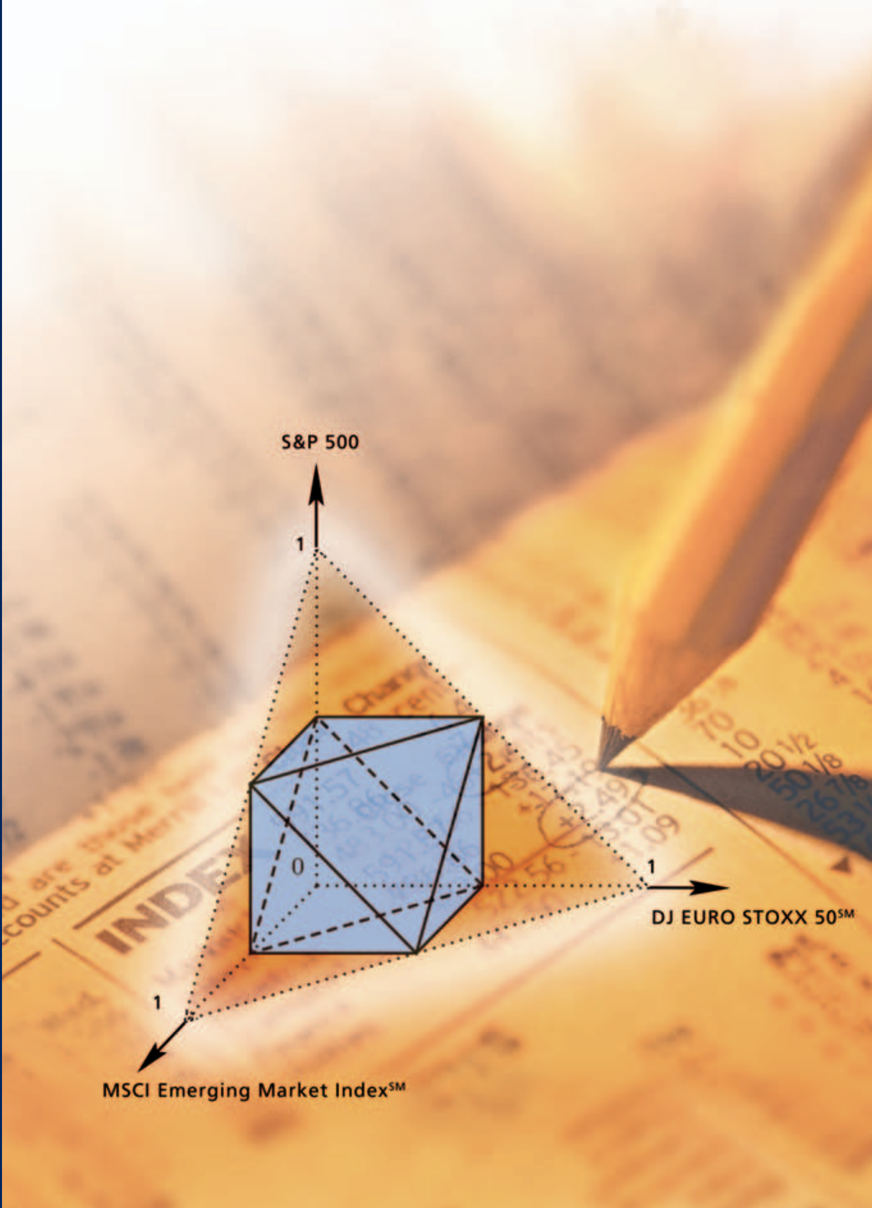
IGOR POUCHKAREV

Performance Evaluation of Constrained Portfolios



Performance Evaluation of Constrained Portfolios

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Performance Evaluation of Constrained Portfolios

Performance evaluatie

van gerestricteerde portefeuilles

PROEFSCHRIFT

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Erasmus Universiteit Rotterdam
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to my wife Natalia

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Contents

Mathematical Notation

$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Z}$	the set of integer numbers including zero
$\mathbb{Z}_+$	the set of positive integer numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of positive real numbers
$\mathbb{C}$	the set of complex numbers
$\|\mathbf{x}\|$	the length (norm) of vector $\mathbf{x}$ in $\mathcal{L}_2$ space
$\sim$	is distributed
$\mathsf{E}[\cdot]$	the expectation operator
$\mathsf{D}[\cdot]$	the dispersion operator
$\mathrm{cov}(X,Y)$	the covariance of random variables X and Y
$X_n \xrightarrow{P} X$	the convergence of a random variable X_n in distribution to X
$\mathcal{U}(a,b)$	the uniform distribution on an open interval (a,b)
$\mathcal{E}(\lambda)$	the exponential distribution with the parameter λ
$\mathcal{N}(\mu,\sigma)$	the normal distribution with mean μ and standard deviation σ
MC	the Monte Carlo method
QMC	the quasi-Monte Carlo method
$\mathfrak{fr}(\cdot)$	the frequency density function
$\mathfrak{Fr}(\cdot)$	the cumulative frequency density function
$\mathfrak{fr}^*(\cdot)$	the estimate of a frequency density function
$\mathfrak{Fr}^*(\cdot)$	the estimate of a cumulative frequency density function

Chapter 1

Introduction

1.1 Domain of the Study

In this dissertation we study the performance evaluation of constrained portfolios. Since indexes of financial markets can be viewed as portfolios, we also address the domain of describing (the performance of) financial markets.

Since seminal papers of Treynor (1965), Sharpe (1966) and Jensen (1968), (1969) much theoretical and empirical research on performance measurement has been done. *Performance evaluation* is an assessment of the investment returns against the return on some benchmark or target, or against returns of (market) peers. The subsequent identification of sources of the evaluated investment return is named *performance attribution*. Thus, we try to answer questions such as: *Is the activity well done?*, *Who is the best?* and *Why did we achieve such a result?*

One of the main motives for academic studies in this area is that the performance of professional investors provides an excellent test for capital market efficiency. On the other hand, professional investors regard performance evaluation as an activity helping to value specific investment decisions and identify the determinants of the risks and superior returns. The development and evolution of performance measures occur together with the development of the theory of financial markets and asset pricing: the development of the *Capital Asset Pricing Model* (CAPM) calls upon several one-factor performance measures such as Treynor's measure (1965), Sharpe's ratio (1966) and (1994), Jensen's alpha (1968) and (1969), the M^2 measure by Modigliani & Modigliani (1997). The development of the *Arbitrage Pricing Theory* (APT) led to multiple-factor measures by Connor & Korajczyk (1986), Lehmann & Modest (1987), Chen, Copeland & Mayers (1987). Recent advances in Value-at-Risk methodology and downside risk stimulate the development of alternative metrics. For further information on the contemporary techniques for performance evaluation and attribution we refer to the comprehensive book of Amenc & Sourd (2003), which provides an excellent overview of contemporary measures and covers the evaluation process in detail.

Despite a broad variety of performance measures the basic evaluation procedure is common for all measures: In the first stage, an appropriate benchmark or a model portfolio is selected. The choice often is determined by an asset pricing model. In the second stage, we calculate the relative performance of the managed portfolio or peer group of investments with respect to this benchmark. On the basis of these relative values we build the ranking and carry out performance attribution.

A *financial market description* reflects the most important aspects of the underlying market performance assessing the market parameters and relations among them qualitatively or quantitatively, and then describing these parameters and relations in a condensed, standardized form.

Financial markets can be described in different ways. A widely used approach for describing financial market dynamics is the use of indexes. For many markets and segments of these markets, indexes are available: if we wish to study the development of a market (or a market segment), we study the appropriate market (or segment) index. A flourishing industry of index providers exists, delivering standardized indexes (be it that different providers generally use different definitions and standards). The choice of an appropriate index has developed into a fine art. Also, providers develop tailor-made indexes for individual clients.

Widespread establishing of financial indexes started in the 1960th. Indexes incepted at that time are capitalization-weighted and consist of country-specific large blue-chip companies. The boom on Asian markets in the 1980th lead to the development of different emerging market series, e.g. MSCI Emerging Market Indexes (1987). In 1992 S&P, in cooperation with Barra, started the Growth and Value subsets of S&P US equity indexes. The main aim was to replicate various investment styles of active portfolio managers. The MSCI followed with introducing in 1997 the Value&Growth indexes for major developed and emerging countries. In the same year the Dow Jones started its Country Style indexes, which provide coverage of the growth and value segments of the US and different European markets. The more recent advances are the introduction of free-float factors (e.g. S&P, MSCI, Deutsche Börse in 2002), inception of enhanced equity style indexes (e.g. 9 style indexes based on the Morningstar's style box by Morningstar in 2002; Global

Value and Growth Index Series by MSCI in 2003), comprehensive hierarchical coverage of (sub)-industries, industry groups, sectors, country markets, and geographical regions (e.g. All Country Sectors by MSCI in 2002). Overall, the major driving force is the intension to provide portfolio managers and investors with better benchmarks tailored not only for market descriptions but also for adequate evaluation of portfolio performance.

The attractiveness of indexing for describing financial markets dynamics has the following reasons:

- *Indexes provide the ultimate summary of markets.* An index condenses all investor sentiments toward companies traded on the market into a single value;

- *Standardization.* By indexing a market we "standardize" the market development. The comparison of different markets or market segments is easy to perform;

- *Indexes are considered to be good substitutes for the market portfolio.* With the development of quantitative methods for optimal investment choice and asset pricing models, the concept of "market portfolio" has gained importance. Often, a properly built index is used as a proxy for the market portfolio.

However, the strong side of indexing is at the same time its weak site: An index or any other average by definition summarizes the price dynamics of individual financial assets and only shows part of the vast amount of information available. The information which is hidden by the use of aggregate indexes is potentially useful.

In the next section we provide the motivation and the main research issues discussed in this thesis. Section 1.3 summarizes the original contributions of our study and Section 1.4 outlines the structure of the thesis.

1.2 Motivation

In conventional performance evaluation methods, a portfolio performance metric is calculated and evaluated either in absolute sense or with respect to a benchmark or a peer group. In practice, however, there can be a discrepancy between the universe used for portfolio construction, and the universe underlying the benchmark portfolios or peer group. As a benchmark usually some market index is adopted. On the other hand, portfolio managers are often restricted to specific asset classes and geographic regions; they may face constraints (some of them institutionally imposed) on the amount of individual investments or investment classes (Henceforth we call such a manager's environment of an investment a *decision-making context*). Therefore, the chosen benchmark may not match with the universe used for portfolio construction and the evaluation will be flawed. A similar discrepancy may exist between an evaluated portfolio and peers.[1]

In addition, the insight obtained by comparison to a benchmark or to a few alternative portfolios is limited. Relative performance values obtained by such a comparison do not reflect the full range of opportunities available under specific investment constraints on the market and, hence, attribution of manager's decisions in the context of a given investment is difficult.

Finally, the definition of investment mandates and constraints will influence the investment performance. We may then be interested to know what the impact of a no-short-selling constraint or an imposed Value-at-Risk constraint is on the performance of the managed portfolio. However, using a benchmark or a peer group, the effect of investment guidelines and constraints cannot be evaluated.

These problems are well recognized across the financial community.[2]

In this dissertation we develop a methodology, which addresses the issues discussed above while taking account of the decision-making context within the performance evaluation process.

1.3 Contribution of the Study

This thesis contributes to the two areas: new theoretical ideas for modelling investment decision-making contexts, and practical applications of these ideas for financial market descriptions and performance evaluations.

In the first area our contributions are the following.

Firstly, in this thesis we present a conceptual framework, which allows to incorporate the decision context of any constrained investment into the performance evaluation process. The main feature that distinguishes our methodology from conventional performance evaluation methods is that it tackles the performance at the decision-making level: the portfolio weights. We consider all possible portfolios that can be constructed given the specific investment objective(s) as well as the prescribed investment constraints, and then evaluate all these portfolios according to (a) selected performance measure(s).

[1] We discuss these issues in Chapter 2 in detail.

[2] For example, in 2002-2004 Hewitt Bacon & Woodrow developed the *SimIAn* (Simulated Investment Analysis), an analyzing and reporting tool for performance evaluation and risk attribution of investment processes via random portfolios.

Consequently, the performance of asset managers can be adjusted for their specific investment objective(s) and constraints and, hence, the conventional evaluation methodology can be extended with "performance-to-possibilities" insights. Our framework is not limited to specific performance measures. On the contrary: it is suitable for almost any performance measure or a combination thereof. Moreover, we leave the choice of the relevant performance attributes to the evaluator who may choose one or more performance measures depending on the performance question(s) to be answered.

Secondly, we show how institutional, legal and self-imposed investment constraints can be translated into constraints on asset weights and how we can formalize the description of investment decision contexts and represent them as polyhedra in the continuous asset weight space. This formalization allows us to use the tools of geometry and linear algebra for the analysis of investment contexts.

Thirdly, throughout this thesis we develop mathematical tools for working with various investment contexts in the continuous asset weight spaces (i.e. continuous portfolio spaces). Obviously, the portfolios and benchmarks we are dealing with on financial markets are discrete. On the contrary, the continuation is the standard approach in most of the mathematical studies, and, hence, many mathematical tools were elaborated over time for analyzing continuous objects. Using the continuity, we provide theoretical ideas which link investment contexts and performance evaluation. In particular, we show that the performance evaluation via random portfolios (or via a random index) represents a special case of evaluation using decision-making contexts, and we also provide an accuracy estimation of such a performance evaluation *et cetera*.[3]

Applying our framework and the corresponding methodology, we contribute to the following topics.

Firstly, using the methodology developed throughout this thesis, we introduce a new way of describing market dynamics through the portfolio opportunity perspective. Instead of focusing on only one portfolio, an index, our view provides a comprehensive perspective on the performance of the variety of portfolios that can be formed given a specific opportunity set and constraints. Thus we obtain a broad view on opportunities available on a specific market. We can also study the dynamics of the portfolio opportunity

[3]The evaluation of mutual funds using random portfolios is discussed in Friend, Blume & Crockett (1970) (Elton, Gruber, Brown & Goetzmann (2003) also discuss in Chapter 24 the results of Friend et al.). The random performance index was introduced by Cohen & Fitch (1966).

set over time. Observed statistics are locations of the distributions, trend, homogeneity of performance values, the quantile in which the index plots, as well as the stability of this quantile over time *et cetera*. As we illustrate in Chapter 6, our descriptions of market dynamics have the power to provide valuable insights for investors and portfolio managers.

Secondly, in the area of evaluating the performance of constrained investments, our methodology provides a natural way to put the absolute and relative performance in the perspective of a specific decision context (i.e. investment mandate, market and business environment) in which the evaluated manager operates. Evaluating performance, the consideration of decision-making contexts extends the conventional metrics with the distributions of performance values for all alternative portfolios with respect to a given performance metric. Such frequency distribution functions can be used for the classification of manager's professionalism by subdividing the possible performances into areas, calculating the percentile as well as providing the information about general trend and dispersion of distribution widths over time, and helping to analyze the influence of various constraints on the exposure of portfolio opportunity set toward performance metrics. In addition, we can evaluate the persistence of a portfolio performance not only with respect to the underlying benchmark, but also with respect to the corresponding opportunities (e.g. looking at the relative percentile values) *et cetera*.

1.4 Thesis Organisation and Outline

The structure of this thesis and the interdependence of the material are shown in Figure 1.1.

Chapter 2 discusses typical problems encountered by performance evaluation of an investment using a benchmark or a peer group first. Afterwards we present an idea how we can bypass these problems by incapsulating the investment decision context into the performance evaluation. Developing our idea, we formulate a new qualitative framework, which extends the performance evaluation process with investment-specific requirements and which allows to calculate the distribution(s) of performance values for all alternative portfolios. The framework and the associated methodology are illustrated with an example. Finally, we list the advantages of our methodology.

Aiming to incorporate the investment decision context into the performance evaluation, we need to define such contexts formally. Chapter 3 starts with the characterization of different kinds of institutional, legal and self-imposed constraints that we encounter in an investment context. We categorize the constraints and specify how most common types of restrictions can

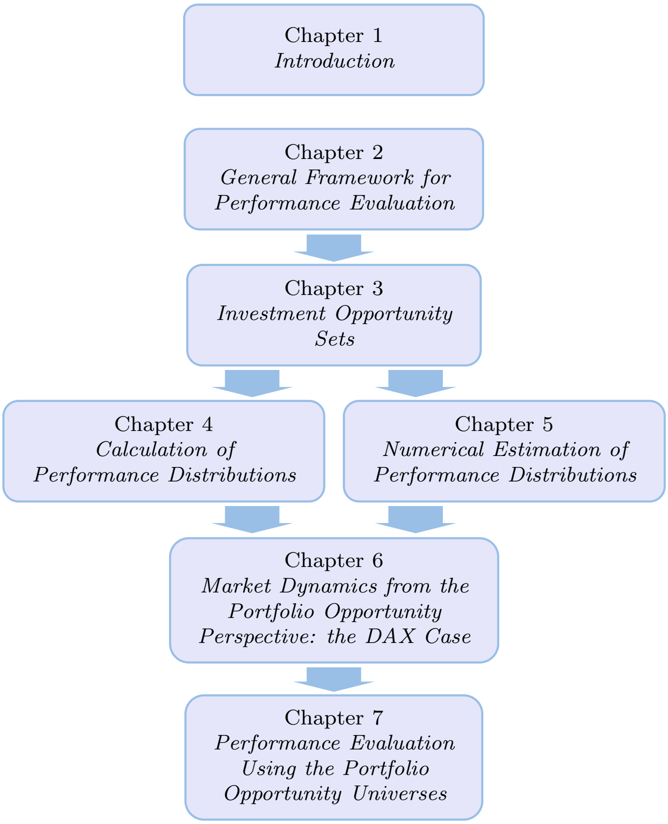


Figure 1.1: The structure of the study. The lines between chapter blocks define the interdependence of the material.

be translated into constraints on asset weights. Based on the asset weights constraints we formalize the description of investment contexts (we call them opportunity sets) and specify the most common kinds formally.

Chapter 4 is devoted to the derivation of explicit analytical formulae for each of our standard opportunity sets. For a given portfolio opportunity set and a linear performance measure we will show how to derive the (cumulative) distribution function, which calculates for any performance value its relative frequency among feasible portfolios. Furthermore, we consider the computational complexity of such density function calculations.

In Chapter 5 we discuss how to numerically estimate the frequency distribution function in case of a very complicated set of investment constraints, especially for non-linear constraints. We review various techniques to form a portfolio opportunity set sample, their convergence speed and efficiency.

Additionally, we consider how quality of estimation can be calculated.

In Chapter 6 we apply our framework for calculating frequency distributions with respect to various return and risk metrics in order to provide a new way of looking at markets and of describing their dynamics over time. In particular, we illustrate how our methodology can be used for providing enhanced market descriptions of the blue-chip segment of the German stock market as represented by the DAX index over the last 14 years.

Chapter 7 demonstrates how portfolio opportunity sets can be used for the evaluation of professional investment managers. Firstly, we consider how a standard evaluation procedure can be extended through the formation of portfolio opportunity set distributions. Afterwards, we consider a very common case when a manager is restricted by a tracking error constraint. We present several applications of how frequency distributions for such an opportunity set can be used to monitor portfolio managers, to calculate the opportunity set-adjusted information ratio *et cetera*.

Chapter 8 summarizes the main ideas and conclusions of the previous chapters and provides directions for future research.

Chapter 2

General Framework for Performance Evaluation

2.1 Why a New Approach

Essentially, performance evaluation of managed portfolios of financial investments consists of two parts:

Firstly, one estimates the performance of the evaluated, e.g. managed, portfolio with the performance that could have been obtained by investing in alternative portfolios with similar characteristics. Reformulating this idea, the portfolio is evaluated with respect to possible alternative portfolios, which are in the line with the investment objective and meeting the specific investment constraints. Also the absolute performance of the evaluated portfolio with respect to a risk-free rate is considered. Calculating these metrics, a clear picture emerges whether the aimed investment objective is met or not.[1]

Secondly, one looks back at all investment decisions made prior to and during the investment period which is evaluated. In simple cases, these are the capital allocation among different asset classes, country and sector allocation, asset selection and definition of an investment portfolio as well as the reshuffling strategy. By looking back at these decisions, one would like to evaluate the optimality of each of the decisions made and to identify the sources of the over- or underperformance.[2]

The crucial observation is that conventional evaluation methods discriminate between

[1] We can characterize the first step through the simple question: How did we do?

[2] In this step we are interested in answering the question: What decisions did we take, why did we make these decisions and what were the consequences?

- What portfolios are in the opportunity set, aiming at the specific investment objective and satisfying the specific constraints

and

- What portfolios could be observed and used for the measurement of investment performance and the evaluation of the decisions made.

In other words, a strong differentiation is made between the universe used for portfolio construction and the universe used for the performance evaluation. Whilst by composing the portfolio the complete opportunity set of investment alternatives is considered, in the latter case a very restricted, general representation of this opportunity set is used: a peer group or (a) benchmark portfolio(s).

Both representations, through a peer group or through a benchmark, have several problems caused by the reduction of the original opportunity set. Evaluating the performance with the help of a peer group, one faces the following difficulties:

- *Finding homogeneous investments is difficult*: Each investor has his own objective(s) while imposing specific investment constraints. So, finding a sufficient number of similar investment portfolios for comparison is problematic. As one pursues a more specific investment portfolio, e.g. concentrated into an industry or a sector, it is often impossible to find even a few peers;

- *Peer grouping is subjective*: The construction of a peer group is always a trade-off between similarity of alternatives to the evaluated investment portfolio and the breadth of the peer group. So, creating a peer group is necessarily subjective;

- *Lack of reliable information about peers*: Even if similar investment portfolios could be found, it is almost impossible to get the insights into these alternative portfolios such as portfolio composition, reshuffling strategy *et cetera*. So the peers can usually be used for the evaluation of relative overall performance only;

- *Survivorship bias in peer groups*: Looking at the historical performance of peers, one considers more or less successful investment portfolios only. Poor investment portfolios drop out of the scene, so one often leaves the poor results out of the performance analysis. To get an idea of how substantial the bias could be, we refer to Malkiel (1995), for example.

Bailey (1992) discusses the peer group approach in detail (in his own words "manager universes") and points out which problems are associated with this approach. Consequently, he outlines the essential criteria for a good benchmark, which could be considered as an appropriate standard metric for performance evaluation of an investment portfolio and/or of a manager.

For such a benchmark portfolio that completely or partially satisfies these criteria, one can use either a custom-built benchmark portfolio or an appropriate standard index. Still, the benchmark approach faces several problems:

- *It completely ignores the managerial input*: For example, we may be interested to know what the impact of a no-short-sales constraint or an imposed Value-at-Risk constraint is on the performance of the managed portfolio. Clearly, using the benchmark approach, the investment guidelines (e.g. different types of constraints as well as their sensitivities) cannot be evaluated;
- *The insight obtained by comparison to a benchmark or to a few alternative portfolios is limited*: Relative performance values obtained by such a comparison do not reflect the full range of opportunities available under specific investment constraints on the market and, hence, manager decision freedom(s) due to an individual investment problem.

Facing these problems, one could combine, of course, both the peer group approach and the benchmark approach but most of the listed problems will not disappear. The reason is that all problems raised by these two approaches have essentially only one source: the substitution of the universe used for portfolio composition through a benchmark or a peer universe. To overcome all these problems we need to go for a source of the problems itself. Essentially, the solution is to *incapsulate the investment decision context into the performance evaluation.*

We developed a qualitative framework, which extends the performance evaluation process with investment-specific requirements. It should be noted that the framework *does not replace* the existing performance measures and techniques but *extends* the evaluation process by incorporating information from different elements of the design stage of the investment. For example, it is easy to combine our framework with evaluation against a benchmark and/or against a peer group. So we can profit from the strengths of standard approaches. And the analysis provided by the framework helps to surpass the problems of the standard approaches. In fact, we strongly recommend such a combined performance evaluation method. (Consequently, definition of a good benchmark as well as selection of an appropriate performance evaluation metric stays *essential* for a reliable, high-quality performance analysis.)

Introducing our framework, we proceed as follows. In the next section we outline the general idea of the framework and overview its structure first. Afterwards we provide a comprehensive description of building blocks and information processing in the framework. Section 3 contains a stylized example of an investment mandate. We apply our framework and the corresponding methodology to evaluate the example investment and present the performance from the portfolio opportunity perspective. Section 4 summarizes benefits from using our methodology.

2.2 Procedural Framework

2.2.1 Ideas behind the methodology and the general framework structure

The crucial observation is that in the existing evaluation approaches the information available on the earlier stages of the investment process (e.g. required asset allocation among different asset classes, tolerated level of risk) is utilized in indirect ways only, or not at all. Therefore, we propose a new procedural framework for performance evaluation, which utilizes the following idea:

> *Considering all possible alternative portfolios that can be constructed given the specific investment objective(s) and also given the prescribed investment constraints, we evaluate all these alternative portfolios according to (a) selected performance measure(s). Simultaneously the performance of the investment portfolio is calculated and then evaluated against the performance of this complete opportunity set.*

In other words, instead of limiting ourselves to (a) benchmark(s) or a peer group we propose to explore the *whole* set of portfolio formation opportunities. (Henceforward we call such a set the *portfolio opportunity set.*) Instead of confining ourselves to evaluating the performance of the investment against a benchmark or a peer group performance, we estimate the *distribution* of the performance values (e.g. distribution of realized returns, and/or of variances, and/or of Sharpe and information ratios etc.) of *all* portfolios from the relevant opportunity set. (Henceforward we call such portfolios also the *feasible* portfolios.)

The development of these distributions yields a picture of the variety of portfolios that can be composed under specific investment requirements, i.e. all possible decision alternatives available to a manager at the portfolio design time. So we can estimate the optimality of the selected portfolio or strategy.

The development of the dispersion of these distributions provides a picture of the development of the "feasible" market dynamics over time. In addition, it offers the possibility to check the persistence of the manager's or of the strategy selection ability.

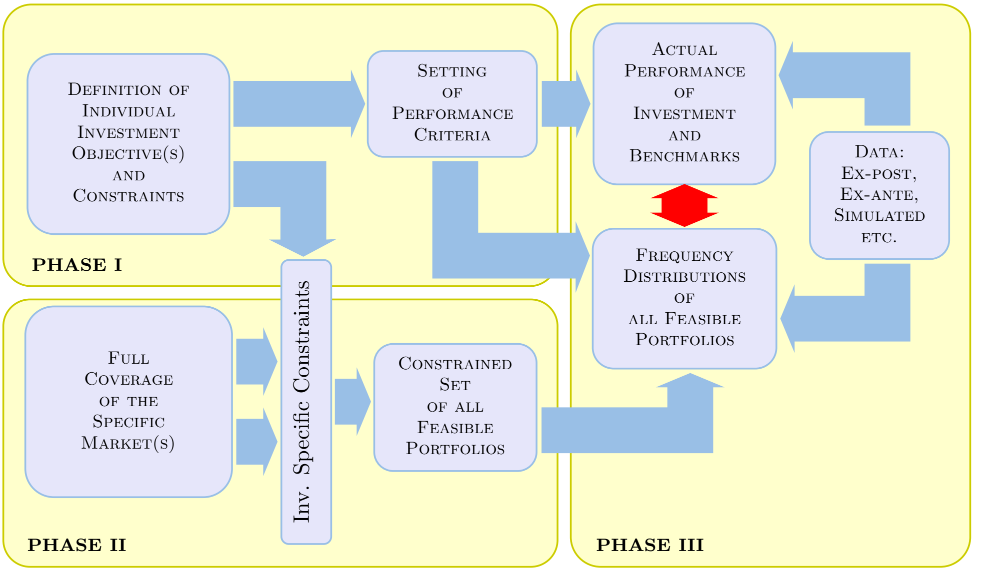


Figure 2.1: General Framework for the Evaluation of Individual Investment Performance: Main Building Components

The general framework shown in Figure 2.1 embraces this general idea into a "skeleton" model and formalizes thereby our approach to performance evaluation using all portfolio opportunities. In the figure the rounded rectangles schematically represent the main building blocks of the framework. The thick arrows display the information flows among these blocks, defining the sequential order of tasks.

Each evaluation process consists of seven building components subdivided in three consecutive phases:

PHASE I: Formulation of investment requirements;

PHASE II: Determination of the portfolio opportunity set;

PHASE III: Analysis of investment performance.

We consider each of these three phases in detail in the next sections.

2.2.2 Phase I: Formulation of investment requirements

Considering an investment process as a managed, rational activity, we start by setting out what a specific investment should achieve and what the constraints are under which it must operate. At the early stage of definition such descriptions are often conceptual and informal. For example, we specify only the level of return we target at and level of risk(s) we could tolerate. After the initial feasibility study, the detailed investment objective report is produced. This document specifies precisely the investment objective(s) and relevant (portfolio) attributes, e.g. the investment horizon, client's constraints on asset allocation, limits of exposure to each source of risks, regulatory body requirements, and reporting procedure. For example, the investment portfolio seeks to outperform the one-year LIBOR rate by 200 basis points investing in governmental and corporate bonds with rating not lower than "A" *et cetera.*[3]

Such a comprehensive investment objective report gives a portfolio manager a concrete mandate for the investment implementation. Consequently, aiming to incorporate the managerial context into performance evaluation, the report is the initial point we start with. In Figure 2.1 the issue of such a mandate is reflected in the framework by the "Definition of Individual Investment Objective(s) and Constraints" block. For performance evaluation such a comprehensive investment specification is important for two reasons.

Firstly, it defines formally and unambiguously the investment objective(s) simultaneously by establishing performance evaluation guidelines. (The guidelines specify a benchmark and performance metrics along with a measurement procedure and reporting standards.) Therefore, in our framework we reflect the existing practices through derivation of the "Setting of Performance Criteria" block from the "Definition of Individual Investment Objective(s) and Constraints" component. Because the framework does not restrict us to any performance metric, we consider by "Setting of Performance Criteria" block the measures that are established in an investment specification report. Additionally, we can consider any additional performance measure(s), which can be useful for internal portfolio management as well as for external controlling of the manager's activities. The latter measures are also identified and specified in the "Setting of Performance Criteria" step. For example, the prescribed performance measure is the realized return and we may be also interested in the Sharpe and the information ratios for the investment. Overall, this step consists of the usual setting of performance measures.

Secondly, an investment objective report determines the opportunity set,

[3] For more details on a formal, quality-based investment cycle we refer to Ho (1995), for example.

which is used by a portfolio manager to compose a portfolio by implementing the investment. So if we want to follow our idea and evaluate a manager against his complete opportunity universe, then we need to consider the same investment input as the manager does. Therefore, in the second step of Phase I we translate the given investment guidelines into certain types of constraints, which formally describe the portfolio opportunity set. Such a formal description is represented through the "Specific Constraints" block. Essentially, the description consists of:

- *screening constraints*, which select financial assets appropriate for the specified investment;
- *selection constraints*, which impose a weight constraint $\alpha_i \leq w_i \leq \beta_i$ on the investment in any feasible financial asset. Additionally, we can restrict some combinations of assets through $\alpha_{A_s} \leq \sum_{j \in A_s} w_j$ for $A_s \subseteq i = 1, 2, \ldots, n$ and/or $\sum_{j \in B_t} w_j \leq \beta_{B_t}$ for $B_t \subseteq i = 1, 2, \ldots, n$.

(The detailed description of different constraint types and their representation is discussed in the next chapter.)

2.2.3 Phase II: Determination of the portfolio opportunity set

The major task in Phase II is to apply the constraints from the previous step on a financial asset universe and create a formal description of the investment opportunity set.

The "Full Coverage of the Specific Market" block represents the fast range of financial assets available for investment on a specific market. For example, if we consider the German market, these assets are the 399 securities of the Prime Standard segment[4] plus shares of the General Standard segment, call/put options written on any of these shares, exchange-traded certificates, future contracts, swaps *et cetera*. The infinite number of portfolios, which could be composed from these assets, also belongs to the full set of investment alternatives available on the German market. Clearly, many financial assets and portfolios from such a full set can be inconsistent with the investment objective and constraints. So we translate such a full set into a "Constrained Set of all Feasible Portfolios" in two consecutive steps.

In the first step we apply the screening constraints from our formal constraint set. The screening constraints work on the asset level only. Consequently, we select from the full set the financial assets that are appropriate for the evaluated investment. For example, if a fund is constrained to invest in German equities from the Prime Standard segment exclusively, then in this

[4] As of 15 September 2003.

first step we reduce the full set of German financial assets and portfolios to the 399 securities comprised in this segment. (Henceforward we call such a set of assets appropriate for investment the *feasible asset set.*)

In the second step we apply the selection constraints from the "Specific Constraints" set. Essentially, this is a transformation of any selection constraint into concrete constraints on the feasible asset set from the previous step. For example, if an additional constraint is to limit the amount to be invested in shares of any one issuer to 10% of the capital and short sales are prohibited, then the formal representation of these constraints in the "Specific Constraints" is $0 \leq w_i \leq 0.1$ for any stock i. And we translate this constraint into the following constraints defined on the feasible asset set:

$$0 \leq w_i \leq 0.1 \quad \forall i = 1, 2, \ldots, 399 \text{ and } i \in \text{Prime Standard} \tag{2.1}$$

$$\sum_{i=1}^{399} w_i = 1.0 \tag{2.2}$$

Constraints (2.1) and (2.2) define the formal description of the portfolio opportunity set of our sample investment. Overall, such a description is the result of Phase II. In Figure 2.1 it is represented through the "Constrained Set of all Feasible Portfolios" block.

2.2.4 Phase III: Analysis of investment performance

Given the set of all feasible portfolios, we can actually start the performance evaluation. Phase III is threefold.

Firstly, we calculate the actual performance of the investment portfolio and all additional benchmarks using the performance criteria specified in the "Setting of Performance Criteria" block. An additional input for this step is prices and various statistics of feasible financial assets. We can use different type of these data such as *ex-post* time series, forecasts for a specific time interval, bootstrapped data etc. In general, the evaluation in this step is not different from the standard one we see in any decent textbook about investment.

Secondly, we estimate the distribution of performance values for every criterion from the "Setting of Performance Criteria" block over the whole feasible portfolio set defined in Phase II. In other words, we calculate the performance of every possible portfolio from the portfolio opportunity set, and then create the distribution of these performance values. We repeat such a distribution estimation for all specified performance criteria. Clearly, the main problem is that while the number of feasible financial assets is finite, the number of feasible portfolios that can be composed from these assets is uncountable infinite. How to estimate the distribution for a feasible portfolio opportunity set with respect to a performance measure will be discussed in detail in the next two chapters.

Finally, we plug the performance results of our first step into the calculated distribution(s). Further analysis is very similar to the analysis on a peer group. Of course, the crucial difference is that we evaluate the performance of a (managed) portfolio or a strategy against its opportunity universe.

The framework and the corresponding methodology of evaluation from the portfolio opportunity perspective were successfully applied in different financial areas. Part III of the thesis provides a variety of real-world applications such as financial market descriptions, cross-sectional comparison of different industries, evaluation of investments under constraints *et cetera*.

The next section illustrates our framework and the methodology with an example.

2.3 Evaluation of an example investment using the framework

2.3.1 Investment mandate and its implementation

Let us consider the following investment mandate:[5]

INVESTMENT OBJECTIVE:

To invest in equities issued by companies incorporated in Germany and belonging to the "Banks" industry sector as classified by the Frankfurter Wertpapierbörse (FWB).

INVESTMENT GUIDELINES:

- Companies should be incorporated in Germany;
- Only shares of companies included in the Prime Standard segment of the Frankfurter Wertpapierbörse (FWB) and traded on the FWB are eligible for investment;
- Only securities which average market capitalization is greater than € 1 billion, are eligible for investment;

[5]Please note that this example is provided for the purpose of illustration only. Therefore, the financial asset space is restricted to shares of a few companies, and the diversification effects, e.g. cross-countries and cross-sectional, are not taken into account.

- Companies should have at least 20% of share capital in a free float;
- 100% of the capital should be invested in stocks;
- Short sales are precluded.

The performance of the investment portfolio should be reported monthly. The reported statistics are:

- Portfolio structure;
- Absolute performance over the month (in %);
- Sharpe ratio (risk-free rate is LIBOR 1M).

Given this concrete investment mandate, we assume a portfolio manager deciding to use a simple passive strategy investing into feasible assets according to their market capitalization. We will illustrate how our framework is used to evaluate the manager's performance under the defined mandate during a sample period of one month, being December 2003.

2.3.2 Phase I: Formulation of investment requirements

Definition of Individual Investment Objective(s) and Constraints

Our sample mandate is a concrete implementation of the "Definition of Individual Investment Objective(s) and Constraints" block. So the mandate text is a starting point for the evaluation. It should be noted that we do not need to perform any steps in this block in our sample. In practice, however, we process at this stage a vaste range of different documents and guidelines (e.g. legal, tax-related, company-specific) in order to compile a description similar to one given in 2.3.1.

Setting of Performance Criteria

The major task in this block is to identify the required performance metrics. Clearly, these are:

- Absolute performance over December 2003 (in %);
- Sharpe ratio with respect to LIBOR 1M at the end of December 2003.

The information about the portfolio structure is used to report the specific portfolio implemented by the manager. We will use this metric at a later stage.

Specific Constraints

By the implementation of this block we translate the restrictions given in the investment mandate into constraints, which formally describe the manager's opportunity set (i.e. alternatives available to the manager for investment under our mandate). In our case the imposed investment constraints are:

Screening constraints

1. Companies should be incorporated in Germany;
2. Only stocks, common and preferred, are eligible for investment;
3. Only shares of companies included in the Prime Standard segment of the Frankfurter Wertpapierbörse (FWB) and traded on the FWB are eligible for investment;
4. Only stocks classified by FWB as belonging to the "Banks" industry sector are eligible for investment;
5. Only securities which average market capitalization is greater than € 1 billion, are eligible for investment;
6. Companies should have at least 20% of share capital in a free float;

Selection constraints

7. $0 \leq w_i \leq 1$ where w_i is the capital invested in any company i feasible under constraints 1–6.

Using these constraints we will derive a formal description of the opportunity set for our sample mandate in the next phase.

2.3.3 Phase II: Determination of the portfolio opportunity set

The major task in Phase II is to impose the formal constraints from the previous step on a financial asset universe and to create a formal description of the investment opportunity set under the concrete mandate.

Full Coverage of the Specific Market

The "Full Coverage of the Specific Market" block in our case represents about 70 thousands financial instruments traded on German exchanges at the end of November 2003. This vast range includes 4,817 domestic and 6,208 foreign ordinary shares or shares with restricted transferability for equities; 16,912 domestic and 1,147 foreign bonds, indices, warrants on equities, options and futures. In addition, the market offers 5,151 domestic and 989 foreign bonds, a few thousands of mutual funds and ETFs plus a huge number of diverse derivative instruments.[6]

Constrained Set of all Feasible Portfolios

Firstly, we apply the screening constraints 1–6 from our constraint set.

Applying constraint#1 to the full financial instrument set, we reduce the set to the 4,817 domestic stocks, subscription rights and stocks-featured securities plus 16,912 domestic warrants, 5,151 domestic bonds *et cetera*. The screening constraint#2 excludes all equity-based instruments, derivative instruments and funds but stocks. There were exactly 929 stocks traded in different market segments at 1 December 2003. Constraint#3 restricts our stock set to the 354 shares of the Prime Standard segment of the FWB. From the remaining 354 shares only 5 stocks are classified as "Banks" (as specified by constraint#4):

Code	**Share**	**ISIN**	**Free-float** (in %)	**Market cap.** (in Mio €)
ARL	AAREAL BANK AG	DE0005408116	59.25%	525.28
HVM	BAY.HYPO-VEREINSBANK AG	DE0008022005	69.28%	8,390.84
CBK	COMMERZBANK AG	DE0008032004	74.20%	6,593.87
DBK	DEUTSCHE BANK AG	DE0005140008	100.00%	34,707.61
IKB	IKB DT.INDUSTRIEBANK AG	DE0008063306	51.23%	794.35

Constraint#5, which requires stocks with market capitalization larger than €1 billion, filters the Aareal Bank AG and the IKB Deutsche Industriebank AG from the list. The remaining stocks, the Bayerische Hypo-Vereinsbank AG, the Commerzbank AG and the Deutsche Bank AG, meet constraint#6.

Secondly, we transform our selection constraint#7 into concrete constraints on our three feasible assets. Essentially, we get the following constraints:

$$0 \leq w_{HVM} \leq 1 \tag{2.3}$$

$$0 \leq w_{CBK} \leq 1 \tag{2.4}$$

$$0 \leq w_{DBK} \leq 1 \tag{2.5}$$

$$w_{HVM} + w_{CBK} + w_{DBK} = 1 \tag{2.6}$$

[6] We refer to (Deutsche Börse Group 2004a) for a detailed overview of the full range of financial instruments available on German exchanges.

where w_{HVM}, w_{CBK}, and w_{DBK} define the amount of capital invested into shares of the Bayerische Hypo-Vereinsbank AG, the Commerzbank AG and the Deutsche Bank AG correspondingly. Constraints (2.4)–(2.6) formally describe the portfolio opportunity set of our sample investment. Figure 2.2 represents this opportunity set graphically.

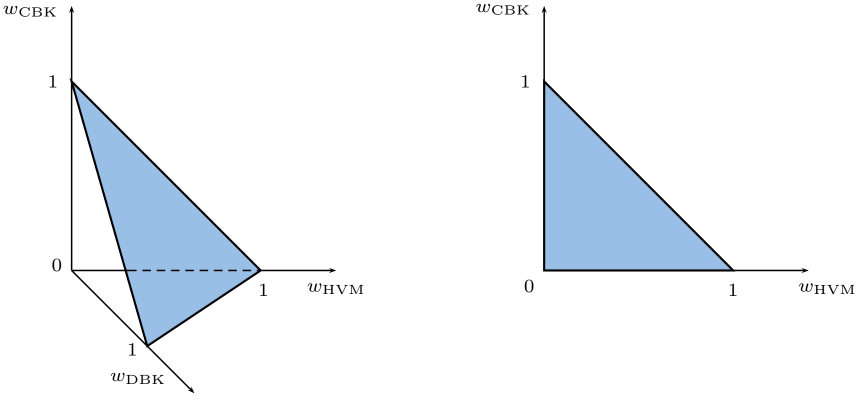


Figure 2.2: Graphical representation of the portfolio opportunity set for the sample investment mandate in $\mathbb{R}^3$ (left) and $\mathbb{R}^2$ (right). Each point in the colored triangle defines a feasible portfolio; in the right figure the third asset weight, w_{DBK}, is defined as $w_{DBK} = 1 - w_{HVM} - w_{CBK}$.

2.3.4 Phase III: Analysis of investment performance

Actual Performance of Investment and Benchmarks

Given the concrete manager's investment implementation, we start the evaluation with calculating actual performance through December 2003 using the specified criteria.

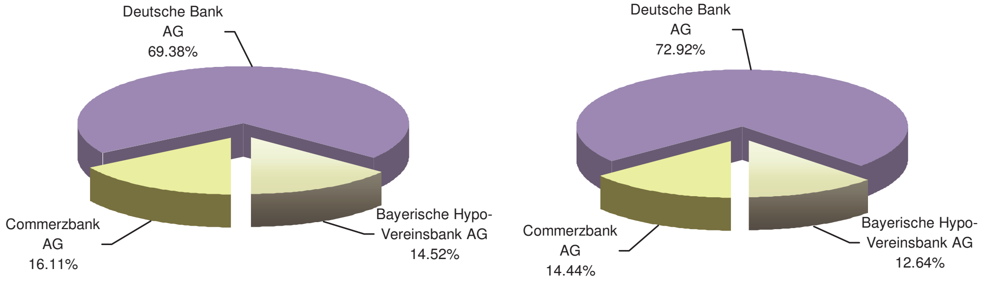


Figure 2.3: Structure of the manager's portfolio on 28 November 2003(left), and on 30 December 2003 (right).

Figure 2.3 shows the manager's portfolio structure and weighting of the feasible stocks over the evaluated period. Further prescribed criteria are the absolute performance over December 2003 and the Sharpe ratio. Table 2.1 presents the required performance values for the manager's portfolio together with performances of the DAX® index and the equally-weighted benchmark.

Realized Return (in % per period)

Portfolio or Benchmark	Dec-03	1 Year	2 Years	3 Years
Manager's portfolio	7.49%	56.25%	-16.08%	-31.34%
DAX index	5.85%	37.08%	-23.16%	-38.37%
Equally-weighted	0.98%	63.10%	-18.70%	-42.79%

Risk (in % per month)

Portfolio or Benchmark	*StdDev*			*Sharpe ratio*		
	1 Year	2 Years	3 Years	1 Year	2 Years	3 Years
Manager's portfolio	12.38%	13.19%	12.25%	0.3428	-0.0082	-0.0456
DAX index	7.53%	9.76%	9.17%	0.3609	-0.0863	-0.1283
Equally-weighted	17.17%	16.79%	14.84%	0.2999	0.0147	-0.0490

Table 2.1: Performance of our portfolio as of December 2003.

Essentially, this step implements the conventional performance evaluation with respect to selected benchmarks and performance criteria.

Frequency Distribution(s) of all Feasible Portfolios

Afterwards, we calculate the distributions of performance values for every criterion from our "Setting of Performance Criteria" section. Figure 2.4 shows the estimated distributions. (For the time being, we skip the issue how to calculate such a frequency distribution having a formal description of an opportunity set.)

As Figure 2.4 shows, the calculated frequency distributions provide a complete description of the manager's "playing field". The advantages of combining the usual performance evaluation methods with the full range of available investment opportunities are listed in the next section. Part II of the thesis

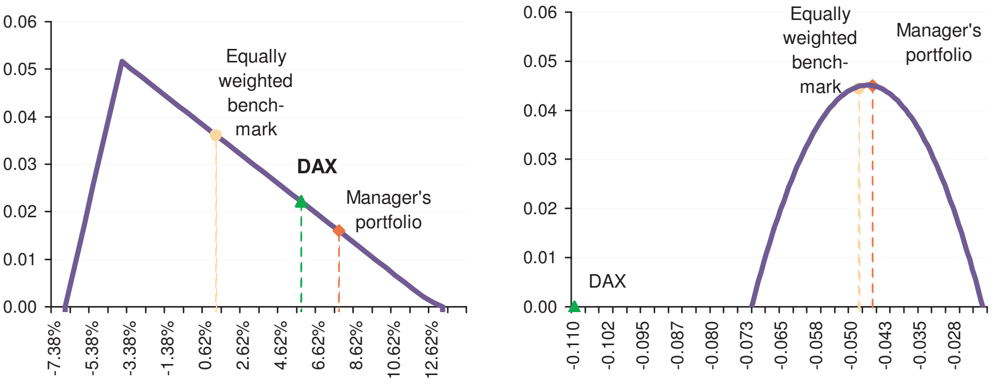


Figure 2.4: Frequency distributions of the realized December 2003 return, in %, (left), and of the Sharpe ratios over the same period (right) for our sample investment opportunity set.

provides a variety of the real-world applications of the framework, which use the proposed methodology for market description, performance evaluation *et cetera*.

2.4 Advantages of the methodology

The key element of the proposed methodology is that it allows to incorporate a given decision context into the evaluation process. Consequently, the defined framework offers promising advantages in two areas.

Firstly, by incorporating information from the design stage of the investment, the framework provides a comprehensive view on the full range of opportunities available under specific requirements and constraints. So the framework is perfectly suitable for descriptions of a specific financial market or a segment thereof.

Secondly, taking into account this full range of opportunities and simultaneously using the existing techniques, we can evaluate the performance of an individual investment more precisely. Particularly, exploring the set of portfolio opportunities, our approach gives us the following important insights:

- The framework provides a perspective on the *ex post* outcomes of the variety of portfolios that can be formed given some opportunity set and constraints. The outcomes may be measured in terms of the return realized over some period as well as in terms of average returns, (downside) risk measures and combined metrics such as the Sharpe or the information ratio, for example. So the proposed methodology is superior with respect to a peer group because it compares to real possible alternative investments and not to "synthetic" ones. In addition, the methodology can provide information about the opportunity set portfolios in any necessary perspective;

- Constructing an opportunity set of a specific market or of a segment, the result is an "ehhanced" description of this financial market or segment. Since the framework does not restrict/enforce us to use any specific performance measure, we can describe the development of the market/segment from different perspectives;
- We can analyze the influence of constraints on (i) the set of portfolio opportunities and on (ii) the outcomes (e.g. on the realized return and/or on the Sharpe ratio distribution). Therefore, our methodology offers to clients and institutions, which engage professional managers, a capability to impose investment restrictions more rationally in terms of both the portfolio opportunities and the outcomes. In addition, our approach facilitates the estimation of the costs of any constraint, which the clients might have formulated;
- In the conventional view, the quality of market representation by an index (e.g. the AEX®, the DAX®or the IBEX 35®) is assumed given, regardless of the performance attributes considered. The new methodology helps to evaluate the market index itself *vis á vis* the portfolio opportunity set. In particular, the location of the market index may be plotted in the frequency distribution of the selected performance measure over the portfolio opportunity set. The quantile in which the index plots, indicates how many (feasible) portfolios have outperformed the index in terms of the selected performance measure (realized return, e.g.). In this way it can be judged, whether an index is representative for the market under consideration or not.[7]

Additionally, by the evaluation of a specific investment our approach has the following benefits:

- The framework facilitates to examine the relative performance systematically. The performance can be measured relative to (i) a benchmark, (ii) a market portfolio, and above all relative to (iii) all portfolios in the opportunity set. Depending on the distribution of performance values of the portfolio opportunities, an outperformance of 1% point by an in-

[7]The adhered criterion for representativeness is not the degree of market coverage measured in terms of capitalization (the usual view) but the degree of the coverage of the portfolio formation opportunity set in the selected performance measure.

vestment portfolio over a benchmark may be either large or small, for example;

- Given a set of *ex-post* specific characteristics of an investment portfolio, our methodology determines portfolio compositions, which have also induced these specific characteristics. In other words, we can determine how much flexibility we have to deviate from the "norm" composition without changing portfolio characteristics. (From *ex-ante* perspective it defines what different portfolio compositions are *expected* to induce these specific characteristics and how they expect to perform.)

2.5 Summary and Conclusions

Conventional evaluation methods strongly differentiate between the universe which is used for portfolio construction, and the universe which is used for the performance evaluation. Whilst by composing the portfolio we consider the complete opportunity set, in the latter case we use a very restricted, general representation of this opportunity set: a peer group or (a) benchmark portfolio(s). It is well recognized that such a reduction of the original opportunity set causes several problems.

Essentially, the solution is to incapsulate the investment decision context into the performance evaluation. We developed a qualitative framework, which extends the performance evaluation process with investment-specific requirements. The main idea of the framework/methodology is to consider all possible alternative portfolios that can be constructed given the specific investment objective(s) and also given the prescribed investment constraints. We evaluate all these alternative portfolios according to (a) selected performance measure(s). Simultaneously the performance of the investment portfolio is calculated and then evaluated against the performance of this complete opportunity set.

The advantages of considering feasible portfolio opportunities for performance evaluation are manifold. The key element is that we evaluate a given investment with respect to the full range (distribution) of performances for all available opportunities. The various benefits become more evident by empirical investigations of stock market indexes, mutual funds, strategies etc as presented in the third part of this thesis.

Discussing the framework, we skipped the issue of how to calculate such a frequency distribution of opportunities having a formal description of an opportunity set. The next three chapters are devoted to this question.

Chapter 3

Investment Opportunity Sets

3.1 Investment opportunities and constraints

Like other human activities, an investment process should be viewed in the context of its environment. This environment affects the functionality of the investment process by imposing various explicit and implicit restrictions. Moreover, we consider the investment process as a managed, rational activity. So we can also impose several constraints in order to influence and guide the investment process of interest. The consequence is that if we would like to follow our idea and evaluate a manager against his complete opportunity universe, then we need to consider all restrictions. In other words, *all imposed constraints are necessary for describing the portfolio opportunity set for investments.*

In addition, we argue that *the consideration of investment restrictions formally expressed as constraints on asset weights provides a thorough description of manager's opportunity sets.* The rationale behind this statement is the following. A portfolio manager receives a concrete investment mandate. Using his knowledge, experience and available tools, the manager selects a portfolio composition (and possibly a reshuffling strategy), back-tests and finally implements his portfolio. The selected portfolio, which consists of specific asset weights, is a final product of the manager's decision process. Therefore we can formally describe the opportunity set of any investment through certain types of constraints on weights of financial assets.

Determined to represent portfolio opportunity sets through weight constraints, we look firstly in the next section on a general characterization of investment restrictions. Then we list and classify the most common types of investment constraints; each of these constraint types is translated into a certain weight constraint. Afterwards we consider mathematical representations of several common types of portfolio opportunity sets.

3.2 Common types of investment constraints

Considering sources of a manager's or a portfolio investment restrictions, we distinguish three different kinds of investment constraints (Spronk 1982):

- *Hard constraints*: These constraints are externally imposed and portfolio managers have to respect them. The managers are *obliged* to choose portfolios and strategies, which satisfy *all* of the hard constraints imposed. Examples of hard constraints are legal investment restrictions, which governmental regulation bodies impose in order to force financial institutions to control financial risks. The restrictions formulated in investment mandates also belong to this kind of constraints;
- *Soft constraints*: These constraints are self-imposed restrictions, which should guarantee the quality of decision-making and secure/enlarge chances of achieving the investment objective or/and reducing the chances of failure. The managers *aim* to respect such constraints but are *not obligated* to do so. An example is the following soft constraint of the Credit Suisse Japan Equity Fund (Credit Suisse Funds 2003*a*, Credit Suisse Funds 2003*b*):[1]

  > "The Fund will invest, under normal market conditions, at least 80% of its net assets, plus any borrowings for investment purposes, in equity securities of companies located in or conducting a majority of their business in Japan. This percentage requirement will not be applicable during periods when the Fund pursues a temporary defensive strategy, as discussed below. The Fund's 80% investment policy is non-fundamental and may be changed by the Board of Directors (the "Board") of the Fund to become effective upon 60 days' notice to shareholders of the Fund prior to any such change..";

- *Game-type constraints*: These constraints represent a special kind of "enforced" self-imposed restrictions. The constraints are determined by the intension of portfolio managers to take account of other participants, e.g. investors, other portfolio managers. An example of a game-type constraint is the rule that a mutual fund may not invest into another

[1] Publicly available equity fund guidelines of Credit Suisse Group (see "Statement of additional information" for any equity fund at the CS Asset Management site http://www.csam.com) consist of around 80 pages prescribing fund policies towards asset allocation among equities, options, warrants, necessary rating for assets, valuation procedure *et cetera*. Most of these constraints are self-imposed.

fund managed by the same investment company (cf. Credit Suisse Funds (2003*b*), for example).[2]

When we talk about low homogeneity of real-world investments, the hard constraints are usually responsible for the cross-sectional diversity through different investment classes. For example, whilst hedge fund managers enjoy almost full freedom of choice, pension fund managers handle in a pretty restrictive environment. Furthermore, the low homogeneity of investments in the same investment class is mostly determined by differences in self-imposed restrictions. Sharing the core knowledge about the market, the managers rely on their company's business practices, control procedures for risk, on access to information and to human and capital resources. Such internal managerial environments play a central role in the formulation of soft constraints.

Another aspect is that each of the three constraint groups has different impact on the investment process and, consequently, on the performance evaluation. Whilst the hard constraints cannot be changed, portfolio managers may adjust the soft constraints using the feedback from the performance analysis. That is, the managers may tighten or relax the value of one or more constraints. So, by replacing the original bounds, we may create a number of different opportunity sets for the original investment mandate. In particular, this procedure is very helpful for estimating the influence of investment constraints on changes in the opportunity set and on the performance of an investment. (Such investigations are usually called sensitivity analysis.)

In this chapter we specify how investment constraints determine portfolio opportunity sets. In that, we do not differentiate between hard, soft and game-type of restrictions and consider any constraint to be satisfied once imposed. In other words, we are interested in how the portfolio opportunity set will look having a specific set of investment constraints. (The analysis of the influence of constraints on the shape of an opportunity set and on the performance for an investment is a separate chapter of this thesis.) The following subsections list the most common types of investment constraints in the same order as considered by portfolio managers through stages of the general investment cycle.

3.2.1 Screening Constraints

Investment constraints based on screening can be defined in an excluding or in an including way. *Excluding screening constraints* filter out companies

[2] Actually, many of self-imposed mutual fund restrictions are of the game-type, which limit actions of fund managers in order to prevent diverse agency problems.

and financial instruments with undesirable activities or values. For example, typical excluding constraints are restrictions forbidding to

- buy shares of "special situation companies", i.e. companies involved in an acquisition, merger, reorganization or in a bankruptcy;
- invest in corporate or governmental debt securities rated lower than the four highest grades, BB/Ba through D/C, by Standard&Poor's or Moody's;
- purchase non-publicly traded or illiquid securities *et cetera*.

On the contrary, *including screening constraints* select the appropriate financial instruments or companies for investment. A typical example is to constrain investment to

- common and preferred shares of companies from the S&P 500 index;
- debt securities rated within the four highest grades, AAA/Aaa through BBB/Baa, by Standard&Poor's or Moody's rating services *et cetera*.

However, in many cases no strict distinction – whether a restriction is an excluding or an including screening constraint – can be made.

Generally, these types of constraints have an absolute context: *"no"* or *"may be"* and can be considered as a preliminary filter to reduce the investment opportunity set to instruments that are possible alternatives for investment. The translation of screening constraints into weight constraints is very straightforward. Having a set of assets and a excluding constraint, we check each asset whether it satisfies the constraint. If it satisfies, then we add the following "excluding" weight constraint to our formal description of the opportunity set:

$$w_s = 0 \tag{3.1}$$

This constraint prevents us from investing into the asset s any amount of capital because this asset has a property, which is not feasible for the investment. In the case of an including screening constraint the modus operandi is opposite: we add the "equal to zero" weight constraint for all assets, which do not satisfy the positively defined constraint.

When we apply all screening constraints, we get the set of financial instruments in which the manager of this specific investment may invest and which he uses to compose a portfolio. Henceforth, we will call instruments from such a set *feasible assets*.

3.2.2 Selection Constraints

Another constraint part is the *selection* constraints, which impose restrictions reducing the number of combinations, i.e. portfolios that can be built given the set of feasible financial assets.

Restrictions on Short Sales

Prohibiting or quantitative restrictions on short sales are the most ubiquitous of the selection constraints. For example, many pension funds avoid selling short completely due to their chosen investment policy. Almost all equity funds may only engage in short sales "against the box" or in order to hedge similar long positions. The no-short-sales constraint is also one of the most binding constraints (Clarke, de Silva & Thorley 2002).

The complete disallowing of short sales is implemented through imposing a weight constraint:

$$0 \leq w_i \leq 1 \tag{3.2}$$

on any feasible financial asset. In the case of a quantitative restriction on the short sales quantity, the constraint above could be relaxed to:

$$\begin{aligned} -\phi &\leq w_i \\ -\phi &\leq \sum_i w_i \quad \text{and} \quad \phi \in \mathbb{R}_+ \end{aligned} \tag{3.3}$$

where ϕ represents a maximal proportion of capital allowed to be raised by short sales.

Restrictions on a Maximal Capital Exposure

Another often used restriction is a constraint on a maximal proportion of capital, which could be invested in one financial instrument. The main reason for this constraint is the enforcement of diversification. (For example, under the current US Internal Revenue Code of 1986, in order to qualify as a "regulated investment company" an investment company should limit its investments so that (i) not more than 25% of its capital is invested in the financial instruments of a single issuer, and (ii) not more than 5% of the capital should be invested in the securities of a single issuer and the company owns not more than 10% of voting shares for at least for 50% of the total capital.) The restriction on a maximal capital proportion invested in a single asset is implemented through addition of weight constraints

$$w_i \leq \beta_i \tag{3.4}$$

into the formal description of the opportunity set for all assets to be restricted.

An extension of this restriction type is the constraint on a maximal capital proportion, which could be invested into:

- different financial markets, e.g. equities, fixed income securities, derivatives;
- different securities of a single issuer;
- the securities of issuers conducting their principal business activity in the same industry or sector;
- the emerging-market securities, the American Depositary Receipts ("ADRs") or the European Depositary Receipts ("EDRs"), debt instruments rated below investment grade *et cetera*.

Such restrictions on a maximal capital exposure to a specific area are implemented through a "summed" weight constraint:

$$\sum_{j \in A_k} w_j \leq \gamma_k, \quad A_k \subseteq \{1, 2, \ldots, n\} \quad \text{and} \quad \gamma_k \in \mathbb{R}_+ \tag{3.5}$$

where set A_k denotes all instruments, which belong to the restricted area k, and γ_k represents the maximal proportion of capital, which could be invested into this specific financial market area. For each of such restrictions a separate weight constraint is introduced.

Restrictions on a Minimal Capital Exposure

Sometimes a restriction on a minimal proportion of capital, which should be invested in one or several financial instruments, is imposed. The rationale behind this constraint is that we would like to invest into certain financial instruments. For example, these are securities selected through analysis of individual companies, large companies playing a substantial role in a specific industry of a sector etc. Such minimal exposure intensions can be inserted in the formal description using the following weight constraint:

$$\alpha_i \leq w_i \tag{3.6}$$

where α_i represents the minimal amount that a manager is "enforced" to invest into the company i.

Similar to the maximal exposure, minimal capital exposures to a specific area, e.g. industry, financial sector, geographic area, can be implemented through a weight constraint:

$$\varsigma_k \leq \sum_{j \in B_k} w_j, \quad B_k \subseteq \{1, 2, \ldots, n\} \quad \text{and} \quad \varsigma_k \in \mathbb{R}_+ \tag{3.7}$$

where set B_k denotes all instruments, which belong to the restricted area k. For each of such minimal exposure restrictions a separate weight constraint should be introduced.

Restrictions According to the Risk Profile

Risk management plays a central role in a professional investment management. Therefore, investment mandates may prescribe tolerable levels of different kinds of risk. Some risks, e.g. political risk, currency risk, can be substantially lowered by hedging or handled implicitly through imposing constraints on maximal capital exposure to a risky area. Others, such as market risk or reinvestment risk, could be controlled through restrictions on risk profiles. The shape of a weight constraint restricting a specific risk exposure depends on the exact formula estimating portfolio risk from the risk levels of individual assets.

Using the variance as an example, we illustrate how a risk constraint could be translated into a formal constraint of an opportunity set. The restriction with respect to variance is formulated as the following weight constraint:

$$\sum_i \sum_j \mathrm{cov}(i,j) \cdot w_i \cdot w_j \leq \zeta \quad \text{and} \quad \zeta \in \mathbb{R}_+ \tag{3.8}$$

where ζ is a tolerable level of variance, and $\mathrm{cov}(i,j)$ is the covariance between stocks i and j.

Another widespread risk constraint is the tracking error volatility/variance (TEV). An imposed limit on TEV restricts manager's exposure to risk whilst a specific over-performance of a benchmark, e.g. an index, is targeted. The TEV restriction is stated as the following weight constraint:

$$\sum_i \sum_j \mathrm{cov}(i,j) \cdot (w_i - b_i) \cdot (w_j - b_j) \leq \psi \quad \text{and} \quad \psi \in \mathbb{R}_+ \tag{3.9}$$

where ψ determines a tolerable level of TEV, b_i is the proportion of the security i in the targeted benchmark, and w_i is the proportion of the security i in a feasible portfolio.[3]

Generally, the recipe is to write a formula for a portfolio risk expressed through risks of individual securities and then create a formal constraint in

[3] It should be noted that imposing a single TEV constraint is not efficient because it forces managers to optimize in the excess-return space only and hence does not account for the total risk of a tracking portfolio (Roll 1992). However, this can be corrected through imposing an additional constraint on the total portfolio risk especially when the TEV value is low or a targeted benchmark is relatively inefficient (Jorion 2003).

form of an inequality, having this formula on the left-hand side and the tolerable level of risk on the right-hand side.

3.3 The Geometry of Some Common Opportunity Sets

Once we translate all prescribed investment restrictions into certain types of constraints on asset weights, we get a system of inequalities and equalities of the form

$$\begin{aligned} f_s(w) &\leq 0 \qquad s \in \mathcal{I}_{\leq} \\ f_t(w) &= 0 \qquad t \in \mathcal{E}_{=} \end{aligned} \tag{3.10}$$

and

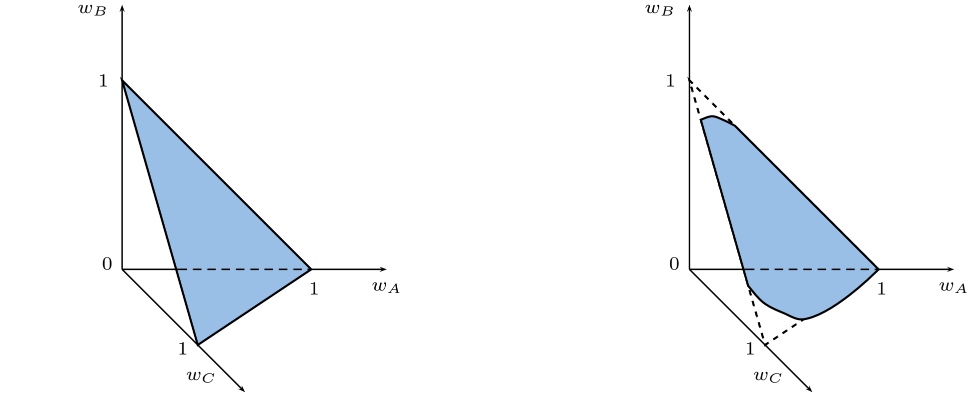
$$\sum_i w_i = 1$$

where $\mathcal{I}_{\leq}$ denotes the set of inequality constraints on asset weights, and $\mathcal{E}_{=}$ denotes the set of equality constraints.[4] Any of the $f_{(\cdot)}(w)$ (in-)equalities may be nonlinear. Similar to the convex optimization, we call any portfolio with weights satisfying all the (in-) equalities from $\mathcal{I}_{\leq}$ and $\mathcal{E}_{=}$ a *feasible* portfolio. Otherwise the portfolio is termed *unfeasible*. The set of all feasible portfolios is called the *feasible portfolio set* or the *portfolio opportunity set*. Figure 3.1 shows two examples of formal opportunity set descriptions.

The system (3.10) is the most general mathematical model of the manager's opportunity set for a specific investment. When we consider such general models, we have to take into account two important issues: system feasibility and non-redundancy (see Exhibit 3.1 for detailed discussion). Without loss of generality, we will assume henceforth that our formal descriptions are feasible and non-redundant.

The geometric form of a portfolio opportunity set depends on polynomial degrees and on the number of (in-)equalities in $\mathcal{I}_{\leq}$ and $\mathcal{E}_{=}$. In this thesis we only consider several common types of opportunity sets in detail. However, we provide various ideas how one of these standard types or a combination thereof can be used to handle any possible system of asset weight constraints.

[4] We represent inequalities with the right side different from zero, i.e. $a_1 w_1^{\alpha} + a_2 w_2^{\beta} + \ldots + a_n w_n^{\upsilon} \leq c$, as $a_1 w_1^{\alpha} + a_2 w_2^{\beta} + \ldots + a_n w_n^{\upsilon} - c \leq 0$. The same notation is used for equalities from $\mathcal{E}_{=}$.



$$\begin{aligned} -w_i &\leq 0 \quad i = A, B, C \\ w_i - 1 &\leq 0 \\ w_A + w_B + w_C - 1 &= 0 \end{aligned}$$

$$\begin{aligned} -w_i &\leq 0 \quad i = A, B, C \\ w_i - 1 &\leq 0 \\ w_A + w_B + w_C - 1 &= 0 \\ 0.25w_A^2 + 0.4w_B^2 + 0.5w_C^2 - 0.4w_B w_C - 0.25 &\leq 0 \end{aligned}$$

Figure 3.1: The investment set consists of three assets: A, B and C. The opportunity set is restricted by a "no-short-sales" constraint only (left); on the opportunity set in (right) an additional constraint on the tolerable risk is imposed. (We use the variance as the risk metric. The variances are $\sigma_A^2 = 0.25$, $\sigma_B^2 = 0.4$, and $\sigma_C^2 = 0.5$. The correlation coefficients are $\rho_{A,B} = 0$ and $\rho_{B,C} = -0.5$. The risk of a portfolio is restricted by $\sigma_P^2 \leq 0.25$.) The systems below the graphs show the formal description of the corresponding portfolio opportunity sets.

Exhibit 3.1 Feasibility and Redundancy of Portfolio Opportunity Set Models

Given a formal description of an investment opportunity set, it may not be clear if we could compose any feasible portfolio at all when following the imposed constraints. Checking, whether a given description defines a feasible portfolio opportunity set, i.e. whether the opportunity set is non-empty, may be difficult. In general, the feasibility checking procedure strongly depends on the exact shape of the system (3.10) for a specific investment. For example, when $\mathcal{I}_{\leq}$ and $\mathcal{E}_{=}$ consist of linear (in-)equalities only, recognizing feasibility of a given description is at least as difficult as solving the following linear programming problem:

$$\begin{aligned} \text{minimize} \quad & \xi \\ \text{subject to} \quad & f_s(w) - \xi \leq 0 \quad \forall\, s \in \mathcal{I}_{\leq} \\ & f_t(w) - \xi \leq 0 \quad \forall\, t \in \mathcal{E}_{=} \\ & -f_t(w) - \xi \leq 0 \quad \forall\, t \in \mathcal{E}_{=} \\ & \sum_i w_i \leq 1 \ \text{ and } \ \xi \geq 0 \end{aligned} \tag{3.11}$$

Including non-linear constraints into an opportunity set description affects both, the complexity and the convergence, of solving the auxiliary problem (3.11) for checking feasibility. To the knowledge of the author, no general method exists for solving non-linear problems. However, several methods (e.g. reduced-gradient methods, penalty and barrier approaches, interior-point algorithms) were recently developed to handle specific classes of (3.11) efficiently. (The books of Nash & Sofer (1996), Bertsekas (1995), Boyd & Vandenberghe (2004) provide a comprehensive overview of contemporary methods for non-linear convex programming.)

Resulting formal descriptions of opportunity sets can also contain constraint redundancies due to various reasons, e.g. large number of assets, different sources of investment constraints, automatic translation of imposed restrictions into weights constraints. For example, assume that we have an opportunity set consisting of securities A, B, C, and D. After the translation of investment guidelines restrictions we get the following formal description of our opportunity set S:

$$\begin{aligned}
&0 \leq w_A \leq 0.3 \\
&0 \leq w_B \leq 0.3 \\
&w_A + w_B \leq 0.7 \\
&w_A + w_B + w_C + w_D = 1
\end{aligned}$$

Clearly, the third constraint is redundant and it could be removed from the description. So transformed opportunity set S' is equal to S. It should be noted that the redundancy elimination is desirable but not necessary: redundancies just increase the size of an opportunity set description and lead to computational inefficiency by estimation of frequency distributions. (Redundancy elimination pays off in considerable computational time-reduction by opportunity sets consisting of reasonable-wide range of assets accomplished by a large number of investment constraints.)

The redundancies in linear (in-)equalities can be recognized and then eliminated relatively easily. Such presolvers are a standard part of software packages for solving LPs, and over the last ten years a wide range of sophisticated presolve techniques was developed. The first class of redundancy-reduction techniques identifies and then eliminates different linear dependencies of equalities and of weights. For this purpose, we transform all linear inequalities into equalities by introducing *slack weights* (slack weights are nothing else but "artificial" variables representing differences between left-hand side values of inequalities and the right-hand side zeros). Afterwards, the following theorem (an adapted version of a theorem from Bertsimas & Tsitsiklis (1997)) is applied:

Theorem 3.3.1. *Let* $P = \{\mathbf{x} \mid \mathbf{A}\,\mathbf{x} = \mathbf{b},\, \mathbf{x} \geq 0\}$ *be a non-empty polyhedron, and* $\mathbf{A}$ *be a* $m \times n$ *matrix with rows* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. *Assume also that* $rank(\mathbf{A}) = k < m$ *and that rows* $\mathbf{a}'_1, \mathbf{a}'_2, \ldots, \mathbf{a}'_m$ *are linearly independent. If we define a polyhedron* P' *as*

$$P' = \{\mathbf{x} \mid \mathbf{a}'_1\,\mathbf{x} = b_1,\, \mathbf{a}'_2\,\mathbf{x} = b_2,\, \ldots,\, \mathbf{a}'_k\,\mathbf{x} = b_k\}$$

Then $P = P'$.

Proof. see the proof of Theorem 2.5 in Bertsimas & Tsitsiklis (1997). □

It should be noted that Theorem 3.3.1 requires a numerically stable Gaussian elimination routine.

The second class of redundancy-reduction techniques includes various methods working directly on the linear (in-)equalities from (3.10). The methods eliminate empty and singleton (in-)equalities and weights, tighten bounds on weights, fix or free weights *et cetera*. The extended discussion about different presolve techniques can be found in Andersen & Andersen (1995), Gondzio (1997) and Gould & Toint (2004).

The reduction of redundancies in nonlinear weight constraints is a more hardcore problem. However, several techniques used in linear cases also can be applied for nonlinear constraints.

3.3.1 Opportunity Sets for Unrestricted Asset Weights

The case in which asset weights are unrestricted, is considered to be the most general in the financial literature. A positive weight for an asset means that we are buying this asset; a negative one means that we are selling this asset short. Having n feasible assets for an investment, the portfolio opportunity set is equal to $\mathbb{R}^{n-1}$. (We lose a degree of freedom due to the constraint $\sum_{i=1}^{n} w_i = 1$.) Clearly, as the weight of any of $n-1$ assets has a range $(-\infty, +\infty)$, we cannot calculate frequency distributions.

Furthermore, it can be shown that frequency distributions exist only in the case when all asset weights lay in a closed range $[\alpha_i, \beta_i]$, $\alpha_i, \beta_i \in \mathbb{R}$ and $\alpha_i \leq \beta_i$. This requirement is not very binding as it seems: in almost all cases the capital, which could be invested or sold short, is restricted. Therefore, the closedness of weight ranges is provided. Moreover, the unrestricted weights reflect the case of full freedom over investment choice what is clearly not coinciding with evaluation of constrained investment performance.

Due to these reasons we do not consider the case of completely unrestricted asset weights further; instead we focus on cases when investments are made under diverse restrictions.

3.3.2 Opportunity Sets with a Short-Sales Restriction only

The case, when only no-short-sales constraints are imposed, is considered as a one of major situations in the modern portfolio management theory and asset pricing. The formal description of the portfolio opportunity set consisting of n feasible assets is given by the system:[5]

$$\begin{aligned} &0 \leq w_i \leq 1 \quad \forall\, i = 1, 2, \ldots, n \\ &\sum_{i=1}^{n} w_i = 1 \end{aligned} \tag{3.12}$$

Transforming the opportunity set description into the form (3.10), we get the following description:

$$\begin{aligned} -w_i &\leq 0 \quad \forall\, i = 1, 2, \ldots, n \\ w_i - 1 &\leq 0 \quad \forall\, i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} w_i - 1 &= 0 \end{aligned}$$

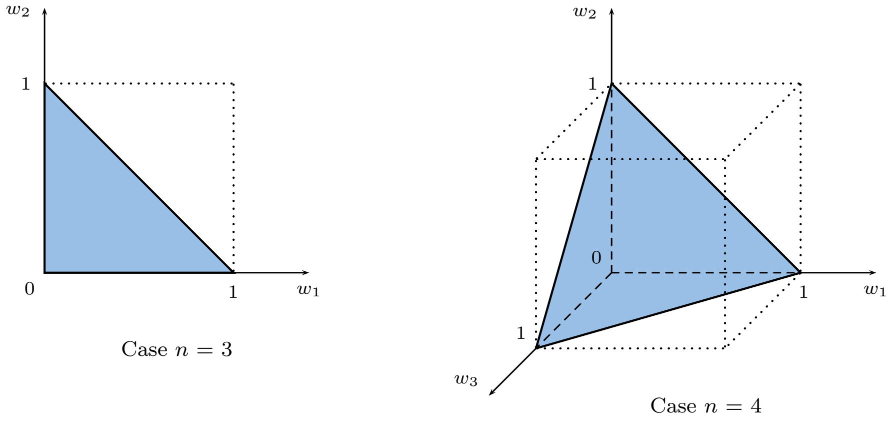


Figure 3.2: Cases $n = 3$ and $n = 4$. The dotted lines border $[0, 1]^{n-1}$-cubes; parts of these cubes that satisfy the inequality $\sum_{i=1}^{n} w_i \leq 1$, i.e. build up the two- and three-dimensional basic simplexes, are filled with color.

The opportunity set is a closed polyhedron called a basic simplex in $\mathbb{R}^{n-1}$. Figure 3.2 shows the portfolio opportunity sets, which consist of three and four assets and are restricted by the no-short-sales constraint only.

[5] In (3.12) the inequalities $w_i \leq 1$ are redundant; we leave them in the opportunity set description for the sake of better understanding.

3.3.3 Opportunity Sets with Restrictions on Individual Weights and Linear Summary Constraints

The situation when all asset weights lay in closed ranges $[\alpha_i,\, \beta_i],\ \ \alpha_i,\, \beta_i\ \in\ \mathbb{R}$ and $\alpha_i \le \beta_i$, we consider as the most general case for the constrained investment. In this case the maximal capital exposure toward any asset is restricted along with the maximum capital, which could be raised by selling short any of assets. Setting $\alpha_i \ge 0$, we can also incorporate restrictions on minimal capital exposure toward selected assets.[6]

The formal description of such portfolio opportunity sets consisting of n feasible assets is given by the system:

$$\begin{aligned} &\alpha_i \le w_i \le \beta_i \quad \forall\, i = 1, 2, \ldots, n, \quad \alpha_i, \beta_i \in \mathbb{R} \quad \text{and} \quad \alpha_i \le \beta_i \\ &\sum_{i=1}^{n} w_i = 1 \end{aligned} \tag{3.13}$$

However, it is more convenient to analyze and use such opportunity sets when their formal descriptions are transformed into the special representation, which we call "standard form" (see Exhibit 3.2). We use the descriptions in standard form to analyze opportunity sets and calculate required frequency distributions. Afterwards, results of our analysis and calculations are transformed using inverse operations in order to present results in original form. An opportunity set description in standard form is defined as:

$$\begin{aligned} &0 \le w_i \le \varphi_i \quad \forall\, i = 1, 2, \ldots, n, \quad \varphi_i \in \mathbb{R}_+ \\ &\sum_{i=1}^{n} w_i = 1 \end{aligned} \tag{3.14}$$

or it can be re-written into the form (3.10) as:

$$\begin{aligned} -w_i &\le 0 \quad \forall\, i = 1, 2, \ldots, n \\ w_i - \varphi_i &\le 0 \quad \forall\, i = 1, 2, \ldots, n, \quad \varphi_i \in \mathbb{R}_+ \\ \sum_{i=1}^{n} w_i - 1 &= 0 \end{aligned}$$

[6]Indeed, section 3.3.2 discusses a particular case with all $\alpha_i = 0$ and $\beta_i = 1$. The reason for this extra separation is that opportunity sets with a restriction on short-sales only are widely considered in the financial literature as the main case representing a constrained investment activity. In addition, such opportunity sets have a "degenerate" nature, which made them easy to analyze.

Exhibit 3.2 Transformation of Opportunity Set Descriptions into Standard Form

In order to transform descriptions from form (3.13) into the standard form (3.14) we need to perform the following steps.

Firstly, if a weight w_i has a lower bound α_i other than zero, we subtract the value of α_i from all parts of the constraint and obtain:

$$0 \leq w_i - \alpha_i \leq \beta_i - \alpha_i$$

Replacing weights and upper bounds by $\hat{w}_i = w_i - \alpha_i$ and $\hat{\beta}_i = \beta_i - \alpha_i$ we obtain:

$$0 \leq \hat{w}_i \leq \hat{\beta}_i \quad \forall\, i = 1, 2, \ldots, n$$
$$\sum_{i=1}^{n} \hat{w}_i = 1 - \sum_{i=1}^{n} \alpha_i$$

Secondly, all constraints should be divided by $1 - \sum_{i=1}^{n} \alpha_i$. Replacing again weights and upper bounds by

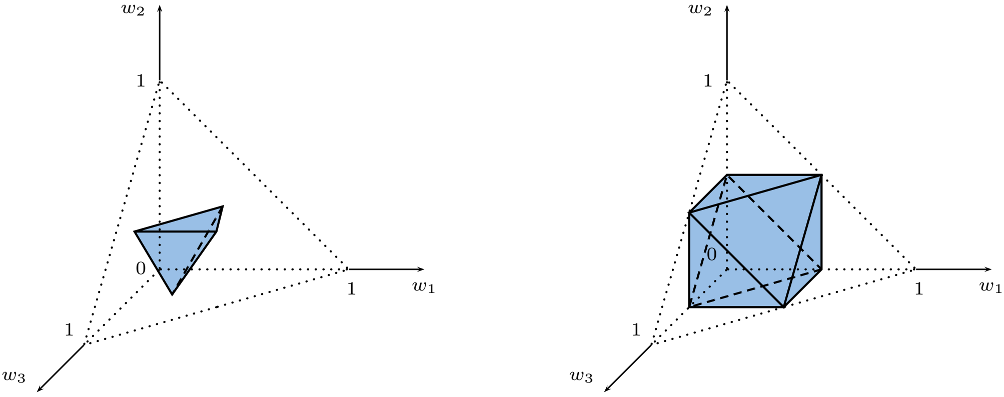
$$\tilde{w}_i = \frac{\hat{w}_i}{1 - \sum_{i=1}^{n} \alpha_i} \text{ and } \tilde{\beta}_i = \frac{\hat{\beta}_i}{1 - \sum_{i=1}^{n} \alpha_i}$$

we obtain a description in the standard form (3.14).

Henceforth, analyzing our general case we consider formal descriptions of opportunity sets in the standard form (3.14) only.

The geometric form of portfolio opportunity sets of the type (3.14) depends on the coefficients φ_i. For example, if all φ_i are equal to $1/n$, where n denotes the number of feasible assets, then the portfolio opportunity set contains only one feasible portfolio $\mathbf{p} = \langle w_1 = 1/n,\, w_2 = 1/n,\, \ldots,\, w_n = 1/n \rangle$. In the case that all φ_i are equal and lay in the interval $(1/n,\, 1/(n-1)]$, portfolio opportunity sets are simplexes dual to the basic simplex. In the case that all φ_i are equal and lay in the interval $(1/(n-1),\, 1/(n-2)]$, portfolio opportunity sets are basic simplexes with cutoff vertices and so on. Figure 3.3 shows the portfolio opportunity sets consisting of four assets and restricted by $\varphi_i = 1/3$ and $\varphi_i = 0.5$ respectively.

The exact geometric form of opportunity sets is not relevant to the further analysis nor suitable for working with these sets. Actually, the important fact is that descriptions are closed convex polyhedra. The systems of linear (in-)equalities of the form (3.14) represent these opportunity set polyhedra in terms of linear (in-)equalities. An alternative representation of a closed polyhedron can be given in terms of its extreme points (also called vertices).



$$\begin{aligned} -w_i &\leq 0 \quad i = 1,2,3,4 \\ w_i - \tfrac{1}{3} &\leq 0 \\ w_1 + w_2 + w_3 + w_4 - 1 &= 0 \end{aligned}$$

$$\begin{aligned} -w_i &\leq 0 \quad i = 1,2,3,4 \\ w_i - 0.5 &\leq 0 \\ w_1 + w_2 + w_3 + w_4 - 1 &= 0 \end{aligned}$$

Figure 3.3: Geometrical forms of portfolio opportunity set consisting of 4 assets and restricted by $0 \leq w_i \leq 0.3(3)$, $i = 1,2,3,4$ (left), and $0 \leq w_i \leq 0.5$, $i = 1,2,3,4$ (right). The dotted lines border basic simplexes; parts of these simplexes that satisfy inequalities $0 \leq w_i \leq 0.3(3)$, $i = 1,2,3,4$ and $0 \leq w_i \leq 0.5$, $i = 1,2,3,4$ are filled with color.

We will use the latter representation.

Enumerating of all vertices of a convex polyhedron using its representation given by a system of linear inequalities is $\mathcal{NP}$-complete in the general case. However, there are several algorithms, which can enumerate vertices efficiently, especially for some subclasses of polyhedra (e.g. Avis & Fukuda (1992), Fukuda, Liebling & Margot (1997)).[7]

Insertion into opportunity set descriptions of the type (3.13) additional "summed" linear constraints such as

$$\sum_{j \in A_k} w_j \leq \gamma_k, \ \ A_k \subseteq \{1,\, 2,\, \ldots,\, n\} \quad \text{or} \quad \varsigma_l \leq \sum_{j \in B_l} w_j, \ \ B_l \subseteq \{1,\, 2,\, \ldots,\, n\}^{\,8}$$

changes the geometrical form of the feasible portfolio opportunity space.[9] Still, such extended descriptions can be theoretically handled by the same methodology as the descriptions (3.13). The only difficulty is that "summed"

[7] The Mathematica Information Center provides the package "VertexEnumeration", which implements the Avis-Fukuda algorithms for enumerating all vertices of a convex polytope given by a system of linear inequalities. (For more information see http://library.wolfram.com/infocenter/MathSource/440/.)

[8] cf. section 3.2.2

[9] Given the assumption that such a constraint is not redundant.

constraints can vastly increase the computational complexity of calculating frequency distributions. The reason is the potentially exponential increase in the number of vertices of a feasible opportunity set polyhedron. We will stop at this point and return to this issue in the next chapter.

3.3.4 Opportunity Sets with Non-linear Restrictions

Of course, we can further generalize opportunity set descriptions by incorporating non-linear constraints into (3.13). Addition of any non-linear constraint (e.g. restriction on the risk level or on the tracking error) complicates handling such opportunity sets and calculating of frequency distributions enormously. For example, if we add a restriction on the portfolio variance, the feasible space of portfolio opportunities will be the intersection of a closed convex polyhedron representing linear constraints and the isovariance ellipse representing maximal tolerable level of risk.

Generally, opportunity set descriptions with non-linear constraints can be handled by simulation only. The recipe is to find the enclosed polyhedron of the type (3.13), and reject portfolios, which do not "meet" the non-linear constraints by evaluation.

3.4 Summary and Conclusions

We distinguish three different kinds of investment constraints. These are hard constraints, which are externally imposed and cannot be changed; soft constraints, which are self-imposed restrictions aimed to improve the decision-making and reduce the chance of failure; and game-type constraints which are the result of self-imposed restrictions aimed to take account of other parties in the investment field.

At a very general level investment constraints can be subdivided into two classes: screening constraints and selection constraints. Investment constraints based on screening can be defined in a negative or in a positive way. Negative screening constraints filter out companies and financial instruments with undesirable activities or values. On the contrary, positively defined screening constraints select the appropriate financial instruments or companies for investment. Selection constraints impose restrictions on proportions, which can be invested in one or another asset. The most common kinds of selection constraints are restrictions on short-sales, restrictions on a maximal and/or a minimal capital exposure toward individual assets or combinations thereof as well as restrictions according to the risk profile.

Once we translate all prescribed investment restrictions into certain types of constraints on asset weights, we get a system of inequalities and equalities, which formally describe a feasible portfolio opportunity set. The three standard opportunity sets considered through this thesis are:

- Opportunity sets with a constraint on short-sales only;
- Opportunity sets with constraints on individual asset weights and linear summary constraints thereof;
- Opportunity sets with non-linear constraints.

The first kind of opportunity sets are geometrically basic simplexes. Such sets can be easily described and analyzed. The second kind of sets are closed convex polyhydra; their geometric form strongly depends on the upper bounds of weights. We will use tools from the analytical geometry and linear algebra for calculating frequency distributions with respect to linear performance measures for the first two kinds of portfolio opportunity sets. The next chapter is completely devoted to developing mathematical tools for examining such sets.

Imposing non-linear constraints such as diverse restrictions on a risk exposure enormously increase the complexity of representation and handling such opportunity sets. Portfolio opportunity sets of such kind will be considered in Chapter 5.

Chapter 4

Calculation of Performance Distributions

4.1 The Intuition behind the Calculation Methodology

The crucial stage in the framework is the calculation of frequency distributions with respect to all specified performance measures for a formally defined portfolio opportunity set. That is, having at our disposal

- a portfolio opportunity set for an investment (e.g. one of the type defined by the system (3.13))

and

- one or several metrics used for evaluation of *ex ante* or *ex post* performance (e.g. realized return over some horizon, Sharpe ratio, Jensen's alpha)

we are interested to know how many of the feasible portfolios take on a particular performance value for each of specified performance criteria. In other words, we aim at calculating frequency distributions of feasible portfolios in terms of the values of a performance measure.[1]

In the space of asset weights a set of portfolios, all of which have the same value for a specific performance metric, builds up the so-called *iso-surfaces*. For example, if we consider the realized return as a performance measure, the sets of portfolios with the same returns form iso-return hyperplanes; in

[1]Indeed, such frequency distributions can be easily transformed into probabilistic (cumulative) density functions by using the definition of geometric probability.

case of the Sharpe ratio, the sets of portfolios with the same ratio values form iso-ratio hyperellipses. Consequently, *the task of calculating the frequency distribution for a performance metric is equal to finding the volume of the cross-section between the feasible portfolio set and the corresponding iso-surfaces.* We illustrate this observation with a small example.

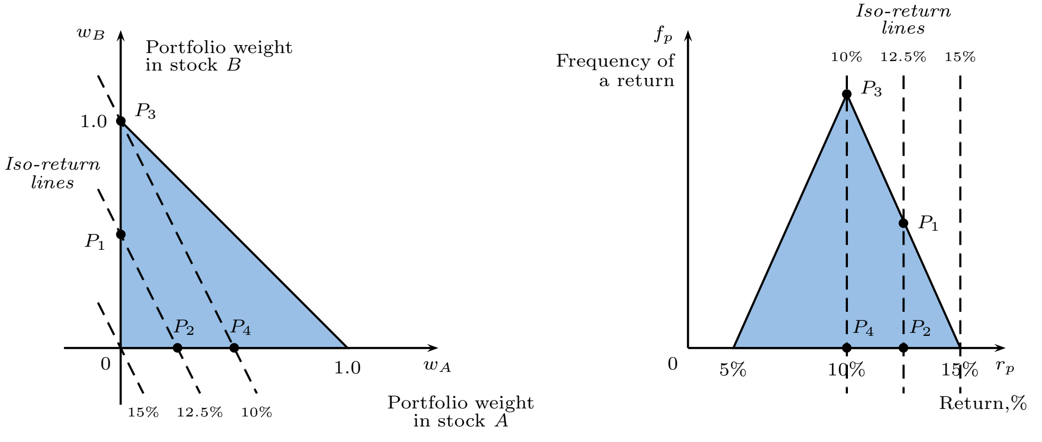


Figure 4.1: Portfolio opportunity set consisting of three assets with a restriction on short-sales (left). The dashed lines show iso-return lines for the realized return measure. The shaded triangle (right) shows the frequency distribution of feasible portfolios with respect to realized returns. The lengths of segments $[P_1, P_2]$ and $[P_3, P_4]$ in (right) reflect the volume of cross-sections between 12.5% iso-return hyperplane and the portfolio opportunity set, and between 10% iso-return hyperplane and the opportunity set in (left).

Consider the following portfolio opportunity set consisting of three securities:[2]

$$\begin{aligned} -w_i &\leq 0 \quad \forall\, i = 1, 2, 3 \\ w_i - 1 &\leq 0 \quad \forall\, i = 1, 2, 3 \\ \sum_{i=1}^{3} w_i - 1 &= 0 \end{aligned}$$

and the realized return as a performance metric. Let the returns for asset A, B, and C be 5%, 10%, and 15% respectively. The left graph of Figure 4.1 shows the portfolio opportunity set and the iso-return lines for 10%, 12.5%,

[2]The inequalities $w_i - 1 \leq 0$ are redundant; we leave them in the description for the sake of a better understanding of upper bounds for asset weights.

and 15%. The right graph of Figure 4.1 plots the frequency distribution for realized returns. The lengths of segments $[P_1, P_2]$ and $[P_3, P_4]$ in the frequency distribution in the right graph are equal to the volume of the cross-sections between 12.5% iso-return hyperplane and the portfolio opportunity set, and between 10% iso-return hyperplane and the opportunity set in the left graph of Figure 4.1.

Commonly, our aim through this chapter is to determine an explicit analytical formula for each of the standard opportunity sets from section 3.3. That is, for a given portfolio opportunity set and a performance measure we show how to derive the corresponding function $\mathfrak{fr}(t) \to \mathbb{R}_+$, which calculates for any performance value t its relative frequency among all feasible portfolios. Determined to derive closed-end formulae, we look in the next section on linear performance measures first. Then we review some properties of closed polyhedra focusing on calculation of the polyhedron volumes using their vertices. Afterwards we investigate whether it is possible to compute the cross-section volumes for all standard portfolio opportunity sets defined in the previous chapter, and, if possible, how to do this. Finally, we briefly look at non-linear performance measures as applied to our standard opportunity sets.

4.2 Linear Performance Measures

4.2.1 The Geometrical Form and Representations of Linear Performance Measures

Linear performance measures mean that the performance of a portfolio τ_p is a linear combination of performance values of the constituting assets, i.e.

$$\tau_p = \tau_1 w_1 + \tau_2 w_2 + \ldots + \tau_n w_n$$

where τ_i is the performance of asset i. Examples of such measures are different kinds of returns: realized, expected, average, differential.

Generally, iso-surfaces for linear performance metrics are hyperplanes in the asset weight space. When one considers portfolio opportunity sets consisting of n feasible assets and a linear performance measure τ, the formal description of the set of portfolios having the same performance value t is given by the following equation:

$$\tau_1 w_1 + \tau_2 w_2 + \ldots + \tau_{n-1} w_{n-1} + \tau_n w_n = t \tag{4.1}$$

We can rewrite the last equation in matrix form as

$$\boldsymbol{\tau}^T \mathbf{w} = t \tag{4.2}$$

Equation 4.1 defines an $n-1$-dimensional hyperplane in $\mathbb{R}^n$ due to the budget constraint. Sometimes it is more convenient to analyze such sets if we remove the last dependent weight from equation (4.1):

$$\begin{aligned} \tau_1 w_1 + \tau_2 w_2 + \ldots + \tau_{n-1} w_{n-1} + \tau_n \left(1 - \sum_{i=1}^{n-1} w_i\right) &= t \\ (\tau_1 - \tau_n) w_1 + (\tau_2 - \tau_n) w_2 + \ldots + (\tau_{n-1} - \tau_n) w_{n-1} &= t - \tau_n \end{aligned}$$

Denoting differential performances and the right-hand side of the equation by $\hat{\tau}_i = \tau_i - \tau_n$ and $\hat{t} = t - \tau_n$, we obtain the ordinary equation of a full-dimensional hyperplane in $\mathbb{R}^{n-1}$:

$$\hat{\tau}_1 w_1 + \hat{\tau}_2 w_2 + \ldots + \hat{\tau}_{n-1} w_{n-1} = \hat{t} \tag{4.3}$$

We rewrite the last equation in matrix form:

$$\hat{\boldsymbol{\tau}}^T \hat{\mathbf{w}} = \hat{t} \tag{4.4}$$

Henceforth, analyzing our standard opportunity sets we consider iso-hyperplanes in both (4.2) and (4.4) forms depending on in which dimension, $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$, the analysis is performed. However, when we use the form (4.4), we leave out the preprocessing transformations.

4.2.2 General Approach for Deriving an Analytical Solution

We postpone the discussion about portfolio opportunity sets and linear performance measures till the end of this section and will concentrate on some important geometrical and algebraic aspects of high-dimensional polyhedra and their volume calculation. (We also refer to Appendix A, which contains an elementary introduction into convex geometry.)

General Remark. When we discuss portfolio opportunity sets, we consider the asset weight space, or w-space for short. A point in the w-space corresponds to a portfolio, which is usually denoted as a vector $\mathbf{w} = \langle w_1, w_2, \ldots, w_n \rangle$ (or $\mathbf{p}_t = \langle w_1, w_2, \ldots, w_n \rangle$ if we look at a specific portfolio t). Consequently, iso-hyperplanes for a linear performance measure τ are denoted in the w-space through $\boldsymbol{\tau}^T \mathbf{w} = t$ (or $\hat{\boldsymbol{\tau}}^T \hat{\mathbf{w}} = \hat{t}$ if we remove the dependent constraint). Discussing the general mathematical properties of portfolio opportunity sets, we use the usual x-coordinate space $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$, depending on the context. In this case the hyperplanes are denoted through $\mathbf{c}^T \mathbf{x} = d$. The reason for the double notation is our intension to differentiate between the abstract mathematics of derivation of the (cross-section) volume formulae and the core financial context of this thesis.

We start our overview of existing mathematical tools for calculating polytope volumes with a proposition, which states the volume formula for an arbitrary simplex.

Proposition 4.2.1. *Let* $\mathbf{v}_i$*,* $1 \leq i \leq n+1$*, be given vectors in* $\mathbb{R}^n$ *and let* P *be a polyhedron defined on these vectors as*

$$P = \{\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{v}_i, \ \sum_{i=1}^{n+1} \lambda_i = 1 \text{ and } \lambda_i \geq 0\}$$

The (unsigned) volume of P *is*

$$\mathrm{Vol}(P) = \frac{1}{n!} \left|\det(\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \ldots, \mathbf{v}_{n+1} - \mathbf{v}_1)\right|$$

So, the volume of a simplex can be expressed/computed easily. A direct method for exact volume calculation of a closed polyhedron is to decompose it into a finite number of polytopes whose volumes can be expressed through closed-end formulae.[3] In particular, the following proposition expresses a volume formula for a polytope through signed volumes of certain simplexes, where at each vertex of the original polytope the simplexes are formed one at a time.

Proposition 4.2.2. *Let* $\mathbf{v}_i$*,* $1 \leq i \leq s$ *and* $s \geq n+1$*, be given vectors in* $\mathbb{R}^n$*, and let* P *be a simple polytope defined on these vectors as*

$$P = \{\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^{s} \lambda_i \mathbf{v}_i, \ \sum_{i=1}^{s} \lambda_i = 1 \text{ and } \lambda_i \geq 0\}$$

For a given halfspace $S' = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} \leq d\}$ *with the boundary hyperplane* $\mathbf{c}^T\mathbf{x} = d$ *in a general position to* P*, the volume of* P *in* S' *is defined by*

$$\mathrm{Vol}(P \cap S') = \sum_{\mathbf{v}_i \in S'} \mathrm{Vol}(\Delta_{\mathbf{v}_i}) \cdot \psi$$

where $\Delta_{\mathbf{v}_i}$ *is the simplex formed by the faces of* P *incident to the vertex* $\mathbf{v}_i$ *and the hyperplane* $\mathbf{c}^T\mathbf{x} = d$*, and* $\psi \in \{0, 1\}$*.*

That is, when we have a closed polyhedron, all vertices of which have exactly n incident edges, we can define the volume of this polyhedron using the formula for the simplex volume. Simplexes are built around all vertices of the polyhedron: for a vertex $\mathbf{v}_i$ the corresponding simplex $\Delta_{\mathbf{v}_i}$ is formed by faces incident to the vertex i and a specific additional hyperplane. This additional hyperplane is placed in such a way that it is not parallel to any

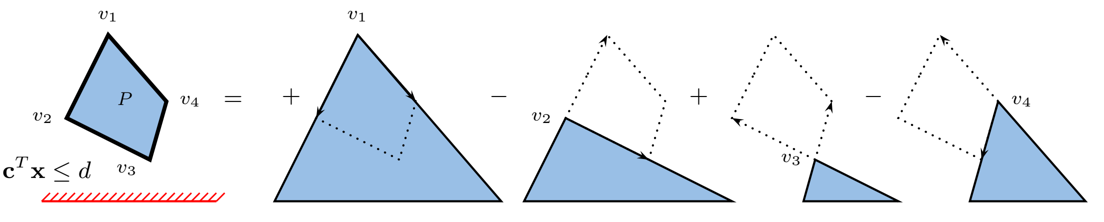
[3] See, for example, the technical report of Büeler, Enge & Fukuda (1998), which provides a compact overview of contemporary techniques for calculating volumes of high-dimensional polytopes.

of the edges of the polyhedron. (Henceforth, we will say that the hyperplane is in *a general position* to the polyhedron.) Clearly, the volume for each of such simplexes can be calculated and/or expressed analytically. Each of the volumes is signed, i.e. will be positive or negative. (In Proposition 4.2.2 we use the binary function ψ therefore.) The sum of all signed simplex volumes gives the volume of the polyhedron.

We illustrate the idea with an example in $\mathbb{R}^2$ shown in Figure 4.2. The volume of the polyhedron P can be computed as:

$$\mathrm{Vol}(P) = +\mathrm{Vol}(\mathbf{v}_1) - \mathrm{Vol}(\mathbf{v}_2) + \mathrm{Vol}(\mathbf{v}_3) - \mathrm{Vol}(\mathbf{v}_4)$$

where Δ's are defined with respect to the hyperplane $\mathbf{c}^T\mathbf{x} = d$. The sign of each particular simplex volume can be determined by counting the number of edges of the considered vertex which "reverse" the direction. For example, for $\mathbf{v}_1$ in Figure 4.2 it is equal to zero, for $\mathbf{v}_2$ it is equal to 1, and for $\mathbf{v}_3$ and $\mathbf{v}_4$ the number is 2 and 1 respectively. The value of the function ψ can be defined as $\psi = (-1)^{\text{number of "reversed" edges}}$, so in the case of $\mathbf{v}_1$ $\psi(\mathbf{v}_1) = (-1)^0 = 1$, $\psi(\mathbf{v}_2) = (-1)^1 = -1$ *et cetera.*



$$\mathrm{Vol}(P) = +\,\mathrm{Vol}(\Delta_{\mathbf{v_1}}) - \mathrm{Vol}(\Delta_{\mathbf{v_2}}) + \mathrm{Vol}(\Delta_{\mathbf{v_3}}) - \mathrm{Vol}(\Delta_{\mathbf{v_4}})$$

Figure 4.2: Illustration of volume calculation of a polytope through summing the signed volumes of simplexes formed at the polytope vertices with respect to a hyperplane.

The example in Figure 4.2 shows a hyperplane that does not intersect the polytope. However, Proposition 4.2.2 also states that if we select a hyperplane which intersects the considered polytope, we can use the same method of signed Δ–volumes to compute the volume of the polytope part laying in the halfspace $\mathbf{c}^T\mathbf{x} \le d$. The only difference is that we need to sum the signed volumes of the simplexes formed by polytope vertices laying in that halfspace only, i.e. vertices that satisfy the inequality $\mathbf{c}^T\mathbf{v}_i \le d$. Figure 4.3 illustrates this generalization.

Proposition 4.2.2 can easily be extended to non-simple polytopes in $\mathbb{R}^n$: For each vertex $\mathbf{v}_i$ we use the lexicographic perturbation method to choose

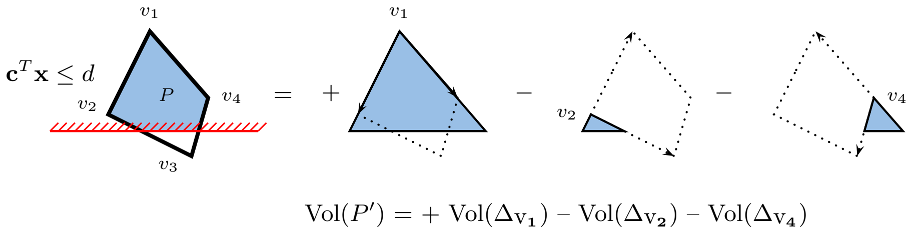


Figure 4.3: Illustration of volume calculation of a polytope part laying in the upper halfspace using summing signed volumes of simplexes formed at polytope vertices with respect to a separating hyperplane.

all n combinations of hyperplanes incident to $\mathbf{v}_i$.

Till now we have looked only at the volume calculation for a closed polyhedron or a part thereof. Indeed, given a polyhedron P and a hyperplane H, we are interested in the calculation of the cross-section volume of P and H. When we "fix" P and the $\mathbf{c}$ vector of the hyperplane equation, the cross-section volume depends only on the d value. Henceforth, we denote such cross-section volumes by $\mathrm{Vol}(P \cap H \mid d)$ in order to emphasis the dependency of such a volume from d by given P and $\mathbf{c}$.

The following theorem states how an analytical formula for cross-sectional volumes $\mathrm{Vol}(P \cap H \mid d)$ can be derived from an analytical expression for the volume formula from Proposition 4.2.2. The intuitive idea is represented in Figure 4.4. Namely, given a polytope P and a hyperplane $H = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} \leq d\}$, we can calculate the volume of P laying above the hyperplane (denoted by $\mathrm{Vol}(P' \mid d)$ in Figure 4.4). When we "shift" the hyperplane, let us say by δ, the volume $\mathrm{Vol}(P' \mid d + \delta)$ increases. As the value of δ approaches to zero, $\delta \to 0$, the increase in the volume, from $\mathrm{Vol}(P' \mid d)$ to $\mathrm{Vol}(P' \mid d + \delta)$ approaches the cross-section volume multiplied by h, i.e.

$$\mathrm{Vol}(P' \mid d + \delta) - \mathrm{Vol}(P' \mid d) \to \mathrm{Vol}(P \cap H \mid d) \cdot h,$$

where h is the distance between two hyperplanes $\mathbf{c}^T\mathbf{x} = d$ and $\mathbf{c}^T\mathbf{x} = d+\delta$. In other words, *the cross-section volume is the derivative of the polytope volume where the "changing" variable is the value of d*, the right-hand side of the hyperplane equation. Theorem 4.2.1 formally defines the relation between the polytope volume and the cross-section volume.

Theorem 4.2.1. *Let $\mathbf{v}_i$, $1 \leq i \leq s$ and $s \geq n + 1$, be given vectors in $\mathbb{R}^n$, and let P be a polyhedron defined on these vectors as*

$$P = \{\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^{s} \lambda_i \mathbf{v}_i,\ \sum_{i=1}^{s} \lambda_i = 1 \text{ and } \lambda_i \geq 0\}$$

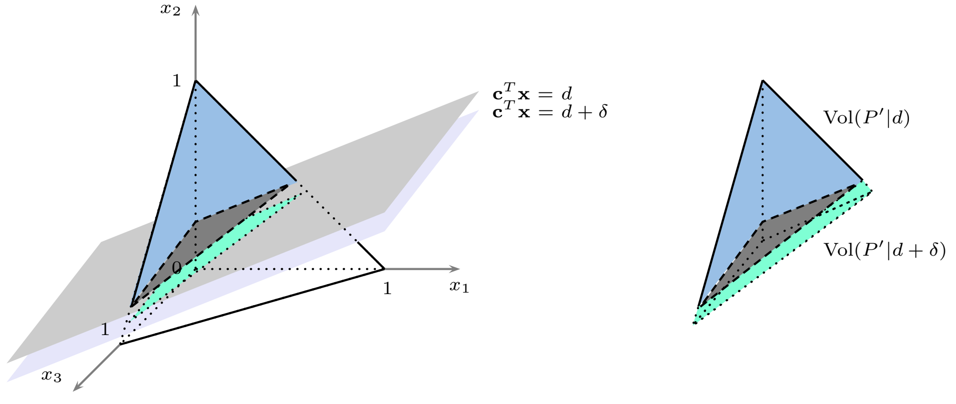


Figure 4.4: Graphical representation of the idea for the derivation of an analytical formula for the cross-sectional volume $\mathrm{Vol}(P \cap H \mid d)$, $H = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = d\}$, on the example of the basic simplex P in $\mathbb{R}^3$. Using Proposition 4.2.2 we can calculate the volume of P laying in the upper halfspace (We denote this simplex part by P'). This volume is shaded in both pictures. When we shift the hyperplane H by a small amount δ, the volume of P' increases as the right subfigure demonstrates. As the distance between two hyperplanes, $\mathbf{c}^T\mathbf{x} = d$ and $\mathbf{c}^T\mathbf{x} = d + \delta$ decreases and approaches to zero, the increase in the volume, $[\mathrm{Vol}(P' \mid d+\delta) - \mathrm{Vol}(P' \mid d)]/h$, approaches to the cross-sectional volume $\mathrm{Vol}(P \cap H \mid d)$, where h is the distance between two hyperplanes. (In the right subfigure the lower light-colored triangle emerges with the upper, dark-colored triangle.)

If a hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} = d\}$ *is in general position to* P*, then the* $n-1$*-dimensional volume of the intersection of* P *and* H *is*

$$\mathrm{Vol}(P \cap H \mid d) = \frac{\partial \mathrm{Vol}(P'|d)}{\partial d} \cdot \|\mathbf{c}\|$$

Proof. We can define the $\mathrm{Vol}(P \cap H \mid d)$ as

$$\mathrm{Vol}(P \cap H \mid d) = \lim_{\delta \to +0} \frac{\mathrm{Vol}(P' \mid d+\delta) - \mathrm{Vol}(P' \mid d)}{h} \tag{4.5}$$

where h is the distance between the hyperplane $\mathbf{c}^T\mathbf{x} = d + \delta$ and the hyperplane $\mathbf{c}^T\mathbf{x} = d$, and P' is the part of the polyhedron P laying in the halfspace $S' = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{c}^T\mathbf{x} \leq d\}$ with the boundary hyperplane $\mathbf{c}^T\mathbf{x} = d$.

The distance between the hyperplane $\mathbf{c}^T\mathbf{x} = d + \delta$ and the hyperplane $\mathbf{c}^T\mathbf{x} = d$ is equal to:

$$h = \frac{(d+\delta) - d}{\|\mathbf{c}\|} = \frac{\delta}{\|\mathbf{c}\|} \tag{4.6}$$

Substituting (4.6) into (4.5) we get:

$$\begin{aligned} \mathrm{Vol}(P \cap H \mid d) &= \lim_{\delta \to +0} \frac{\mathrm{Vol}(P' \mid d+\delta) - \mathrm{Vol}(P' \mid d)}{\delta} \cdot \|\mathbf{c}\| \\ &= \frac{\partial \mathrm{Vol}(P'|d)}{\partial d} \cdot \|\mathbf{c}\| \end{aligned} \tag{4.7}$$

□

Returning to the asset weight space, we observed that different types of portfolio opportunity sets represent different kinds of polytopes. And linear performance measures "collect up" the portfolios with the same performances into hyperplanes. These iso-hyperplanes are oriented according to performances of individual assets. Therefore, Proposition 4.2.2 and Theorem 4.2.1 provide us with necessary mathematical tools to deal with the derivation of analytical formulae for frequency distributions for given classes of portfolio opportunity sets and linear performance measures.

Summarizing the results of this subsection, we can define a general approach to the task of deriving analytical formulae for frequency distributions for our standard portfolio opportunity sets from Chapter 3:

1. Consider the task of *enumerating* all vertices of portfolio opportunity set polytopes of a specific kind, i.e. with restrictions on short-sales only, with restrictions on individual asset weights *et cetera*.[4] (Discussing opportunity sets, we also call vertices of the opportunity set polytopes *extreme portfolios*.) Having n assets in an opportunity set of a specific kind, either the $n-$ or $n-1$-dimensional w-space may be used for formulation of the enumeration procedure and further analysis;

2. Define the general analytical formula for the (partial) volume of opportunity set polytopes. Use for this the volume decomposition formula from Proposition 4.2.2 and vertices from Step 1 to build the decomposition simplexes;

3. Derive the general closed-end formula for frequency distributions defined as a function of a specific performance value t for a specific kind of opportunity set polytopes and of performances of individual feasible assets. According to Theorem 4.2.1 an explicit formula for frequency distributions is the derivative of the volume formula from Step 2 multiplied by the norm of the hyperplane coefficient vector.

[4] In this thesis under *enumerating* we mean the task of running through all of the vertices one at a time, examine the content of each vertex and perform some calculations using the current vertex vector. Other authors also refer to this activity as vertex *listing* or *generating* task.

4.2.3 Opportunity Sets with a Short-Sales Restriction only

A portfolio opportunity set of this kind consisting of n feasible assets is a basic simplex in $\mathbb{R}^{n-1}$ asset weight space. The n vertices of the simplex are enumerated easily: $(0,0,\ldots,0)$, $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, $\ldots$, $(0,0,\ldots,1)$. That is to say, each vertex represents an extreme portfolio consisting of a single asset.

When we look at a linear performance measure τ, one specific performance level is "attached" to each simplex vertex, and it is nothing else than the performance of an individual asset. Additionally, these individual asset performances determine the orientation of iso-hyperplanes: the coefficient vector $\hat{\boldsymbol{\tau}}$ of the iso-hyperplane family $\hat{\boldsymbol{\tau}}^T\hat{\mathbf{w}} = \hat{t}$ is defined as $\hat{\boldsymbol{\tau}} = \langle \hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_{n-1}\rangle$ (cf. Section 4.2.1).

We use these two observations to extend the volume formula from Proposition 4.2.2 expressing $\Delta_{\mathbf{v}_i}$-subvolumes through vertex performances only. For the sake of simplicity, let us examine the problem of defining the (partial) volume of the basic simplex with respect to any hyperplane in general terms first. We can formulate the following theorem.[5]

Theorem 4.2.2. *Let $\mathbf{v}_i$, $1 \le i \le n$, be given vectors in $\mathbb{R}^{n-1}$ and let P be a simplex defined on these vectors as*

$$P = \{\mathbf{x} = \langle x_1, x_2, \ldots, x_{n-1}\rangle \in \mathbb{R}^{n-1} \mid \mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i,\ \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \ge 0\}$$

Let $H = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid \mathbf{c}^T\mathbf{x} = d\}$ be a hyperplane in a general position to P, and $\mathbf{c}^T\mathbf{v}_i = d_i$, $i = 1, 2, \ldots, n$. For a halfspace $S' = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid \mathbf{c}^T\mathbf{x} \le d\}$ with the boundary hyperplane H the volume of $P' = P \cap S'$ is defined as

$$\mathrm{Vol}(P' \mid d) = \mathrm{Vol}(P) \cdot \sum_{d_k \le d} \left[(d - d_k)^{n-1} \prod_{i=1, i \ne k}^{n} \frac{1}{(d_i - d_k)} \right]$$

Proof. Without loss of generality we can assume that $d_1 < d_2 < \ldots < d_n$. That is, if we make a parallel shift of hyperplane H by increasing the right-hand side d, the vertices of our simplex P will be "passed" in ascending order.

For any given d the volume $\mathrm{Vol}(P' \mid d)$=$\mathrm{Vol}(P \cap H \mid d)$ is then defined by the formula

$$\mathrm{Vol}(P \cap H \mid d) = \sum_{\mathbf{v}_i : \mathbf{c}^T\mathbf{v}_i \le d} \mathrm{Vol}(\Delta_{\mathbf{v}_i} \mid d) \cdot (-1)^{i+1} \tag{4.8}$$

[5] We define this general theorem in terms of the basic simplex in $\mathbb{R}^{n-1}$ in order to be consistent with the opportunity set defined on n feasible assets.

Figure 4.5 illustrates the formula expansion for three different values of the right-hand side of H in $\mathbb{R}^2$.

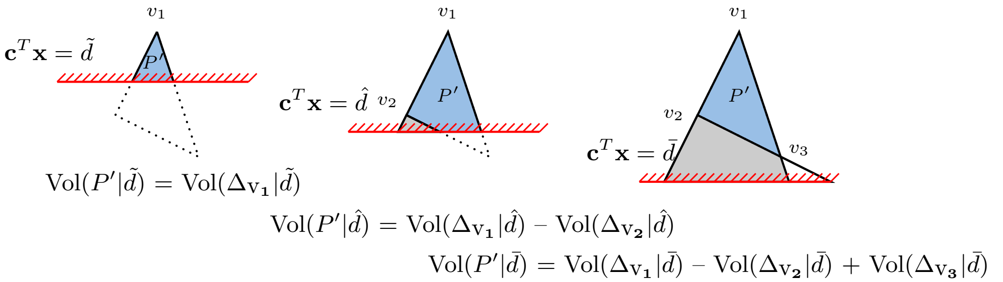


Figure 4.5: Expansion of the formula (4.8) for three different values of the right-hand side of a hyperplane H and a simplex P in $\mathbb{R}^2$.

Consider now the simplex formed on a vertex $\mathbf{v}_k$, $\Delta_{\mathbf{v}_k}$. (This simplex is formed by the faces incident to the vertex $\mathbf{v}_k$ and the hyperplane H.) Beyond $\mathbf{v}_k$ other vertices of $\Delta_{\mathbf{v}_k}$ can be defined as

$$\mathbf{v}_k + \lambda_i (\mathbf{v}_i - \mathbf{v}_k), \quad i = 1, 2, \ldots, n \text{ and } i \neq k$$

On the other hand, these vertices lay on H, i.e.

$$\begin{aligned} \mathbf{c}^T (\mathbf{v}_k + \lambda_i(\mathbf{v}_i - \mathbf{v}_k)) &= d \\ \mathbf{c}^T\mathbf{v}_k + \lambda_i \left(\mathbf{c}^T\mathbf{v}_i - \mathbf{c}^T\mathbf{v}_k\right) &= d \\ \lambda_i (d_i - d_k) &= d - d_k \\ \Rightarrow \lambda_i = \frac{d - d_k}{d_i - d_k} \end{aligned}$$

The volume of $\Delta_{\mathbf{v}_k}$ can be expressed by λ_i's as

$$\mathrm{Vol}(\Delta_{\mathbf{v}_k} \mid d) = \left| \prod_{i=1,\, i\neq k}^{n} \lambda_i \right| \cdot \mathrm{Vol}(P) = \left| \frac{(d - d_k)^{n-1}}{\prod_{i=1,\, i\neq k}^{n} (d_i - d_k)} \right| \cdot \mathrm{Vol}(P)$$

Assuming that $d_1 < d_2 < \ldots < d_n$ gives us

$$(-1)^{k+1} \cdot \mathrm{Vol}(\Delta_{\mathbf{v}_k} \mid d) = \frac{(d - d_k)^{n-1}}{\prod_{i=1,\, i\neq k}^{n} (d_i - d_k)} \cdot \mathrm{Vol}(P) \tag{4.9}$$

Substituting (4.9) into (4.8) we obtain the final formula. □

Using the last theorem we can formulate an analytical expression for calculating frequency distributions on short-sales restricted opportunity sets with respect to any linear performance measure. (It should be noted that we need to substitute into the formula of Theorem 4.2.2 the hyperplane equation of form (4.3). However, the "differential" performances $\hat{\tau}_i$ and $\hat{t}$ are reduced to the original performance values of individual assets after the substitution into the formula.)

Corollary 4.2.3. *Given an investment opportunity set consisting of n feasible assets and a linear performance measure τ, the relative frequency of portfolios having a specific performance t is given by*

$$\mathfrak{fr}(t) = \frac{1}{(n-2)!} \cdot \|\hat{\boldsymbol{\tau}}\| \cdot \sum_{\tau_k \leq t} \left[(t - \tau_k)^{n-2} \cdot \frac{1}{\prod_{i=1,\, i \neq k}^{n} (\tau_i - \tau_k)} \right]$$

where $\tau_1, \tau_2, \ldots, \tau_n$ are τ-performances of individual assets, and $\hat{\boldsymbol{\tau}}$ is the vector consisting of performances $\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_{n-1}$.

Proof. The proof follows directly from Theorem 4.2.1 and Theorem 4.2.2. □

Using this corollary we can formulate the following algorithm for computing the frequency distribution function for a linear performance measure τ:

1. Compute the τ performance values of individual assets, i.e. $\tau_1, \tau_2, \ldots, \tau_n$;
2. Sort performances τ_i in ascending order, i.e. define the order statistics $\tau_{(1)}, \tau_{(2)}, \ldots, \tau_{(n)}$;
3. Compute the norm of $\hat{\boldsymbol{\tau}}$ using the differential performances with respect to the performance $\tau_{(n)}$, i.e.

$$\|\hat{\boldsymbol{\tau}}\| = \sqrt{(\tau_{(1)} - \tau_{(n)})^2 + (\tau_{(2)} - \tau_{(n)})^2 + \ldots + (\tau_{(n-1)} - \tau_{(n)})^2};$$

4. Compose the expression for $\mathfrak{fr}(t)$ using all $\tau_{(i)} \leq t$, $\|\hat{\boldsymbol{\tau}}\|$ and the dimension n, and compute the required frequency for a specific performance t.

The algorithm and the closed-end formula of Corollary 4.2.3 are illustrated with a small example in the next section.

One should take notice that the function $\mathfrak{fr}(t)$ from Corollary 4.2.3 defines a frequency distribution function in such a way that

$$\int_{-\infty}^{+\infty} \mathfrak{fr}(t) dt = \mathrm{Vol}(P)$$

where P is the basic simplex representing a portfolio opportunity set. So, the values of $\mathfrak{fr}(t)$ strongly depend on the dimension of the corresponding opportunity set. For this reason in cases when we need to compare opportunity sets of different dimensions, we can “normalize” the frequency density function through the volume of the opportunity set polyhedron:

$$\mathfrak{fr}(t) = (n-1) \cdot \|\hat{\boldsymbol{\tau}}\| \cdot \sum_{\tau_k \le t} \left[(t-\tau_k)^{n-2} \cdot \frac{1}{\prod_{i=1,\, i\neq k}^{n} (\tau_i - \tau_k)} \right] \tag{4.10}$$

Henceforth, we will call such a function a *normalized frequency density function*. Similar to how it is usually done in the probability theory, we can also define the cumulative frequency density function as

$$\mathfrak{Fr}(t) = \frac{1}{(n-1)!} \cdot \sum_{\tau_k \le t} \left[(t-\tau_k)^{n-1} \cdot \frac{1}{\prod_{i=1,\, i\neq k}^{n} (\tau_i - \tau_k)} \right] \tag{4.11}$$

And the next expression represents the normalized version of the cumulative frequency density function:

$$\mathfrak{Fr}(t) = \sum_{\tau_k \le t} \left[(t-\tau_k)^{n-1} \cdot \frac{1}{\prod_{i=1,\, i\neq k}^{n} (\tau_i - \tau_k)} \right] \tag{4.12}$$

It should be emphasized that normalized versions of the frequency density function and the cumulative density function are probabilistic. So we can work with them using standard statistical methods and techniques.

4.2.4 Deriving an Analytical Solution for the 1M-return Frequency Distribution for our Example Investment

We look again at our example opportunity set from section 2.3. Our investment opportunity set consists of three stocks, the Bayerische HypoVereisbank AG, the Commerzbank AG, and the Deutsche Bank AG, and the portfolio opportunity set is the basic simplex in $\mathbb{R}^2$ (cf. Figure 2.2). Clearly, only one of the performance measures, the absolute performance over December 2003, is linear. Let us build the analytical expression for both the non-cumulative and the cumulative frequency density functions.

The performance values for our three feasible stocks are shown in the table in Figure 4.6. Consequently, the iso-return hyperplanes are defined by the following equation system:

$$\begin{cases} -0.0638\, w_{HVM} - 0.0366\, w_{CBK} + 0.1296\, w_{DBK} &= t \\ w_{HVM} + w_{CBK} + w_{DBK} &= 1 \end{cases}$$

where t denotes a specific absolute performance value. Taking the w_{DBK} as the dependent weight, we rewrite the iso-hyperplane equations as:

$$-0.1934\,\hat{w}_{HVM} - 0.1662\,\hat{w}_{CBK} = \hat{t}$$

The corresponding portfolio opportunity set and three example iso-hyperplanes for t equal to -6.38%, -3.66% and 12.96% together with the portfolio opportunity set are presented in Figure 4.6. Furthermore, the norm of $\hat{\boldsymbol{\tau}}$ is equal to:

$$\|\hat{\boldsymbol{\tau}}\| = \sqrt{(-0.1934)^2 + (-0.1662)^2} \approx 0.255$$

HVM-FF	CBK-FF	DBK-FF
-6.38%	-3.66%	12.96%

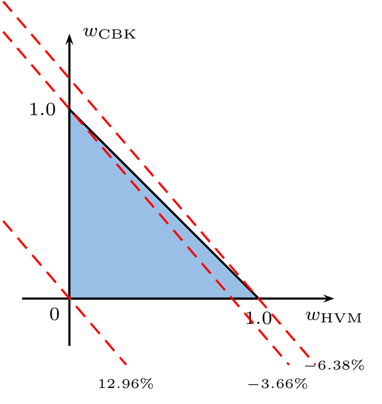


Figure 4.6: Absolute Performance over December 2003, in %, (left), and the portfolio opportunity set for the sample investment together with three iso-return hyperplanes, which pass through vertices of the opportunity set simplex (right).

According to Corollary 4.2.3, the value of $\tau_{(1)} = \tau_1 = -6.28\%$, $\tau_{(2)} = \tau_2 = -3.66\%$ and $\tau_{(3)} = \tau_3 = 12.96\%$. These three values subdivide the domain of $\mathfrak{fr}(t)$ into four intervals. So the frequency density function is a spline (smooth piecewise-polynomial function) defined as:

$$\mathfrak{fr}(t) = \begin{cases} \mathfrak{fr}_0(t) = 0 & t < \tau_{(1)} \\ \mathfrak{fr}_1(t) = \mathfrak{fr}_0(t) + 0.255 \cdot \left[\frac{t-\tau_{(1)}}{(\tau_{(2)}-\tau_{(1)})(\tau_{(3)}-\tau_{(1)})}\right] & \tau_{(1)} \le t < \tau_{(2)} \\ \mathfrak{fr}_2(t) = \mathfrak{fr}_1(t) + 0.255 \cdot \left[\frac{t-\tau_{(2)}}{(\tau_{(1)}-\tau_{(2)})(\tau_{(3)}-\tau_{(2)})}\right] & \tau_{(2)} \le t < \tau_{(3)} \\ \mathfrak{fr}_3(t) = \mathfrak{fr}_2(t) + 0.255 \cdot \left[\frac{t-\tau_{(3)}}{(\tau_{(1)}-\tau_{(3)})(\tau_{(2)}-\tau_{(3)})}\right] & t \ge \tau_{(3)} \end{cases}$$

The corresponding cumulative frequency density function is defined as:

$$\mathfrak{Fr}(t) = \begin{cases} \mathfrak{Fr}_0(t) = 0 & t < \tau_{(1)} \\ \mathfrak{Fr}_1(t) = \mathfrak{Fr}_0(t) + \frac{1}{2} \cdot \left[\frac{(t-\tau_{(1)})^2}{(\tau_{(2)}-\tau_{(1)})(\tau_{(3)}-\tau_{(1)})} \right] & \tau_{(1)} \leq t < \tau_{(2)} \\ \mathfrak{Fr}_2(t) = \mathfrak{Fr}_1(t) + \frac{1}{2} \cdot \left[\frac{(t-\tau_{(2)})^2}{(\tau_{(1)}-\tau_{(2)})(\tau_{(3)}-\tau_{(2)})} \right] & \tau_{(2)} \leq t < \tau_{(3)} \\ \mathfrak{Fr}_3(t) = \mathfrak{Fr}_2(t) + \frac{1}{2} \cdot \left[\frac{(t-\tau_{(3)})^2}{(\tau_{(1)}-\tau_{(3)})(\tau_{(2)}-\tau_{(3)})} \right] & t \geq \tau_{(3)} \end{cases}$$

Figure 4.7 shows the frequency density function and the cumulative frequency density function as well as their normalized versions.

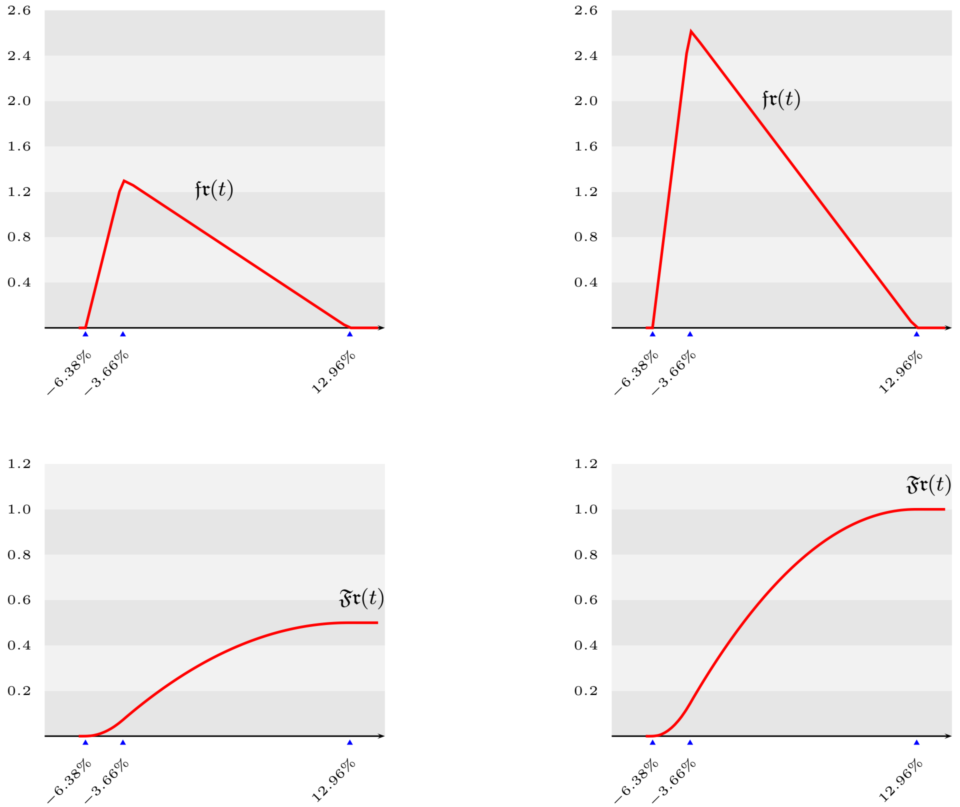


Figure 4.7: Frequency density function $\mathfrak{fr}(t)$ and its cumulative function $\mathfrak{Fr}(t)$ of the realized over December 2003 return (left), and normalized versions of both functions (right).

Looking at Figure 4.6 we can see why $\mathfrak{fr}(t)$ and $\mathfrak{Fr}(t)$ have this shape. Iso-hyperplanes with performances under -6.38% do not intersect the sample portfolio opportunity set, i.e. none of the feasible portfolio performed worse than -6.38%.[6] Therefore, $\mathfrak{fr}(t)$ and $\mathfrak{Fr}(t)$ are both equal to zero on the interval $t < \tau_{(1)} \equiv (-\infty, -6.38\%)$. The absolute performance of -6.38% has only one portfolio consisting of HVM stock solely. On the next interval $\tau_{(1)} \leq t < \tau_{(2)}$ the volume of the simplex laying right to such an iso-hyperplane (cf. Figure 4.6) is determined by the vertex $(1,0)$ only. So we have one term in $\mathfrak{fr}(t)$ and $\mathfrak{Fr}(t)$ formulae. On the third interval $\tau_{(2)} \leq t < \tau_{(3)} \equiv [-3.66\%, 12.96\%)$ the volume is defined by two vertices: $(1,0)$ and $(0,1)$. So, $\mathfrak{fr}(t)$ and $\mathfrak{Fr}(t)$ formulae contain two terms and so on.

4.2.5 Opportunity Sets with Restrictions on Individual Weights and Linear Summary Constraints

The geometric form of portfolio opportunity sets of this kind strongly depends on the upper bounds φ_i, which restrict the proportion of capital that could be invested in any single asset (cf. Section 3.3.3). So, considering opportunity set polytopes, our first challenge is to define a procedure/algorithm, which can enumerate all polytope vertices. It should be emphasized that in this part we will analyze the opportunity sets consisting of n assets in $\mathbb{R}^n$ asset weight space. That is, $n-1$-dimensional polytopes in $\mathbb{R}^n$ will be considered. Again, we start with a general discussion of the mathematics behind this kind of opportunity sets.

Lemma 4.2.4. *Let P be a polytope in $\mathbb{R}^n$ defined as*

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq \varphi_i, \forall\, i : i = 1, 2, \ldots, n \ \textit{and}\ \varphi_i \geq 0,\ \textit{and}\ \sum_{i=1}^{n} x_i = 1\}$$

A vector $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle \in \mathbb{R}^n$ is a vertex of polytope P if and only if:

1. *$\exists j \in \{1, 2, \ldots, n\}$ such that $\forall\, i \neq j$ the values v_i are equal ether to 0 or to φ_i;*
2. *$v_j = 1 - \sum_{i=1\, i\neq j}^{n} v_i$;*
3. *$v_j \leq \varphi_j$.*

Proof.
"⇒" Suppose to the contrary that a vector $\mathbf{v}$ satisfies all three conditions

[6] Please note that we discuss an *ex post* performance evaluation in our example. In an *ex ante* case we will have a specific portfolio frequency density for *ex ante* (i.e. expected) performance values.

and it is *not* a vertex of polytope P. Without loss of generality we can assume that

$$\mathbf{v} = \langle \varphi_1,\, \varphi_2,\, \ldots,\, \varphi_k,\, 1 - \sum_{i=1}^{k} \varphi_i,\, 0,\, \ldots,\, 0 \rangle.$$

If $\mathbf{v}$ is not a vertex of P, then according to the definition of polyhedron vertices, there exist two vectors, $\mathbf{a} \in P$ and $\mathbf{b} \in P$, both different from $\mathbf{v}$, such that $\mathbf{v}$ is a convex combination of $\mathbf{a}$ and $\mathbf{b}$, i.e. $\mathbf{v} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, $0 \le \lambda \le 1$. That is, $v_i = \lambda a_i + (1-\lambda) b_i \;\; \forall\, i = 1, 2, \ldots, n$.

Taking $i = 1$, we have $v_1 = \lambda a_1 + (1-\lambda) b_1$ and $v_1 = \varphi_1$. In addition, $a_1, b_1 \in [0, \varphi_1]$. This implies that either $a_1 = \varphi_1$ (if $\lambda = 0$), or $b_1 = \varphi_1$ (if $\lambda = 1$), or that $a_1 = b_1 = \varphi_1$ (if $0 < \lambda < 1$) in order to satisfy our two conditions. The same argument could be applied for all $i = 2, \ldots, k$. For $k+1$ it is true due to the condition that $\sum a_i = \sum b_i = 1$. So, we have $\mathbf{v} = \mathbf{a}$, or $\mathbf{v} = \mathbf{b}$, or $\mathbf{v} = \mathbf{a} = \mathbf{b}$. Thus, contradicting our assumption, vector $\mathbf{v}$ is a vertex of P when it satisfies all three conditions of the theorem.

"$\Leftarrow$" Suppose to the contrary that a vector $\mathbf{v}$ is a vertex of polytope P but it *does not satisfy* the first of the three conditions (if either the second or the third condition is violated, then the vector $\mathbf{v}$ is not in P, so these conditions must be satisfied). Without loss of generality we can assume that

$$\mathbf{v} = \langle \varphi_1,\, \varphi_2,\, \ldots,\, \kappa_k,\, 1 - \sum_{i=1}^{k-1} \varphi_i - \kappa_k,\, 0,\, \ldots,\, 0 \rangle.$$

where $0 < \kappa_k < \varphi_k$. That is, the first condition is violated by the k-th element of $\mathbf{v}$.

We "construct" two points as follows:

$$\mathbf{v}' = \langle \varphi_1,\, \varphi_2,\, \ldots,\, \kappa_k - \epsilon,\, 1 - \sum_{i=1}^{k-1} \varphi_i - \kappa_k + \epsilon,\, 0,\, \ldots,\, 0 \rangle.$$

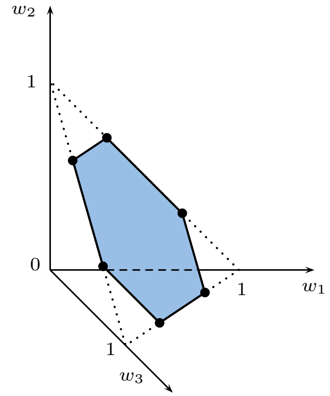
and

$$\mathbf{v}'' = \langle \varphi_1,\, \varphi_2,\, \ldots,\, \kappa_k + \epsilon,\, 1 - \sum_{i=1}^{k-1} \varphi_i - \kappa_k - \epsilon,\, 0,\, \ldots,\, 0 \rangle.$$

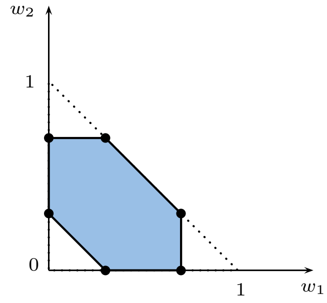
where $0 < \epsilon \le \varphi_k - \kappa_k$.

Clearly, $\mathbf{v}'$ and $\mathbf{v}'' \in P$. And, in addition, the vector $\mathbf{v}$ is a linear combination of $\mathbf{v}'$ and $\mathbf{v}''$, i.e. $\mathbf{v} = (\mathbf{v}' + \mathbf{v}'')/2$. Consequently, this contradicts our assumption that $\mathbf{v}$ is a vertex of P. □

Figure 4.8 illustrates the lemma by showing all vertices of a portfolio opportunity set, which consist of three feasible assets with $\varphi_1 = \varphi_2 = \varphi_3 = 0.7$.



$$
\begin{aligned}
-w_i &\leq 0 \quad i = 1, 2, 3 \\
w_i - 0.7 &\leq 0 \\
w_1 + w_2 + w_3 - 1 &= 0
\end{aligned}
$$



(w_1, w_2, w_3)
$(0.7, 0.3, 0.0)$
$(0.7, 0.0, 0.3)$
$(0.3, 0.7, 0.0)$
$(0.0, 0.7, 0.3)$
$(0.3, 0.0, 0.7)$
$(0.0, 0.3, 0.7)$

Figure 4.8: Portfolio opportunity set consisting of three assets in $\mathbb{R}^3$ (left) and $\mathbb{R}^2$ (right) weight spaces. The maximal capital proportion, which could be invested in any asset, is restricted by 0.7. The thick dots "mark" the vertices of the opportunity set polytope, and the corresponding labels show their values. The system below the left subfigure represents the formal description of the portfolio opportunity set, the list below the right subfigure enumerates all the vertices as if they are generated by an enumeration algorithm.

Algorithmically, the task of enumerating vertices of such portfolio opportunity sets is very similar to the dynamic programming technique (see Cormen, Leiserson & Rivest (1994), for example) and to the family of mixed-radix algorithms for generating combinatorial patterns (Knuth 2004). So, the generation routine for vertices can be implemented easily. In general, the number of vertices of an portfolio opportunity set is related to the number of assets asymptotically as $\mathcal{O}(n \cdot 2^n)$. In practice, however, the number of vertices for reasonably-large opportunity sets is manageable. (For example, an asset opportunity set consisting of 50 feasible assets and having an exposure toward a single asset at most 15% of the capital has about $699 \cdot 106$ vertices.) Larger investment opportunity sets are more amenable to sophisticated numerical techniques than to deriving and using an explicit analytical formula.[7] (The numerical estimation of frequency distributions is the topic of the next chapter.)

Having the knowledge about vertices of polytopes of this kind, our next task is to derive a formula which defines the polytope volume with respect to a given hyperplane. Clearly, we would like to use the volume decomposition techniques from Proposition 4.2.2. The necessary condition for using the proposition formula is that all polytope vertices are simple. In our case, if the complement element j of a vertex is equal to its lower or upper bound, i.e.

$$v_j = 1 - \sum_{i=1\, i\neq j} v_i = 0 \quad \text{and} \quad v_j = 1 - \sum_{i=1\, i\neq j} v_i = \varphi_j$$

this requirement is violated because the vertex $\mathbf{v}$ has n incident faces. Therefore, we first will consider the case when all vertices are simple, i.e. $0 < v_j = 1 - \sum_{i=1\, i\neq j} v_i < \varphi_j$ for the complement element. Afterwards, we will generalize the derived volume formula for covering also polytopes with non-simple vertices in an additional theorem.

Theorem 4.2.5. *Let P be a polytope in $\mathbb{R}^n$ defined as*

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq \varphi_i, \forall\, i : i = 1, 2, \ldots, n \text{ and } \varphi_i \geq 0, \text{ and } \sum_{i=1}^{n} x_i = 1\}$$

and all vertices of P are simple. Let $H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} = d\}$ be a hyperplane in a general position to P. For a halfspace $S' = \{\mathbf{x}\mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} \leq d\}$ with the boundary hyperplane H the volume of $P' = P \cap S'$ is defined as:

$$\mathrm{Vol}(P' \mid d) = \frac{\sqrt{n}}{(n-1)!} \cdot \sum_{\mathbf{v}:\mathbf{c}^T\mathbf{v} \leq d} \left[\frac{\left(d - c_j \cdot (1 - \sum_{i=1\, i\neq j}^{n} v_i) - \sum_{i=1\, i\neq j}^{n} c_i \cdot v_i\right)^{n-1}}{\prod_{i=1\, i\neq j}^{n} (c_i - c_j)} \right]$$

where j is the complement coordinate/element of vertex $\mathbf{v}$.

Proof. The theorem formula expresses the volume of polytope P through signed volume of simplexes built at vertices of P. As the first step we consider the volume calculation of a single simplex built at a given vertex $\mathbf{v}$ with respect to a given hyperplane H. Without loss of generality we can assume that $\mathbf{v}$ is:

$$\mathbf{v} = \langle 1 - \sum_{i=2}^{k} \varphi_i,\, \varphi_2,\, \ldots,\, \varphi_k,\, 0,\, \ldots,\, 0 \rangle.$$

[7] Beyond the time complexity the issue is the numerical stability of calculating such analytical formulae.

That is, the first element is the complement to 1 (i.e. index j refer to the first coordinate of vector $\mathbf{v}$), the elements from 2 to k take maximal values, and the others are equal to zero.

As defined, all vertices of P including $\mathbf{v}$ have $n-1$ incident faces and $n-1$ incident edges. The signed volume of the simplex at $\mathbf{v}$ with respect to H, $\Delta_{\mathbf{v}}$, can be expressed by the lengths of these $n-1$ incident edges as:

$$\mathrm{Vol}\Big(\Delta_{\mathbf{v}|}H\Big) = \prod_{i=2}^{n} \lambda_i \cdot \mathrm{Vol}({}_{n-1}\Delta) \tag{4.13}$$

where λ_i denotes the length of the edge i of $\Delta_{\mathbf{v}}$ and ${}_{n-1}\Delta$ denotes the $n-1$-dimensional basic simplex in $\mathbb{R}^n$.

For the sake of simplicity for computing λ_i, we perform the following transformation of coordinate systems first:

$$\begin{aligned} y_1 &= x_1 + \sum_{i=2}^{k} \varphi_i \\ y_i &= x_i - \varphi_i, \quad i = 2, 3, \ldots, k \\ y_i &= x_i, \qquad\quad i = k+1, k+2, \ldots, n \end{aligned} \tag{4.14}$$

That is, we make a parallel shift of our opportunity set polytope P across first k axes in such a way that vertex $\mathbf{v}$ moves into the point $(1, 0, 0, \ldots, 0)$ in y-coordinate system. Figure 4.9 provides an illustration of such a parallel shift for an example opportunity set in $\mathbb{R}^3$.

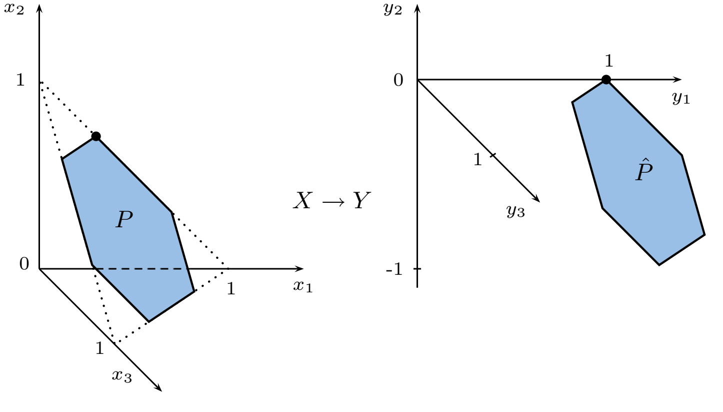


Figure 4.9: The $X \to Y$ transformation of coordinate systems for the sample opportunity set from Figure 4.8 at vertex $(0.3, 0.7, 0)$. The vertex is "marked" with a thick dot.

Respectively, hyperplane H in the new coordinate system would be:

$$\mathbf{c}^T\mathbf{x} = d \quad \Rightarrow \quad c_1(y_1 - \sum_{i=2}^{k} \varphi_i) + c_2(y_2 + \varphi_2) + \ \ldots \ + c_k(y_k + \varphi_k) + \\ + c_{k+1}y_{k+1} + \ldots + c_n y_n = d$$

Reorganizing the equation, we obtain:

$$\sum_{i=1}^{n} c_i y_i = d + c_1 \sum_{i=2}^{k} \varphi_i - \sum_{i=2}^{k} c_i \varphi_i \tag{4.15}$$

Replacing the right-hand side value of (4.15) by $\hat{d}$, we get the new hyperplane equation:

$$\hat{H} = \sum_{i=1}^{n} c_i y_i = \hat{d}$$

Beyond $\mathbf{v}$ other vertices of $\Delta_{\mathbf{v}}$ are laying on $\hat{H}$ and are defined in the y-coordinate system as

$$\mathbf{v}_i = \langle y_1, 0, \ldots, 0, y_i, 0, \ldots, 0 \rangle \qquad i = 2, 3, \ldots, n$$

So we can find these vertices using the following equation systems:

$$\begin{cases} y_1 + y_i &= 1 \\ c_1 y_1 + c_i y_i &= \hat{d} \end{cases} \qquad \forall\, i = 2, 3, \ldots, n$$

Solving the systems, we find that the lengths of $\Delta_{\mathbf{v}}$ edges are:

$$\lambda_i = y_i = \frac{\hat{d} - c_1}{c_i - c_1} \qquad \forall\, i = 2, 3, \ldots, n \tag{4.16}$$

The other value which we need for formula (4.13) is the expression for $\mathrm{Vol}({}_{n-1}\Delta)$, i.e. the volume of the $n-1$-dimensional basic simplex in $\mathbb{R}^n$. We can find the exact formula for $\mathrm{Vol}({}_{n-1}\Delta)$ using the following observation that

$$\mathrm{Vol}({}_n\Delta) = \frac{1}{n!}. \tag{4.17}$$

That is, the volume of the basic simplex in $\mathbb{R}^n$ can be found using Proposition 4.2.1. On the other hand, the volume is a product of the face volume of the basic simplex, i.e. $\mathrm{Vol}({}_{n-1}\Delta)$, with the simplex height and the extra dimension factor:

$$\mathrm{Vol}({}_n\Delta) = \frac{1}{n} \cdot \frac{1}{\sqrt{n}} \cdot \mathrm{Vol}({}_{n-1}\Delta) \tag{4.18}$$

Therefore, the analytical expression of $\mathrm{Vol}({}_{n-1}\Delta)$ is:

$$\mathrm{Vol}({}_{n-1}\Delta) = \frac{\sqrt{n}}{(n-1)!} \tag{4.19}$$

Substituting $\hat{d}$, (4.16) and (4.19) into (4.13), we obtain the final formula for the volume of the simplex build at $\mathbf{v}$ with respect to given H:

$$\mathrm{Vol}(\Delta_{\mathbf{v}} \mid H) = \frac{\sqrt{n}}{(n-1)!} \cdot \frac{\left(d - c_1(1 - \sum_{i=2}^{k} \varphi_i) - \sum_{i=2}^{k} c_i\varphi_i\right)^{n-1}}{\prod_{i=1\, i\neq j}^{n} (c_i - c_1)} \tag{4.20}$$

Equation (4.20) expresses the signed volume formula for $\Delta_{\mathbf{v}}$ in terms of coefficients of H, dimension n, and vector $\mathbf{v}$ solely.

The extension of (4.20) to define the $\mathrm{Vol}(P' \mid d)$ is straightforward: we add simplex volumes for all vertices, which lay in halfspace S':

$$\mathrm{Vol}(P' \mid d) = \frac{\sqrt{n}}{(n-1)!} \cdot \sum_{\mathbf{v}:\mathbf{c}^T\mathbf{v} \leq d} \left[\frac{\left(d - c_j(1 - \sum_{i\neq j} \varphi_i) - \sum_{i\neq j} c_i\varphi_i\right)^{n-1}}{\prod_{i=1\, i\neq j}^{n} (c_i - c_j)} \right]$$

where j denotes the index of the complement element in a vertex, and i's are vertex elements, which take on the upper bound values φ_i.

According to Lemma 4.2.4 all coordinates v_i but the complement coordinate v_j of every vertex of P are equal either to 0 or to φ_i. So, we replace in sums $\sum_{i\neq j} \varphi_i$ and $\sum_{i\neq j} c_i\varphi_i$ the upper bounds with v_i's and obtain the theorem formula. □

The volume formula is true for the general case when the complement coordinate value is coinciding with lower of upper bound.

Corollary 4.2.6. *Let P be a polytope in $\mathbb{R}^n$ defined as*

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq \varphi_i, \forall\, i : i = 1, 2, \ldots, n \text{ and } \varphi_i \geq 0, \text{ and } \sum_{i=1}^{n} x_i = 1\}$$

Let $H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} = d\}$ be a hyperplane in a general position to P. For a halfspace $S' = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} \leq d\}$ with the boundary hyperplane H the volume of $P' = P \cap S'$ is defined as:

$$\mathrm{Vol}(P' \mid d) = \frac{\sqrt{n}}{(n-1)!} \cdot \sum_{\mathbf{v}:\mathbf{c}^T\mathbf{v} \leq d} \left[\frac{\left(d - c_j(1 - \sum_{i=1\, i\neq j}^{n} v_i) - \sum_{i=1\, i\neq j}^{n} c_i v_i\right)^{n-1}}{\prod_{i=1\, i\neq j}^{n} (c_i - c_j)} \right]$$

Proof. We need to proof the formula only for the case where P contains one or more non-simple vertices. Clearly, in this case we can slightly "disturb" the faces of P, let us say by an amount ϵ, in such a way that all vertices of P become simple. Then the formula will be correct. Decreasing ϵ, the volume of the modified polytope approaches the volume of the original polytope, and in the limit converges with it. Therefore, the formula is also correct for the general case. □

Returning to the discussion of opportunity sets, we can easily translate these general results into the concept of frequency distributions. We start with the definition of extreme portfolios in the w-space for the $[0, \varphi_i]$-restricted opportunity sets.

Corollary 4.2.7. *Given a portfolio opportunity set consisting of n feasible assets with investment restrictions:*

$$\begin{aligned} -w_i &\leq 0 \qquad \forall\, i = 1, 2, \ldots, n \\ w_i - \varphi_i &\leq 0 \qquad \forall\, i = 1, 2, \ldots, n \quad \textit{and} \quad \varphi_i \geq 0 \\ \sum_{i=1}^{n} w_i - 1 &= 0 \end{aligned}$$

a portfolio **p** *is an extreme portfolio if:*[8]

1. *all asset weights but one in the portfolio vector* **p** *are equal to either 0 or φ_i;*
2. *an asset weight w_j is the complement to 1 of all asset weights from 1), i.e.*

$$w_j = 1 - \sum_{i=1\, i \neq j}^{n} w_i \quad \textit{and} \quad w_j \leq \varphi_j$$

Proof. Immediate from Lemma 4.2.4. □

Using Theorem 4.2.5 and Corollary 4.2.7 we can formulate an analytical expression for calculating frequency distributions for $[0, \varphi_i]$-restricted opportunity sets with respect to any linear performance measure.

[8] As a side remark we state that the set of extreme portfolios contains the portfolios with the best and the worst performances with respect to a linear performance measure. Similarly to the linear optimization theory we can state that given a portfolio opportunity set and a linear performance measure:

- The portfolio with the *best* performance is one of the extreme portfolios;
- The portfolio with the *worst* performance is one of the extreme portfolios.

Corollary 4.2.8. *Given an investment opportunity set consisting of n feasible assets and a linear performance measure τ, the relative frequency of portfolios having a specific performance t is given by*

$$\mathfrak{fr}(t) = \frac{\sqrt{n}}{(n-2)!} \cdot \|\boldsymbol{\tau}\| \cdot \sum_{\mathbf{p}:\ \boldsymbol{\tau}^T\mathbf{p} \le t} \left[\left(t - \tau_j \left(1 - \sum_{i=1\, i\neq j}^{n} w_i \right) - \sum_{i=1\, i\neq j}^{n} \tau_i w_i \right)^{n-2} \cdot \frac{1}{\prod_{i\neq j}(\tau_i - \tau_j)} \right]$$

where $\tau_1, \tau_2, \ldots, \tau_n$ are τ-performances of individual assets, $\mathbf{p}=\langle w_1, w_2, \ldots, w_n\rangle$'s are extreme portfolios of the corresponding portfolio opportunity set, and j denotes the index of the complement coordinate of a **p** *vector.*

Proof. Immediate from Theorem 4.2.5 and Corollary 4.2.7. □

Using the corollary we can formulate the following algorithm for computing the frequency distribution functions $\mathfrak{fr}(t)$ and/or $\mathfrak{Fr}(t)$ for a linear performance measure τ:

1. Compute the τ performance values of individual assets, i.e. $\tau_1, \tau_2, \ldots, \tau_n$. Use these performances to calculate the norm $\|\boldsymbol{\tau}\|$;
2. Generate a new extreme portfolio **p** according to Corollary 4.2.7;
3. Evaluate the new extreme portfolio. If $\boldsymbol{\tau}^T\mathbf{p} \le t$, add the corresponding term to the expression for $\mathfrak{fr}(t)$ and/or $\mathfrak{Fr}(t)$ using values $\tau_1, \tau_2, \ldots, \tau_n$, the dimension n, and $\|\boldsymbol{\tau}\|$;
4. If all extreme portfolios are evaluated, then stop. Otherwise return to step 3.

The algorithm and the closed-end formula of Corollary 4.2.8 are illustrated with a small example in the next section.

Normalized versions of $\mathfrak{fr}(t)$ and $\mathfrak{Fr}(t)$ can be computed by calculation first the volume of given portfolio opportunity set (For that we can use in the formula the largest τ_i as t value), and then dividing the value of the non-normalized version of $\mathfrak{fr}(t)$ or $\mathfrak{Fr}(t)$ by this volume.

4.2.6 Deriving an Analytical Solution for the 1M-return Frequency Distribution for our Example Investment if $\varphi_{HVM} = \varphi_{CBK} = \varphi_{DBK} = 0.7$

We look again at our example opportunity set from section 2.3. The investment opportunity set consists of three stocks, the Bayerische HypoVereisbank AG, the Commerzbank AG, and the Deutsche Bank AG, but in contrast to Section 4.2.4 here we consider the portfolio opportunity set where the maximum capital that can be invested in one asset is restricted by 0.7. Consequently, the portfolio opportunity set is the same as in Figure 4.8. The only performance measure we are evaluating is the absolute performance over December 2003. Let us build the analytical expression for both, the non-cumulative and the cumulative, frequency density functions.

Extreme portfolio	w_{HVM}	w_{CBK}	w_{DBK}	**Return**
$\mathbf{p}_1$	0.7	0.3	0	-5.56%
$\mathbf{p}_2$	0.3	0.7	0	-4.47%
$\mathbf{p}_3$	0.7	0	0.3	-0.58%
$\mathbf{p}_4$	0	0.7	0.3	1.33%
$\mathbf{p}_5$	0.3	0	0.7	7.16%
$\mathbf{p}_6$	0	0.3	0.7	7.98%

Table 4.1: Extreme portfolios and their absolute performances over December 2003 (in %).

The performance values for our three feasible stocks are:

HVM-FF	**CBK-FF**	**DBK-FF**
-6.38%	-3.66%	12.96%

Consequently, $\boldsymbol{\tau} = \langle -0.0638, -0.0366, 0.1296 \rangle$, and the iso-return hyperplanes are defined by the following equation system:

$$\begin{cases} -0.0638\, w_{HVM} - 0.0366\, w_{CBK} + 0.1296\, w_{DBK} & = & t \\ w_{HVM} + w_{CBK} + w_{DBK} & = & 1 \end{cases}$$

where t denotes a specific absolute performance value. The extreme portfolios together with their absolute performances are listed in Table 4.1.

According to Corollary 4.2.8, the values of $\mathbf{p}_1$ through $\mathbf{p}_6$ subdivide the domain of $\mathfrak{fr}(t)$ into seven intervals. So the frequency density function is a spline defined as:

$$\mathfrak{fr}(t) = \begin{cases} \mathfrak{fr}_0(t) = 0 & t < \mathbf{p}_1 \\ \mathfrak{fr}_1(t) = \mathfrak{fr}_0(t) + \frac{\sqrt{3}}{1!} \cdot 0.149 \cdot \left[\frac{t-\tau_2(1-\varphi_1)-\tau_1\varphi_1}{(\tau_1-\tau_2)(\tau_3-\tau_2)}\right] & \mathbf{p}_1 \le t < \mathbf{p}_2 \\ \mathfrak{fr}_2(t) = \mathfrak{fr}_1(t) + \frac{\sqrt{3}}{1!} \cdot 0.149 \cdot \left[\frac{t-\tau_1(1-\varphi_2)-\tau_2\varphi_2}{(\tau_2-\tau_1)(\tau_3-\tau_1)}\right] & \mathbf{p}_2 \le t < \mathbf{p}_3 \\ \mathfrak{fr}_3(t) = \mathfrak{fr}_2(t) + \frac{\sqrt{3}}{1!} \cdot 0.149 \cdot \left[\frac{t-\tau_3(1-\varphi_1)-\tau_1\varphi_1}{(\tau_1-\tau_3)(\tau_2-\tau_3)}\right] & \mathbf{p}_3 \le t < \mathbf{p}_4 \\ \mathfrak{fr}_4(t) = \mathfrak{fr}_3(t) + \frac{\sqrt{3}}{1!} \cdot 0.149 \cdot \left[\frac{t-\tau_3(1-\varphi_2)-\tau_2\varphi_2}{(\tau_1-\tau_3)(\tau_2-\tau_3)}\right] & \mathbf{p}_4 \le t < \mathbf{p}_5 \\ \mathfrak{fr}_5(t) = \mathfrak{fr}_4(t) + \frac{\sqrt{3}}{1!} \cdot 0.149 \cdot \left[\frac{t-\tau_1(1-\varphi_3)-\tau_3\varphi_3}{(\tau_2-\tau_1)(\tau_3-\tau_1)}\right] & \mathbf{p}_5 \le t < \mathbf{p}_6 \\ \mathfrak{fr}_6(t) = \mathfrak{fr}_5(t) + \frac{\sqrt{3}}{1!} \cdot 0.149 \cdot \left[\frac{t-\tau_2(1-\varphi_3)-\tau_3\varphi_3}{(\tau_1-\tau_2)(\tau_3-\tau_2)}\right] & t \ge \mathbf{p}_6 \end{cases}$$

The corresponding cumulative frequency density function is defined as:

$$\mathfrak{Fr}(t) = \begin{cases} \mathfrak{Fr}_0(t) = \mathfrak{fr}_0(t)/\mathfrak{fr}_6(t) & t < \mathbf{p}_1 \\ \mathfrak{Fr}_1(t) = \mathfrak{fr}_1(t)/\mathfrak{fr}_6(t) & \mathbf{p}_1 \le t < \mathbf{p}_2 \\ \mathfrak{Fr}_2(t) = \mathfrak{fr}_2(t)/\mathfrak{fr}_6(t) & \mathbf{p}_2 \le t < \mathbf{p}_3 \\ \mathfrak{Fr}_3(t) = \mathfrak{fr}_3(t)/\mathfrak{fr}_6(t) & \mathbf{p}_3 \le t < \mathbf{p}_4 \\ \mathfrak{Fr}_4(t) = \mathfrak{fr}_4(t)/\mathfrak{fr}_6(t) & \mathbf{p}_4 \le t < \mathbf{p}_5 \\ \mathfrak{Fr}_5(t) = \mathfrak{fr}_5(t)/\mathfrak{fr}_6(t) & \mathbf{p}_5 \le t < \mathbf{p}_6 \\ \mathfrak{Fr}_6(t) = \mathfrak{fr}_6(t)/\mathfrak{fr}_6(t) & t \ge \mathbf{p}_6 \end{cases}$$

Figure 4.10 graphically shows the frequency density function and the cumulative frequency density function as well as their normalized versions.

4.3 Opportunity Sets with Non-linear Restrictions and Non-linear Performance Measures

We examine non-linear performance measures as applied to linear opportunity sets first. The task of calculating the volume of the cross-section between a portfolio opportunity set and non-linear iso-surfaces is equal to the task of finding the surface volume of a non-linear (convex) body, which is "cut off" by a specific n-dimensional angle. For example, in case of Sharpe ratio, sets of portfolios with the same ratios form hyperellipses. Consequently, the corresponding mathematical problem is to find an analytical formula, which defines the surface volume of ellipse part which is "cut off" by an angle.

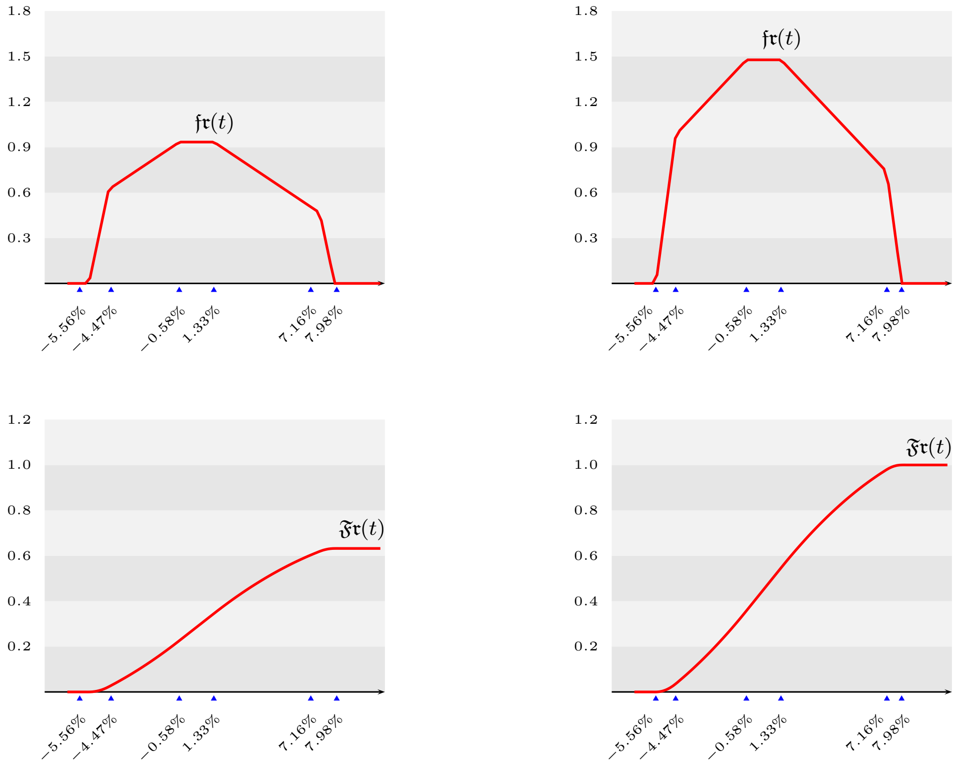


Figure 4.10: Frequency density function $\mathfrak{fr}(t)$ and its cumulative function $\mathfrak{Fr}(t)$ of the realized over December 2003 return (left), and normalized versions of both functions (right).

In general, analytical formulae of this kind are very cumbersome to derive even for three-dimensional elliptical surfaces. (Clearly, extending opportunity sets to incorporate non-linear investment restrictions further complicates the derivation of the analytical formulae.) Also, the form of the volume formula depends on the shape of the performance measure polynomial. Therefore, the reasonable approach in such cases is to look for a *numerical* solution, which allows to "trade-off" the solution exactness against the complexity. So, relaxing the requirement of precise values and looking for reasonably-good approximations for $\mathfrak{fr}(t)$ and $\mathfrak{Fr}(t)$ only, we can reduce the complexity sharply and generalize the computational process simultaneously. Finding a numerical estimation of the (cumulative) frequency density function $\mathfrak{fr}(t)$ ($\mathfrak{Fr}(t)$) is the topic of the next chapter.

4.4 Summary and Conclusions

In the space of asset weights a set of portfolios, all of which have the same value for a specific performance metric, builds up so-called iso-surfaces. Consequently, the task of calculating the frequency distribution for a performance metric is equal to finding the volume of the cross-section between the portfolio opportunity set and the corresponding iso-surfaces.

The general approach for deriving analytical formulae for (cumulative) frequency distributions is to enumerate all extreme portfolios of a specific type of opportunity sets (i.e. vertices of opportunity set polytopes), and then to decompose the opportunity set polytope into finite number of opportunity sub-sets that are built at the extreme portfolios one at a time.

Using the developed methodology, we have derived the general formula for the (cumulative) frequency distribution for opportunity sets with a short-sales restriction and linear performance measures. The frequency distribution function is a spline with critical points at performances of individual feasible assets. Also for portfolio opportunity sets with restrictions on individual weights the frequency density function is a similar spline. However, for the latter opportunity sets the computation is more difficult and time-consuming because the asymptotic increase in the number of extreme portfolios.

Using non-linear performance measures and/or imposing non-linear constraints such as diverse restrictions on a risk exposure enormously increase the complexity of deriving analytical formulae for frequency distributions. So, it is reasonable to look in such cases for a numerical solution. The latter is the topic of the next chapter.

Chapter 5

Numerical Estimation of Performance Distributions

5.1 Important Aspects of a Numerical Estimation Procedure

As discussed in the previous chapter, a closed form formula for a (cumulative) frequency density function(s) derived analytically with respect to the selected performance measure(s) is the primary choice. Unfortunately, in case of a very complicated set of investment constraints the formulation of the frequency distribution function in analytical form is difficult or even impossible to derive. Similar to integration in multi-dimensional space, the required frequency distributions can be estimated numerically. The procedure is based on statistical sampling: we estimate the distribution of performance values of the full opportunity set through the distribution of performance values of a reasonably-large sample. The important aspects of a numerical estimation are:

- How to form the sample of random portfolios considering a portfolio opportunity set of a particular kind;
- The quality of the estimate $\mathfrak{fr}^*(t)$ with respect to the original $\mathfrak{fr}(t)$ function, i.e. what is the accuracy of estimation. Additionally, we are interested in, which factors influence the accuracy and how it can be increased.

Beyond these two points, several other technical aspects arise, e.g. how the frequency histogram can be transformed into the estimate $\mathfrak{fr}^*(t)$, what is the optimal histogram bin size, and how the values between estimation points can be interpolated.

We consider core aspects of an estimation procedure in the successive sections. In the next section we review various techniques to form a portfolio opportunity set sample, their convergence speed and efficiency. We also introduce the discrepancy notion. Sections 3 through 5 discuss the sampling techniques in detail. The numerical estimation is more or less independent of the evaluated performance measure, so looking at each technique, we first consider our standard case with the single no-short-sales restriction, i.e. $0 \leq w_i \leq 1$ $\forall\, i$. Afterwards, we generalize sampling methods to other types of opportunity sets. At the end of each of these sections we analyze the quality of estimation. In particular, we are interested in how we could compute confidence intervals for our estimates. Section 6 deals with some additional aspects of the numerical computation of (cumulative) frequency density function(s) such as optimal histogram bin size and curve fitting techniques.

It should be noted that each of these topics is a “thing in itself” and that we consider only the core issues and techniques, which are relevant for our needs.

5.2 Forming a Portfolio Opportunity Set Sample

5.2.1 Approaches to Form a Sample

Clearly, the sample consists of feasible portfolios, which should be uniformly distributed over the opportunity set. We can use one of the following alternatives to form such a sample:

- Regular grid (implicitly implies the use of a quadrature rule);
- Straightforward Monte Carlo method;
- Quasi-random sequences (e.g. Halton’s or Sobol’s low discrepancy sequences). This method is also often referred as the quasi-Monte Carlo.

Using a regular grid, we obtain an estimation with a *guaranteed* error bound. Namely, this approach gives us a guarantee that the error does not exceed a specific upper bound. Generally, it can be shown that no grid estimators exist with an error bound better than $\mathcal{O}\left(N^{-1/s}\right)$, where N is the number of grid points, and s is the dimension of the estimated domain (Bakhvalov, Zhidkov & Kobelkov 2000).[1] Thus, the clear deficiency of a grid method is

[1] Essentially, the Bakhvalov’s theorem states that given a deterministic quadrature, there exists a s-dimensional integrand with r continuous bounded derivatives such that the convergence rate has the order $\mathcal{O}\left(N^{-r/s}\right)$.

that the number of grid points increases exponentially with s having a specific level of accuracy required.[2]

The Monte Carlo method provides a faster convergence with respect to a required level of accuracy. In general, the convergence rate has the order $\mathcal{O}\left(N^{-1/2}\right)$ and is independent of the problem dimension (Sobol' 1994). Thus, the method is asymptotically much faster than the grid method with respect to the accuracy level required. However, the Monte Carlo uses a random sampling and therefore it provides a probabilistic error bound only. This means that the error bound is *expected* with some (high) probability but not guaranteed.

The quasi-Monte Carlo method is a modification of the Monte Carlo method with the error bound $\mathcal{O}\left(f(s) \cdot (\ln N)^s \cdot N^{-1}\right)$, where $f(s)$ is a function depending on the domain dimensionality only. The bound is deterministic because the sample consists of well-chosen deterministic points. In general, the quasi-Monte Carlo is faster than the grid and comparable to the standard Monte Carlo in yielding a prescribed level of estimation accuracy.

Another distinguish feature which characterizes the quality of a sampling method is the notion of *discrepancies*. The next section addresses this theoretical concept in more detail.

5.2.2 On Discrepancy Bounds for Sampling Methods

Computing a frequency density function $\mathfrak{fr}(t)$ numerically, the kernel of the procedure is the estimation of partial volumes of a portfolio opportunity set P, $\text{Vol}(P_\delta)$, for each of the equal performance intervals having a length δ. (We use a histogram to calculate how many random portfolios fail within each of the performance intervals. That is, we estimate the ratios of $\text{Vol}(P_\delta)$ relative to the complete volume of P.) Therefore, our task is nothing else but numerical integration in multidimensional space. Let us consider the numerical integration over the normalized s-dimensional integration domain $[0,1]^s$, i.e. over the $[0,1]^s$-hypercube, in general terms.

The general condition of the (quasi-) Monte Carlo method(s) is that having an infinite sequence $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, …, of vectors, the following equality should hold for a reasonable class of integrands $f(\cdot)$ on the $[0,1]^s$:

$$\lim_{N\to\infty} \frac{1}{N} \cdot \sum_{i=1}^{N} f(\mathbf{x}_i) = \int_{[0,1]^s} f(\mathbf{z})d\mathbf{z} \tag{5.1}$$

[2] We consider only the grid method, which is one of the broad family of lattice methods. Not all of lattice methods are featured by the "curse of dimensionality" to the same extent as the regular grid does. Presently, lattice methods are subject of active research.

Consequently, using a finite sequence $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_N$, we can obtain an approximation of the integral value. If the sequence $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, ... is uniformly distributed in the $[0,1]^s$, then (5.1) holds for all Riemann-integrable functions (Niederreiter 1992). So, we need a measure of uniformity or non-uniformity for sequences of vectors.

Discrepancy is a quantitative measure how inhomogeneously a sequence of n-dimensional vectors is distributed in the unit hypercube $[0,1]^s$. Formally, for a sequence X consisting of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \in [0,1]^s$ the discrepancy is defined as:

$$D_N^{(s)}(X) = \sup_{\mathbf{y}\in[0,1]^s} \left| S_N(\Pi_{\mathbf{y}}) - N \cdot V_{\Pi_{\mathbf{y}}} \right| \tag{5.2}$$

where $\mathbf{y}$ denotes a point in the unit hypercube, $\Pi_{\mathbf{y}}$ is the s-dimensional parallelepiped delimited by coordinate planes and $\mathbf{y}$ as the upper-right corner, $S_N(\Pi_{\mathbf{y}})$ denotes the number of $\mathbf{x}_i \in \Pi_{\mathbf{y}}$, and $V_{\Pi_{\mathbf{y}}}$ is the volume of $\Pi_{\mathbf{y}}$.[3] Sobol' (1969) and (1981) showed the following discrepancy bounds for different methods:

Regular grid (as defined in section 5.3):	$D_N^{(s)} = \mathcal{O}\left(N^{1-\frac{1}{s}}\right)$
Monte Carlo method:	$D_N^{(s)} = \mathcal{O}\left(\sqrt{N}\right)$
Quasi-Monte Carlo method (Sobol sequences):	$D_N^{(s)} = \mathcal{O}\left(\ln^{s-1} N\right)$

For an in-depth discussion of theoretical concepts behind the low-discrepancy sequences and the analysis of different sampling methods we refer to Sobol' (1967), (1969), Niederreiter (1992), and Tezuka (1995).

The discrepancies and estimation error bounds are strongly related: sequences of vectors with small discrepancy *guarantee* small errors. (For example, Niederreiter (1992) provides the error bounds for the quasi-Monte Carlo method in terms of the bounded variation of the integrand and of the star discrepancy. We also discuss this relation in section 5.5.3 devoted to the calculation of accuracy of quasi-Monte Carlo methods.) The discrepancy of the grid method is deteriorating very fast and approaches the order N with the increase of the dimensionality. Therefore, the convergence of the grid method is very slow and is inferior to the Monte Carlo method already for dimensions $n > 3$. Consequently, our preferences are clear:[4] we would rather use

[3] It should be noted that in the literature different kinds of discrepancy measures are considered: the star discrepancy, the extreme discrepancy, the isotropic discrepancy *et cetera*. We choose the measure used by Sobol' to analyze his sequences. This measure is very similar to the star discrepancy: $D_N^{(s)} = N \cdot D_N^*$. Thus, the star discrepancy is a normalized version of $D_N^{(s)}$. The analysis based on the star discrepancy is given in Niederreiter (1992).

[4] More precisely: at least from the theoretical point of view.

the QMC or the MC to form a sample. The grid method is the last choice. However, for the sake of simplicity we analyze in details three sampling methods as applied to different opportunity sets in the following sections in the opposite order.

5.3 Regular Grid

Forming a representative sample based on a regular grid is what often seems to be a natural and good choice for estimating a frequency density function. But the method is inferior in all aspects (i.e. computability and convergence speed with respect to a specific estimation error) to the other sampling methods.

Let us consider the standard case when the opportunity set is restricted by no-short-sales constraint only. Given n assets, their weights form a basic $n-1$ dimensional simplex and a regular grid can be defined as

$$\langle w_1 = \frac{\varrho_1}{M},\ w_2 = \frac{\varrho_2}{M},\ \ldots,\ w_{n-1} = \frac{\varrho_{n-1}}{M} \rangle$$

where $\varrho_1, \varrho_2, \ldots, \varrho_{n-1}$ independently take on values $0, 1, 2, \ldots, M$ and are subject to the following constraint:[5]

$$\sum_{i=1}^{n-1} \frac{\varrho_i}{M} \leq 1$$

The constant $M+1$ defines the number of regular grid points across each dimension of our weight simplex. From the financial point of view, we subdivide the available capital into equal investable lots, e.g. 5% of the whole capital, and then build our portfolio by investing these lots into different assets. Of course, we can invest more than one lot into an asset or none at all.[6]

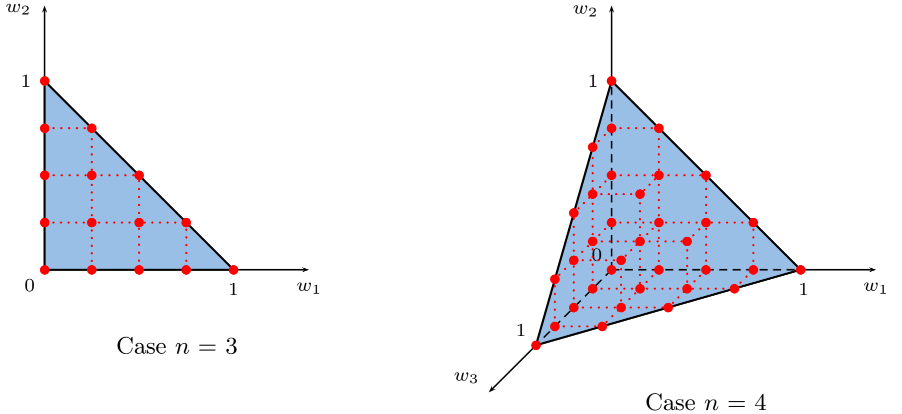
Figure 5.1 shows regular grid points for the opportunity set consisting of 3 assets (left) and 4 assets (right), which have the minimal investable lot equal to 0.25 or 25% (i.e. $M = 1.0/0.25 = 4$ and, thus, we have $M + 1 = 5$ grid points across each dimension in this particular case).

Estimating performance distributions through a regular grid, we have to calculate the performance value in each grid point, though in any order. The total number of grid points, N, is related to n and M parameters as:

$$N\,(n, M) = \frac{(n-1+M)!}{M! \cdot (n-1)!} \tag{5.3}$$

[5]Obviously, the latest asset weight for such grid is defined as $w_n = 1 - \sum_{i=1}^{n-1} \frac{\varrho_i}{M}$.

[6]i.e. $w_i \in \{0.0, 0.05, 0.10, 0.15, \ldots, 1.0\}\ \forall\, i = 1, 2, \ldots, n$



Case $n = 3$ Case $n = 4$

Figure 5.1: The regular grid defined on the opportunity set, which is restricted by the no-short-sales constraint and consists of three assets (left) and four assets (right). The minimal investable lot is equal to 0.25 or 25% in both cases (i.e. $M = 4$). The basic simplex formed by $n - 1$ independent asset weights is filled with color on both subfigures.

The distribution estimation accuracy ε for the grid method is proportional to the inverse value of M to the power of 2. The sample size is related to the number of assets asymptotically as:

$$N(n, M) = \mathcal{O}(n \cdot (n+1) \cdot (n+2) \cdot \ldots \cdot (n - 1 + M)) = \mathcal{O}\left(n^M\right)$$

Consequently, $M = \mathcal{O}(\log_n N)$. Thus, we obtain that the accuracy depends on the problem dimensionality n and the sample size N as:

$$\varepsilon = \mathcal{O}\left(\log_n^{-1} 2N\right) \tag{5.4}$$

Therefore, aiming for a reasonably good estimation for (cumulative) frequency distributions, the regular grid method is extremely computationally inefficient. Because of the time complexity, the method cannot be used whenever we have more than a handful of assets in the opportunity set. Another reason for not using the grid method is its poor discrepancy bound when compared to the two other methods.

Henceforth, we consider only the Monte Carlo and quasi-Monte Carlo methods as applied to the estimation of frequency density functions $\mathfrak{fr}(t)$. Both, the MC and QMC, have their advantages and drawbacks.

5.4 Straightforward Monte Carlo Approach

The Monte Carlo method (MC) is the most famous technique among the numerical methods. The general idea of the approach is to solve problems by random sampling. Formally, assume that we need to compute the quantity a. Selecting a random variable ξ with $\mathsf{E}[\xi] = a$, the Monte Carlo method is defined as:

$$\frac{1}{N}\sum_{i=1}^{N}\xi_i \xrightarrow{P} a \tag{5.5}$$

ξ_i represent independent trials of ξ, and as $N \to \infty$ the mean of ξ_i stochastically converges to a.[7]

In our case we estimate frequency distributions by choosing N feasible portfolios randomly from an evaluated opportunity set, and then calculate the performance metric(s) for these selected portfolios. The distribution of performance values of the sampled portfolios approximates the frequency distribution for the full opportunity set. Clearly, the bigger the sample size N, the greater is the accuracy of the MC estimation. (The accuracy of the MC method is discussed in section 5.4.3 in details.) The key issue of the MC method is how to generate the stream of random portfolios over a given opportunity set. We start with our standard case with the no-short-sales constraint only.

5.4.1 Opportunity Sets with a Short-Sales Restriction only

Let us consider a portfolio opportunity set consisting of n assets. The opportunity set is the $n-1$-dimensional basic simplex, and we need to generate random points (i.e. sample portfolios) over it. The commonly used acceptance-rejection generation procedure is very inefficient due to reasons given in Exhibit 5.1. So we consider in details two alternative transformation techniques to generate random portfolio vectors that are uniformly distributed over the surface and interior of the multidimensional basic simplex of asset weights. Both algorithms are described in Rubinstein & Melamed (1998). However, our description of uniform spacing technique is based on Devroye (1986).

Exhibit 5.1 A Note of Caution on Using the Acceptance-Rejection Generation Techniques

An often used strategy to generate uniformly distributed random points over bounded, regular region S is to generate random points over the multidimen-

[7] This description has been adapted from Sobol' (1998).

sional $[0,1]$ hypercube; then use an affine transformation to translate the generated points into uniformly distributed over hyper-rectangle or hypercube including S. Generating a new random point, we accept or reject it depending whether the point is inside or outside the investigated region S. So at first sight we can adopt the following simple acceptance-rejection strategy:

1. Generate weights $w_1, w_2, \ldots, w_{n-1}$ as $n-1$ independent $[0,1]$ uniformly distributed random numbers;
2. Check if the condition $\sum_{i=1}^{n-1} w_i \leq 1$ is valid for these weights. If not, then repeat the first step;
3. Create a new random portfolio $\mathbf{p}_t$ with asset weights

$$\mathbf{p}_t = \langle w_1,\, w_2,\, \ldots,\, w_{n-1},\, 1 - \textstyle\sum_{i=1}^{n-1} w_i \rangle.$$

The fundamental (and disastrous) disadvantage of this acceptance-rejection technique is that the proportion of feasible random portfolios in the generated sequence, i.e. random weight vectors which pass the second step, decreases strongly with an increase of the number of assets in the opportunity set. The following proposition defines this ratio explicitly.

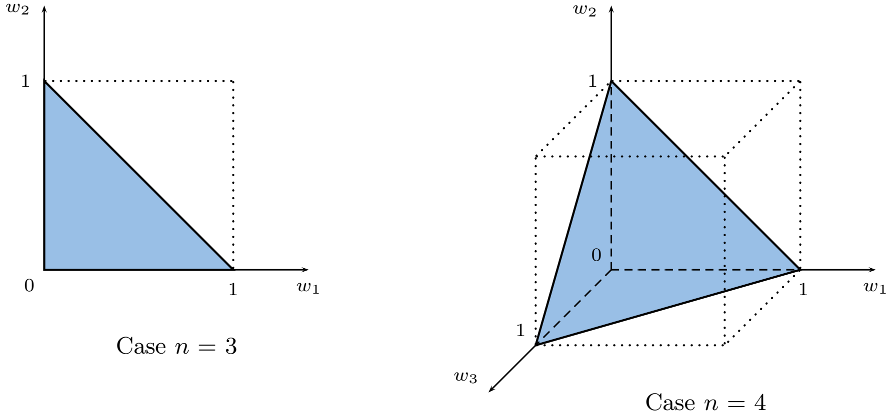
Proposition. *Let $U = \langle u_1, u_2, \ldots, u_{n-1} \rangle$ be an $n-1$-dimensional $[0,1]$ uniformly distributed random variable. The expected number of iterations of the algorithm that are needed to produce m points such that $u_1 + u_2 + \ldots + u_{n-1} \leq 1$ is $(n-1)! \cdot m$.*

Proof. The fraction of $[0,1]^{n-1}$ uniformly distributed random points, that pass the $u_1 + u_2 + \ldots + u_{n-1} \leq 1$ rejection test, is equal to the volume ratio of two polytopes:

$$\frac{\text{Volume of the basic } \mathbb{R}^{n-1}\text{simplex}}{\text{Volume of the } [0,1]^{n-1}\text{-cube}}$$

or

$$\frac{\mathrm{Vol}\Big(\Big\{\langle u_1, u_2, \ldots, u_{n-1}\rangle \mid 0 \leq u_i \leq 1,\, i = 1,2,\ldots,n-1 \text{ and } \sum_{j=1}^{n-1} u_j \leq 1\Big\}\Big)}{\mathrm{Vol}(\{\langle u_1, u_2, \ldots, u_{n-1}\rangle \mid 0 \leq u_i \leq 1,\, i = 1,2,\ldots,n-1\})}$$



Case $n = 3$

Case $n = 4$

Proof (continued). The figures above illustrate particular cases for $n = 3$ and $n = 4$, i.e. $[0,1]^{n-1}$-cubes and basic simplexes in $\mathbb{R}^2$ and $\mathbb{R}^3$. The dotted lines border $[0,1]^{n-1}$-cubes, and the basic simplexes are filled with color.

The volume of the $[0,1]^{n-1}$-cube is always 1 and the volume of the basic simplex is equal to $1/(n-1)!$. Clearly, as the $1/(n-1)!$ fraction of points is the only accepted one, the acceptance-rejection algorithm should generate $(n-1)! \cdot m$ points to produce m points uniformly distributed in the basic $\mathbb{R}^{n-1}$ simplex. □

Generating Random Portfolios Using Uniform Spacings

Let U_1, U_2, ..., U_{n-1} be $[0,1]$ uniformly distributed random variables and let $U_{(1)}$, $U_{(2)}$, ..., $U_{(n-1)}$ be an ordering for a sample of these variables such that

$$U_{(1)} \le U_{(2)} \le \ldots \le U_{(n-1)}$$ [8]

We define two additional values: $U_{(0)} = 0$ and $U_{(n)} = 1$. The statistics

$$S_i = U_{(i)} - U_{(i-1)},\ 1 \le i \le n$$

are called the *uniform spacings* for this sample.

Theorem 5.4.1. *$\langle S_1, S_2, \ldots, S_{n-1}\rangle$ is uniformly distributed over the $n-1$-dimensional basic simplex Δ in $\mathbb{R}^{n-1}$:*

$$\Delta = \{\langle x_1, x_2, \ldots, x_{n-1}\rangle \in \mathbb{R}^{n-1} \mid x_i \ge 1,\ i = 1, 2, \ldots, n-1 \textit{ and } \sum_{i=1}^{n-1} x_i \le 1\}$$

and $\langle S_1, S_2, \ldots, S_n\rangle$ is uniformly distributed over the $n-1$-dimensional basic simplex ${}_{n-1}\Delta$ in $\mathbb{R}^n$:

$${}_{n-1}\Delta = \{\langle x_1, x_2, \ldots, x_n\rangle \in \mathbb{R}^n \mid x_i \ge 1,\ i = 1, 2, \ldots, n \textit{ and } \sum_{i=1}^{n} x_i = 1\}$$ [9]

Proof. see Devroye (1986) □

Using this theorem we can formulate the following algorithm to generate random portfolio vectors that are uniformly distributed over the portfolio opportunity set consisting of n assets. The algorithm uses a standard generator of random numbers uniformly distributed on the interval $[0,1]$.

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[8] Such an ordering is also called the order statistics for the sample of U_1, U_2, ..., U_{n-1}.

[9] i.e. the random variable is uniformly distributed over the surface of the basic simplex Δ in $\mathbb{R}^n$

1. Generate $U_1, U_2, \ldots, U_{n-1}$ as $n-1$ independent $[0,1]$ uniformly distributed random variables (i.e. $U_i \sim \mathcal{U}(0,1),\ i = 1, 2, \ldots, n-1$);
2. Sort U_i in ascending order, i.e. define the order statistics $U_{(1)}, U_{(2)}, \ldots, U_{(n-1)}$;
3. Create a new random portfolio $\mathbf{p}$ with asset weights $\langle w_1, w_2, \ldots, w_n \rangle$ where $w_i = U_{(i)} - U_{(i-1)}$, $1 \le i \le n$, $U_{(0)} = 0$ and $U_{(n)} = 1$. Add this new random portfolio to the sample;
4. If the sample is big enough, then stop.[10] Otherwise return to step 1.

The algorithm is very compact and can be implemented easily. For generation of $[0,1]$ uniformly distributed random variables we can use any of widely available standard routines, e.g. the “Numerical Recipes in C” book (Press, Flannery, Teukolsky & Vetterling 1993) provides three different routines ran0(), ran1() and ran2(). The fast Mersenne Twister routine (Matsumoto & Nishimura 1998) is another alternative.

The only drawback of the uniform spacings approach is that in the second step we need to sort the uniform variates. For opportunity sets consisting of large number of assets, the sorting can substantially slow down the generation of random portfolios due to the time complexity $\mathcal{O}(n \lg n)$. The next subsection provides another algorithm, which can be more time efficient in case of large number of assets.

Generating Random Portfolios Using the Exponential Approach

An alternative approach can also be used to generate random portfolio vectors:

1. Generate $E_1, E_2, \ldots, E_n$ as n independent exponentially distributed random variables with the exponential distribution parameter $\lambda = 1$ (i.e. $E_i \sim \mathcal{E}(1),\ i = 1, 2, \ldots, n$);
2. Create a new random portfolio $\mathbf{p}$ with asset weights $\langle w_1, w_2, \ldots, w_n \rangle$ where
$$w_i = \frac{E_i}{\sum_{j=1}^{n} E_j}, \quad i = 1, 2, \ldots, n$$
Add this new random portfolio to the sample;
3. If the sample is big enough, then stop. Otherwise return to step 1.

[10] We are discussing this issue in the next section.

Theorem 5.4.2. *Let $E_1, E_2, \ldots, E_n$ be independent exponentially distributed random variables with $\lambda = 1$. Then the $n-1$-dimensional random variable $\langle x_1, x_2, \ldots, x_{n-1}\rangle$,*

$$w_i = \frac{E_i}{\sum_{j=1}^{n} E_j}, \quad i = 1, 2, \ldots, n-1$$

is distributed over the $n-1$-dimensional basic simplex Δ in $\mathbb{R}^{n-1}$:

$$\Delta = \{\langle x_1, x_2, \ldots, x_{n-1}\rangle \in \mathbb{R}^{n-1} \mid x_i \geq 1,\ i = 1, 2, \ldots, n-1 \text{ and } \sum_{i=1}^{n-1} x_i \leq 1\}$$

and $\langle x_1, x_2, \ldots, x_n\rangle$ is uniformly distributed over the $n-1$-dimensional basic simplex ${}_{n-1}\Delta$ in $\mathbb{R}^n$:

$${}_{n-1}\Delta = \{\langle x_1, x_2, \ldots, x_n\rangle \in \mathbb{R}^n \mid x_i \geq 1,\ i = 1, 2, \ldots, n \text{ and } \sum_{i=1}^{n} x_i = 1\}$$

Proof. see (Karlin & Taylor 1975) or (Rubinstein & Melamed 1998) □

In the first step we use any standard generator of random numbers uniformly distributed on the interval $[0, 1]$ with an additional transformation of these uniform variates into exponentially distributed random variables.[11] In the second step we create a random portfolio just by dividing the first n variables by the sum of all variables generated in first step. According to Theorem 5.4.2, the random portfolio vectors generated in this way are uniformly distributed over the portfolio opportunity set consisting of n assets.

For a large number of assets the exponential approach is faster than the uniform spacings algorithm because the computational overhead to standard $[0, 1]$ uniform routine consists of the transformation of uniform variates into exponential variables, and dividing of each exponential variable by the sum only.[12] That is, the time complexity of these extra calculations is $\mathcal{O}(2n)$ only.

5.4.2 Extending Random Portfolio Generation to Different Types of Constraints

The geometric form of portfolio opportunity sets depends on imposed investment constraints. Presently there exist several universal algorithms, e.g. by Smith (1984), Borovkov (1994), which can generate a random variable X over any bounded region $P \subset \mathbb{R}^n$ with a given density f (e.g. uniformly distributed

[11] We use the Inverse-Transform method for the last operation.

[12] The former operation is equivalent to taking natural logarithm, i.e. calling $\ln(U_i)$ function.

over P in our case). However, these methods are substantially slower than the transformation techniques used for the no-short-sales case.

The general idea of all algorithms is to construct an ergodic Markov chain, recursively generating a sequence of points all in P and whose stationary distribution is uniform. If the initial point of the chain is uniformly distributed over P and the transition kernel is symmetric, then the points of the generated sequence will be independent and uniformly distributed over P asymptotically. The most known algorithms from the family are the *Hit-and-run* algorithm by Smith (1984), (1993) and the *Sequential Direction* algorithm[13] by Telgen (Berbee, Boender, Kan, Scheffer, Smith & Telgen 1987) and Ritov (1989).We adopt the last one to generate random portfolios for the general case opportunity sets.

Generating Random Portfolios Using the Sequential Direction Approach

For the sake of simplicity let us consider the opportunity sets with restrictions on individual weights and linear summary constraints (cf. section 3.3.3) first. Such an opportunity set P is defined as:

$$\begin{aligned} & 0 \le w_i \le \varphi_i \quad \forall i = 1, 2, \dots, n \;\text{ and }\; \varphi_i \in \mathbb{R}_+ \\ & \sum_{i=1}^{n} w_i = 1 \\ & \sum_{j \in A_k} w_j \le \gamma_k \quad A_k \subseteq \{1, 2, \dots, n\} \;\text{ and }\; \gamma_k \in \mathbb{R}_+ \\ & \varsigma_l \le \sum_{j \in B_l} w_j \quad B_l \subseteq \{1, 2, \dots, n\} \;\text{ and }\; \varsigma_l \in \mathbb{R}_+ \end{aligned} \tag{5.6}$$

A portfolio opportunity set of this kind consisting of n feasible assets forms a $n-1$-dimensional polytope. Removing the second budget constraint from the system (5.6), we obtain a full-dimensional opportunity set polytope in $\mathbb{R}^{n-1}$. Henceforth, we denote coefficients of the transformed system through a "hat" sign above the corresponding scalars and vectors. The sequential direction algorithm that generates random portfolios uniformly distributed over such a transformed opportunity set $\hat{P} \subset \mathbb{R}^{n-1}$ is formulated as follows:

1. Choose uniformly a feasible random point $\mathbf{p}_0 = \langle w_1^{(0)}, w_2^{(0)}, \dots, w_{n-1}^{(0)} \rangle$ in a given portfolio opportunity set $\hat{P}$;

[13] It is also called the *Coordinate Direction* algorithm. The Sequential Direction algorithm is a slight modification of the Hit-and-Run algorithm proposed by Smith (1984).

2. Draw a straight line L through the point $\mathbf{p}_t$ (the subscript $t = 0, 1, 2, \ldots$ denotes the actual step of the chain) in a direction taken consequently from the direction set $D = \{e_1, e_2, \ldots, e_{n-1}, -e_1, -e_2, \ldots, -e_{n-1}\}$ (i.e. $e_1, e_2, \ldots, e_{n-1}, -e_1, -e_2, \ldots, -e_{n-1}, e_1, e_2, \ldots$), where e_i are the elements of an orthogonal basis in $\mathbb{R}^{n-1}$;
3. Compute the next element of the chain $\mathbf{p}_{t+1}$ by choosing a point on $L \cap \hat{P}$ uniformly. Formally:

$$\mathbf{p}_{t+1} \sim \mathcal{U}(L \cap \hat{P})$$

$$L \cap \hat{P} = \{\mathbf{x} \mid \mathbf{x} = \lambda \mathbf{p}_t + (1-\lambda)\mathbf{d}_t,\ \lambda \in [0, 1]\}$$

   where

$$\mathbf{d}_t = \mathbf{p}_t + a \cdot e_j$$

   and the scalar a such that $\mathbf{d}_t$ is the endpoint of L laying on the boundary of the portfolio opportunity set $\hat{P}$;
4. Create a new random portfolio extending $\mathbf{p}_{t+1} = \langle w_1^{(t+1)},\ w_2^{(t+1)},\ \ldots,\ w_{n-1}^{(t+1)} \rangle$ with the last depending weight as

$$\langle w_1^{(t+1)},\ w_2^{(t+1)},\ \ldots,\ w_{n-1}^{(t+1)},\ w_n^{(t+1)} = 1 - \sum_{i=1}^{n-1} w_i^{(t+1)} \rangle$$

   Add this new random portfolio to the sample;
5. If the sample is big enough, then stop. Otherwise return to step 2.

The transition kernel from $\mathbf{p}_t$ to $\mathbf{p}_{t+1}$ of the chain is defined in the step 3 of the algorithm. As we "move" along the orthogonal vectors only, the computation of the endpoint $\mathbf{d}_t$ of L is efficient due to usual sparsity of the constraints system $\hat{P}$: Substituting the values of $\mathbf{p}_t$ except the value for w_j element into $\hat{P}$, we just need to define the tightest bound for $w_j^{(t)}$ variable in order to find the $\mathbf{d}_t$ point. Consequently, $\mathbf{p}_{t+1}$ can be found easily.

Furthermore, the routine leaves the two important issues open:

- How to find an initial point of the chain, i.e. $\mathbf{p}_0$;
- How many steps of the chain are necessary in order to guarantee that all "one-dimensional" projections of the chain will be uniformly distributed, i.e. the chain converges to an invariant uniform distribution.

The simplest solution to the first issue is to "pack" a given opportunity set into a simple "envelope", for example a cube or a simplex, and then sampling uniformly until a feasible point will be found. A more sophisticated technique for this preprocessing step with polynomial time is discussed in Lovász (1998).

The second issue is also discussed in Lovász (1998). The article proves that Markov chains constructed by hit-and-run algorithms converge to the uniform distribution in $\mathcal{O}\left(n^3\right)$ steps. Thus, having n assets in the opportunity set, we need to generate at least $\mathcal{O}\left(n^3\right)$ random portfolios in order to guarantee the uniformity of the sample. (It should be noted that another bound is imposed through the required accuracy of estimation. We consider it in the next section in detail.)

The generalization of the discussed approach for handling other types of investment constraints such as bounds on risk, for example, is very straightforward: We determine $\mathbf{d}_t$ endpoint of L having just additional (may be non-linear) inequalities in the opportunity set description. Because we are moving across orthogonal directions, the number of inequalities needed to evaluate may increase, but the complexity does not rise substantially and we can generate random portfolios rather efficiently.

5.4.3 Confidence Interval for a Frequency Density Function Estimate

When estimating a frequency density function $\mathfrak{fr}(t)$ numerically, it is natural to ask what errors may occur. Formally, we are interested to quantify the following expression:

$$\Pr\left[\mid \mathfrak{fr}^*(t) - \mathfrak{fr}(t) \mid \leq \varepsilon\right] = \rho \tag{5.7}$$

where $\mathfrak{fr}^*(t)$ is an estimate of $\mathfrak{fr}(t)$. The expression defines a ρ-percent confidence interval of estimate for $\mathfrak{fr}(t)$. It is intuitively clear that the estimation accuracy ε depends on the size of the portfolio opportunity set sample: the bigger the sample used the better should be our estimate for $\mathfrak{fr}(t)$. For the sake of simplicity we analyze linear performance measures first and then generalize the result.

As discussed in Chapter 4, given a portfolio opportunity set P and a performance measure τ, the frequency distribution function $\mathfrak{fr}(t)$ represents the volumes of the cross-section between P and the corresponding τ-isosurfaces. Furthermore, using a histogram to tabulate the performance values from the sample, we estimate how many of random portfolios "fail" within each of the performance intervals. Let us consider such an interval for performances between t and $t+\delta$. This interval corresponds to the volume of P laying between two hyperplanes $\boldsymbol{\tau}^T\mathbf{w} = t$ and $\boldsymbol{\tau}^T\mathbf{w} = t + \delta$ (cf. Figure 4.4). We denote this partial volume of P by $\mathrm{Vol}(P_\delta)$. Both values, $\mathfrak{fr}(t)$ and $\mathrm{Vol}(P_\delta)$, are related. Namely, for small values of δ the following approximation holds:

$$\mathfrak{fr}(t) \approx \frac{\mathrm{Vol}(P_\delta)}{h} \tag{5.8}$$

where h is the distance between the hyperplanes $\boldsymbol{\tau}^T \mathbf{w} = t$ and $\boldsymbol{\tau}^T \mathbf{w} = t + \delta$. We will use this approximation later. For the time being, we concentrate on the analysis of probabilities of random portfolios "hitting" P_δ.

We consider a random portfolio W as an independent random variable, the outcome of which can be classified either as a "success" when the performance of W fails between t and $t+\delta$, or a "failure" otherwise. That is, the probability density function is given by:

$$\begin{aligned} &\Pr\left[t \leq \boldsymbol{\tau}^T W \leq t + \delta\right] = p \\ &\Pr\left[\boldsymbol{\tau}^T W < t \;\text{ or }\; \boldsymbol{\tau}^T W > t + \delta\right] = 1 - p \end{aligned} \tag{5.9}$$

Then W is a multidimensional Bernoulli random variable. Random portfolios are uniformly distributed over the portfolio opportunity set whatever algorithm we use for forming our sample. Therefore, the probability p is equal to the ratio of $\mathrm{Vol}(P_\delta)$ to the volume of the portfolio opportunity set $\mathrm{Vol}(P)$, i.e.

$$p = \frac{\mathrm{Vol}(P_\delta)}{\mathrm{Vol}(P)} \tag{5.10}$$

Consequently, we can also consider our portfolio opportunity set sample of N random portfolios as N independent realizations/trials of the Bernoulli variable (5.9). In these N realizations let N_δ be the number of realizations with "success" When N is large, the law of large numbers implies that N_δ is approximately normally distributed with the mean $N \cdot p$ and the variance $N \cdot p(1-p)$ (Shiryaev 2004, Ross 2001), or, equivalently,

$$\frac{N_\delta - N \cdot p}{\sqrt{N \cdot p \cdot (1-p)}} \sim \mathcal{N}(0,1) \tag{5.11}$$

Hence,

$$\Pr\left[\left|\frac{N_\delta - N \cdot p}{\sqrt{N \cdot p \cdot (1-p)}}\right| \leq \chi\right] = 2\Phi(\chi) \tag{5.12}$$

where $\Phi(\chi)$ is the Laplace function

$$\Phi(\chi) = \frac{1}{\sqrt{2\pi}} \int_0^\chi e^{-\frac{y^2}{2}} dy$$

and χ is the quantile of $\Phi(\chi)$.[14]

[14] The Laplace function $\Phi(\chi)$ specifies values of the standard normal distribution function for the domain $[0, +\infty)$.

Expressions (5.12) and (5.7) are closely related, obviously. For any specific probability level, e.g. 99% or 95%, the following inequality holds:

$$\left| \frac{N_\delta - N \cdot p}{\sqrt{N \cdot p \cdot (1-p)}} \right| \leq \chi$$

Then,

$$\left| \frac{N_\delta}{N} - p \right| \leq \frac{\chi \cdot \sqrt{p \cdot (1-p)}}{\sqrt{N}}$$

Replacing p through the right-hand side of (5.10), we obtain:

$$\left| \frac{N_\delta}{N} - \frac{\mathrm{Vol}(P_\delta)}{\mathrm{Vol}(P)} \right| \leq \frac{\chi \cdot \sqrt{\mathrm{Vol}(P_\delta) \cdot [\mathrm{Vol}(P) - \mathrm{Vol}(P_\delta)]}}{\sqrt{N} \cdot \mathrm{Vol}(P)} \tag{5.13}$$

Using approximation (5.8) and the inequality $\mathrm{Vol}(P) - \mathrm{Vol}(P_\delta) \leq \mathrm{Vol}(P)$, we rewrite (5.13) as

$$\left| \frac{N_\delta}{N} - \frac{\mathfrak{fr}(t) \cdot h}{\mathrm{Vol}(P)} \right| \leq \frac{\chi \cdot \sqrt{\mathfrak{fr}(t) \cdot h \cdot \mathrm{Vol}(P)}}{\sqrt{N} \cdot \mathrm{Vol}(P)}$$

or, equivalently,

$$\left| \frac{N_\delta}{N} \cdot \mathrm{Vol}(P) \cdot \frac{1}{h} - \mathfrak{fr}(t) \right| \leq \frac{\chi \cdot \sqrt{\mathfrak{fr}(t) \cdot \mathrm{Vol}(P)}}{\sqrt{N} \cdot \sqrt{h}} \tag{5.14}$$

The multiplication factor $(N_\delta/N) \cdot \mathrm{Vol}(P)$ on the left-hand side of (5.14) represents nothing else but an estimate for $\mathrm{Vol}(P_\delta)$. We denote it by $\mathrm{Vol}^*(P_\delta)$.[15] Therefore, we have

$$\left| \mathrm{Vol}^*(P_\delta) \cdot \frac{1}{h} - \mathfrak{fr}(t) \right| \leq \frac{\chi \cdot \sqrt{\mathfrak{fr}(t) \cdot \mathrm{Vol}(P)}}{\sqrt{N} \cdot \sqrt{h}}$$

or, equivalently,

$$|\mathfrak{fr}^*(t) - \mathfrak{fr}(t)\,| \leq \frac{\chi \cdot \sqrt{\mathfrak{fr}(t) \cdot \mathrm{Vol}(P)}}{\sqrt{N} \cdot \sqrt{h}}$$

Finally, we can formulate the expression for the confidence interval of the estimate of a (cumulative) frequency density function as:

$$\Pr\left[|\mathfrak{fr}^*(t) - \mathfrak{fr}(t)\,| \leq \frac{\chi \cdot \sqrt{\mathfrak{fr}(t) \cdot \mathrm{Vol}(P)}}{\sqrt{N} \cdot \sqrt{h}} \right] = 2\Phi(\chi) \tag{5.15}$$

[15] We estimate these partial volumes $\mathrm{Vol}^*(P_\delta)$ for equal intervals δ by composing a frequency histogram.

In the term defining the confidence interval bounds in formula (5.15) we have the value of $\mathfrak{fr}(t)$. However, instead of this unknown value we can use either a rough estimate (e.g. the volume of a cross-section of the $[0,1]^n$-cube) or a reasonably big constant by computing the bounds.

The approximation (5.8) is still valid in case of "slicing" of a portfolio opportunity set by a non-linear performance measure. Therefore, the last formula for the estimation of confidence interval for frequency density functions is also valid in general.

5.4.4 Computing a Confidence Interval for a Numerical Solution for our Example Investment

In this section we compute a numerical estimation of the frequency density function for our example opportunity set from section 2.3. The portfolio opportunity set consists of three stocks restricted by the no-short-sales constraint only, and it is the basic simplex in $\mathbb{R}^2$ (cf. Figure 2.2). The performance measure we are evaluating is the absolute performance over December 2003.

Expression (5.15) defines the confidence interval for a numerical estimate of the frequency density function given a specific confidence level, a histogram bin size, and the number of random portfolios in a sample. We consider the inverse problem: given a specific confidence level, a histogram bin size, and a *specific estimation accuracy*, we would like to find the size of a sample, which "guarantees" with the specified probability the required accuracy of estimation.

Let the bin size δ be equal to 0.5%. The vector formed by performances is defined as $\boldsymbol{\tau} = \langle -0.0638, -0.0366, 0.1296 \rangle$, so the norm of $\hat{\boldsymbol{\tau}}$ is equal to $\|\hat{\boldsymbol{\tau}}\| = 0.255$. Consequently, distances between hyperplanes defining a histogram bin are equal to $h \approx 0.0196$. Requiring the 99% confidence level and the accuracy $\varepsilon = 0.1$, we obtain the expression:

$$\Pr\left[|\mathfrak{fr}^*(t) - \mathfrak{fr}(t)| \leq \frac{\chi \cdot \sqrt{\mathfrak{fr}(t) \cdot \mathrm{Vol}(P)}}{\sqrt{N} \cdot \sqrt{h}} = \varepsilon \right] = 2\Phi(\chi)$$

Hence,

$$N = \frac{\chi^2 \cdot \mathfrak{fr}(t) \cdot \mathrm{Vol}(P)}{\varepsilon^2 \cdot h}$$

For the 99% level, $\chi^2 = 5.4119$. And, finally,

$$N = \frac{5.4119 \cdot \sqrt{2} \cdot 0.5}{0.1^2 \cdot 0.0196} = 19524$$

So, we need to sample at least 19524 random portfolios in order to achieve the estimation accuracy $\varepsilon = 0.1$ with the probability of 99%. Figure 5.2 shows the frequency density function as computed in section 4.2.4, the 99% confidence interval, and the estimated values of $\mathfrak{fr}(t)$ using given parameters as well as a sample of 25 thousand random portfolios.

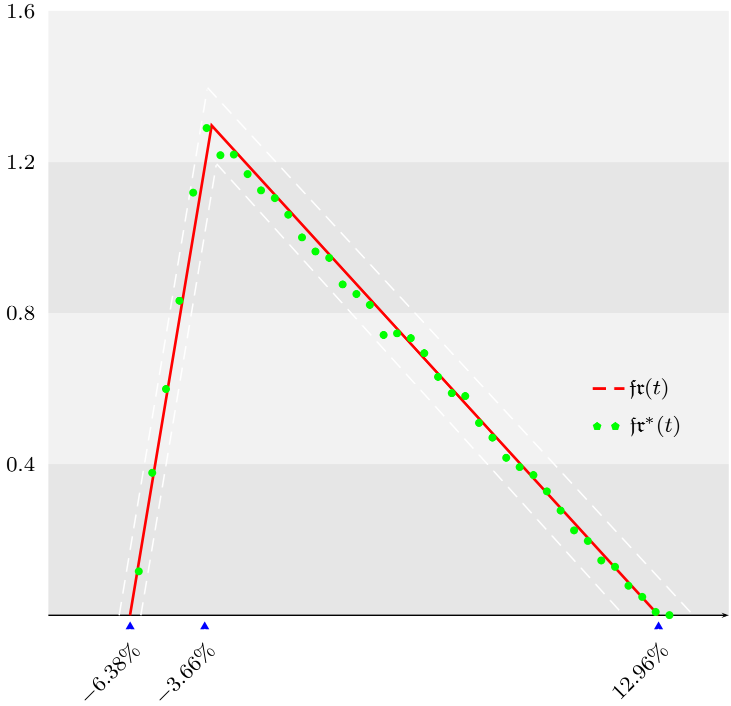

Figure 5.2: Frequency density function $\mathfrak{fr}(t)$ of the realized over December 2003 return. The thick dots show the estimated values of $\mathfrak{fr}^*(t)$, and the thin dashes lines border the 99% confidence interval for the estimation error of 0.1.

5.5 Quasi-Monte Carlo Approach

The general condition for (5.1) to hold is the uniformity of the chosen sequence. Thus, approximating the right-hand integral of (5.1) through a finite sequence/sample, the uniformity of the chosen sequence is the key element, which determines the quality of approximation. The idea behind the quasi-Monte Carlo (QMC) is that we can use some deterministic number sequences, which are in some sense evenly distributed, to compose a sample (For that reason the term "low-discrepancy" sequences is more suitable than the QMC to reflect the nature of this subject). Presently, there exist several algorithms to construct such sequences of well-chosen points. The most famous are:

- Halton-Hammersley sequences (Halton 1960);
- Faury sequences (Faury 1982);
- Sobol' sequences (Sobol' 1967, Sobol' 1969);
- Niederreiter sequences (Niederreiter 1992).

For a more detailed theoretical treatment of the mentioned algorithms, implementation details and code together with diverse applications of QMC in finance we refer to Jackel (2002) and Glasserman (2004). For a rigorous and comprehensive discussion of the theory of low-discrepancy sequences we refer to the excellent book of Niederreiter (1992). The recent developments in the area can be found at the server entirely devoted to QMC: http://www.mcqmc.org.

Let us consider how low-discrepancy sequences can be used to compose a sample of random portfolios and, thus, estimates $\mathfrak{fr}^*(t)$ of frequency density functions can be computed.

5.5.1 Opportunity Sets with a Short-Sales Restriction only

Given n assets, the opportunity set is the $n-1$-dimensional basic simplex, and we need to generate the low-discrepancy points over it. All existing algorithms assume the transformation of the integration domain to the $[0,1]^s$ region (i.e. $[0,1]^{n-1}$ in our case), and provide standard routines, which generate the low-discrepancy sequences over the unit hypercube. As generating the low-discrepancy points over the unit cube and then accepting only basic simplex points is very inefficient (cf. Exhibit 5.1), we have to find a way to generate the low-discrepancy points directly over the basic simplex without any "rejection" losses.

The exponential approach from section 5.4.1 can be easily adapted for such a transformation of a low-discrepancy sequence distributed over the unit hypercube to a low-discrepancy sequence distributed over the basic simplex as follows:

1. Construct the next n-dimensional point $\mathbf{x}_t = \langle x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)} \rangle$ of a $[0,1]^n$ low-discrepancy sequence (e.g. Sobol', or Halton or Faure sequence);
2. Create a new random portfolio $\mathbf{p}_t$ with asset weights $\langle w_1^{(t)}, w_2^{(t)}, \ldots, w_n^{(t)} \rangle$

where

$$w_i^{(t)} = \frac{\ln(x_i^{(t)})}{\sum_{j=1}^{n} \ln(x_j^{(t)})}, \quad i = 1, 2, \ldots, n$$

Add this new random portfolio to the sample;

3. If the sample is big enough, then stop. Otherwise return to step 1.

The key element of the procedure is that by such a transformation of a $[0,1]^n$ low-discrepancy sequence, the discrepancy bound on that sequence will still be valid after the transformation.

For the practical implementation of the algorithm we can use any of available standard implementations of low-discrepancy sequences. For example, one alternative is to use the C routines for Sobol' numbers from (Press et al. 1993) together with the primitive polynomials modulo 2 provided by Jackel (2002) on the accompanying CD. (This implementation of Sobol' numbers with regularity breaking initialization shows no substantial decline in uniformity up to 100 dimensions.) The libseq C++ library of Ilja Friedel and Alexander Keller (Caltech Multi-Res Modeling Group) provides an implementation of fast generation of various QMC sequences.[16] For more information on further available software packages we refer to Glasserman (2004).

5.5.2 Extending QMC Portfolio Generation to Different Types of Constraints

The geometric shape of opportunity sets strongly depends on the type and number of constraints in the description of those sets. To the knowledge of the author, there exists no methodology or an algorithm, which can generate a low-discrepancy sequence of any kind over arbitrary bounded regions (even when defined through linear inequalities only).

Therefore, the general recipe in this case is to “pack” a given portfolio opportunity set into a parallelepiped or a simplex, generate low discrepancy points over this “envelope”, and accept only those points that satisfy all constraints of the opportunity set. The major drawback of this approach is that the number of rejected portfolios can be huge with respect to the number of feasible portfolios (cf. Exhibit 5.1). However, for many opportunity sets from practice (e.g. opportunity sets of mutual funds of Credit Suisse Group mentioned in Chapter 3) the method works quite well and it has a rather tolerable acceptance/rejection ratio.

[16] The library is freely available at http://www.multires.caltech.edu/software/libseq/

5.5.3 Calculating the Accuracy of a Numerical Estimation through QMC

Generally, the error of estimation of the right-hand integral of (5.1) through a finite low-discrepancy sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ is given through the Koksma-Hlavka inequality:

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) - \int_{[0,1]^n} f(\mathbf{z}) d\mathbf{z} \right| \leq V(f) \cdot D_N^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \tag{5.16}$$

where $D_N^*(\cdot)$ is the star discrepancy of the sequence, and $V(f)$ is the Hardy and Krause's variation of the integrand f on $[0,1]^s$ (Niederreiter 1992). That is, the error is split into two factors: variation $V(f)$, which is entirely defined by the "shape" of the integrand, and discrepancy $D_N^*(\cdot)$, which completely depends on the quality (uniformity) of the sequence used for the estimation.

Inequality (5.16) defines a tight worst-case bound on the QMC error.[17] In practice, unfortunately, this bound could not be used for error estimation because the evaluation of both factors, $V(f)$ and $D_N^*(\cdot)$, is extremely difficult (even harder than the numerical integration itself). Therefore, one often used strategy is to randomize the QMC points. The consequence of this step is that we can use the MC intervals to measure errors but retain all low-discrepancy sequence features. And randomizing QMC points wisely, we can even improve the accuracy of our estimation.[18] Another strategy is to use $\mathcal{L}^2$-norm instead of $\mathcal{L}^\infty$. The $V(f)$ and $D_N^*(\cdot)$ with respect to $\mathcal{L}^2$-norm can be computed relatively easily. For example, the $\mathcal{L}^2$-star discrepancy is computed as

$$D_N^{\mathcal{L}^2 *} = \sqrt{\left(\frac{1}{3}\right)^s - \frac{2}{N} \sum_{i=1}^{N} \prod_{k=1}^{s} \left(\frac{1 - (x_k^{(i)})^2}{2} \right) + \frac{1}{N^2} \sum_{i,j=1}^{N} \prod_{k=1}^{s} \left(1 - \max(x_k^{(i)}, x_k^{(j)})\right)}$$

where $x_k^{(i)}$ denotes k-th element of i-th point, i.e. $\mathbf{x}_i$ vector, of a low-discrepancy sequence. To the best knowledge of the author, we can use $V^{\mathcal{L}^2}(f)$ and $D_N^{\mathcal{L}^2 *}$ to compute the error bound. However, in this case we obtain the mean square error. The detailed discussion of discrepancies and the corresponding error bounds with respect to different norms are given in Hickernell (1998).

Furthermore, a simpler and practical consideration is that we need to choose the length of a low-discrepancy sequence proportional to $2N - 1$. The reason lays in the way how points of a sequence "fill" the integration area.

[17] And, on the contrary to the Monte Carlo, this bound is not probabilistic.

[18] We refer to Owen (1998) for this subject. Other papers concerning the scrambled QMC sequences can be downloaded from Owen's webpage: http://www-stat.stanford.edu/∼owen/.

In addition, considering (5.16) we noted that the estimation error depends on the integrand variation $V(f)$ and discrepancy $D_N^*(\cdot)$. In our case, we have a given portfolio opportunity set and, thus, $V(f)$ cannot be changed. So, $D_N^*(\cdot)$ is a decisive element. Looking at star discrepancies for the MC and QMC, $\mathcal{O}\left(N^{-0.5}\right)$ and $\mathcal{O}\left((\ln N)^s \cdot N^{-1}\right)$ respectively, the QMC method has advantages over the MC (i.e. lower errors) when the dimensionality of a problem is low. For medium- and high-dimensional problems the MC method is often a better choice.[19]

5.5.4 Computing a Numerical Solution for our Example Investment Using the quasi-Monte Carlo

We compute a numerical estimation of the frequency density function for our example opportunity set from section 2.3 using Sobol' low-discrepancy numbers (cf. section 5.4.4 for the MC-based estimation). The example portfolio opportunity set consists of three stocks restricted only by the no-short-sales constraint and, thus, it is the basic simplex in $\mathbb{R}^2$. Again, the evaluated performance measure is the absolute performance over December 2003.

Let the bin size δ be equal to 0.5%. The vector formed by performances is defined as $\boldsymbol{\tau} = \langle -0.0638, -0.0366, 0.1296\rangle$. By applying the standard Monte Carlo estimation and requiring the estimation accuracy $\varepsilon = 0.1$ with the probability of 99%, we computed in section 2.3 that we need to sample 19524 random portfolios at least. Figure 5.3 shows the frequency density function as computed in section 4.2.4 the 99% confidence interval (as in Figure 5.2), and the estimated values of $\mathfrak{fr}(t)$ using 25 thousand portfolios sampled using the exponentially-transformed Sobol' numbers.

Comparing Figure 5.2 and Figure 5.3, we see that differences in estimates $\mathfrak{fr}^*(t)$ are marginal, possibly with a slightly better QMC-estimate.

5.6 Further Aspects of the Estimation Procedure(s)

Finally, we briefly consider several technical aspects of the estimation procedure. One of them is the question how to choose the histogram bin size.

The histogram bin size δ is directly related to distance h between iso-surfaces (cf. section 5.4.4, for example). Furthermore, expression (5.15) defines the confidence interval of the MC estimates of a (cumulative) frequency density function. Reformulating that expression for N and denoting

[19] Despite a better asymptotic behavior, the function $g(N) = (\ln N)^s \cdot N^{-1}$ is increasing on the interval $(0, N = e^s)$, and is decreasing on the interval $(N = e^s, +\infty)$.

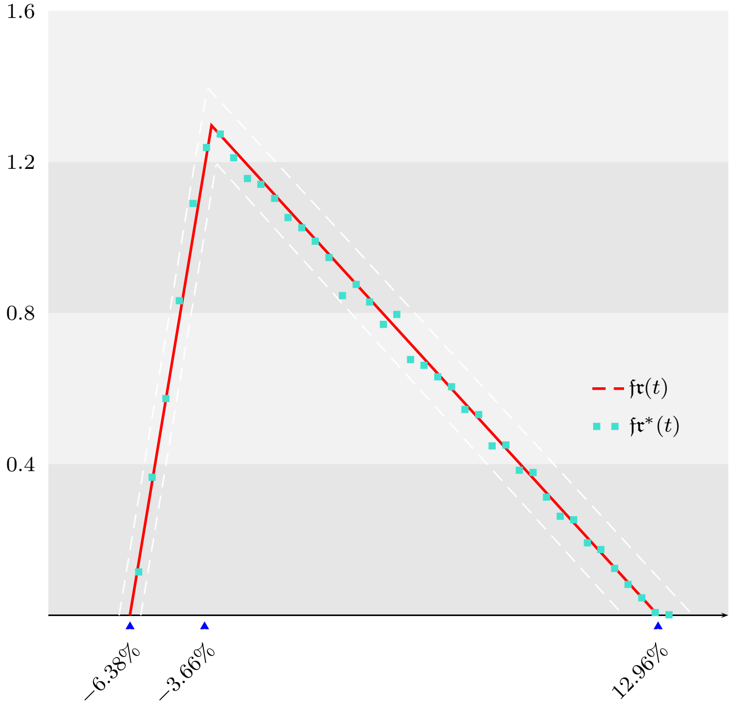

Figure 5.3: Frequency density function $\mathfrak{fr}(t)$ of the realized over December 2003 return. The small squares show the values of $\mathfrak{fr}^*(t)$ estimated using Sobol' numbers, and the thin dashes lines border the 99% confidence interval for the estimation error of 0.1 as calculated for the Monte Carlo estimation in section 5.4.4.

$|\mathfrak{fr}^*(t) - \mathfrak{fr}(t)|$ by ε, we obtain the following equation:

$$N = \frac{\chi^2 \cdot \mathfrak{fr}(t) \cdot \mathrm{Vol}(P)}{\varepsilon^2 \cdot h}$$

On the one side, fixing a number of portfolios in a sample, the denominator of the right-hand side of this expression could be seem as a trade off between the accuracy and the number of bins in the histogram (the reverse of values δ and h). On the other side, requiring a specific estimation accuracy, the expression can be seem as a trade off between the increased approximation "accuracy", which comes from increasing the number of bins (i.e. smaller values for δ's and h's), and the increased cost of estimation (i.e. the required number of portfolios in the sample).

Generally, determination of the optimal number of bins in a histogram, which was used to estimate a distribution, is one of the problems in nonparametrical statistics. Scott (1979) has shown that the optimal histogram bin size is:

$$\delta = 3.49 \cdot \sigma \cdot N^{-1/3}$$

where σ is the standard deviation of the distribution. The expression is optimal in the sense that it asymptotically minimizes the mean squared error between estimates and distribution densities.

Of course, estimating frequency density functions $\mathfrak{fr}(t)$ and $\mathfrak{Fr}(t)$, we pursue a different objective. However, the above expression can be used as a "rule of thumb" for defining or approximating the histogram bin size. Empirical investigations indicate that the bin size calculated using this rule is rather suitable for most cases.

5.7 Summary and Conclusions

Numerical estimation is a universal but rather slow technique for calculating frequency distributions. The procedure is based on statistical sampling: we estimate the distribution of performance values of the full opportunity set through the distribution of performance values of a reasonably-large sample.

We can use one of the following alternatives to form such a sample: the Monte Carlo method or a low-discrepancy sequence (the quasi-Monte Carlo). Looking at the Monte Carlo technique, the most universal approach for sample random portfolios is the Sequential Direction algorithm, which constructs an ergodic Markov chain. The approach is universal in any sense: we can estimate frequency distributions for portfolio opportunity sets restricted by both linear and non-linear constraints with respect to any kind of performance measure. In addition, for the no-short-sales case the further two efficient transformation approaches, the Uniform Spacings and the Exponential, can be used. Furthermore, the error of estimation is easily computed as a confidence interval.

The low-discrepancy sequences are another technique for numerical estimation of frequency distributions. The general recipe in this case is to "pack" a given portfolio opportunity set into a parallelepiped or a simplex, generate low discrepancy points over this "envelope", and accept only those points that satisfy all constraints of the opportunity set. As low-discrepancy sequences are fully deterministic, so is the corresponding error bound. However, the worst-case error is very difficult to compute in practice.

Generally, we experienced that Monte Carlo-based methods are more suitable for estimating frequency distributions. They are easier to analyze, implement and use. They are also more efficient than the low-discrepancy sequences in many cases. However, the latter are worth using as an alternative estimation mechanism and further investigating in more detail.

Chapter 6

Market Dynamics from the Portfolio Opportunity Perspective: the DAX Case[1]

6.1 Introduction and Motivation

Financial market dynamics can be described in different ways (cf. Chapter 1). A widely used approach for describing financial market development is the use of indexes. For many markets and segments of these markets, indexes are available: if we wish to study the development of a market (or a market segment), we study the appropriate market (or segment) index. A flourishing industry of index providers exists, delivering standardized indexes (be it that different providers generally use different definitions and standards). The choice of an appropriate index has developed into a fine art. Also, providers develop tailor-made indexes for individual clients. Another approach for describing financial market dynamics is to look at the performance of portfolios in a peer group such as mutual funds that invest in the same market or market segment. Intuitively, most investors know that an index or a peer group only provide a limited view on the market dynamics. And of course, an index or any other average by definition summarizes the price dynamics of individual financial assets and only shows part of the vast amount of information available. The information which is hidden by the use of indexes is potentially useful.

The objective of this chapter is to apply our methodology (i.e. frequency

[1]This chapter is a revised version of the article Hallerbach, Hundack, Pouchkarev & Spronk (2005).

distributions with respect to various return and risk metrics) in order to provide a new way of looking at markets and of describing their dynamics over time. The distinctive feature of our approach is that we choose the *perspective of available portfolio opportunities.* When aiming at specific investment objectives and satisfying specific investment constraints, a universe of feasible portfolios, a portfolio opportunity set, can be identified. This set signifies a level playing field for any portfolio that aims for the same investment objectives and respects the same constraints. Next, we observe the dynamics of each feasible portfolio in this opportunity set. Finally, we condense the information of the whole portfolio opportunity set into a compact and informative market description.

The rationale behind describing the market dynamics from the portfolio opportunity perspective is fourfold.

Firstly, the advantage of our approach is that *our market description can be made commensurate with any specific investment environment* (such as an investment mandate, defined by goals and constraints). Many investors are interested to see the potential (including the upside and downside) of a market or a segment from the viewpoint of their specific investment circumstances. Therefore, our view from the portfolio opportunity perspective provides valuable insights into such “window of opportunities” available on a market/segment at any specific time.

Secondly, the performance of an investment portfolio is usually evaluated *vis á vis* the performance of a benchmark. So, when evaluating an investment, we can calculate the relative performance with respect to the benchmark. But *how substantial* is a relative over- or under-performance for a specific period? Our way of viewing a market can answer this question, because we can compare the performance of a particular investment portfolio with the performances of all portfolio opportunities. Moreover, when comparing two different portfolio strategies within the considered market over time, we can “norm” these relative performances against the performances of all portfolio opportunities. (For an enhanced illustration we refer to Chapter 7.)

Thirdly, we can consider an (equity) market or a market segment and the mutual funds that invest into this market (segment) using the portfolio opportunity perspective. Comparing these two views, we can evaluate whether mutual funds exploit the available investment opportunities in a successful way (for an illustration we refer to Pouchkarev, Spronk & Steenbeek (2005).

Fourthly, our way of viewing a market can help to assess the relative importance of different investment activities on that market. For example, by estimating the view with respect to different sectors and confronting it with the view with respect to the securities, we can assess the relative importance of the sector allocation versus the selection of individual securities for any period of time. In addition, this allows us to appraise changes in their importance ratio over time. (We refer to Kritzman & Page (2003) for an enhanced discussion of evaluation of the hierarchy of investment choice).

In this chapter particularly, we illustrate how our methodology can be used for providing enhanced market descriptions on example of the blue-chip segment of the German stock market as represented by the DAX® index over the last 14 years.[2] Imposing the same investment restrictions as the DAX does, we invest with varying weights in baskets of all the index components. We show the estimated dynamics of this investment universe over the last 14 years from different perspectives, i.e. in terms of realized returns, average returns, and different kinds of risk measures.

The organization of the chapter is as follows. Section 6.2 outlines the new methodology and describes our data set. Section 6.3 reviews the DAX index and its methodology and presents the conventional view on the index development over the last 14 years. The next sections are devoted to the empirical results from the portfolio opportunity approach. We apply our methodology and provide the description of the German market dynamics between January 1990 and June 2004 in terms of various return (Section 6.4) and risk (Section 6.5) metrics from the perspective of investing in DAX stocks. Section 6.6 gives a combined risk-return perspective. Having the description of the investment universe of the DAX, we analyze the performance of the index and other benchmark portfolios. Section 6.7 concludes.

6.2 Methodology and Data

6.2.1 Description of the Methodology

Our goal is to view the dynamics of the large cap segment of the German equity market from the perspective of portfolio opportunities available under the same investment constraints as implied by the definition of the DAX. By applying the rules and constraints as used by the DAX to select stocks (see chapter appendix 6.A), we obtain the same set of 30 stocks that compose the index. Consequently, the portfolio opportunity set consists of all portfolios that can be composed from these stocks with weights subject to the following constraints:

[2] DAX® is a registered trademark of Deutsche Börse AG. It is written with the symbol ® in the title, and at the first occurrence in this chapter, and as DAX thereafter.

$$0 \leq w_i \leq 0.15, \quad i = 1, 2, \ldots, 30 \quad \text{such that} \quad \sum_{i=1}^{30} w_i = 1 \tag{6.1}$$

Clearly, given these weight constraints, the number of portfolios in the opportunity set is infinite. Therefore, we numerically estimate the required frequency distribution of performance values from a reasonably-large sample.[3] In particular, we use the following procedure:

I. In each simulation step we sample ten million feasible random portfolio weight vectors for stocks of the DAX. We use the adapted uniform spacing algorithm (cf. section 5.4.1) to generate random portfolios:[4]

   1. Generate $U_1, U_2, \ldots, U_{29}$ as 29 independent $[0, 1]$ uniformly distributed random variables (i.e. $U_i \sim \mathcal{U}(0, 1)$, $i = 1, 2, \ldots, 29$);
   2. Sort U_i in ascending order, i.e. define the order statistics $U_{(1)}, U_{(2)}, \ldots, U_{(29)}$;
   3. Create a new random portfolio with asset weights $\langle w_1, w_2, \ldots, w_{30} \rangle$ where $w_i = U_{(i)} - U_{(i-1)}$, $1 \leq i \leq 30$, $U_{(0)} = 0$ and $U_{(30)} = 1$;
   4. Check if the condition $w_i \leq 0.15$ is valid for all weights. In not, then repeat steps 1-3. Otherwise add this new portfolio to the sample;
   5. If the sample is big enough, then stop. Otherwise return to step 1.

   Each sampled weight vector defines a feasible portfolio and is an alternative for investing in the German large cap segment. The sampled portfolios are uniformly distributed over the DAX portfolio opportunity set;

II. For these sampled portfolios, as well as for the actual DAX index and the equally weighted benchmark, we calculate different portfolio return characteristics: realized discrete (percentage) returns over different periods, the arithmetic average rate of returns, standard deviations, and

[3] Given the DAX portfolio opportunity set, there are several ways to compute the required distribution(s). One way is to use the analytical formula from section 4.2.5, simulation is another. However, we use the numerical estimation because of computation of frequency distributions for various non-linear risk metrics.

[4] The first three steps of the algorithm are equal to the uniform spacing approach described in section 5.4.1. We use an additional acceptance-rejection step in order to filter the portfolios, which satisfy the DAX upper bound constraint, i.e. $w_i \leq 0.15$. In should be noted that only about 23% of portfolio are rejected in step 4. So, the generation algorithm is rather efficient.

semi-standard deviations. Except for realized returns, these statistics are estimated using 36 monthly observations prior to the actual evaluation period. (For example, for evaluating the market during May 2004, the stock returns from May 2001 through April 2004 are used.);

III. We estimate the frequency distributions of the selected performance measures over the whole portfolio opportunity set;

IV. The time window is shifted one month forward and the next simulation is carried out.

Thus, steps I-III provides a "snapshot" of existing opportunities available on the German blue-chip equity market over a certain time period. The information content of these steps is the location and form of the computed distributions of performance values. Step IV provides a picture of the development of the market dynamics over time by showing the development of the dispersion of these distributions.

6.2.2 Dataset

We used the "Short Information to the Equity Indices of Deutsche Börse" as well as reports on the DAX index published by the Deutsche Börse AG to comprise the list of stocks constituting the DAX from December 1987 till present. The input data consists of monthly observations on the DAX stocks from January 1987 through June 2004. We used closing prices at the last trading day of each month to compute discrete (percentage) returns;[5] until May 1999 inclusive, these are closing prices traded on the floor at the Frankfurt Stock Exchange, afterwards Xetra®-prices are used (This coincides with the DAX calculation methodology, see Deutsche Börse Group (2004b) for details).

The stock data and correction factors for all corporate actions (e.g. stock splits, capital increases, dividends) were downloaded from the Karlsruher Kapitalmarkt Datenbank (KKMDB) and cross-checked against the data available through the Bloomberg and Thompson Datastream databases. The prices are corrected through operation blanche for all corporate actions including dividends as well as for the Euro introduction (see Sauer (1991), for more details).

For a few stocks e.g. Deutsche Telekom, Deutsche Post, only short historical time series exist before the index inclusion. Therefore, we performed the

[5]We use ultimo month prices, so our evaluation horizon is one month. We do not use within-month averaged prices because this would generate various statistical biases in the return series, see for example Wilson, Jones & Lundstrum (2001) and Hallerbach (2003).

regression of such stocks on the DAX level and other index constituents in order to extend their time series to have 36 monthly price observations before the index inclusion.[6]

An important issue in our simulation procedure is how to handle changes in the index composition. Regular changes are carried out once a year in the third Friday in September. However, the changes in the DAX are quite irregular due to mergers, new admissions, deletions, *et cetera*, which should be reflected in the index shortly after their occurrence. After changes in the structure of the DAX are decided upon by the Deutsche Börse, they give a 4-6 weeks notice before implementation. Therefore our strategy is to hold the security deleted from the index until the start of the replacement month and then replace it by the new one. For example, on July 23, 2001, Dresdner Bank was exchanged in the index against MLP. When we evaluate the performance of the feasible portfolios at the last trading day of June 2001, we have Dresdner Bank as one of the DAX stocks. For the evaluation month July 2001 the Dresdner Bank stock drops from the stock set and, thus, from the portfolios. Instead, MLP will be used as a new stock in the DAX stock set to form portfolios.

6.3 Conventional View on the DAX Index

The DAX is the major benchmark index of the German equity market. It consists of the 30 largest German companies in 14 different sectors[7] that have the largest free-float capitalization and the highest exchange liquidity level. The objective of the index is to reflect the financial capital dynamics of the German blue-chip companies from the main sectors of the national economy

[6] We use a recursive regression approach to fill in missing observations. When there is one or more missing observations (for example when a stock is incorporated in the DAX), we start with a window of 36 return observations on the stock i and the index I, which begins immediately after the missing return data point for the stock. Within this window, we use OLS regression to estimate the parameters α_i and β_i of the market model:

$$r_{i,t} = \alpha_i + \beta_i \cdot r_{I,t} + \epsilon_{i,t} \quad t = 1, 2 \ldots, 36$$

Given these parameters and the available index return for the first observation $t = 0$ before the start of the window, we construct the stock return, conditional on the index return:

$$r_{i,t} \mid r_{I,t} = \alpha_i + \beta_i \cdot r_{I,t} \quad t = 0$$

Finally, we randomly sample one residual term from the 36 residuals in the estimation window and add that to $r_{i,0} \mid r_{I,0}$. We substitute this return for the missing return $r_{i,0}$. We can repeat the procedure by moving the data window backwards in time.

[7] As of June 30, 2004, see also chapter appendix 6.A.

(Over the last five years the average index capitalization accounts for more than 70% of the total capitalization of the German stock market).

The DAX is a free float capitalization-weighted performance index. It is based on the Laspeyres' index formula, with the base date December 30, 1987 and a base value of 1,000. The DAX index is considered to be a good representative of the dynamics of the German large caps segment. It is formulated strictly and clearly, and can be easily "reproduced". For these reasons the index is widely used as a benchmark of the German stock market, and also as underlying for diverse derivative products such as ETC's, ETF's, options and futures. For the actual index formula, constraints and the index composition as of June 30, 2004 we refer to chapter appendix 6.A. For the detailed index methodology, the actual composition and index values we refer to the site of the Deutsche Börse, http://www.exchange.de.

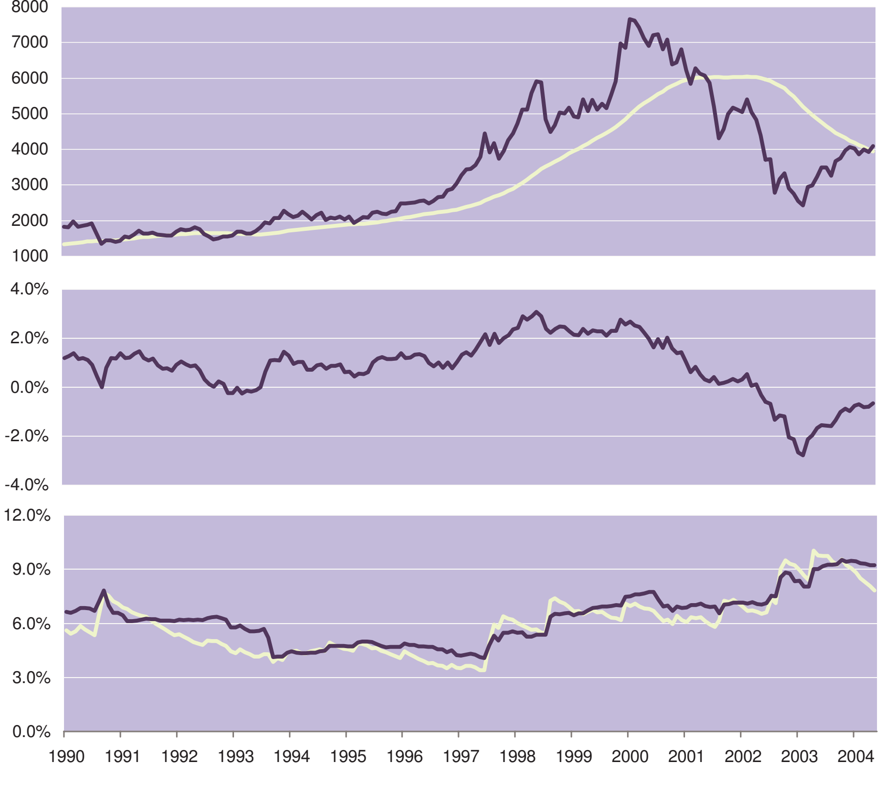

Figure 6.1: The development of the DAX. Monthly and 36-month moving average index level from January 1990 through June 2004 (top), two-year moving averages of monthly-realized returns (middle) and 36-month equally- and exponentially-weighted standard deviation of returns representing the index risk (bottom, dark line and light lines respectively).

	Return (in %)				**Risk** (Std Dev,% ann.)				**Sharpe Ratio**			
Index	*1Y*	*3Y*	*5Y*	*10Y*	*1Y*	*3Y*	*5Y*	*10Y*	*1Y*	*3Y*	*5Y*	*10Y*
DAX	26.7	-32.7	-24.1	102	18.1	32.1	28.6	25.1	0.371	-0.097	-0.047	0.078
MSCI	27.0	-31.0	-24.5	95	17.5	31.3	28.0	24.1	0.386	-0.094	-0.051	0.074

Table 6.1: Performance and risk metrics of the DAX and MSCI Germany indexes as of June 30, 2004. (Source: Thompson Datastream & MSCI).

The development of the DAX over the last fourteen-and-half years is shown in Figure 6.1 and Table 6.1. The top graph in Figure 6.1 shows the monthly index values over the period January 1990 through June 2004. The light line plots the 36-month moving average of index values. The graph in the middle plots the two-year moving average of monthly realized returns.[8] The bottom graph shows two different types of standard deviation of the monthly returns. The dark line represents the conventional equally-weighted standard deviation and the light line represents the standard deviation based on the exponentially weighted moving average scheme (EWMA) with a decay factor of 0.95.[9]

The top graph and performance figures in Figure 6.1 show that the general trend of the German equity market was upward over the sample period. This is also reflected by the fact that the averages of monthly-realized returns lay above the zero level almost for the entire sample period, as the middle graph shows.

The DAX index has been developed in line with the international markets. It experienced a rapid growth from 1997 due to the upward effects of the ICT bubble in the US until that bubble burst in March 2000. As the top graph shows, in that period we clearly see three intervals of extreme growth interrupted for a short time by the Asia crisis in the summer of 1997 and the Russian default in August 1998. After March 2000, together with other developed equity markets, the German stock market was falling sharply until March 2003, when it reached the level of 1995. After that, the market has experienced an impressive renaissance: from March 2003 till June 2004 the DAX surged from 2,202.96 to 4,080.08 points, or 85%, and it was the best-

[8]The last business day of each month is used.

[9]The EWMA scheme allows to register changes in the variance (standard deviation) faster and avoids clustering effects caused by shocks. We refer to (J.P.Morgan 1996) for further details about the EWMA procedure.

performing market index in Europe.

6.4 Return Dynamics from the Portfolio Opportunity Perspective

We start the discussion of the results by looking at the distributions of one-year realized returns. Figure 6.2 considers one such a distribution, for returns realized from January 1, 2003 through December 31, 2003 by portfolios from the DAX opportunity set.

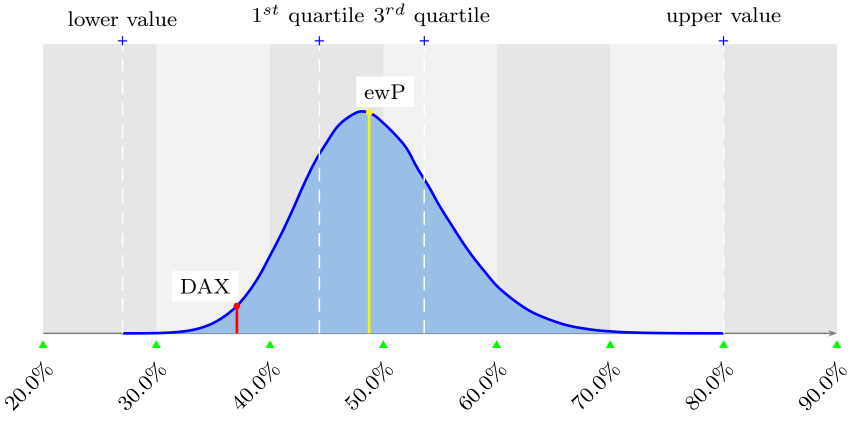

Figure 6.2: The distribution of returns realized from January 1, 2003 through December 31, 2003 by feasible portfolios of DAX stocks. The dashed white lines mark the upper and lower distribution values as well as the 1st and 3rd quartiles. In addition the performances of the DAX and the equally-weighted benchmark are plotted.

This particular distribution has a lower value of 27.17% and an upper value of 80.47%. Two additional dashed white lines in the graph show the 1st and the 3rd quartiles of the distribution, 44.35% and 53.6% return respectively. The position of the DAX realized return and the return of the equally-weighted benchmark (ewP) are also plotted in the distribution. Given an investment universe and the portfolio forming constraints as defined in section 6.2, the distribution represents the *full range of investment opportunities* with respect to the realized return for the year 2003. More precisely, in that period *any* feasible portfolio returned between 27.17% and 80.47%. Thus, from the viewpoint of portfolio opportunity perspective our DAX investment mandate had the lower potential of 27.17% and the upper potential of 80.47%; so under this mandate it was impossible to lose money. Furthermore, the distribution reveals interesting insights into the performance of the DAX. The index performed strongly over the year 2003 and ended up 37% relative to its level at the beginning of the year.

However, looking at realized returns from the portfolio opportunity perspective, the index performance is poor. By taking some portfolio *completely randomly* from our portfolio opportunity set, we expect to outperform the DAX index in *approximately 96% of the cases.* The performance of the ewP offered a better performance in this respect.

Figure 6.3 graphically shows 15 distributions of one-year realized returns over the whole sample period from 1990 through 2004. For each sub-period the corresponding pictogram represents the distribution of returns realized over the calendar year for the DAX portfolio opportunity set. That is, the 1990 pictogram shows the distribution of returns realized by feasible portfolios from January 1, 1990 through December 31, 1990; the 1991 pictogram represents the distribution of returns realized by feasible portfolios from January 1, 1991 through December 31, 1991 and so on. For the year 2004 the corresponding pictogram shows the distribution of returns realized from January 1, 2004 through June 30, 2004. Figure 6.4 represents the same distributions as Figure 6.3 by means of box plots. Considering the development of one-year realized return distributions we can make the following observations:

1. *General upward trend.* From the portfolio opportunity perspective the upside potential of our investment mandate prevails over the downside potential;

2. *Heterogeneity of stock returns has increased.* Heterogeneity is defined here loosely as the magnitude of the differences between the cross-sectional stock returns. For the period from 1996 onwards we observe that the heterogeneity of stock returns in the large cap segment of the German equity market has increased strongly. While from 1990 to 1995 the spreads between the upper and lower values of distributions for the considered portfolio opportunity set were about 25% in terms of one-year realized returns, from 1996 on such spreads increased to approximately 40%. We look at this issue in more detail by considering the monthly return figure;

3. *DAX stability with respect to the considered investment mandate is decreased.* While till 1998 the performance of the DAX index was quite persistent with respect to our portfolio opportunity set and the index is located in the interquartile ranges, from 1999 onwards the location of the DAX is mostly in the 1st or 3rd quartiles. Our conjecture is that this fact reflects the relatively extreme performance of the large

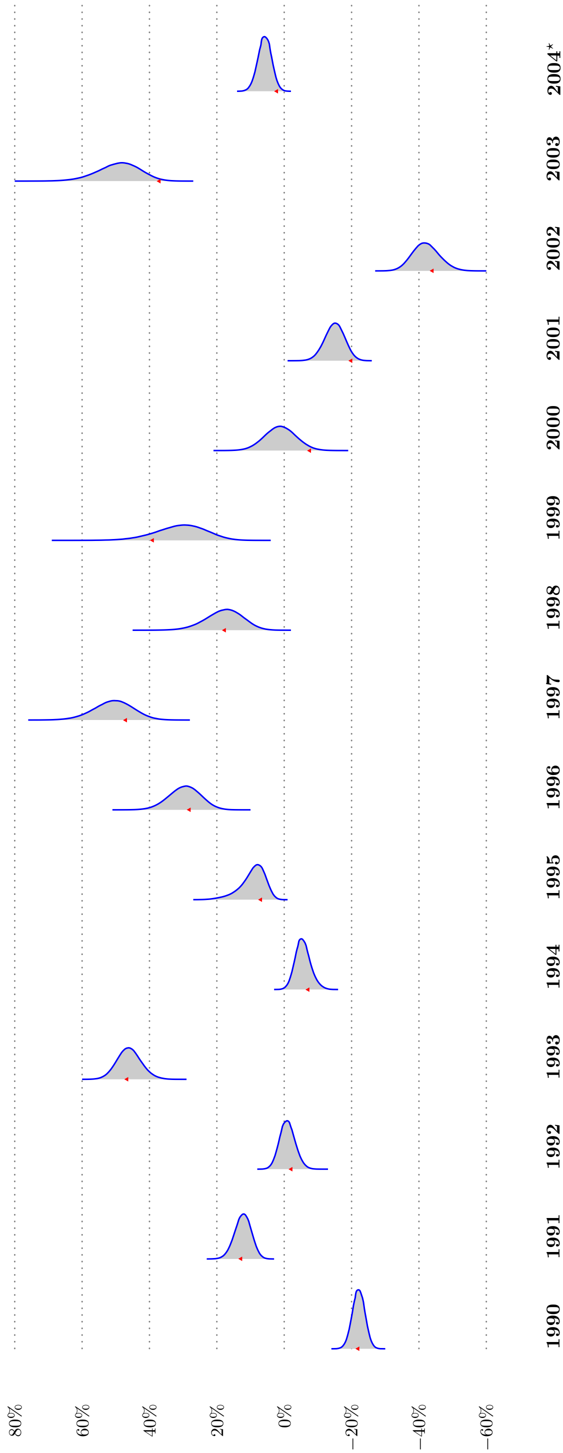

Figure 6.3: The distributions of portfolio realized returns from 1990 through 2004. For each sub-period the corresponding pictogram represents the distribution of returns realized over the calendar year for the DAX portfolio opportunity set. The position of the DAX is marked by a small triangle.

capitalization stocks in the index;

4. *Superior portfolios.* It seems that over the last five years, a strategy tilting towards an equally-weighted portfolio would have been clearly superior to the DAX, seen as a strategy.

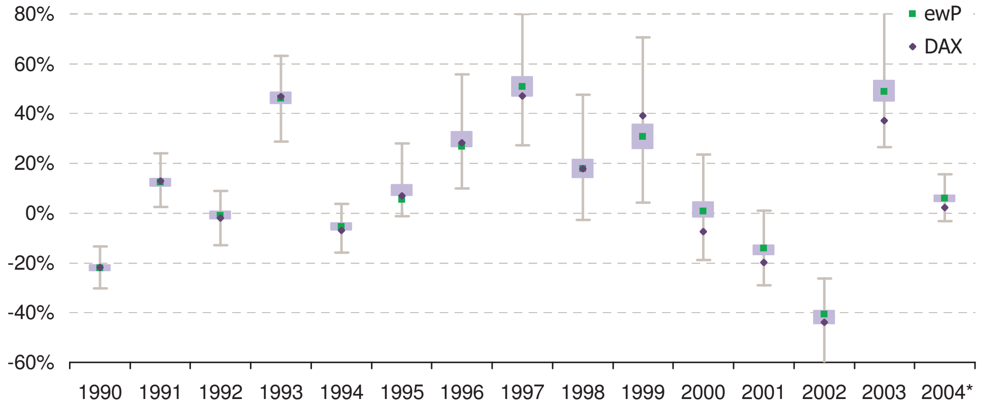

Figure 6.4: The distributions of portfolio realized returns from 1990 through 2004 by box plots. The length of the fat bar represents the return range of the middle 50% of all portfolios. The thin bar represents the range capturing all generated portfolios. Additionally, the diamonds indicate the development of the DAX realized returns, and the dots indicate the returns realized by the equally weighted portfolio.

At this point it is important to note that these insights are produced because of the portfolio opportunity perspective and that these insights were not obvious at all by looking at the index development alone.

Panel A of Figure 6.5 graphically shows the distributions of monthly realized returns for portfolios of our DAX opportunity set over the whole sample period from January 1990 through June 2004. The distributions are represented by box plots: for each month end, the return distribution is mapped on a vertical bar. The length of the fat bar represents the return range of the middle 50% of all portfolios. The thin bar represents the return range capturing all generated portfolios. In addition, the rhombus dots represent the corresponding monthly realized returns of the DAX index. The box plots clearly reflect the development of the rate-of-return distributions during the sample period.

From January 1990 to approximately June 1997, the distributions display similar *form* and *range*. Also the *changes in the month to month distribution*

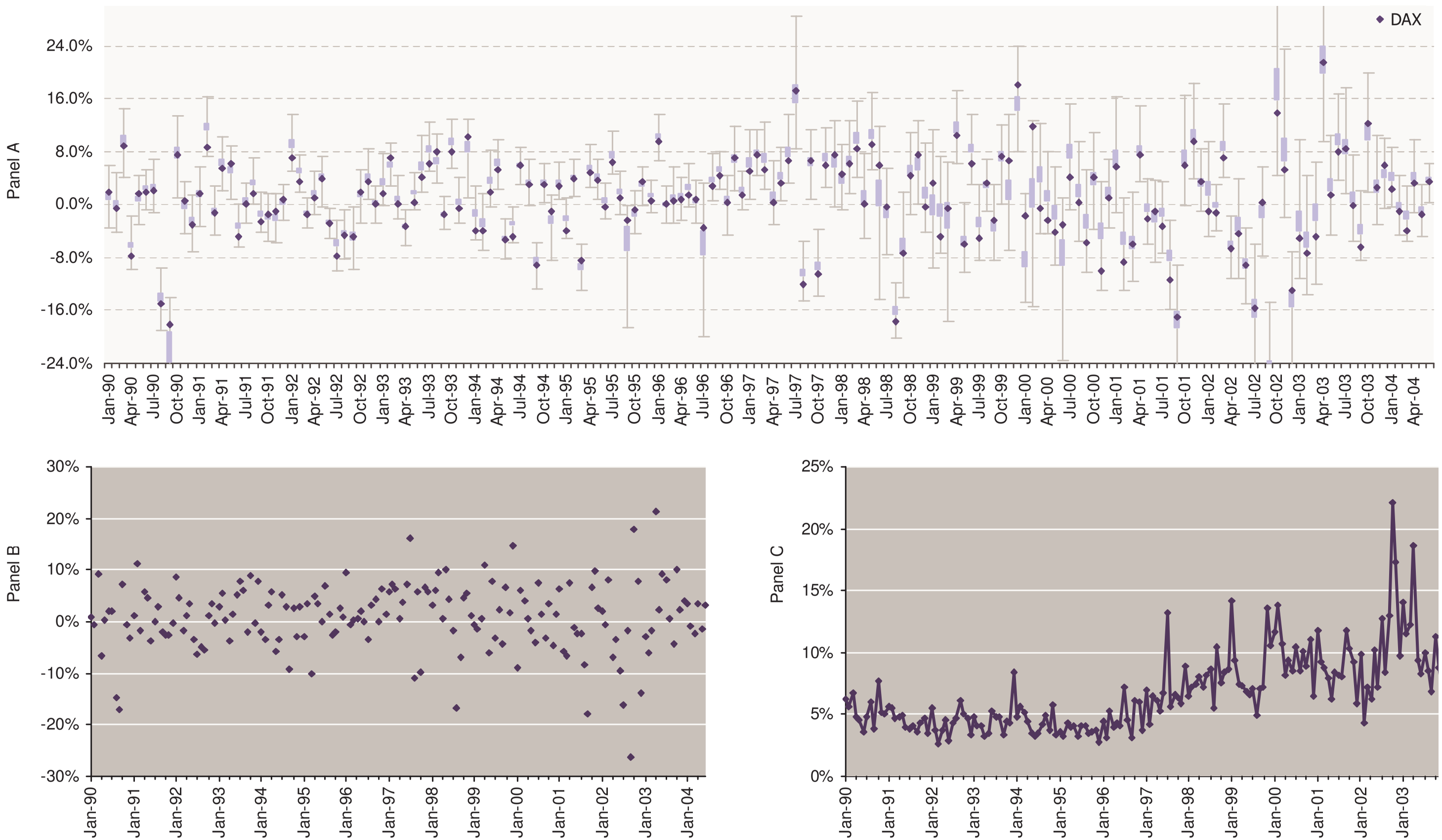

Figure 6.5: Panel A shows the distribution of monthly realized returns in feasible portfolios of our portfolio opportunity set from January 1990 through June 2004. The length of the fat bar represents the return range of the middle 50% of all portfolios. The thin bar represents the range capturing all generated portfolios. Additionally, the dark rhombus dots plot the monthly realized returns of the DAX index. Panel B graphs the cross-sectional averages of monthly returns of the DAX stocks for the same period, and Panel C plots the cross-sectional standard deviations of these returns.

locations are similar.[10] From July 1997 onwards these characteristics of the distributions have changed substantially:

- The return range of feasible portfolios has increased from 8-10% to 12-16%;
- The month-to-month volatility of returns in cross-section has increased.

To focus on the changes in the *locations* of the distributions, Panel B of Figure 6.5 displays for each month the cross-sectional average of the returns on the 30 stocks in the DAX. So each dot represents the return on the equally-weighted portfolio in the respective month. The graph suggests that the cross-sectional averages are more loosely distributed from 1997 to 2004 than in the first half of the sample period.

To focus on the changes in the *ranges* of the return distributions, we computed for each month the cross-sectional standard deviation of the stock returns. These cross-sectional standard deviations are depicted in Panel C of Figure 6.5. The graph clearly shows that the return dispersion across stocks has strongly increased over time. Hence, the market has become more heterogeneous over time. Note that Panel A and Panel C both reveal the increased heterogeneity of the German large cap segment. However, only Panel A presents the market heterogeneity from the full portfolio opportunity perspective. In Panel C only one portfolio is considered: the equally-weighted portfolio.

6.5 Risk Dynamics from the Portfolio Opportunity Perspective

Let us now consider the risk development of the DAX investment universe from the portfolio opportunity perspective. In the form of box plots, Figure 6.6 shows the distribution of 36-month trailing standard deviations in feasible portfolios for DAX stocks from January 1990 through June 2004. The length of the middle box (i.e. the fat bar) represents the range of the middle 50% of all portfolios. The thin bar represents the range capturing all generated portfolios. Additionally, the dark line plots the 36-month moving standard deviations of the DAX values, and the light line plots the values for the equally-weighted portfolio.

Figure 6.7 presents two graphs of the dynamics of portfolio opportunities in terms of alternative market risk measures. Panel A of this figure shows

[10] An exception is formed by the four distributions for realized returns over August-September 1990, September 1995, and August 1996.

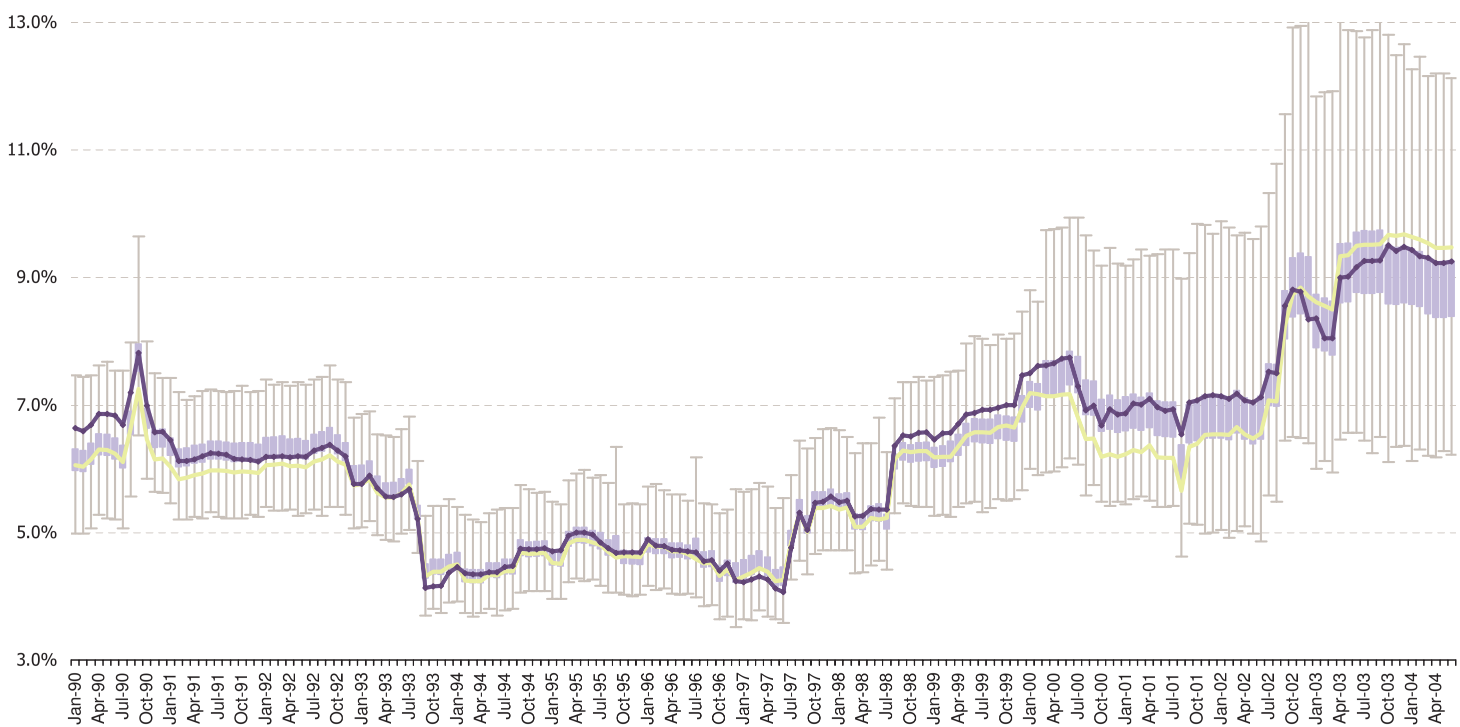

Figure 6.6: The box plot shows the distribution of 36-month standard deviations in feasible portfolios for DAX stocks from January 1990 through June 2004. The length of the middle box (i.e. fat bar) represents the return range of the middle 50% of all portfolios. The thin bar represents the range capturing all generated portfolios. Additionally, the dark line plots the 36-month moving standard deviations of the DAX values, and the light line plots the values for the equally-weighted portfolio.

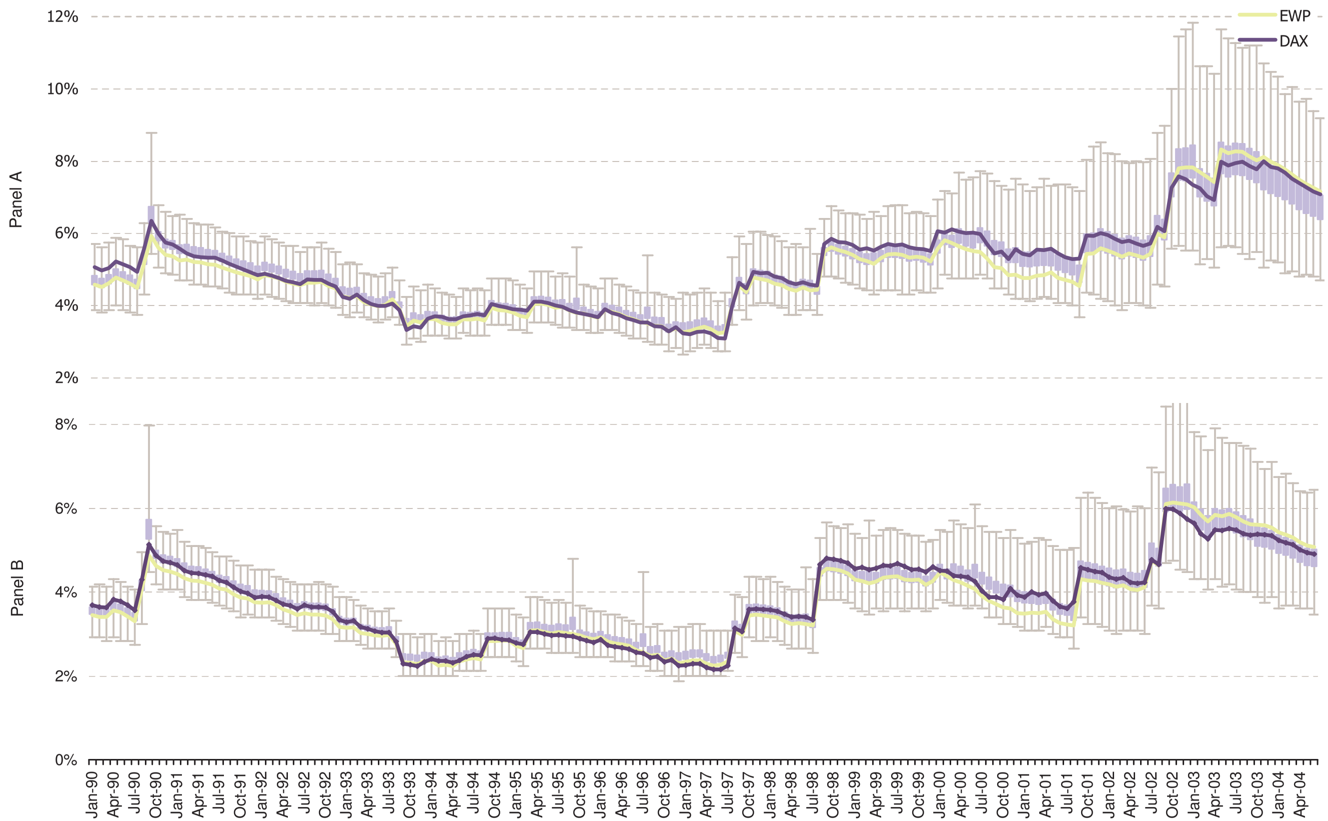

Figure 6.7: Panel A shows the distribution of 36-month exponentially-weighted standard deviations in feasible portfolios for DAX stocks from January 1990 through June 2004. Panel B graphs the distribution of portfolio semi-variances for the same period. The dot on the bar represents the median value of the distribution. The length of the fat bar represents the semi-variance range of the middle 50% of all portfolios. The thin bar represents the range capturing all generated portfolios. Additionally, the dark line plots the 36-month semi-variances of the DAX values, and the light line plots the positive semi-variance of the equally-weighted portfolio.

the distributions of 36-month exponentially weighted standard deviations in feasible portfolios for DAX stocks from January 1990 through June 2004. Panel B plots the distributions of portfolio semi-variances for the same period. In both graphs the boxes represent the middle 50% of all portfolios, and the thin bars identify the ranges capturing all generated portfolios. Additionally, the dark line plots the 36-month exponentially-weighted standard deviations and semi-standard deviations of the DAX index, and the light line plots the exponentially-weighted standard deviations and semi-standard deviations of the equally-weighted portfolio. The risk graphs reveal the following facts:

1. *Increase of risk.* As both graphs suggest, the common market risk level has increased substantially. While till 1997 the standard deviation level of monthly returns for portfolios from the DAX opportunity set was about 4-7%, from 1998 onwards this range was "pushed" by various periods of euphoria alternating with depression and crises to the 6-11% range. The same trend is observable for the downside part of the risk;

2. *Heterogeneity in risk levels has increased.* Observing the full risk range (and/or the middle 50% box length) of feasible portfolio distributions over time, we see that this range has almost doubled over the last years when compared with the level of the early 90s;

3. *DAX risk profile has changed from moderate to high-risk.* As graphs in Figure 6.7 and Figure 6.6 show, till 1997 the index was "medium risk" when compared with the risk of the other portfolios in the opportunity set. Afterwards, the DAX has moved into the high-risk area.

We look now at these facts in more detail. As the bottom graph of Figure 6.7 showed, the risk level of the DAX strongly increased from 1998 onwards. Thus, looking only at a risk graph of the index, we can observe the tendency towards higher risk. However, the observation on basis of a single index or portfolio does not tell whether the riskiness of the entire portfolio population is rising. Viewing a specific investment universe from the portfolio opportunity perspective provides such evidence. In the particular case of DAX, we see that a general trend of increasing market risk is present (cf. Figure 6.7). Furthermore, the overall increase of the risk level is coincident with the observation that the month-to-month volatility of returns has increased.

Another observation is that the heterogeneity of portfolios with respect to risk has become stronger. Generally, the heterogeneity with respect to a measure (e.g. return or standard deviation) is characterized by the range of the corresponding distribution. In particular, the range of the distribution of standard deviations is determined by the covariances of stocks. Changes

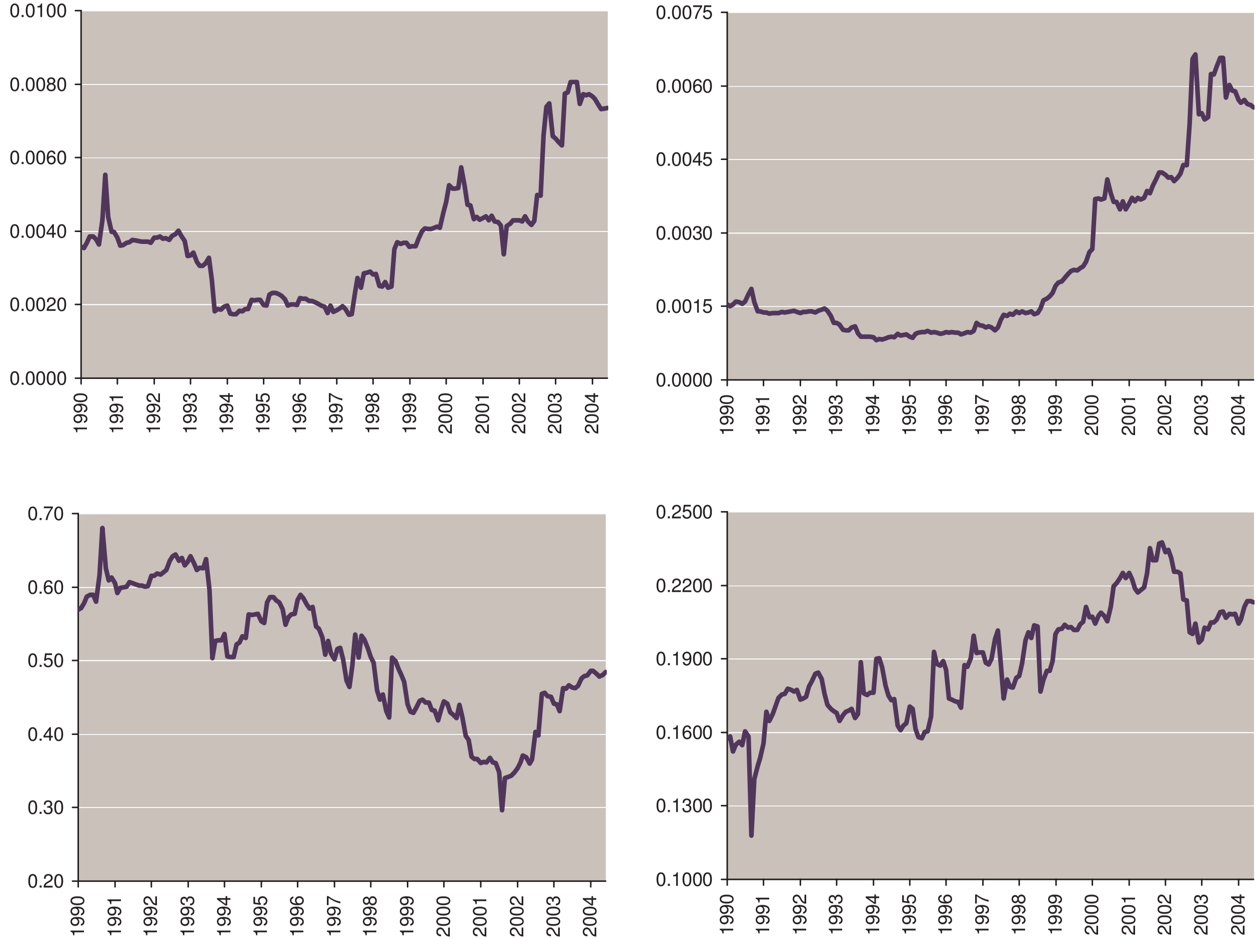

Figure 6.8: Statistics of the covariance matrix structure for the DAX stocks from 1990 through 2004. The top graphs show the averages and standard deviations of the 36-month trailing covariance matrix entries (left and right graph respectively); and the bottom graphs present the averages and standard deviations of the corresponding correlation matrix entries (left and right graph respectively).

in this range over time reflect how changes in the covariance structure of the investment universe (in our case the 30 DAX stocks) influence the potential riskiness of a specific investment (in our case the riskiness of the DAX opportunity set). As Figure 6.7 suggests, the risk heterogeneity has almost doubled over the last years.

To get more insight into the development of the covariance structure, we calculated at the end of each month of the sample period the covariance matrix and the corresponding correlation matrix using the preceding 36 monthly return observations of the DAX stocks. The top two graphs in Figure 6.8 show the trailing averages (on the left) and standard deviations (on the right) of the covariance matrix entries. The increase of the average covariance values coincides with the observation that the common risk level is increased over time (the level of the variance/covariance entries have risen substantially). And the rise in the standard deviation of the covariance terms represents the strengthened heterogeneity of the stock risks.

The two bottom graphs in Figure 6.8 show the corresponding averages and standard deviations of the correlation matrix entries. The left graph showing the average correlations suggests that stocks are more loosely correlated from 1997 to 2004 than in the first half of the sample period. However, we observe some “recovery” in the correlations in the last 2-3 years. The right graph depicting the standard deviations of the correlations shows that the heterogeneity of correlation entries has strongly increased over time. That is, the stocks move more discordantly. Again, Figure 6.8 and Figure 6.7 present this information from different perspectives.

6.6 Combined View on Risk-Return Dynamics

When considering the performance of a portfolio opportunity set, one may look at one performance measure at a time as we did in the two preceding sections. Obviously, one may also wish to consider two (or more) performance measures at a time. The risk-return is an obvious domain to show the frequency distribution for a given portfolio opportunity set. In Figure 6.9, we show the DAX portfolio opportunity set as of December 2003 in terms of the frequency distribution which has the ’average return’-’exponentially-weighted standard deviation’ plane as domain. The right diagram shows the projection of the left diagram on the ’return’-’standard deviation’ space that is often used in the portfolio selection literature (The vertical axis shows average return and the horizontal axis shows standard deviation.) In both diagrams the positions of the DAX index and the equally weighted index are indicated. Figure 6.9 clearly shows that during this particular month, neither the DAX nor the equally-weighted portfolio are doing very well: both of them are dominated by a large proportion of the feasible portfolios. The DAX is doing worse than the equally-weighted portfolio. Actually, there are hardly any portfolios that are dominated by the DAX; on the contrary, the DAX is dominated by most portfolios.

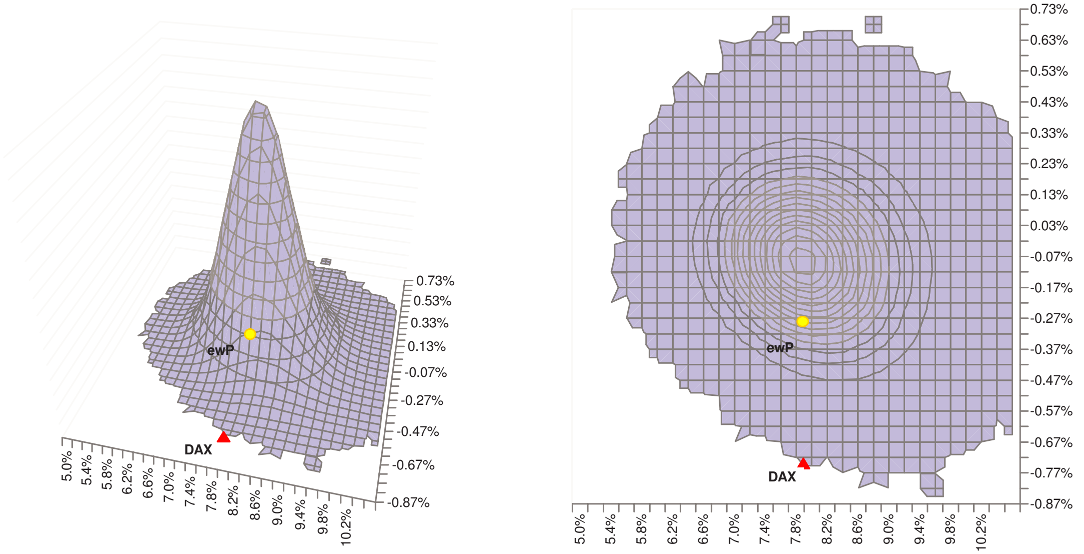

Figure 6.9: Average return-exponentially-weighted standard deviation frequency distribution of portfolios from the DAX portfolio opportunity set as of December 2003. The right diagram shows the projection of the left diagram on the standard return-standard-deviation space that is often used for portfolio selection. In both diagrams the positions of the DAX index and the equally weighted benchmark (ewP) are marked by a triangle and by a dot respectively.

To show the dynamics of the risk-return distributions over time, we show the "boxes" that are defined by the limits of the risk-return diagrams as depicted in the right hand side of Figure 6.9. This is demonstrated in Figure 6.10 and Figure 6.11. In both graphs, the standard deviations of the portfolio returns are estimated exponentially-weighted. In this way, changes in volatility are accounted for in a timely fashion. In Figure 6.10, however, the realized returns are computed over a one-year time span, whereas in Figure 6.11 returns are averaged over a 3-year period (to obtain somewhat smoother averages). The risk-return box plots confirm what the distributions of the portfolio standard deviations already revealed in Figure 6.6 and Figure 6.7: that the range of portfolio risk in the opportunity set has increased over time. Moreover,

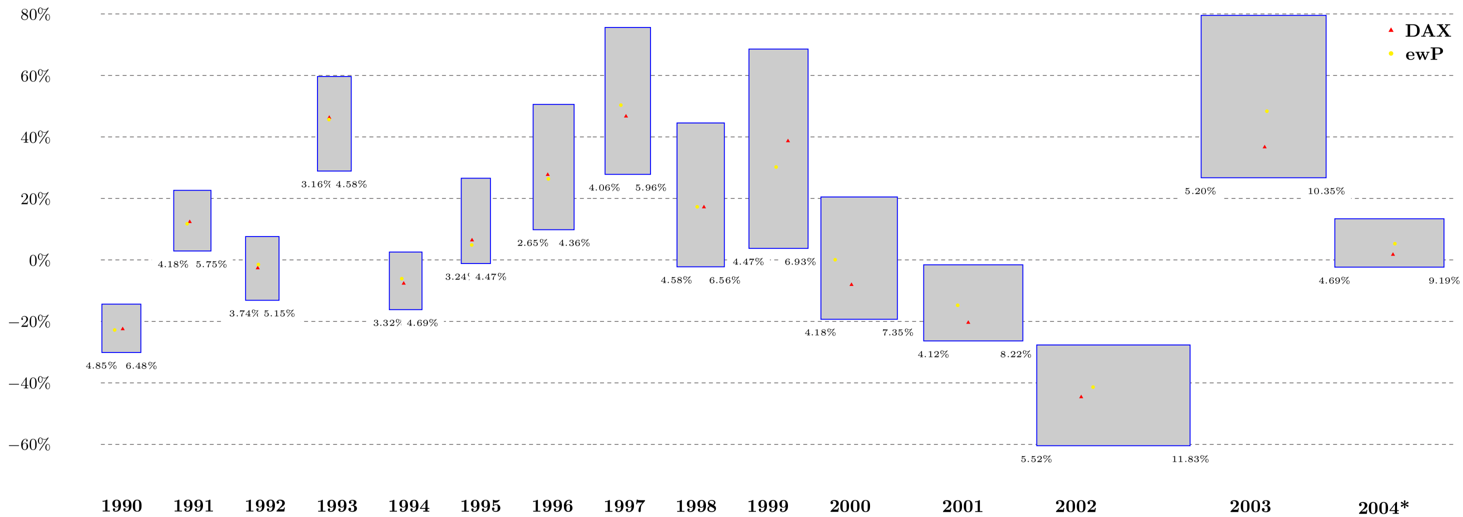

Figure 6.10: Two-dimensional distribution of yearly realized returns (y-axis) and 36-month exp.-weighted standard deviations (x-axis) in feasible portfolios for DAX stocks from January 1990 through June 2004. The position of the DAX index is marked by a small triangle, and the position of the equally-weighted benchmark is marked by a small dot. The numbers below boxes give minimal and maximal values of the standard deviation range for the corresponding period.

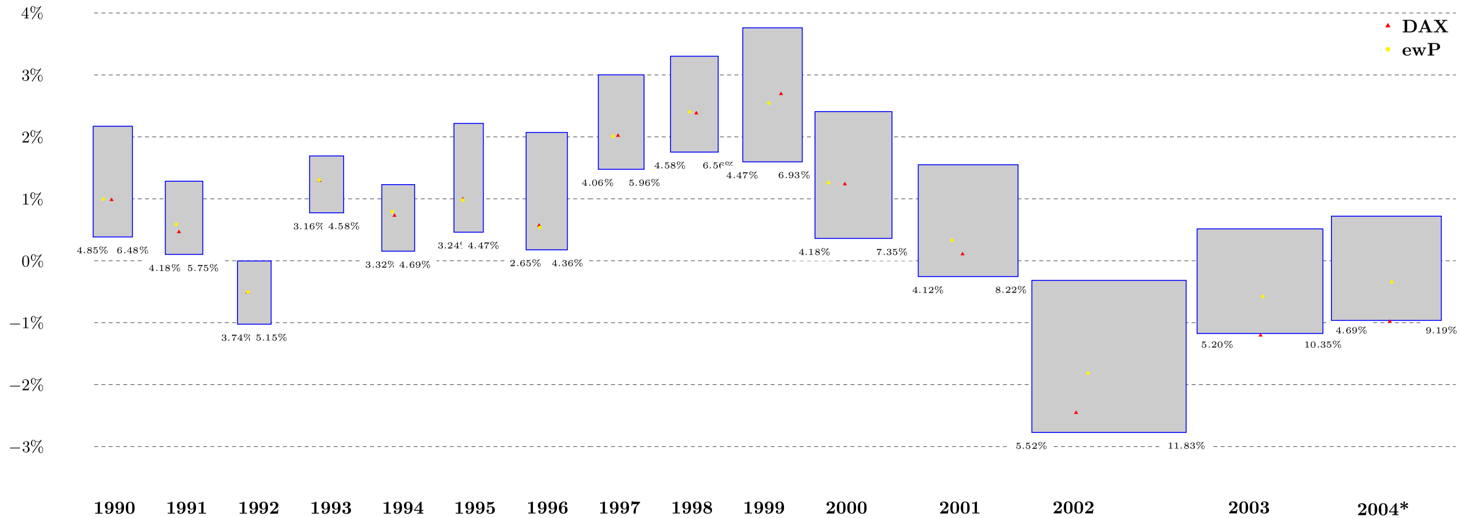

Figure 6.11: Two-dimensional distribution of 36-month average returns (y-axis) and 36-month exp.-weighted standard deviations (x-axis) in feasible portfolios for DAX stocks from January 1990 through June 2004. The position of the DAX index is marked by a small triangle, and the position of the equally-weighted benchmark is marked by a small dot. The numbers below boxes give minimal and maximal values of the standard deviation range for the corresponding period.

they provide additional insights because of adding the average return dimension. In addition to the conclusions concerning the one-dimensional measures as discussed in the two preceding sections, one can conclude that over each of the 14 years considered, both the equally weighted index and the DAX were dominated by many other portfolios.

The observed effects can be explained by an interplay between:

- The returns on individual stocks, dispersed in cross-section;
- The weights of the stocks in a portfolio;
- The correlations between the stock returns.

At this point, we do not want to draw any further conclusions from this empirical phenomenon. However, the portfolio opportunity approach can be a fruitful basis for describing market dynamics and analysis.

6.7 Summary and Conclusions

In this chapter, we introduced and illustrated a novel way of looking at stock market performance. Our approach starts from constructing the relevant portfolio opportunity set. This is the universe of all feasible portfolios that respect prespecified investment constraints. For any portfolio in this set we can compute the desired performance metric: realized return, average return, standard deviation of return, and so on. We can collect this information and represent it by means of a frequency distribution. Comparing these distributions over time (and the changes in their location and form) provides us with information about the market dynamics. We have illustrated this approach for the German stock market and the DAX index.

Studying market dynamics from this portfolio opportunity perspective provides many advantages over the conventional view on the index. We here highlight two: enhanced market description and relative performance evaluation. Firstly, instead of focusing on only one portfolio (the index) it provides a comprehensive perspective on the performance of the *variety* of portfolios that can be formed given some opportunity set and constraints. So we obtain a broad view on opportunities available on a specific market. In addition, we can study the dynamics of the portfolio opportunity set over time. For the German stock market we found that both the level of risk and the heterogeneity of the portfolio opportunities increased over time. We were able to support that observation by investigating the dynamics in the covariance and correlation structures of the stock returns over time.

Secondly, in the conventional view, the quality of market representation by an index is assumed given, regardless of the performance attributes considered. The new methodology, instead, helps to evaluate the market index itself *vis à vis* the portfolio opportunity set. In particular, the location of the market index may be plotted in the frequency distribution of the selected performance measure over the portfolio opportunity set. The quantile in which the index plots indicates how many (feasible) portfolios have outperformed the index in terms of the selected performance measure (realized return, e.g.). In this way it can be judged whether an index is representative for the market under consideration or not. The adhered criterion for representativeness is not the degree of market coverage measured in terms of capitalization (the usual view) but the degree of coverage of the portfolio formation opportunity set. Over the period 1990-2004 studied, we find that the DAX is often outperformed by many portfolios in the portfolio opportunity set in terms of realized returns. The changes over time in the width of the performance metric distributions also provide a means to "norm" the performance of investment portfolios. After all, a wide distribution allows for large differences in performance whereas in a narrow distribution even very small differences in performance are relevant.

Furthermore, apart from enhancing the market description (and extending the description to also include sector information), and apart from further investigating applications to performance evaluation, we note that the methodology may help in discovering promising investment strategies that comply with specific constraints. This would complement the real-time and *ex post* use of the methodology with an *ex ante* use.

6.A Short Description and Methodology of the DAX Index

The DAX is the major index of the German equity market. The objective of the index is to represent the price dynamics of the largest German blue-chip companies that are available worldwide for investing. To achieve broad and fair market representation, the following selection criteria and rules are imposed:

- Companies should be incorporated in Germany, i.e. have Germany as their legal domicile;
- Only shares of companies listed in the Prime Standard segment of the Frankfurt Stock Exchange (FWB) and traded continuously on Xetra® system can be included into the index;

- The index contains 30 stocks that have the largest free-float market capitalization and the highest turnover/liquidity within the last review period of twelve months; [11]
- Shares of companies should have at least 5% of share free float;
- Market capitalization of each stock is limited to 15% of the total index capitalization. (If the free-float capitalization of a company exceeds the limit, then the number of shares is lowered to 15% of the index capitalization on the chaining date.)

The DAX is a free-float capitalization-weighted performance index.[12] It is based on the Laspeyres' index formula and is calculated as follows:

$$\mathrm{DAX}_t = K_{t_1} \cdot \frac{\sum_{i=1}^{30} \left(p_{i(t)} \cdot q_{i(t_1)} \cdot f\!f_{i(t_1)} \cdot c_{i(t)} \right)}{\sum_{i=1}^{30} \left(p_{i(0)} \cdot q_{i(0)} \right)} \cdot 1000$$

where

0	–	December 30, 1987
t_1	–	day of last index chaining
K_{t_1}	–	chain index factor
$c_{i(t)}$	–	actual adjustment factor of stock i
$f\!f_{i(t_1)}$	–	free-float factor (from June 24, 2002)
$p_{i(0)}$	–	price of individual stock i as at December 30, 1987
$q_{i(0)}$	–	number of shares of individual stock i as at December 30, 1987
$p_{i(t)}$	–	actual price of individual stock i
$q_{i(t_1)}$	–	number of shares of individual stock i as at review date

Factors $c_i(\cdot)$ are used to adjust for dividends and equity capital changes between the last and the next chaining days. On the date the Eurex stock-index futures fall due, i.e. the third Friday of the quarter end month, the DAX is calculated for the last time using the actual factors $c_i(\cdot)$. This day is set-up as a new chaining day and the Xetra® closing prices are used for chaining procedure: all $c_i(\cdot)$ are set to 1 and the number of shares of each company q_i is updated. Simultaneously, the index-chaining factor K is adjusted in order

[11] Deutsche Börse uses these two statistics to compose the monthly equity ranking. Applying the 35/35 rule on this ranking as well as using different absolute and relative measures such as frequencies of price determination, sector affiliation, patterns of traded value, statistics of buy-sell spreads, etc. the Management Board of Deutsche Börse decides upon the composition of the DAX.

[12] Deutsche Börse uses free-float instead of full market capitalization from June 24, 2002 on.

to avoid an index breakup. (The factor K is used for adjustment after index composition change as well.)

The base date of the DAX is December 30, 1987. The value of the index was set to 1000 at that date. The index is updated in real time every 15 seconds during market hours. Furthermore, a daily settlement price (based on intraday midday auction prices) and the closing index level (based on last traded prices) are calculated once a day. Based on these data the daily report is published after 6 pm.[13] The index data are available at the German exchange home page (Deutsche Börse, http://www.exchange.de) and in major financial databases such as Reuters (.DAX) and Bloomberg (DAX). The index is used as an underlying for options (ODAX) and future contracts (FDAX) traded at Eurex, warrants traded on the cash market, and a variety of exchange-traded funds (ETF's) and exchange-traded certificates (ETC's) offered on the Xetra®.

The development of the DAX index over the period January 1990 through June 2004 is shown in Figure 6.12.

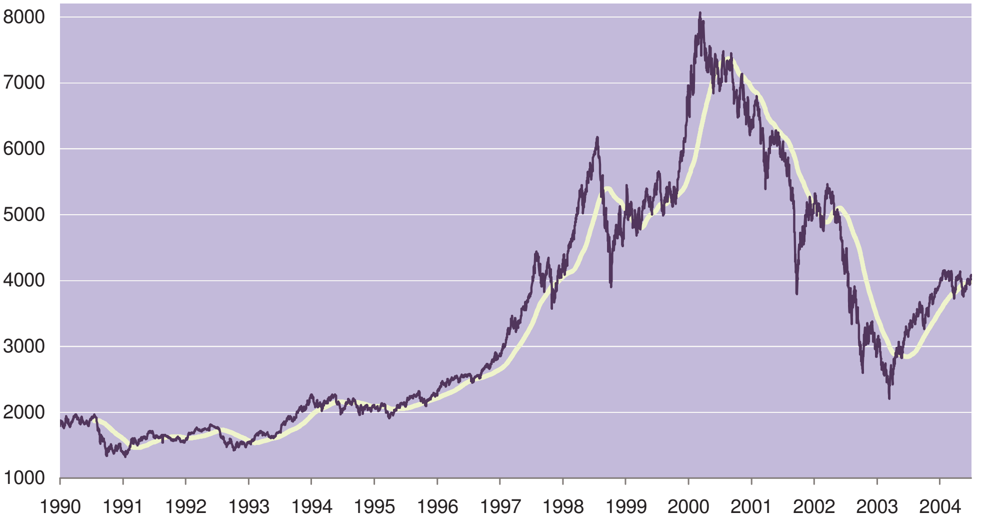

Figure 6.12: The development of the DAX index daily and the 200-day moving average from January 1990 through June 2004.

As of June 30, 2004 the DAX index consisted of the 30 largest German companies in 14 sectors. Of the sectors, Industrial is responsible for about 13.58% of the index market capitalization, followed by Automobile (13.25%), Utilities (12.67%) and Banks (12.05%). The capitalization of each sector is

[13] Additionally, the DAX price index level is calculated once a day after a closing bell.

shown in Figure 6.13. The largest company in the index is Siemens (11.52%), followed by E.On and Deutsche Bank (9.10% and 8.35% respectively). The complete DAX composition as of June 30, 2004 and the historical index changes are summarized in Table 6.2 and Table 6.3 respectively.

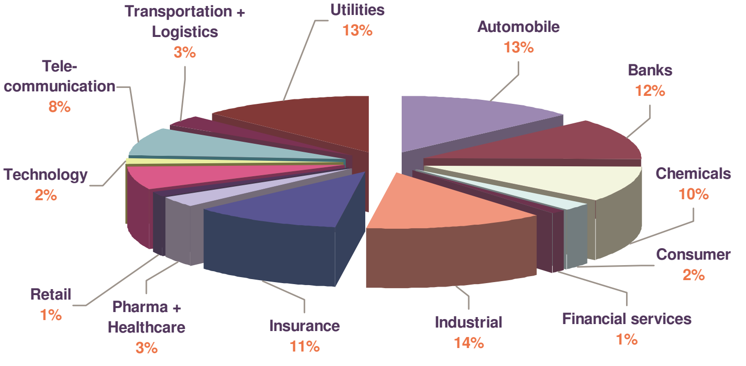

Figure 6.13: Capitalization of each sector in the DAX index as of 30. June 2004.

Company	**ISIN**	**Closing price** €	**Free float**	**Market cap** Bl.€	**Weight**
ADIDAS-SALOMON	DE0005003404	98.13	100.0%	4,460.38	1.04%
ALLIANZ	DE0008404005	89.03	87.8%	30,072.83	7.04%
ALTANA	DE0007600801	49.43	49.9%	3,463.05	0.81%
BASF	DE0005151005	43.99	100.0%	24,342.90	5.70%
BAY.HYPOVEREINSBN.	DE0008022005	14.62	81.6%	8,782.16	2.05%
BAY.MOTOREN WERKE	DE0005190003	36.37	53.4%	12,084.65	2.83%
BAYER	DE0005752000	23.70	94.3%	16,322.48	3.82%
COMMERZBANK	DE0008032004	14.48	81.4%	7,046.78	1.65%
CONTINENTAL	DE0005439004	39.64	100.0%	5,370.76	1.26%
DAIMLERCHRYSLER	DE0007100000	38.40	81.0%	31,502.88	7.37%
DEUTSCHE BANK	DE0005140008	64.58	94.9%	35,686.07	8.35%
DEUTSCHE BÖRSE	DE0005810055	41.77	100.0%	4,670.01	1.09%
DEUTSCHE POST	DE0005552004	17.75	37.4%	7,389.30	1.73%
DEUTSCHE TELEKOM	DE0005557508	14.44	57.0%	34,550.86	8.08%
E.ON	DE0007614406	59.30	94.8%	38,901.75	9.10%
FRESENIUS MED. CARE	DE0005785802	60.99	49.2%	2,100.50	0.49%
HENKEL KGAA	DE0006048432	70.18	100.0%	4,167.82	0.98%
INFINEON TECHOLOG.	DE0006231004	11.03	81.8%	6,742.42	1.58%
LINDE	DE0006483001	45.24	67.7%	3,652.70	0.85%
LUFTHANSA	DE0008232125	11.18	89.9%	4,605.03	1.08%
MAN	DE0005937007	29.98	71.8%	3,033.72	0.71%
METRO	DE0007257503	38.96	44.3%	5,593.90	1.31%
MUNCHENER RÜCKV.	DE0008430026	89.10	80.6%	16,493.35	3.86%
RWE	DE0007037129	38.65	75.4%	15,245.03	3.57%

(continued)

Company	**ISIN**	**Closing price** €	**Free float**	**Market cap** Bl.€	**Weight**
SAP	DE0007164600	136.26	65.4%	28,127.08	6.58%
SCHERING	DE0007172009	48.43	89.4%	8,401.38	1.97%
SIEMENS	DE0007236101	59.11	93.5%	49,236.27	11.52%
THYSSENKRUPP	DE0007500001	14.01	80.0%	5,766.39	1.35%
TUI	DE0006952005	15.70	68.6%	1,922.98	0.45%
VOLKSWAGEN	DE0007664005	34.72	68.8%	7,650.88	1.79%

Table 6.2: The DAX constituting stocks and their weighting in the index as of June 30, 2004. (Source: Deutsche Börse AG)

Date of change	*Announ-ced*	*Companies*	
		deleted	*new*
03.09.90	22.05.90	FELDMÜHLE NOBEL	METALLGESELLSCHAFT
		NIXDOFT	PREUSSAG
15.09.95	18.07.95	DEUTSCHE BABCOCK	SAP
22.07.96	06.01.96	KAUFHOF*	METRO
23.09.96	16.07.96	CONTINENTAL	MÜNCHENER RÜCK
18.11.96	16.07.96	METALLGESELLSCHAFT	DEUTSCHE TELEKOM
19.06.98	26.05.98	BAY.VEREINSBANK*	BAY.HYPOVEREINSBANK
		BAY.HYPO-& WECH.-BANK	ADIDAS-SALOMON
21.12.98	05.11.98	DAIMLER*	DAIMLERCHRYSLER
		THYSSEN*	THYSSEN-KRUPP
01.01.99	22.10.98	*switched from DEM to Euro*	
22.03.99	03.02.99	DEGUSSA*	DEGUSSA-HÜLS
20.09.99	20.07.99	HOECHST	FRESENIUS MED.CARE
14.02.00	10.02.00	MANNESMANN	EPCOS
19.06.00	10.05.00	VEBA	EON
		VIAG	INFINEON
18.12.00	14.11.00	DEGUSSA-HÜLS*	DEGUSSA (Fusion with SKW)
19.03.01	14.02.01	KARSTADT QUELLE	DEUTSCHE POST
23.07.01	26.06.01	DRESDNER BANK	MLP
23.09.02	13.08.02	DEGUSSA	ALTANA
23.12.02	12.11.02	EPCOS	DEUTSCHE BÖRSE
22.09.03	19.08.03	MLP	CONTINENTAL

Table 6.3: Review of the DAX over January 1990 – June 2004. The star sign * marks the merger companies. (Source: Deutsche Börse AG)

For further details about the DAX formula, index guidelines, correction factors, and the actual index composition we refer to Deutsche Börse Group (2004b), (2004c) and to the official site of the Frankfurt Stock Exchange http://www.exchange.de.

Chapter 7

Performance Evaluation Using the Portfolio Opportunity Universes

7.1 Introduction

The objective of this chapter is to provide two applications of our framework and demonstrate how portfolio opportunity sets (we also call them *portfolio opportunity universes*) can be used for the evaluation of professional investment managers. Firstly, we consider how a standard evaluation procedure can be extended through the formation of portfolio opportunity set distributions. In particular, we examine the evaluation of a managed portfolio using the Sharpe ratio. Secondly, we consider a very common case when a manager is restricted by a tracking error constraint. For such managers we discuss how the corresponding portfolio opportunity sets can be calculated. These descriptions of opportunity universes together with the location of the active portfolio and of the benchmark allow to verify whether portfolio managers respect the given tracking error constraint on the run, to test the portfolio efficiency for a given mandate etc. Then, it is also possible to calculate the distribution of information ratios. In particular, we consider how opportunity sets can be used to incorporate information embedded in portfolio opportunity sets into the evaluation process. For that we propose to compute the opportunity set-adjusted information ratios.

Of course, the use of portfolio opportunity sets for performance evaluation is much broader than the particular applications discussed in this chapter. Our main objective is to demonstrate how such universes can extend the existing performance evaluation procedures and metrics. The section devoted to the information ratio demonstrates, for example, how we can introduce an additional step into the standard evaluation procedure. In this step the "raw" realized returns are "normalized" in a special way to the contemporary opportunity sets. The ratio is then calculated on those adjusted returns. This general idea can be used for other performance measures straightforwardly.

Throughout the chapter we work primarily with the realized returns as performance metric. The rationale behind this is the following. A portfolio manager receives a concrete investment mandate. Using his knowledge, experience and available tools, the manager selects a portfolio composition (and possibly a reshuffling strategy), back-tests and finally implements his portfolio. The concrete portfolio allocation (i.e. portfolio weights) is a final product of manager's thoughts, work and skills. Usually, this information stays proprietary and is not disclosed. For that reason, the major part of performance measures uses the realized returns as a basis for performance analysis.

We proceed as follows. Section 7.2 defines how our general framework extends the performance evaluation. We discuss first how portfolio opportunity sets put the performance into perspective for a given and specific managerial environment. Then we list the important aspects revealed by the opportunity sets. In Section 7.3 we consider in detail how one- and multiple-period performance evaluation can be extended via portfolio opportunity sets. We illustrate the formal concepts with distributions of Sharpe ratios using the same investment mandate as in Chapter 6 and considering a specific managed portfolio. Section 7.4 formally defines the portfolio opportunity sets with a tracking error constraint and reveals the methodology for calculating frequency distributions for such opportunity sets. Afterwards, we present several applications of computed opportunity set distributions. In particular, we discuss how frequency distributions of realized returns can be used to monitor portfolio managers and how we can normalize the information ratio for particular market developments. Section 7.5 concludes.

7.2 How relative is the relative performance?

The primary task of a professional investment manager is to provide superior returns relative to a (passive) benchmark or a reference index. Consequently, the professional quality of the manager is judged on the basis of excess returns with respect to the selected benchmark. Furthermore, the persistence of these returns is of special interest. However, the question remains: how relative are these excess returns (or risk-adjusted metrics such as information ratio). That is, we would also like to know how difficult it was to achieve the attained overperformance over a given period. Figure 7.1 provides an illustration of this issue.

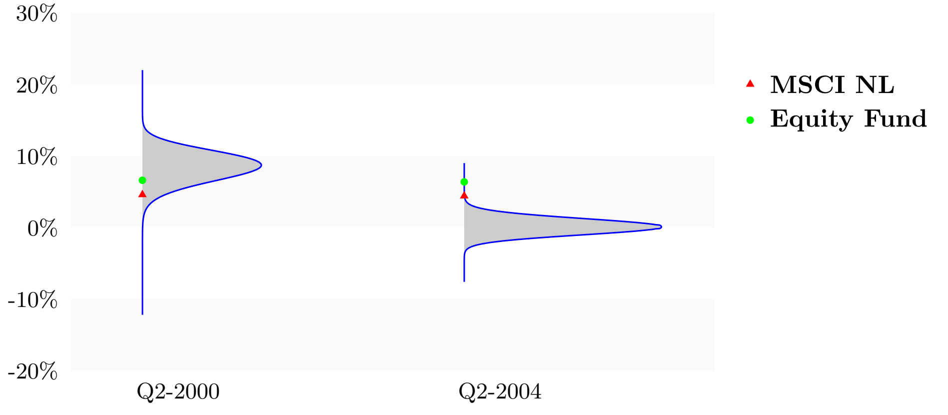

Figure 7.1: The portfolio opportunity sets of quarterly realized returns for an equity fund, the actual return of the equity fund and of the MSCI Netherlands equity index over the same periods. The excess return of the fund relative the benchmark index is of a different nature over these two sample periods. While the overperformance of 2% over the 2nd quarter 2000 seems to be relatively easy achievable (more than 50% of portfolios from the fund opportunity provided the excess of more than 2% over the MSCI index), the superior return over the 2nd quarter 2004 is indeed extraordinary.

When considering only the performance of an equity fund and its benchmark, the MSCI Netherlands index, over the second quarters in 2000 and 2004 (as shown in Figure 7.1), it is not possible to differentiate between these two 2% excess returns. Generally, in order to get more insights into how difficult it was to achieve a certain realized or excess return one would usually look at the peers (i.e. equity funds investing in the Netherlands in this particular case). But then one would face various problems associated with a peer group-based evaluation (cf. Section 2.1).[1]

If we apply our framework, we can calculate the frequency distributions of performance values with respect to various performance measures. The enormous advantage is that using portfolio opportunity universes provides a natural way to put the relative performance for a specific managerial input

[1]Indeed, some large investment companies such as Russell (http://www.russell.com) consider a multi-manager approach as one the pillars of their investment activity. On a continuous basis Russell's analysts try to identify managers who most likely outperform the targeted index and/or peers, and, thus, provide the best excess returns. (In addition, diversifying among the top 25% of managers should improve the persistence of excess returns.)

(i.e. investment mandate) and/or for an investment environment in which the evaluated manager operates in perspective. In Figure 7.1 we plotted the corresponding portfolio opportunity sets for both quarters. As can be observed, the 2% excess returns of our equity fund relative to the MSCI Netherlands index have a different nature and it makes sense to valuate these excess returns differently.[2] Generally, two issues matter:

- The *location* of the evaluated (managed) portfolio and the corresponding benchmark in the opportunity set distribution, calculated with respect to a given performance measure. The two top graphs in Figure 7.2 for the time periods t_1 and t_2 illustrate this issue. The evaluated portfolio and the corresponding benchmark have the same Sharpe ratios over two investment periods t_1 and t_2. Having the same overperformance with respect to the benchmark over these two periods, the portfolio manager clearly provides a better "performance quality" as he/she "beat" most of the alternative portfolios in period t_2;

- The *width* of the frequency distribution for a given opportunity set over the evaluated period, which represents market opportunities with respect to a specific performance metric for a given investment mandate. Considering the two bottom graphs in Figure 7.2 (i.e. for investment periods t_3 and t_4), we definitely would value the excess performance over the benchmark in period t_4 as a more substantial one.

In order to "clean" the performance value(s) of a portfolio from the influence of these two effects (e.g. if we would like to compare performances over two different periods) we can "normalize" the performance(s) by subtracting the cross-sectional mean and then dividing by the cross-sectional standard deviation of the corresponding distribution of performance values.[3] When we consider, for example, the distribution of realized returns for the period t_3 in Figure 7.2, the normalized performance $\bar{\bar{r}}_p$ (i.e. normalized realized t_3 return) of a portfolio p is:

$$\bar{\bar{r}}_p = \frac{r_p - \mu_{XS}}{\sigma_{XS}} \tag{7.1}$$

where μ_{XS} is the mean of the realized return distribution (i.e. mean of the cross-sectional distribution in t_3), and σ_{XS} its standard deviation. We use such a particular "normalization" of realized returns when we consider the

[2]It should be noted that we do not attribute the Q2-2004 return to "skill" or "luck" but emphasize the fact that it was much more difficult to achieve the 2% excess return in that quarter.

[3]Under the term "normalize" we mean to make things (e.g. distribution of performances) of the same type (e.g. all have the same mean and standard deviation).

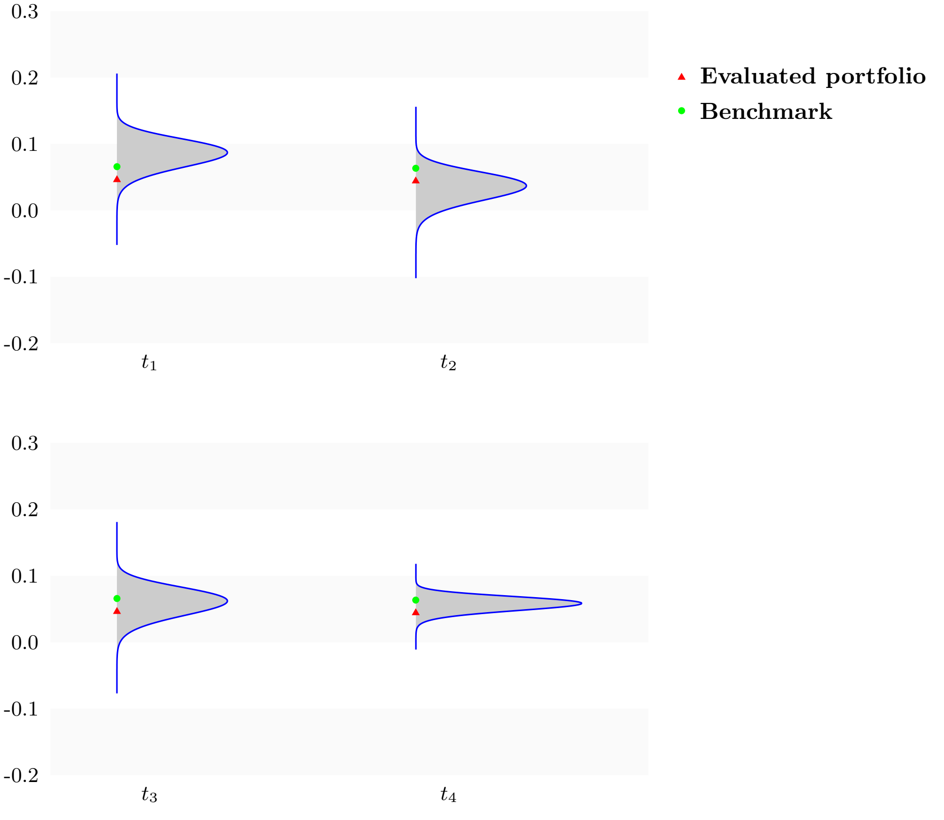

Figure 7.2: The graphs in this figure point to the two important aspects when a performance evaluation is extended through portfolio opportunity set distributions In our case these are distributions of Sharpe ratios for a hypothetical investment mandate. Two top graphs reveal the effect of the location of the evaluated portfolio and the benchmark in the distribution of Sharpe ratios for portfolios from the opportunity set. Two bottom graphs show the effect of the distribution width. Clearly, in both cases we would consider the results over t_2 and t_4 as of a better quality performance results.

information ratio in Section 7.4.

7.3 Evaluation of Investments under Constraints

7.3.1 General Procedure

Our general idea is that instead of limiting ourselves to evaluating the performance of the investment against a benchmark or a peer group, we first estimate the distribution(s) of the performance values (e.g. distribution of realized returns, and/or of variances, and/or of Sharpe and information ratios etc.) of all portfolios from the relevant portfolio opportunity set. We also calculate the actual performance of the evaluated investment portfolio and all benchmarks using the required performance measures. Combining the distributions with the performances of our portfolio and benchmark, we obtain comprehensive performance data which could be analyzed in various ways.

We distinguish between two types of evaluation:

- *One-period performance evaluation.* We consider at place the evaluation over a specific time horizon (e.g. as defined in the investment mandate). In this case the location and the form of the performance distributions as well as the positions of the portfolio and benchmark(s) are of main interest. Both the location and form over such a period yield a picture of the variety of portfolios that can be composed under specific investment requirements, (i.e. all possible decision alternatives available to a manager at the portfolio design time);
- *Multi-period performance evaluation.* Here we consider the development of the dispersion of these distributions over multiple evaluation periods. The dynamics of the distributions of performance values, the portfolio and benchmarks offers, for example, the possibility to check the persistence of the manager or of the strategy selection ability, skills *et cetera.*

Clearly, the performance evaluation using the portfolio opportunity universes is very similar to the ideas revealed in Chapter 6. But instead of describing the market dynamics we compute the frequency distributions of performance values for our investment mandate and "plug-in" the performances of the evaluated portfolio, the underlying benchmark(s), and of the peers (if any) into these distributions. The following sections provide various illustrations of such extended evaluations.

7.3.2 One period Performance Evaluation

Let us consider the use of performance opportunity universes for the usual *ex post* performance evaluation first. Given a performance measure(s) (e.g. realized return, Sharpe ratio etc.), the combination of the distribution of performance values and the performances of the evaluated portfolio and benchmark provides us with the following information, which enhances the usual performance numbers:

- *Minimum and maximum performance values.* These two values determine boundaries for performances, which were achievable under our investment mandate. That is, the values define the "window of alternative performances", which were available at the market;

- *Continuous frequency distribution function* $\mathfrak{fr}(t)$. Consequently, the whole area of feasible portfolio alternatives can be subdivided into performance areas, which can be used for classification of portfolio manager professionalism. One very common division is the following quartiles:

Percentile	*Performance*
75	Excellent
50	Good
25	Below average
0	Poor

- *Percentile values for the evaluated portfolio and benchmark.* We can calculate the percentiles for the evaluated portfolio and benchmark. These percentiles give us the exact proportion of alternative portfolios, which under- and overperformed the portfolio or benchmark;

- *Score of the excess performance over the benchmark.* Using the normalization procedure discussed at the end of Section 7.2, we can compute the excess-performance coefficient (we call this coefficient the *score*) for our portfolio over the corresponding benchmark. Scores have the range approximately between -2 and 2. Thus, the score is a fair measure of relative over- or underperformance, which is invariant to the distribution width.

Of course, this list can be easily extended.

As an example we consider the performance of a managed portfolio investing into the large caps segment of the German equity market. This portfolio has a simple investment mandate restricting the manager to investing into the DAX stocks with a maximum proportion of capital invested in one asset equal to 15% (i.e. $w_i \leq 0.15$). We use the same monthly price observations as in Chapter 6 (cf. Section 6.2.2) and evaluate the performance as of December 31, 2000. As a performance metric we use the Sharpe ratio based on the return observations over the previous 3 years.

The managed portfolio had a Sharpe ratio equal to 0.1738 against 0.1601 of the benchmark (the DAX index). That is, the portfolio provided 17.38 basis points of return over the risk-free rate for each 100 basis points of risk. Furthermore, comparing to the underlying benchmark, the performance of our portfolio was clearly better than of the benchmark because it provided 1.37 basis points of return additionally while having the same level of risk.

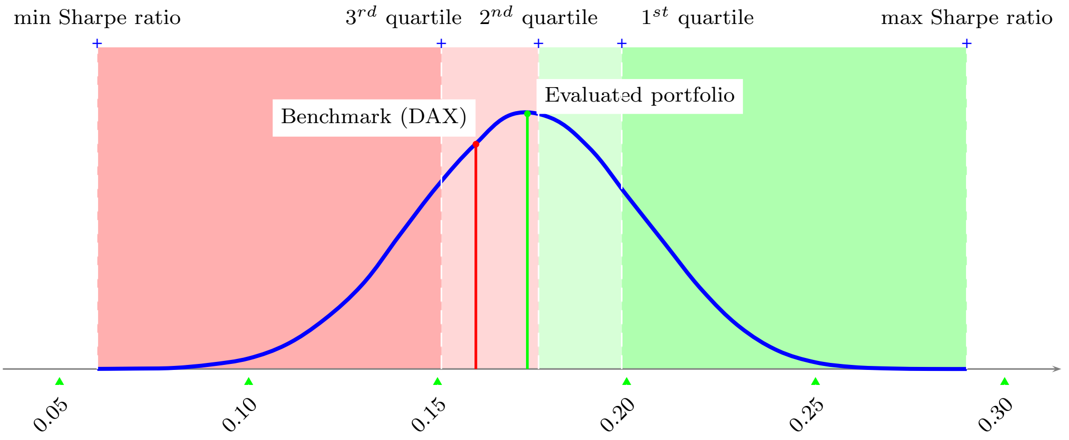

Figure 7.3: The distribution of monthly Sharpe ratios as of December 31, 2000 by feasible portfolios of DAX stocks. The ratios are based on 3-year monthly observations. The dashed white lines mark the upper and lower distribution values as well as the 1st, 2nd and 3rd quartiles. In addition the performances of the benchmark (DAX) and the managed portfolio are plotted. The different colors for quartiles help to attribute the performance grade.

Figure 7.3 shows the distribution of performance values for opportunity set portfolios with respect to the Sharpe ratio as of December 31, 2000. Using this distribution of performance values of all feasible portfolios, we can derive the following performance information:

- *Minimum and maximum performance values.* The maximum Sharpe ratio for our portfolio opportunity set for the evaluated period was equal to 0.2897, and the minimum ratio was 0.0612. These two boundaries define achievable performances (i.e. Sharpe ratios) under our investment mandate. An immediate consequence of these values is the observation that over the evaluated period our investment mandate provided positive premium with respect to the risk-free rate whatever portfolio was composed;

- *Continuous frequency distribution function* $\mathfrak{fr}(t)$. In Figure 7.3 the whole area of feasible portfolio alternatives is subdivided in quartiles. The white lines show the quartile boundaries, the quartile areas are filled with colors: green for the 1st quartile, light green for the 2nd, light red for the 3rd and red for the 4th. Although the performance of our portfolio not only in absolute, but also in relative context (with respect to the DAX) seems to be good, from the perspective of other available alternatives it falls in the "Below average" category only. Thus, a large amount of opportunities (more than 50%) was left unused;

- *Percentile values for the evaluated portfolio and benchmark.* The evaluated portfolio and benchmark outperformed 42.98% and 25.99% of its "mandate peers" correspondingly. Again, these results position the performance values of the portfolio and benchmark inside the realized market environment very precisely;
- *Score of the excess performance over the benchmark.* The difference between the Sharpe ratios for the evaluated portfolio and benchmark is quite large. And the distribution in Figure 7.3 shows how large is it in a relative context. The normalized excess-performance coefficient (score) of our portfolio over the corresponding benchmark is equal to 0.3728. The comparison with the distribution shows that the score reflects the overperformance degree very plausibly.

Obviously, the presented performance data are only a small part of various insights provided by extending the evaluation through portfolio opportunity sets.

Till this point we considered the *ex post* performance evaluation exclusively. However, the portfolio opportunity set is invariant from the perspective, *ex post* or *ex ante*, we are interested in. What changes is the used asset statistics. In the *ex post* case these are the realized ones, in the *ex ante* case these are expected ones. Therefore, from *ex ante* perspective we obtain the distribution(s) of *expected* performance values. Consequently, all statistics discussed above in this section have expected context and the frequency distributions are used to explore which specific characteristics (e.g. return/risk) are expected to be induced by different portfolio compositions and how these compositions are expected to perform.

Clearly, in both *ex ante* and *ex post* cases our methodology allows to analyze the performance on a completely new comprehensive level. The use of portfolio opportunity sets does not replace the existing performance measures but extends them in a very natural way with very interesting insights. It is worth noting that our methodology can be applied to *any* performance measure (the Sharpe ratio is just one of them) or *any multi-dimensional combination* thereof (e.g. differential return with respect to the risk free rate across one dimension and volatility across another).

7.3.3 Multi-period Performance Evaluation

Having several single-period performance distributions at our disposal, we are interested in the development of these distributions over time. Given a performance measure, such a multi-period evaluation can be enhanced via portfolio opportunity sets with the following information:

- *General trend & dispersion of distribution widths.* The development of the portfolio opportunity sets provides information about the general trend and exposure dynamics (e.g. toward return, risk *et cetera*) of the underlying investment mandate over time;
- *Sensible mandate formulation.* We can analyze the influence of various constraints on our portfolio opportunity set and on the exposures (e.g. toward the realized return and/or volatility and/or the Sharpe ratio distributions). Therefore, to clients and institutions, which engage professional managers, the methodology offers a capability to impose investment restrictions more rationally in terms of both the portfolio opportunities and the outcomes. We can also facilitate the estimation of the costs of any constraint which the clients might have formulated;
- *Portfolio performance persistence.* We can consider the performance persistence of the managed portfolio not only with respect to the underlying benchmark, but also with respect to the portfolio opportunity universes (e.g. looking at the relative percentile values) as well as using the normalized version of the relative overperformance: the score coefficient (cf. Section 7.3.2);
- *Benchmark profile.* The quality of a mandate representation through a benchmark is often assumed given, regardless of the performance attributes considered. We can evaluate the benchmark itself *vis á vis* the portfolio opportunity sets. In particular, the percentile indicates how many (feasible) portfolios have outperformed the benchmark in terms of the selected performance measure. In this way it can be judged whether a benchmark is representative for the market under consideration or not.

Obviously, this list of statistics can be extended as the need arises.

For an illustration we consider once again the investment mandate from Section 7.3.2. Figure 7.4 graphically shows 15 distributions of Sharpe ratios from 1990 through 2004 for our sample investment mandate. For each sub-period the corresponding pictogram represents the distribution of Sharpe ratios at the end of a calendar year; the ratios are estimated using the 36

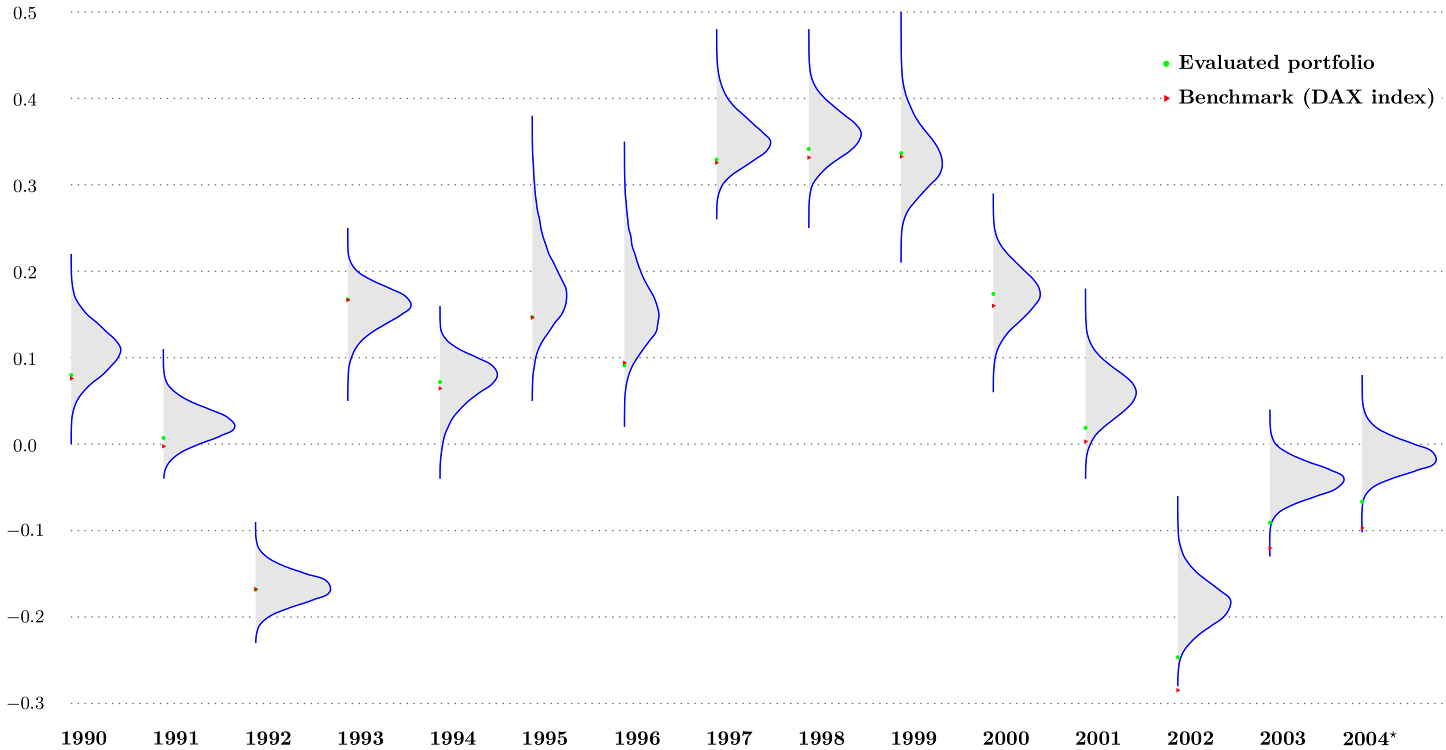

Figure 7.4: The distribution of Sharpe ratios from 1990 through 2004 for our sample portfolio opportunity set. For each sub-period the corresponding pictogram represents the distribution of Sharpe ratios at the end of a calendar year. The ratios are estimated using the 36 monthly observations prior the end of the corresponding year. That is, the 1990 pictogram shows the distribution of Sharpe ratios realized by feasible portfolios from January 1, 1988 through December 31, 1990; the 1991 pictogram represents the distribution of ratios realized by feasible portfolios from January 1, 1989 through December 31, 1991 and so on. The performance of the evaluated portfolio and benchmark (the DAX index) are marked by a thick dot and a small triangle, respectively.

monthly observations prior the year end.[4] That is, the 1990 pictogram shows the distribution of Sharpe ratios by feasible portfolios from January 1, 1988 through December 31, 1990; the 1991 pictogram represents the distribution of returns realized by feasible portfolios from January 1, 1989 through December 31, 1991 and so on. With respect to these particular distributions we can derive the following information for the statistics listed previously:

- *General trend & dispersion of distribution widths.* The major observation is that our investment mandate provided positive return premium most of the years observed. In addition, although the location could change dramatically from year to year (for example, at the end of 2001 our mandate provided positive return-risk ratios for almost all portfolios; at the end of 2002 not one portfolio could manage to do it), the distribution width was not substantially changing over the whole period 1990-2004. Comparing this fact with the observations that the market homogeneity with respect to the returns and risks dramatically increased in the late 90th (cf. Section 6.4 and Section 6.5), we can conclude that the cost of return in terms of risk was stable over time;
- *Sensible mandate formulation.* We have conducted several experiments in order to estimate sensitivities for different kinds of return and risk with respect to various investment constraints. One of the result was, for example, that the exposures to risk-adjusted returns of our mandate can be improved by about 10% diversifying between German and Dutch stocks (in the last case we used the constituents of the MSCI Netherlands index) and enforcing the more even allocation of capital across different sectors;
- *Portfolio performance persistence* and *Benchmark profile.* Considering the performance persistence of the managed portfolio with respect to the benchmark, we can observe that the portfolio (i.e. our manager) outperformed the index in terms of risk-adjusted return by a large amount from the year 2000 on. However, in the same period the portfolio performed worse with respect to the corresponding portfolio opportunity sets (cf. Figure 7.4) than in the period between 1990 and 2000. This is due to the observations that the benchmark stability with respect to the portfolio opportunity sets of returns decreased simultaneously with the change of the benchmark risk profile from moderate to high risk (cf. also with discussion in Sections 6.4-6.6).

[4] Thus, only one-third of observations have been renewed at each year end and we observe the smoothed transition of distributions over time.

Again, in this section we considered how the *ex post* multi-period performance evaluation can be enhanced via portfolio opportunity sets. From the *ex ante* perspective we would have considered the distribution(s) of *expected* performance values from some multiple scenarios.

7.4 Evaluation of Tracking Error-Restricted Investments

With strongly increased interest for investing into ETFs and other index-linked investment products we would also be interested in the evaluation of such investment mandates and investments. The next section provides valuable insights into evaluation of such kind of investments.

7.4.1 Opportunity Sets with Tracking Error Constraints

Delegating an investment implementation to a professional manager, investors very often "anchor" their mandate to a specific benchmark or an index, and control the manager's exposure to risk via the tracking error volatility constraint (TEV). The case is widely discussed and analyzed in financial literature. Few references are Roll (1992), Grinold & Kahn (2000), Jorion (2003).

The portfolio opportunity set, which consists of n assets and which is subject to such a TEV-constraint, can be specified through the following formal description:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{cov}(i,j)\cdot(w_i-b_i)\cdot(w_j-b_j)\le\psi^2 \tag{7.2}$$

$$\sum_{i=1}^{n}(w_i-b_i)=0$$

where ψ determines a tolerable level of TEV, b_i is the proportion of the security i in the targeted benchmark, and w_i is the proportion of the security i in a feasible portfolio.

It should be noted that imposing a single TEV constraint is not efficient because it forces managers to optimize in the excess-return space only and hence does not account for the total risk of a tracking portfolio (Roll 1992). However, this can be corrected through imposing an additional constraint on total portfolio risk especially when the TEV value is low or a targeted benchmark is relatively inefficient (Jorion 2003). Another often used technique is to constraint the beta of the portfolio to be equal to 1 with respect to the benchmark. Alexander & Baptista (2004) also discuss how imposing an additional VaR constraint can mitigate the problem pointed by Roll (1992). Although such additional constraints are undoubtedly important, they can be incorporated in the estimation procedure easily and have no impact on the general ideas described in this chapter. Therefore, for the sake of simplicity, we consider the opportunity sets as formally described by (7.2).

It is more convenient to work with such opportunity sets if we remove the last dependent weight from the system (7.2). Reorganizing the second budget equation from (7.2), we have:

$$w_n - b_n = -\sum_{i=1}^{n-1}(w_i - b_i) \tag{7.3}$$

Substituting (7.3) into the first equation of (7.2), we obtain:

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[\mathrm{cov}(i,j) - 2\,\mathrm{cov}(i,n) + \mathrm{cov}(n,n)] \cdot (w_i - b_i) \cdot (w_j - b_j) \leq \psi^2 \tag{7.4}$$

Denoting the expression in the square brackets by $\widehat{\mathrm{cov}}(i,j)$, we obtain the expression for a full-dimensional portfolio opportunity set in $\mathbb{R}^{n-1}$:

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\widehat{\mathrm{cov}}(i,j) \cdot (w_i - b_i) \cdot (w_j - b_j) \leq \psi^2 \tag{7.5}$$

Considering the issue of generating portfolios for TEV-constrained opportunity sets, we use the short form (7.5) explicitly and leave out the preprocessing step.

7.4.2 Estimating frequency distributions of TEV-constrained opportunity sets

We can easily adopt the Sequential Direction algorithm from section 5.4.2 to generate random portfolios uniformly distributed over a TEV-constrained opportunity set $\hat{P}$, where $\hat{P}$ is defined by (7.5). Having n assets in the opportunity set and transforming the set into (7.5) form, the estimation procedure is formulated as follows:

1. Choose uniformly a feasible random portfolio $\mathbf{p}_0 = \langle w_1^{(0)}, w_2^{(0)}, \ldots, w_{n-1}^{(0)}\rangle$ in a given portfolio opportunity set $\hat{P}$ (The simplest solution is to sample uniformly in the neighborhood of the benchmark);

2. Having the last generated portfolio $\mathbf{p}_t = \langle w_1^{(t)}, w_2^{(t)}, \ldots, w_{n-1}^{(t)} \rangle$ (the subscript $t = 0, 1, 2, \ldots$ denotes the actual step of our ergodic Markov chain), determine the "changing" weight $w_j^{(t)}$ as j-th element of $\mathbf{p}_t$ and j is taken consequently from the set $\{1, 2, \ldots, n-1\}$;

3. Substituting all $w_i^{(t)}$ but $w_j^{(t)}$ into the left-hand side of (7.5) and replacing the inequality sigh with equality, we obtain a quadratic equation of only one variable $w_j^{(t)}$. Solving the equation, we obtain two boundary values for $w_j^{(t)}$, $\dot{w}_j^{(t)}$ and $\ddot{w}_j^{(t)}$. Compute the next value of $w_j^{(t+1)}$ by choosing a point on the segment $\left[\dot{w}_j^{(t)}, \ddot{w}_j^{(t)}\right]$ uniformly, i.e.

$$w_j^{(t+1)} \sim \mathcal{U}(\dot{w}_j^{(t)}, \ddot{w}_j^{(t)})$$

4. Create a new random portfolio extending $\mathbf{p}_{t+1} = \langle w_1^{(t+1)}, w_2^{(t+1)}, \ldots, w_{n-1}^{(t+1)} \rangle$ with $w_i^{(t+1)} = w_i^{(t)}$ for all elements but $w_j^{(t+1)}$, a new $w_j^{(t+1)}$ from step 3, and the last depending weight as

$$\langle w_1 = w_1^{(t+1)}, w_2 = w_2^{(t+1)}, \ldots, w_{n-1} = w_{n-1}^{(t+1)}, w_n = 1 - \sum_{i=1}^{n-1} w_i^{(t+1)} \rangle$$

Add this new random portfolio to the sample;

5. If the sample is big enough, then stop. Otherwise return to step 2.

Having n assets in the opportunity set, we need to generate at least $\mathcal{O}\left(n^3\right)$ random portfolios in order to guarantee the uniformity of the sample over a given TEV-constrained opportunity set. The accuracy of a frequency density estimation is the same as for linearly-restricted opportunity sets (cf. Section 5.4.3).

It should be noted that the algorithm is slower than the direct transformation techniques used for opportunity sets with a short-sales constraint only. However, the portfolios can be generated rather efficiently, and we can use large samples (e.g. 1 million portfolios) estimating frequency distributions for TEV-restricted sets. Furthermore, the algorithm can be easily extended to incorporate additional risk constraints as discussed by Jorion (2003) and Alexander & Baptista (2004). In that case, we introduce an additional acceptance-rejection step, which filters out portfolios violating at least one of these supplementary constraints.

7.4.3 Evaluation of TEV-constrained Investments

Obviously, the TEV-constrained opportunity sets can be evaluated in the same way and using the same metrics as discussed in Section 7.3. In this section we focus on additional ways how portfolio opportunity sets can be used for the evaluation of the TEV-constrained investments. In particular, we consider the evaluation using the information ratio (IR). Grinold & Kahn (2000) argue that information ratio is a key statistic measuring performance of an active portfolio manager because "information ratios determine value added". Also in practice, the IR is the most frequently used statistic to measure manager's skills and performance for index- or benchmark-linked investments.[5]

Descriptive Statistics & Evaluation

The TEV-constrained opportunity sets can be evaluated using the absolute or relative context. In the absolute context we evaluate the performance using the realized or projected returns, standard deviations *et cetera*. Thus, the evaluation is done in the same way as we discussed in Section 7.3 with the only exception that we consider portfolios which satisfy a prescribed TEV-constraint. And frequency distributions would be estimated using the procedure in Section 7.4.2.

In the relative context we are interested in the differential returns and/or other metrics based on these differential returns with respect to a given benchmark. Again, frequency distributions of such metrics can be numerically estimated as described in Section 7.4.2.[6] Let us consider frequency distributions of information ratios (IR) for a given TEV-constrained active investment mandate, in particular. Grinold & Kahn (2000) propose a typical distribution of IR for professional managers: According to the authors' criteria, top 25% of managers should provide an IR greater than 0.5. However, the empirical study of Goodwin (1998) on 10-year data shows that "a manager's information ratio should be judged relative to the manager's style universe".[7] Another result of the study is that information ratios are sensitive to the choice of the underlying benchmarks.

Considering the frequency distributions of *ex ante* (e.g. historical) IRs allows us to quantify our expectation toward the possible IR and, thus, toward the possible value-added of active management for a particular investment mandate. For example, having the top quartile of IR-distribution equal to

[5] Formally this is only true when differential return is uncorrelated with return on the underlying benchmark (Treynor & Black 1973). To conform with practice and without loss of generality we here use the simple differential return.

[6] It should be noted that when we consider *ex ante* case the underlying benchmark is a median of the distribution of standard deviations/variances. This is not necessarily true for frequency distributions of other performance metrics.

[7] Goodwin (1998), p.40.

0.2, we could expect that a top manager will have about 20 basis points of overperformance for 100 basis points of active risk (tracking error volatility).

The frequency distribution of the *ex post* IR for a given investment mandate, allows us to evaluate the professional skills of the hired manager realistically. Opportunity set distributions help us judge the professional quality of managers *relative to their universe, investment mandate and the underlying benchmark*.

Monitoring/Controlling Portfolio Managers

The TEV-restricted opportunity sets can be also used to monitor whether portfolio managers respect a given tracking error constraint on the run.

The idea is very simple. Let us consider a TEV-restricted portfolio opportunity set consisting of n assets. At time t when portfolio is composed or adjusted by the portfolio manager, we have knowledge about the benchmark weights and the co-variances of individual securities (the last ones are estimated in the same way as our manager does, e.g. using historical data). From (7.2) we can derive the information about portfolios, which satisfy the TEV-constraint (The simplest method is to form a sample of feasible portfolios as described in Section 7.4.2).

As the next evaluation point $t+1$ arrives, we observe, for example, the *ex post* returns for each of the assets over the period from t to $t+1$. Hence we have the information which portfolio compositions were feasible with respect to the TEV-constraint at time t. Next, we can compute the distribution of performance values (realized returns in our example case) at time $t+1$ combining the returns with the portfolio compositions. And confronting this distribution with the realized return provided by the manager, we can verify whether the constraint was respected at time t if return falls inside the distribution. Clearly, if the return is outside of the distribution range, the constraint was violated. The opposite statement, if return falls inside the distribution then the manager respects the constraint, is not always true. The similar verification can be performed using other metrics.

Opportunity Set-Normalized Information Ratio

Given a portfolio opportunity set, the width of the distribution of realized returns varies substantially over time (cf. Chapter 6). The same applies to the frequency distributions of differential returns. Consequently, as differential returns are conditional on contemporary market dynamics, so is the IR. In some circumstances we would wish to consider an IR which is unconditional of market dynamics.

In Section 7.2 we introduced normalized performances in which values were normalized by subtracting the cross-sectional mean and dividing by the cross-sectional standard deviation of the corresponding distribution of performance values (see equation (7.1)). Utilizing this idea, we can introduce an additional step into the IR calculation procedure in order to compute an unconditional IR.

Let us consider how such unconditional IR (henceforth we denote such information ratios by $\overline{\overline{\mathrm{IR}}}$) can be calculated given realized returns $r_{p,t}$ of an active portfolio P and returns $r_{b,t}$ of the underlying benchmark b over some historical periods $t = 1, 2, \ldots, T$. Calculating the distributions of realized returns for each period t, we can calculate the normalized differential return between the active portfolio and benchmark as

$$\overline{\overline{r}}_t^d = \overline{\overline{r}}_{p,t} - \overline{\overline{r}}_{b,t} = \frac{r_{p,t} - \mu_{XS,t}}{\sigma_{XS,t}} - \frac{r_{b,t} - \mu_{XS,t}}{\sigma_{XS,t}} = \frac{r_{p,t} - r_{b,t}}{\sigma_{XS,t}} \tag{7.6}$$

where $\mu_{XS,t}$ is the mean of the realized return distribution (i.e. mean of the cross-sectional distribution for a time period t), and $\sigma_{XS,t}$ its standard deviation. Afterwards we calculate the $\overline{\overline{\mathrm{IR}}}$ either by dividing the average of these normalized differential returns by their standard deviation, i.e.

$$\mathsf{E}\left[\overline{\overline{r}}_t^d\right] = \frac{1}{T}\sum_{t=1}^{T} \overline{\overline{r}}_t^d$$

$$\mathsf{D}\left[\overline{\overline{r}}_t^d\right] = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(\overline{\overline{r}}_t^d - \mathsf{E}\left[\overline{\overline{r}}_t^d\right]\right)^2}$$

and

$$\overline{\overline{\mathrm{IR}}} = \frac{\mathsf{E}\left[\overline{\overline{r}}_t^d\right]}{\mathsf{D}\left[\overline{\overline{r}}_t^d\right]} \tag{7.7}$$

or using the least-squares regression (Treynor & Black 1973):

$$\overline{\overline{r}}_{p,t}^d = \alpha + \beta \cdot \overline{\overline{r}}_{b,t}^d + \varepsilon_t$$

and

$$\overline{\overline{\mathrm{IR}}} = \frac{\alpha}{\mathsf{D}\left[\varepsilon_t\right]} \tag{7.8}$$

When $\sigma_{XS,t} = \sigma_{XS}$ for all t, the normalized version of IR reduces to conventional IR. The main advantage of $\overline{\overline{\mathrm{IR}}}$ is that it measures normalized differential returns and, thus, is independent of a specific market conditions and investment mandate. It should be also noted that the unconditional information ratio is very similar to the t-statistic.

7.5 Summary and Conclusions

Using portfolio opportunity universes provides a natural way to put in perspective the absolute and relative performance for a specific decision context (e.g. investment mandate) and/or for an investment environment in which the evaluated manager operates. Considering a frequency distribution for a given performance measure, our methodology highlights two issues: where the evaluated (managed) portfolio and the corresponding benchmark are located in the distribution calculated with respect to a given performance measure, and what is the width of this frequency distribution, which represents market opportunities with respect to a specific performance metric for a given investment mandate.

Evaluating performance over one period, the portfolio opportunity sets provide us with the following statistics: minimum and maximum performance values, which define the "window of alternative performances", continuous frequency distribution function $\mathfrak{fr}(t)$, which can be used for classification of portfolio manager professionalism by subdividing the possible performances into areas as well as for calculating the percentile. The function can also be used for computing a score coefficient of the excess performance over the benchmark, which is invariant to the market development. The metric list can be extended.

Evaluating performance over multiple periods, the portfolio opportunity sets provide us with information about general trend and dispersion of distribution widths over time, help to analyze the influence of various constraints on the exposure of portfolio opportunity set toward performance metrics. In addition, we can evaluate the persistence of a portfolio performance not only with respect to the underlying benchmark, but also with respect to the corresponding portfolio opportunity universes (e.g. looking at the relative percentile values) as well as using the normalized version of the relative overperformance, the score coefficient.

Our methodology can be easily adapted to evaluate different types of investments. When we consider TEV-constrained investments, we can evaluate such investments in various ways using the portfolio opportunity sets. In particular, our methodology provides very interesting insights for evaluation using the information ratio. The distribution(s) can help us to quantify the expectation toward the possible IR and, thus, toward the possible value-added of active investment for a given mandate. Opportunity set distributions can also help us to judge the professional quality of managers relative to their IR-universe. In addition, our methodology can be deployed to monitor whether portfolio managers respect a given tracking error constraint on the run. Using the portfolio opportunity sets, we can also calculate a normalized version of information ratio, which is invariant to the market developments.

Chapter 8

Conclusions and Directions for Research

Conventional evaluation methods strongly differentiate between the universe which is used for portfolio construction, and the universe which is used for the performance evaluation. Whilst by composing the portfolio we consider the complete opportunity set, in the latter case we use a very restricted, general representation of this opportunity set: a peer group or (a) benchmark portfolio(s). It is well recognized that such a reduction of the original opportunity set causes several problems. From our point of view, the solution is to incapsulate the investment decision context into the performance evaluation.

In this thesis we present a conceptual framework, which allows to incorporate the decision context of any constrained investment into the performance evaluation process. The main feature that distinguishes our methodology from conventional performance evaluation methods is that it tackles the performance at the decision-making level: the portfolio weights. We consider all possible portfolios that can be constructed given the specific investment objective(s) as well as the prescribed investment constraints, and then evaluate all these portfolios according to (a) selected performance measure(s). The performance of the investment portfolio is calculated simultaneously and then evaluated against the performance of this complete opportunity set. Consequently, our methodology extends the conventional performance metrics with the insights into opportunities existed for a particular investment.

Our framework is not limited to specific performance measures. On the contrary: it is suitable for almost any performance measure or a combination thereof. Moreover, we leave the choice of the relevant performance attributes to the evaluator who may choose one or more performance measures depending on the performance question(s) to be answered.

In the thesis we considered in detail two areas where the portfolio opportunity perspective (as delivered by our framework) provides many advantages over the conventional methods: enhanced market descriptions (as discussed in Chapter 6), and performance evaluation (as discussed in Chapter 7).

In the particular case of enhanced market descriptions we observed the following valuable benefits.

Firstly, instead of focusing on only one portfolio, an index, our view provides a comprehensive perspective on the performance of the variety of portfolios that can be formed given a specific opportunity set and constraints. Thus we obtain a broad view on opportunities available on a specific market. We can also study the dynamics of the portfolio opportunity set over time. Observed statistics are locations of the distributions, trend, homogeneity of performance values *et cetera*.

Secondly, a very important feature is that our market descriptions can be commensurate with any specific investment environment such as an investment mandate defined by goals and constraints.

Thirdly, with regard to the performance attributes considered, the methodology helps to evaluate the market index itself *vis á vis* the portfolio opportunity set. Here key statistics are the quantile, in which the index plots, as well as the stability of this quantile over time.

In the area of evaluating the performance of constrained investments, using portfolio opportunity sets provides a natural way to put the absolute and relative performance in the perspective of a specific decision context (i.e. investment mandate, market and business environment) in which the evaluated manager operates.

Evaluating performance over one period, the portfolio opportunity sets approach provides us with the following statistics: minimum and maximum performance values, which define the "window of alternative performances", continuous frequency distribution function, which can be used for classification of portfolio manager professionalism by subdividing the possible performances into areas as well as for calculating the percentile. The function can also be used for computing a score coefficient of the excess performance over the benchmark, which is invariant to the market development.

Evaluating performance over multiple periods, the portfolio opportunity sets approach provides the information about general trend and dispersion of distribution widths over time, helps to analyze the influence of various constraints on the exposure of portfolio opportunity set toward performance metrics. In addition, we can evaluate the persistence of a portfolio performance not only with respect to the underlying benchmark, but also with respect to the corresponding portfolio opportunity sets (e.g. looking at the relative percentile values) as well as using the normalized version of the relative overperformance, the score coefficient.

The modelling techniques and mathematical tools developed throughout this thesis are very powerful and general; they can be broadly applied for thorough analysis of constrained investments far beyond the area of performance evaluation. Especially we would like to draw attention to the two following aspects.

Firstly, we have shown in this study that institutional, legal and self-imposed investment constraints can be translated into constraints on asset weights. Using this approach we formalize the description of investment decision contexts and represent them as polyhedra in the asset weight space. This allows us to use the tools of geometry and linear algebra in our analysis. In particular, we derived an explicit formula for the frequency distributions of performance values; these values turn out to be equal to the cross-sectional volume of the opportunity set polyhedra. Further analysis of various opportunity sets by their geometrical representations opens wide horizons.

Secondly, in this study we developed and used mathematical tools for working with the continuous asset weight spaces (i.e. continuous portfolio spaces). Obviously, the portfolios and benchmarks we are dealing with on financial markets are discrete. However, it can be very beneficial to consider the space of continuous asset weights, especially in the case when an investment or a portfolio is restricted by some characteristics (e.g. prescribed asset allocation, exposure to risk and return). The continuation is the standard approach in most of the mathematical studies, and, hence, many mathematical tools were elaborated over time for analyzing continuous objects.

Finally, we would like to point out some directions for future research. One natural application of our framework and methodology is the investigation of investment constraints. In this thesis we assumed that any constraint should be satisfied once imposed. That is, we did not differentiate between hard, soft and game-type restrictions. However, beyond the hard constraints, other restrictions often can be relaxed or even removed. But what is the price of imposing or tightening or relaxing a specific constraint? And given specific performance attributes, what is the effect of such an action on investment performance? These issues have become the subject of increased interest of several authors over the last few years. And our framework is an excellent tool for the investigation of the influence of constraints in a more systematic way.

Furthermore, the methodology can be used for comparison and/or discovering promising investment strategies that have to comply with specific constraints.

Last but not the least, the portfolio opportunity sets represent a new tool. And how correctly noticed by one of my colleagues “if you have new spectacles, you can recognize new details but the question remains where to look...”. The portfolio opportunity sets have been developed for the performance evaluation of professional managers acting on behalf of clients. However, over time other applications such as market descriptions and the comparison of strategies emerged. Surely, the above list of possible applications of our methodology is far from being final.

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Samenvatting (Summary in Dutch)

Binnen conventionele performance-evaluatiemethodieken bestaat een discrepantie tussen enerzijds het universum van financiële instrumenten waaruit portefeuilles worden samengesteld, en anderzijds het universum van beleggingsmogelijkheden dat ten grondslag ligt aan de performancemeting. Bij het samenstellen van een portefeuille wordt immers het gehele keuzeveld van individuele beleggingsmogelijkheden betracht, terwijl bij de performancemeting wordt uitgegaan van slechts een algemene representatie van dit keuzeveld in de vorm van hetzij een "peer group" dan wel één of verscheidene benchmark portefeuilles. Dit representatie-probleem kan worden opgelost door het proces van performance-evaluatie te beschouwen binnen de specifieke context waarin de beleggingsbeslissing werd genomen. Dit is het onderwerp van deze dissertatie.

Een algemeen raamwerk voor de performance-evaluatie van gerestricteerde portefeuilles (Hoofdstuk 2)

In deze studie presenteren we een raamwerk waarbij binnen het proces van performance-evaluatie rekening wordt gehouden met doelen en restricties zoals die zijn gesteld aan de voorafgaande beleggingsbeslissing. De onderliggende kerngedachte van het raamwerk is om bij de performance-evaluatie van een beleggingsportefeuille niet slechts enkele maar *alle* mogelijke alternatieve portefeuilles te betrachten, waarbij deze alternatieve portefeuilles op een zodanige wijze zijn samengesteld dat zij alle voldoen aan de geldende beleggingsdoelstellingen en opgelegde beleggingsrestricties. Deze verzameling van (mogelijke) alternatieve portefeuilles noemen we de "(feasible) portfolio opportunity set". Vervolgens evalueren we al deze alternatieve portefeuilles op basis van de geselecteerde performancemaatstaf (of -staven). De performance van de te evalueren portefeuille kan dan worden afgezet tegen de performance van de verzameling van alternatieve portefeuilles.

Het beschouwen van de verzameling van portefeuilles (in plaats van het beschouwen van een conventionele peer group of een benchmark-portefeuille) impliceert een verrijking van het proces van performance-evaluatie. De voordelen van de voorgestelde benadering blijken niet alleen bij het proces van performance-evaluatie, maar ook bij het beschrijven van financiële markten en het evalueren van marktindices, en bij het evalueren van portefeuillestrategieen. Dit wordt nader uiteengezet in het derde deel van deze dissertatie.

Om ons raamwerk te operationaliseren moet een tweetal kernvragen worden beantwoord: (1) hoe kan de context van de beleggingsbeslissing formeel worden gepreciseerd om daaruit de *portfolio opportunity set* te destilleren, en (2) gegeven de formele beschrijving van de *portfolio opportunity set* en gegeven de keuze van een performancemaatstaf, hoe kan de frequentieverdeling van de performancemaatstaf over de *portfolio opportunity set* worden bepaald. Deze vragen worden beantwoord in de hoofdstukken 3, 4 en 5.

Portfolio Opportunity Sets (Hoofdstuk 3)

We onderscheiden drie typen van randvoorwaarden waaraan de beleggingsbeslissing kan zijn onderworpen: (1) randvoorwaarden van het harde type, die van buitenaf aan de belegger worden opgelegd en waaraan niet kan worden getornd; (2) randvoorwaarden van het zachte type, die de belegger zichzelf oplegt en die tot doel hebben om het proces en de uitkomst van de besluitvorming te verbeteren, en tenslotte (3) randvoorwaarden van het spel-type, die de belegger zichzelf oplegt om zodoende rekening te houden met de beslissingen en het gedrag van andere spelers in het beleggingsveld.

In zijn algemeenheid kunnen beleggingsrestricties worden onderverdeeld in twee klassen: screening-restricties en selectie-restricties. Beleggingsrestricties met betrekking tot screening kunnen positief dan wel negatief worden geformuleerd. Negatieve screening-restricties beogen beleggingsmogelijkheden uit te sluiten met onwenselijke kenmerken of eigenschappen; positieve screeningrestricties zijn er juist op gericht om wenselijke beleggingsmogelijkheden te filteren uit het keuzeveld. Selectie-restricties stellen randvoorwaarden aan het bedrag of proportie dat mag worden belegd in een beleggingsobject. De meest voorkomende selectie-restricties zijn restricties gesteld aan short-sales, aan minimale en/of maximale beleggingsproporties (portefeuillegewichten) die gelden voor individuele beleggingsobjecten of groepen van beleggingsobjecten, en randvoorwaarden gesteld aan het risicoprofiel.

We vertalen alle beleggingsrestricties naar restricties op portefeuillegewichten. We verkrijgen dan een stelsel van gelijkheden en ongelijkheden die tezamen formeel een *feasible portfolio opportunity set* definiëren. In deze studie onderscheiden we drie standaard *portfolio opportunity sets*:

- *opportunity sets* onder slechts een short-sales restrictie;
- *opportunity sets* onder restricties op de gewichten van individuele beleggingsobjecten alsmede lineaire combinaties daarvan;
- *opportunity sets* onder niet-lineaire restricties.

Portfolio opportunity sets van het eerste type zijn meetkundig gezien basis simplexen en kunnen als zodanig gemakkelijk worden beschreven en geanalyseerd. *Portfolio opportunity sets* van het tweede type zijn gesloten convexe polyeders; hun vorm wordt bepaald door de bovengrenzen gesteld aan de portefeuillegewichten. Om voor deze eerste twee typen van *portfolio opportunity sets* de frequentieverdelingen van performancemaatstaven te bepalen gebruiken we inzichten uit de analytische meetkunde en lineaire algebra. In het bijzonder is het in dit geval mogelijk om expliciete analytische uitdrukkingen af te leiden voor de genoemde frequentieverdelingen.

Bij het opleggen van niet-lineaire restricties daarentegen (bij *portfolio opportunity sets* van het derde type) neemt de complexiteit van de analyse dermate toe dat we in dit geval zijn gedwongen uit te wijken naar het toepassen van numerieke methoden.

Berekening van Verdelingen van Performance Maatstaven (Hoofdstuk 4)

Beschouwd in de ruimte van beleggingsgewichten vormt een verzameling van toegelaten portefeuilles die elk eenzelfde waarde hebben voor de geselecteerde performancemaatstaf een iso-(hyper-)vlak ("*iso-surface*"). Dit betekent dat het bepalen van de frequentieverdeling van een performancemaatstaf gelijk staat aan het bepalen van het volume van de doorsnede van de *portfolio opportunity set* en de corresponderende iso-vlakken.

De algemene benadering voor het afleiden van analytische uitdrukkingen voor de (cumulatieve) frequentieverdelingen is om eerst alle extreme portefeuilles van een specifiek type van *portfolio opportunity sets* te bepalen (dit zijn de vertices van de *opportunity set* polytopen), en vervolgens de *opportunity set* polytoop te decomponeren in een eindig aantal van *opportunity subsets* die beurtelings zijn verkregen uit de extreme portefeuilles.

Met de ontwikkelde methodologie zijn we in staat om algemene uitdrukkingen af te leiden voor de (cumulatieve) frequentieverdelingen van lineaire performancemaatstaven over *portfolio opportunity sets* onder een short-sales restrictie. De frequentieverdeling heeft de vorm van een *spline* waarvan de kritieke punten worden gevormd door de performance van individuele beleggingsobjecten. Voor *portfolio opportunity sets* gevormd onder restricties op de gewichten van individuele beleggingsobjecten heeft deze frequentieverdeling ook de vorm van een dergelijke *spline*; de achterliggende berekeningen zijn evenwel complexer en daarmee tijdrovender vanwege de asymptotische toename in het aantal van extreme portefeuilles.

In geval van niet-lineaire performancemaatstaven en/of wanneer niet lineaire restricties gelden (zoals restricties aan het risicoprofiel) neemt de complexiteit van de analyse dermate toe dat analytische uitdrukkingen voor frequentieverdelingen onbereikbaar zijn. In deze gevallen moet derhalve worden uitgeweken naar numerieke methoden.

Numerieke Bepaling van Performance Verdelingen (Hoofdstuk 5)

Numerieke methoden kennen een universele toepassing, maar binnen ons raamwerk vormen zij een vrij trage benadering om frequentieverdelingen van performancemaatstaven te bepalen. De numerieke procedure is gebaseerd op het statistisch trekken van steekproeven: we schatten de frequentieverdeling van een performancemaatstaf over de gehele *portfolio opportunity set* aan de hand van de frequentieverdeling over een adequate steekproef van toegelaten portefeuilles.

We kunnen twee wegen bewandelen om een dergelijke adequate steekproef te verkrijgen: volgens de Monte Carlo methode, of volgens een quasi-Monte Carlo methode (gebruikmakend van *low-discrepancy sequences*). De meest universele benadering om binnen de Monte Carlo methode steekproefsgewijs toevallige toegelaten portefeuilles te trekken is om gebruik te maken van het *Sequential Direction* algoritme, dat een ergodische Markov keten vormt. Deze benadering is universeel in alle opzichten: we kunnen de frequentieverdeling bepalen van elke mogelijke performancemaatstaf over *portfolio opportunity sets* die zijn onderworpen aan zowel lineaire als niet-lineaire restricties. Bovendien kunnen in het geval van louter “no short-sales” restricties twee efficiënte transformatieprocedures worden gebruikt (namelijk *Uniform Spacings* en exponentieel). Tenslotte kan het betrouwbaarheidsinterval van de schattingsfout worden bepaald.

Een alternatieve benadering voor het verkrijgen van frequentieverdelingen is gebaseerd op *low-discrepancy sequences*. In dat geval vatten we een gegeven *portfolio opportunity set* samen in een *parallellepipedum* of een simplex, genereren we hierin *low-discrepancy* punten, en accepteren we slechts die punten die voldoen aan alle restricties opgelegd aan de *portfolio opportunity set*. Daar *low-discrepancy sequences* volstrekt deterministisch zijn, is ook de corresponderende foutenmarge deterministisch. De grootst mogelijke fout is in de praktijk evenwel zeer moeilijk te bepalen.

In onze toepassingen blijken Monte Carlo methoden het meest geschikte vehikel om frequentieverdelingen te bepalen. Zij zijn gemakkelijker te analyseren en te implementeren. Daarenboven zijn zij in veel gevallen efficiënter dan *low-discrepancy sequence* methoden. Als alternatieve benadering verdienen laatstgenoemde methoden evenwel nader onderzoek.

Markt Dynamiek vanuit het Perspectief van Portefeuillemogelijkheden: de DAX® Case (Hoofdstuk 6)

De toepassing van ons raamwerk opent een frisse kijk op de performance en dynamiek van aandelenmarkten. De procedure is als volgt. We starten met het bepalen van de relevante *portfolio opportunity set*: het universum van toegelaten portefeuilles die voldoen aan de restricties die gelden voor beleggingsobjecten opgenomen in een marktindex. Voor elk van de portefeuilles binnen deze verzameling kunnen we de gewenste performancemaatstaf bepalen (het gerealiseerde rendement over een bepaalde horizon, het gemiddelde rendement, de standaarddeviatie van het rendement, enz.) en samenvatten in een frequentieverdeling. Wanneer we deze verdelingen opstellen voor verschillende perioden kunnen we deze verdelingen (en met name hun ligging en vorm) onderling vergelijken. Dit verschaft ons belangrijke informatie over de dynamiek in de beschouwde markt. Ook stelt deze benadering ons in staat om de relatieve performance van de marktindex ten opzichte van de *portfolio opportunity set* te analyseren. We illustreren deze benadering van marktbeschrijving en -analyse voor de Duitse aandelenmarkt en de DAX index.

Het analyseren van de performance en dynamiek van een financiële markt aan de hand van *portfolio opportunity sets* biedt verscheidene voordelen boven de gebruikelijke focus op een marktindex. In de eerste plaats zijn we niet langer gebonden aan één specifieke portefeuille (in de vorm van de marktindex) om een markt te beschrijven. We kunnen daarentegen het perspectief verbreden naar de variëteit van portefeuilles die kunnen worden gevormd op basis van een gegeven keuzeveld van beleggingsobjecten onder de voor de beschouwde index geldende restricties. De analyse van deze portefeuillemogelijkheden verschaft ons meer informatie over een markt dan de analyse van de marktindex. Bovendien kunnen we de dynamiek in de portefeuillemogelijkheden in de tijd bestuderen. Voor de Duitse aandelenmarkt vinden we dat in de periode 1990-2004 het risiconiveau van de toegelaten portefeuilles (gemeten aan de hand van de standaarddeviatie van het rendement) is toegenomen. Ook is de heterogeniteit van de portefeuillemogelijkheden (gemeten aan de hand van de spreiding in de gerealiseerde rendementen op de toegelaten portefeuilles) in deze periode toegenomen. We hebben deze observaties kunnen onderbouwen door de dynamiek in de covariantiestructuur van de aandeelrendementen in de tijd aan een nader onderzoek te onderwerpen.

In de tweede plaats kunnen we de marktindex evalueren ten opzichte van alle portefeuillemogelijkheden zoals weergegeven door de *portfolio opportunity set*. Dit staat in schril contrast met de conventionele benadering waarin de marktindex wordt geacht een representatieve afspiegeling te vormen van de corresponderende markt. De marktindex is evenwel slechts één van de mogelijke portefeuilles die onder dezelfde restricties op de markt kunnen worden gevormd. We kunnen nu de marktindex plotten in de frequentieverdeling van de geselecteerde performancemaatstaf over de *portfolio opportunity set*. Het quantiel van deze verdeling waarin de marktindex valt geeft aan hoeveel toegelaten portefeuilles een betere performance hebben behaald in termen van de geselecteerde performancemaatstaf (bv. het gerealiseerde rendement over een bepaalde periode). Op deze wijze kan worden nagegaan in hoeverre een marktindex representatief is voor de beschouwde markt. De mate van representativiteit wordt dan niet (zoals gebruikelijk) uitgedrukt in de fractie van de totale marktkapitalisatie die wordt beslagen door de index, maar afgelezen uit de plaats van de index in de verdeling van de *portfolio opportunity set*. Over de periode 1990-2004 blijkt dat de DAX in termen van gerealiseerd rendement vaak in de lagere quantielen van de *portfolio opportunity set* valt; een groot deel van de toegelaten portefeuilles realiseert dus een hoger rendement dan de DAX.

Niet alleen de performance van een marktindex kan worden geëvalueerd met behulp van *portfolio opportunity sets*, maar ook de performance van ieder willekeurige beleggingsportefeuille. Wel dient in dit geval de *portfolio opportunity set* worden opgesteld conform het onderliggende beleggingsmandaat zodat de verzameling van portefeuillemogelijkheden ook daadwerkelijk toegelaten portefeuilles omvat. De dynamiek in de verdelingen van een performancemaatstaf in de tijd, en met name de wisselende spreiding van deze verdelingen, verschaft ons ook de mogelijkheid om de beleggingsperformance te normaliseren. Bij het normaliseren corrigeren we een performancemaatstaf voor de spreiding in de bijbehorende verdeling van performancemaatstaven over de *portfolio opportunity set*. Bij een kleine spreiding van deze verdeling moeten we immers een grotere waarde toekennen aan verschillen in performance dan bij een grote spreiding. In hoofdstuk 7 gaan we hierop verder in.

In de derde plaats (naast marktbeschrijving en performance-evaluatie) kan onze methodiek worden toegepast op het ontwikkelen en analyseren van veelbelovende beleggingsstrategieën die zijn onderworpen aan specifieke randvoorwaarden. Dit betekent dat de voorgaande *ex post* toepassingen van onze benadering wordt gecomplementeerd met een *ex ante* toepassing.

Performance Evaluatie op Basis van Portfolio Opportunity Sets (Hoofdstuk 7)

Met behulp van *portfolio opportunity sets* kan de absolute en relatieve performance van beleggingsportefeuilles met inachtneming van de geldende beleggingsdoelstellingen en opgelegde beleggingsrestricties (zoals bv. neergelegd in een mandaat) op een natuurlijke wijze in een breder perspectief worden geplaatst. Bij het evalueren van de frequentieverdeling van een performancemaatstaf over de relevante *portfolio opportunity set* zijn twee aspecten van belang: (1) waar zijn de te evalueren portefeuille en (indien van toepassing) de bijbehorende benchmark gepositioneerd in deze verdeling, en (2) wat is de spreiding in performances die deze verdeling laat zien.

We beschouwen eerst de performance gemeten over één periode. Uit de verdeling over de *portfolio opportunity set* kunnen de minimum en maximum performancewaarden worden afgelezen; dit geeft het interval aan waarbinnen de performance van alternatieve toegelaten portefeuilles valt. De gehele continue frequentieverdeling kan worden gebruikt om performance-klassen te formuleren, dan wel om de exacte percentielen van relatieve performance te bepalen. De performance is dan relatief ten opzichte van de gehele *portfolio opportunity set* en niet ten opzichte van een benchmark.

Vervolgens bezien we de performance gemeten over verscheidene perioden. De verandering in ligging en vorm van de frequentieverdelingen in de tijd geeft inzicht in de ontwikkeling van performancemogelijkheden. De performance van een portefeuille afgezet tegen de ligging van de frequentieverdelingen geeft informatie over de persistentie van de behaalde performance. De spreiding van de frequentieverdelingen geeft aan of een behaalde (out-) performance relatief gemakkelijk (grote spreiding) dan wel relatief moeilijk (kleine spreiding) was te realiseren binnen de toegelaten portefeuille-mogelijkheden. We gebruiken de spreiding van de frequentieverdeling om genormaliseerde performancemaatstaven af te leiden, de zogenaamde *score coefficients*. Deze maatstaven maken het mogelijk om de gemeten performance te schonen voor de marktdynamiek (de wisselende heterogeniteit in portefeuillemogelijkheden).

De ontwikkelde methodologie kan op een eenvoudige wijze worden uitgebreid om ook de relatieve performancemeting ten opzichte van een benchmark te accommoderen. Portefeuilles die zijn onderworpen aan een restrictie op de *tracking error volatility* kunnen worden geëvalueerd ten opzichte van *portfolio opportunity sets* die zijn gevormd onder dezelfde randvoorwaarden. In dit geval zijn met name de frequentieverdelingen van de *information ratio* van belang. Analoog aan de werkwijze in hoofdstuk 5 laten we zien hoe deze frequentieverdelingen kunnen worden afgeleid. De door een portefeuillebeheerder behaalde *information ratio* kan wederom worden geëvalueerd ten opzichte van de ligging en vorm van de frequentieverdeling over de *portfolio opportunity set*. We gebruiken de spreiding van de frequentieverdeling om een genormaliseerde *information ratio* te construeren. Deze genormaliseerde *information ratios* zijn geschoond voor marktdynamiek en kunnen in de tijd onderling worden vergeleken. Daarenboven maakt onze benadering het mogelijk om te toetsen of een portefeuillebeheer al dan niet een restrictie op de *tracking error volatility* respecteert. Vanuit een *ex ante* perspectief kunnen we tenslotte de verdeling van potentieel te behalen *information ratios* bepalen, aan de hand waarvan inzicht kan worden verkregen in de mogelijke toegevoegde waarde van actief portefeuillebeheer onder een bepaald mandaat.

Curriculum Vitae

Igor Pouchkarev graduated with honors in Microprocessor techniques in Russia and then obtained the Master's degree in Informatics and Computer Science (GPA 1.1) from the University of Saarbrücken (Germany) in 1999. In May 2000 he joined ERIM Ph.D. program.

Throughout his study he developed an interest in performance evaluation, portfolio and risk management, and asset & liability management. His work in these fields have been presented at international conferences and published in academic journals and books.

During his Ph.D. studies he was affiliated with the Asset & Liability Management department of the United Bank of Switzerland in Zürich, Switzerland.

In 1997-1999 he worked as a research assistant at the Max-Planck Institute for Computer Science, Algorithm and Complexity group, and in 1993-1997 developed software for business process re-engineering at the IDS Prof. Scheer AG in Saarbrücken, Germany.

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Performance Evaluation of Constrained Portfolios

Conventional performance evaluation methods strongly differentiate between the universe which is used for portfolio construction, and the universe which is used for the performance evaluation. Whilst by composing the portfolio we consider the complete opportunity set, in the last case we use a very restricted, general representation of this opportunity set: a peer group or (a) benchmark portfolio(s).

In this thesis we present a conceptual framework, which allows to incorporate the decision-making context of any constrained investment portfolio into the performance evaluation process. The main feature that distinguishes our methodology from conventional performance evaluation methods is that it tackles the performance at the decision-making level: the portfolio weights. We consider all possible portfolios that can be constructed given the specific investment objective(s) as well as the prescribed investment constraints, and then evaluate all these portfolios according to (a) selected performance measure(s). The performance of the investment portfolio is calculated simultaneously and then evaluated against the performance of this complete opportunity set. Consequently, our methodology extends the conventional performance metrics with the insights into the performance of all opportunities that existed at the time of the investment decision.

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