underline{p}_B\Leftrightarrow\left(\lambda_A\mu_B-\lambda_B+\gamma\right)\left(1-\lambda_A-\mu_B\right)+\lambda_A\mu_B\left(\gamma\beta-\lambda_B\right)>0$;

Therefore an equilibrium exists if and only if:

Condition 3.1. $\beta>\max\left\{\mu_B,1,\frac{\mu_B}{\mu_A},\frac{\mu_B}{1-\mu_B}\right\}$

Condition 3.2 $\beta<\frac{1}{\gamma}$

Condition 3.3 $\gamma < \mu_A$

Condition 3.4 $\gamma < \frac{\mu_B}{\beta - 1}$

Condition 3.5 $\gamma < \frac{\mu_B}{\beta(1-\mu_B)} \left(\frac{\beta(1-\mu_B)-\mu_B}{\beta} \right)$

Condition 3.6 $\gamma < \frac{\mu_B}{\beta(1-\mu_B)} \left(\frac{\beta\mu_A - \mu_B}{1-\mu_A} \right)$

Condition 3.7 $\left(\mu_B^2 + \beta\gamma\left(1-\mu_B\right)\right)\left(\mu_A - \gamma\right) > \gamma\mu_A\left(\beta - \mu_B\right)$

Condition 3.8 $\left(\lambda_A\mu_B - \lambda_B + \gamma\right)\left(1 - \lambda_A - \mu_B\right) + \lambda_A\mu_B\left(\gamma\beta - \lambda_B\right) > 0$

Moreover rewriting the no-deviation conditions derived above, we obtain the following additional two inequalities:

Condition 3.9 $\gamma > \frac{(\mu_B - \mu_A\beta)(1-\mu_B)\mu_B}{(1-\mu_B)\beta[\beta\mu_A + \mu_M]}$

Condition 3.10 $\overline{p}_A\left(\mu_B - \mu_A + \lambda_A\mu_A\right) - \gamma v\left(\beta - 1\right) < 0$

We start noting that if conditions $3.1, 3.2, 3.3$ and 3.6 hold then 3.4 holds. To see this note that $\frac{\mu_B}{\beta-1} > \frac{\mu_B}{\beta(1-\mu_B)} \frac{\beta\mu_A - \mu_B}{1-\mu_A}$ if and only if $\left(\beta - \mu_B\right)\left(1 - \beta\mu_A\right) > 0$. Given condition 3.1 $\beta - \mu_B > 0$ and therefore the claim follows if and only if $\left(1 - \beta\mu_A\right) > 0$, which is always satisfied given conditions 3.2 and 3.3. Further, given the condition 3.1 and the fact that $\gamma > 0$ then condition 3.9 holds. Thus, the relevant conditions for the existence of an equilibrium are $3.1, 3.2, 3.3, 3.5, 3.6, 3.7, 3.8, 3.10$. Now, let $\mu_A = \mu_B = \frac{1}{4}$. It follows that: $\beta > \max\left\{\frac{1}{4}, 1, 1, \frac{1}{3}\right\} = 1$, $\beta < \frac{1}{\gamma}$, $\gamma < \min\left\{\frac{1}{4}, \frac{3\beta-1}{12\beta^2}, \frac{\beta-1}{9\beta}\right\}$, $\frac{1}{16} - \frac{1}{4}\beta\gamma - 3\beta\gamma^2 > 0$, $\left(\frac{1}{4}\lambda_A - \lambda_B + \gamma\right)\left(\frac{3}{4} - \lambda_A\right) + \frac{1}{4}\lambda_A\left(\gamma\beta - \lambda_B\right) > 0$ and $\frac{1}{4}\overline{p}_A\lambda_A - \gamma\left(\beta - 1\right) < 0$.

We now assume that $\beta = 2$. It follows that for an equilibrium we need that $\gamma < \min\left\{\frac{1}{4}, \frac{5}{48}, \frac{1}{18}\right\} = \frac{1}{18}$, $\frac{1}{16} - \frac{1}{2}\gamma - 6\gamma^2 > 0$, $\left(\frac{1}{4}\lambda_A - \lambda_B + \gamma\right)\left(\frac{3}{4} - \lambda_A\right) + \frac{1}{4}\lambda_A\left(2\gamma - \lambda_B\right) > 0$ and $\frac{1}{4}\overline{p}_A\lambda_A - \gamma v < 0$. We note that given $\gamma < \frac{1}{18}$, then $\frac{1}{16} - \frac{1}{2}\gamma - 6\gamma^2 > 0$ holds. To see this, note that this function is decreasing in γ and at $\gamma = \frac{1}{18}$ is positive. Further the condition $\left(\frac{1}{4}\lambda_A - \lambda_B + \gamma\right)\left(\frac{3}{4} - \lambda_A\right) + \frac{1}{4}\lambda_A\left(2\gamma - \lambda_B\right) > 0$ is equivalent to $1 - 36\gamma + 128\gamma^2 > 0$. This condition is satisfied for all $\gamma \in \left[0, \frac{1}{32}\right]$. Moreover the condition $\frac{1}{4}\overline{p}_A\lambda_A - \gamma v < 0$ is equivalent to $\gamma\frac{32\gamma - 1}{1 + 24\gamma} < 0$, which is always satisfied for any $\gamma \leq \frac{1}{32}$. This completes the proof. ■

Proof of Proposition 5: We start proving part (1). Recall that $\lambda_A = \frac{\phi_B - \phi_A\mu_B}{\mu_B^2 v + \phi_B(1-\mu_B)}$ and $\lambda_B = \frac{\phi_A}{\mu_A v}$. Let, as above, $\phi_A = \gamma v$ and $\phi_B = \beta\phi_A$. First, keeping μ_B constant, λ_A does not depend on μ_M and therefore $\frac{\partial\lambda_A}{\partial\mu_M} = 0$. Second, an increase in μ_M lowers μ_A and therefore $\frac{\partial\lambda_B}{\partial\mu_M} > 0$. From these observations, it follows that $\frac{\partial\lambda_M}{\partial\mu_M} < 0$. Third, we note that $\frac{\overline{p}_A}{\underline{p}_A} = \frac{1}{1-\lambda_A}$; since $\frac{\partial\lambda_A}{\partial\mu_M} = 0$ it follows that $\frac{\partial\left(\overline{p}_A/\underline{p}_A\right)}{\partial\mu_M} = 0$. Thus, an increase in μ_M leaves σ_A unaltered. Fourth, we note that $\underline{p}_B = \left(\frac{\lambda_A\mu_B - \lambda_B(\mu_B + \mu_M)}{\lambda_A\mu_B}\right)v$; since $\frac{\partial\lambda_A}{\partial\mu_M} = 0$ and $\frac{\partial\lambda_B}{\partial\mu_M} > 0$, it follows that an increase in μ_M lowers $\underline{p}_B$ and therefore widens σ_B. Fifth, since $\underline{p}_M = \frac{\lambda_A\mu_B v + \phi_A - \phi_B}{(1-\lambda_A)(1-\mu_B)}$ does not depend on μ_M, it follows that an increase in μ_M leaves σ_M unaltered. Finally, we note that equilibrium expected profit is $E\pi_1\left(\lambda_B = 1, v; s^2\right) = \lambda_A\mu_B v - \phi_A$ which does not depend on μ_M and μ_B is constant. We now turn to part (2). First, we note that $\frac{\partial\lambda_A}{\partial\mu_M} = \frac{\beta\gamma^2 - \gamma\mu_B^2 - \beta^2\gamma^2 + 2\mu_B\gamma\beta}{\left(\mu_B^2 + \beta\gamma(1-\mu_B)\right)^2}$. Inspection of this equation reveals that

$\frac{\partial \lambda_A}{\partial \mu_M} > 0$ if $\gamma < \frac{\mu_B(2\beta-\mu_B)}{\beta(\beta-1)}$. We now show that this last inequality holds in equilibrium. Condition 3.4 implies that $\gamma < \frac{\mu_B}{\beta-1}$, so if $\frac{\mu_B}{\beta-1} < \frac{\mu_B(2\beta-\mu_B)}{\beta(\beta-1)}$ then the proof follows. Note that $\frac{\mu_B}{\beta-1} < \frac{\mu_B(2\beta-\mu_B)}{\beta(\beta-1)}$ whenever $\frac{(2\beta-\mu_B)}{\beta} > 1$ which is always satisfied by condition 3.1. Thus, $\frac{\partial \lambda_A}{\partial \mu_M} > 0$. Second, since μ_A is constant, λ_B does not depend on μ_M and therefore $\frac{\partial \lambda_B}{\partial \mu_M} = 0$. As a consequence $\frac{\partial \lambda_M}{\partial \mu_M} < 0$. Third, since $\underline{p}_A = (1-\lambda_A)\,\overline{p}_A$, it follows that $\frac{\overline{p}_A}{\underline{p}_A} = \frac{1}{1-\lambda_A}$; thus since $\frac{\partial \lambda_A}{\partial \mu_M} > 0$ it follows that an increase in μ_M increases λ_A, which also implies that it increases $\frac{\overline{p}_A}{\underline{p}_A}$ and therefore widens σ_A. Fourth, since $\underline{p}_M = \frac{\lambda_A(1-\mu_A-\mu_M)v+\phi_A-\phi_B}{(1-\lambda_A)(\mu_A+\mu_M)}$, we can calculate:

$$\frac{\partial \underline{p}_M}{\partial \mu_M} = \frac{\frac{\partial \lambda_A}{\partial \mu_M}(1-\mu_B)(\mu_B v+\phi_A-\phi_B) - (1-\lambda_A)(\lambda_A v+\phi_A-\phi_B)}{[(1-\lambda_A)(1-\mu_B)]^2}$$

Inspection of this derivative reveals that $\partial \underline{p}_M/\partial \mu_M > 0$ if and only if the numerator is positive. Using the expressions for $\frac{\partial \lambda_A}{\partial \mu_M}$ and λ_A we obtain that the condition above is satisfied if and only if $[\mu_B - \gamma(\beta-1)]^2(1-\mu_B) > 0$, which is always satisfied. Thus, an increase in μ_M narrows σ_M. Fifth, using the expression for $\underline{p}_B$ we can derive:

$$\frac{\partial \underline{p}_B}{\partial \mu_M} = \frac{\lambda_B(1-\mu_A)}{\lambda_A^2\mu_B^2}\left[\mu_B\frac{\partial \lambda_A}{\partial \mu_M} - \lambda_A\right]$$

Using the expressions for $\frac{\partial \lambda_A}{\partial \mu_M}$ and λ_A we obtain that

$$\frac{\partial \underline{p}_B}{\partial \mu_M} = \frac{\lambda_B(1-\mu_A)}{\lambda_A^2\mu_B^2}\left[\gamma\left(2\mu_B-\beta-\mu_B^2\right)+\mu_B^2\right]$$

We now prove that $2\mu_B-\beta-\mu_B^2 \le 0$. Suppose, on the contrary, that $2\mu_B-\beta-\mu_B^2 > 0$ or $\beta < \mu_B(2-\mu_B)$; since condition 3.1 requires that $\beta > 1$, then it must be the case that $\mu_B(2-\mu_B) > 1$, which is never satisfied. As a consequence $\frac{\partial \underline{p}_B}{\partial \mu_M} \ge 0$ if and only if $\gamma \le \frac{\mu_B^2}{\beta-\mu_B(2-\mu_B)} = \overline{\gamma}$, otherwise $\frac{\partial \underline{p}_B}{\partial \mu_M} < 0$. Finally, we observe that firm equilibrium profits change with μ_M as follows: $\frac{\partial E\pi_1(\lambda_B=1,v;s^2)}{\partial \mu_M} = v\left[\mu_B\frac{\partial \lambda_A}{\partial \mu_M} - \lambda_A\right]$. Using the previous arguments, it follows that $\frac{\partial E\pi_1(\lambda_B=1,v;s^2)}{\partial \mu_M} \ge 0$ if and only if $\gamma \le \overline{\gamma}$, otherwise $\frac{\partial E\pi_1(\lambda_B=1,v;s^2)}{\partial \mu_M} < 0$. The proof is now complete. ■

Proof of Proposition 7: The proof borrows from some of the results above. In particular, it is readily seen that Lemmas 1 and 2 above also hold if firms can practise price discrimination. We now prove that $\lambda_A + \lambda_M = 1$ cannot be part of an equilibrium. Let us denote firm i's strategy as $s^i = \{(\lambda_A, F_A(p)), (\lambda_M, \widetilde{F}_A(p), \widetilde{F}_B(p))\}$, $\sigma_A, \widetilde{\sigma}_A$ and $\widetilde{\sigma}_B$ be the supports of the price distributions, and $\overline{p}_A, \widetilde{\overline{p}}_A$ and $\widetilde{\overline{p}}_B$ the upper bounds of the supports. We note that $F_A(p), \widetilde{F}_A(p)$ and $\widetilde{F}_B(p)$ must be atomless. The profits to a firm advertising only in segment A would be:

$$E\pi_1\left(\lambda_A = 1, p \in \sigma_A; s^2\right) = p\mu_A[\lambda_A(1 - F_A(p)) + \lambda_M(1 - \widetilde{F}_A(p))] - \phi_A.$$

Likewise, the profit to a firm from advertising in the two segments would be:

$$\begin{aligned} E\pi_1\left(\lambda_M = 1, p_A \in \widetilde{\sigma}_A, p_B \in \widetilde{\sigma}_B; s^2\right) &= \lambda_A p_B \mu_B + \lambda_A \mu_A p_A(1 - F_A(p_A)) + \\ &\quad \lambda_M \mu_A p_A(1 - \widetilde{F}_A(p_A)) + \lambda_M \mu_B p_B(1 - \widetilde{F}_B(p_B)) - \phi_A - \phi_B. \end{aligned}$$

We note that $\overline{p}_A < \widetilde{\overline{p}}_A$ otherwise a firm advertising $\overline{p}_A$ in A would make negative profits. This implies that $\widetilde{\overline{p}}_A = v$. It must also be the case that $\widetilde{\overline{p}}_B = v$. Since $v \in \widetilde{\sigma}_A \cap \widetilde{\sigma}_B$, the proof now follows that of Lemma 3.

It remains to prove that firms cannot make positive profits when $\lambda_j > 0, j = A, B, M$. Let $s^i = \{(\lambda_A, F_A(p)), (\lambda_B, F_B(p)), (\lambda_M, \widetilde{F}_A(p), \widetilde{F}_B(p))\}$ denote firm i's strategy, $\sigma_A, \sigma_B, \widetilde{\sigma}_A$ and $\widetilde{\sigma}_B$ be the supports of the price distributions, and $\overline{p}_A, \overline{p}_B, \widetilde{\overline{p}}_A$ and $\widetilde{\overline{p}}_B$ the upper bounds of the supports. We can write down the payoff to a firm from the different advertising strategies:

$$\begin{aligned} E\pi_1\left(\lambda_A = 1, p \in \sigma_A; s^2\right) &= p\mu_A[\lambda_A(1 - F_A(p)) + \lambda_B + \lambda_M(1 - \widetilde{F}_A(p))] - \phi_A \\ E\pi_1\left(\lambda_B = 1, p \in \sigma_B; s^2\right) &= p\mu_B[\lambda_A + \lambda_B(1 - F_B(p)) + \lambda_M p(1 - \widetilde{F}_B(p))] - \phi_B \\ E\pi_1\left(\lambda_M = 1, p_A \in \widetilde{\sigma}_A, p_B \in \widetilde{\sigma}_B; s^2\right) &= E\pi_1\left(\lambda_A = 1, p_A \in \widetilde{\sigma}_A; s^2\right) + E\pi_1\left(\lambda_B = 1, p_B \in \widetilde{\sigma}_B; s^2\right) \end{aligned}$$

We note first that $F_A(p), F_B(p), \widetilde{F}_A(p)$ and $\widetilde{F}_B(p)$ must be atomless. We now note that $\sigma_A \cap \widetilde{\sigma}_A$ cannot be empty. Otherwise, e.g. if $\overline{p}_A < \widetilde{\underline{p}}_A$ a firm advertising $\overline{p}_A$ in A would gain by increasing its price; if, instead $\widetilde{\overline{p}}_A < \underline{p}_A$, a firm advertising $\widetilde{\overline{p}}_A$ and any $p_B \in \widetilde{\sigma}_B$ in the entire market would gain by increasing the price advertised in A. The same arguments imply that $\sigma_B \cap \widetilde{\sigma}_B$ cannot be empty. Now let $p_1 \in \sigma_A \cap \widetilde{\sigma}_A$ and $p_2 \in \sigma_B \cap \widetilde{\sigma}_B$. Since the firms must be indifferent between advertising p_1 in A, p_2 in B and (p_1, p_2) in the entire market, this implies that firms profits must be zero. This completes the proof. ■

References

[1] Bester, H., 1995, "Random Advertising and Monopolistic Price Dispersion", *Journal of Economics and Management Science,* 3, 545-559.

[2] Bester, H. and E. Petrakis, 1995, "Price Competition and Advertising in Oligopoly", *European Economic Review*, 39, 1075-1088.

[3] Butters, G. (1977), "Equilibrium Distributions of Sales and Advertising Prices", *Review of Economic Studies,* 44, 465-491.

[4] Caminal, R. (1996), "Price Advertising and Coupons in a Monopoly Model", *Journal of Industrial Economics,* 44, 33-52.

[5] Elberfeld, W. and E. Wolfstetter (1999), "A Dynamic Model of Bertrand Competition with Entry," *International Journal of Industrial Organization* 17, 513-525.

[6] Esteban, L., Gil, A. and J. M. Hernández (2001), "Informative Advertising and Optimal Targeting in a Monopoly," *Journal of Industrial Economics* 49, 161-180.

[7] Esteban, L., J. M. Hernández and J. L. Moraga-González (2001), "Customer Directed Advertising and Product Quality," Tinbergen Institute Discussion Paper TI 2001-99/1, The Netherlands.

[8] Grossman G. and C. Shapiro (1984), "Informative Advertising with Differentiated Products," *Review of Economic Studies* 51, 63-82.

[9] Iyer, G., D. Soberman and J. M. Villas-Boas (2002), "The Targeting of Advertising," mimeo.

[10] Robert, J. and D. O. Stahl (1993), "Informative Price Advertising in a Sequential Search Model," *Econometrica* 61-3, 657-686.

[11] Rosenthal, R. W. (1980), "A Model in Which an Increase in the Number of Sellers Leads to a Higher Price," *Econometrica* 48, 1575-79.

[12] Roy, S. (2000), "Strategic Segmentation of a Market," *International Journal of Industrial Organization* 18-8, 1279-1290.

[13] Shaffer, G. and J. Zhang (1995), "Competitive Coupon Targeting," *Marketing Science* 14, 395-416

[14] Shapiro, C. (1980), "Advertising and Welfare: Comment," *Bell Journal of Economics* 11-2, 749-752.

[15] Sharkey and Sibley (1993), "A Bertrand model of pricing and entry," *Economics Letters*, 41-2, 199-206.

[16] Stahl, D. O. (1994), "Oligopolistic Pricing and Advertising," *Journal of Economic Theory* 64, 162-177.

[17] Stegeman, M. (1991), "Advertising in Competitive Markets", *American Economic Review* 81, 210-223.

[18] Varian, H. R. (1980), "A Model of Sales," *American Economic Review* 70, 651-659.