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TECHNICAL NOTE

On the Marginal Cost Approach in Maintenance[1]

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Abstract. In this paper we investigate the conditions under which the marginal cost approach of Refs. 1–3 holds. As observed in Ref. 4, the validity of the marginal cost approach gives rise to a useful framework of single-component maintenance optimization models, which covers almost all models used in practice. For the class of unimodal finite-valued marginal cost functions, we show that these optimization models are easy to solve.

Key Words. Maintenance, marginal cost analysis, Lebesgue integrals, unimodality.

1. Introduction

In a series of papers, Berg (Refs. 1–3) introduces the so-called marginal cost analysis (MCA) method for the study of single-component replacement models with particular application to block or age replacement structures. This readily implementable approach is applied to a large collection of models without adequate introduction of the mathematical setting involved. In particular, we shall be interested in revealing the conditions which validate the approach. At the same time, we will show under which conditions on the so-called marginal cost function this approach can be used to determine the optimal age replacement policy.

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2. Analysis

In an infinite-horizon single-component replacement model with stationary replacement policies characterized by a finite positive parameter T, one can identify for every policy $p(T)$, $T>0$, a nonnegative increasing stochastic process $\{R(t, p(T)): t \geq 0\}$ with right-continuous sample paths. For each $t>0$, the random variable $R(t, p(T))$ represents the total maintenance costs up to time t if policy $p(T)$ is used. Well-known examples of such policies are the block and age replacement policies (Ref. 5). In this paper, we will only consider the class of age replacement policies. Observe that a completely similar analysis can be given for the class of block replacement policies. Each age replacement policy $p(T)$ is now completely determined by the random variables $T \wedge X_i$, $i \geq 1$, with $T \wedge X_i := \min\{T, X_i\}$ and X_i, $i \geq 1$, a sequence of nonnegative, independent and identically distributed random variables. Clearly, the random variable $T \wedge X_i$ represents the elapsed time between the $(i-1)$th and ith replacements. If $\mathbb{E}(Z)$ denotes the expectation of the random variable Z, and if the function $k: [0, \infty) \to [0, \infty)$ given by

$$k(T) := \mathbb{E}(R(T \wedge X_1, p(T)))$$

is finite-valued, it is well known by the regenerative structure of the family of age replacement models (Ref. 6) that the infinite-horizon average costs $c(T)$ of policy $p(T)$ equal

$$c(T) = k(T)/\mathbb{E}(T \wedge X_1). \tag{1}$$

In this paper we always assume that the following conditions hold:

(a) the function $k: [0, \infty) \to [0, \infty)$ is continuous on $(0, \infty)$ and right-continuous at 0;

(b) the distribution function F of the random variable X_1 is continuous on $(0, \infty)$, has support $[0, \infty)$, and satisfies $F(0)=0$ and $F(\infty)=1$.

By condition (b), it is easy to verify that the denominator in (1) is positive and continuously differentiable on $(0, \infty)$. Moreover, by condition (a), this yields that the function $c: (0, \infty) \to [0, \infty)$ is finite-valued and continuous. In the MCA approach, it is now assumed by Berg (Refs. 1–3) without any proof that the existence of the marginal cost function implies that the numerator in (1) is absolutely continuous and can be represented by the sum of a constant and an integral involving this marginal cost function and the tail of the distribution function F. This is based, as can be seen, on a tacit assumption that the function k is continuous. Since the MCA approach provides a useful economic interpretation of the replacement-decision timing and its optimization, it is useful to obtain the precise conditions under which it holds. To start with our analysis, we need the following definition (the terminology is that introduced in Refs. 1–3).

Definition 2.1. Let $g: [0, \infty) \to (-\infty, \infty)$ be a finite-valued function. The upper right Dini derivative $D^+g: [0, \infty) \to [-\infty, \infty]$ at $t \geq 0$ is given by

$$(D^+g)(t) := \limsup_{h \downarrow 0} (g(t+h) - g(t))/h.$$

Moreover, the lower right Dini derivative $D_+g: [0, \infty) \to [-\infty, \infty]$ at $t \geq 0$ is given by

$$(D_+g)(t) := \liminf_{h \downarrow 0} (g(t+h) - g(t))/h.$$

Clearly, the upper and lower right Dini derivatives at $t \geq 0$ always exist, but may possibly be $-\infty$ or ∞. To analyze the numerator, we observe that, for every $h > 0$, the policy $p(T)$ is the same as the policy $p(T+h)$ if $X_1 < T$, and so

$$\begin{aligned} &k(T+h) - k(T) \\ &= \mathbb{E}((R((T+h) \wedge X_1, p(T+h)) - R(T \wedge X_1, p(T)))1_{\{X_1 \geq T\}}) \\ &= \mathbb{E}(R((T+h) \wedge X_1, p(T+h)) - R(T \wedge X_1, p(T)) | X_1 \geq T)(1 - F(T)). \end{aligned}$$

If we define

$$m(T) := \limsup_{h \downarrow 0} (\mathbb{E}(R((T+h) \wedge X_1, p(T+h)) - R(T \wedge X_1, p(T)) | X_1 \geq T))/h,$$

then it follows that

$$(D^+k)(T) = m(T)(1 - F(T)).$$

As observed in Ref. 3, the function $m: [0, \infty) \to [-\infty, \infty]$ measures at T the local increase or decrease in costs between the preventive replacement of a component of age T at time T and the costs of deferring this preventive replacement an infinitesimal time unit later. Therefore, it is not surprising that the function $m: [0, \infty) \to [-\infty, \infty]$ is given the following name (Refs. 1–3).

Definition 2.2. The function $m: [0, \infty) \to [-\infty, \infty]$ is called the marginal cost function.

In his papers, Berg in fact assumes that the right-hand derivative defining the marginal cost function exists, which in particular implies that the lower and upper Dini derivatives must be equal, and so his approach is less general. However, since we use upper Dini derivatives, the marginal cost function is always defined. We now prove that the function k is absolutely continuous.

Theorem 2.1. If the function $k: [0, \infty) \to [0, \infty)$ is continuous on $(0, \infty)$ and right-continuous at 0, and if the marginal cost function $m: [0, \infty) \to [-\infty, \infty]$ is nonnegative and finite-valued for every $t \geq 0$, then it follows that

$$k(T) = k(0) + \int_0^T m(y)(1 - F(y))\, dy,$$

for every $T > 0$. Moreover, the above integral represents a Lebesgue integral. If additionally the function m is continuous almost everywhere with respect to the Lebesgue measure and bounded on $[0, T]$, then the above integral can be interpreted as a Riemann integral.

Proof. Introduce for each $n \in \mathbb{N}$ the function $k_n : [0, \infty) \to [0, \infty)$ given by

$$k_n(t) := n(k(t + 1/n) - k(t)).$$

By our assumptions, k_n is a finite-valued continuous function on $(0, \infty)$, and therefore is Riemann integrable on $[0, T]$ for every $0 < T < \infty$. This yields

$$\int_0^T k_n(y)\, dy = n \int_T^{T+1/n} k(y)\, dy - n \int_0^{1/n} k(y)\, dy,$$

and by the continuity of the function k, we obtain that

$$\lim_{n \to \infty} \int_0^T k_n(y)\, dy = k(T) - k(0). \tag{2}$$

Since the marginal cost function m is nonnegative, it follows that, for every $t \geq 0$,

$$(D^+k)(t) = m(t)(1 - F(t)) \geq 0,$$

and again by the continuity of k and Theorem 1.13(ii) of Ref. 7, the function k is increasing on $(0, \infty)$. Hence, for each $n \in \mathbb{N}$, it follows that k_n is nonnegative, and so we can apply the Fatou lemma (Ref. 8) to (2). As a result, we obtain

$$k(T) - k(0) = \lim_{n \uparrow \infty} \int_0^T k_n(y)\, dy \geq \int_0^T \liminf_{n \uparrow \infty} k_n(y)\, dy. \tag{3}$$

Since the function k is increasing, it follows by Theorem 2.3.9 of Ref. 8 that k is differentiable almost everywhere with respect to the Lebesgue measure, and hence $D^+k = D_+k$, except on a set of Lebesgue measure zero. Using

$$(D^+k)(y) \geq \liminf_{n \uparrow \infty} k_n(y) \geq (D_+k)(y), \qquad \text{for every } y \geq 0,$$

Inequality (3), and Theorem 1.6.5(b) of Ref. 8, this finally implies that

$$k(T) - k(0) \geq \int_0^T (D^+k)(y)\, dy = \int_0^T m(y)(1 - F(y))\, dy.$$

To verify the reverse inequality, we introduce the function $\mathscr{K}$: $[0, \infty) \to [0, \infty)$ given by

$$\mathscr{K}(T) := k(0) + \int_0^T m(y)(1 - F(y))\, dy.$$

For every $t \geq 0$, we obtain by the calculus rules for Dini derivatives (Ref. 7) that

$$\begin{aligned}(D^+(\mathscr{K} - k))(t) &\geq (D^+\mathscr{K})(t) + (D_+(-k))(t)\\ &= (D^+\mathscr{K})(t) - (D^+k)(t),\end{aligned}$$

and this yields by Theorem 2.3.10 of Ref. 8 that

$$(D^+(\mathscr{K} - k))(t) \geq 0,$$

for almost every $t \geq 0$ (with respect to the Lebesgue measure). Moreover, since $\mathscr{K}$ is increasing due to m nonnegative, and hence $(D^+\mathscr{K})(t) \geq 0$ and by assumption $(D^+k)(t) < \infty$ for every $t \geq 0$, we obtain that

$$(D^+(\mathscr{K} - k))(t) > -\infty,$$

for every $t \geq 0$. Since by Theorem 2.3.4 of Ref. 8 the function $\mathscr{K}$ is absolutely continuous and hence continuous, and since by assumption k is continuous, it follows that $\mathscr{K} - k$ is continuous on $(0, \infty)$. The assumptions of Lemma 2, p. 370 of Ref. 9, are now satisfied, and so we may conclude that the function $\mathscr{K} - k$ is increasing. This implies that, for every $T \geq 0$,

$$\begin{aligned}\int_0^T m(y)(1 - F(y))\, dy + k(0) - k(T) &= \mathscr{K}(T) - k(T)\\ &\geq \mathscr{K}(0) - k(0)\\ &= 0,\end{aligned}$$

and this yields the inequality

$$k(T) \leq k(0) + \int_0^T m(y)(1 - F(y))\, dy.$$

The second part of the above result is an immediate consequence of Theorem 1.7.1 of Ref. 8. □

The condition in Theorem 2.1 that k is continuous on $(0, \infty)$ and right-continuous at 0 cannot be omitted. As a counterexample, we mention the function $k: [0, \infty) \to \mathbb{R}$ given by

$$k(t) = \begin{cases} 1, & \text{if } t \geq 1, \\ 0, & \text{if } 0 \leq t < 1. \end{cases}$$

Clearly, for this function

$$(D^+k)(t) = 0, \qquad \text{for every } t \geq 0,$$

and so

$$\int_0^t (D^+k)(y)\, dy = 0, \qquad \text{for every } t \geq 0.$$

However,

$$k(t) = 1, \qquad \text{for every } t \geq 1,$$

and this yields that

$$k(t) \neq k(0) + \int_0^t (D^+k)(y)\, dy, \qquad \text{if } t \geq 1.$$

Moreover, one also needs in Theorem 2.1 the condition that $m(t)$, or equivalently $(D^+k)(t)$, is finite for every $t \geq 0$. As a counterexample, we mention the Cantor function $k: [0, 1] \to \mathbb{R}$ discussed in Section 11.72 of Ref. 9. This function is continuous, increasing, and satisfies $(D^+k)(t) = 0$ for almost every $t \geq 0$ and $(D^+k)(t) = \infty$ on a Lebesgue set of measure zero. Moreover, $k(1) = 1$ and $k(0) = 0$, and so

$$k(1) \neq k(0) + \int_0^1 (D^+k)(y)\, dy.$$

It is easy to adapt the above proof whenever the marginal cost function is finite-valued and uniformly bounded from below on every compact interval. Since in almost all age replacement models the marginal cost function is nonnegative, we did not prove this result in detail. Clearly, for $T = 0$ the expected costs in a degenerate cycle of length zero are given by $k(0)$, and this value is assumed to be positive. By the above result, the objective function of any age replacement model belonging to the above family has the following representation involving the marginal cost function. This representation gives an intuitive appealing interpretation of the objective function and is mentioned in Ref. 10 and Ref. 4 without a proper mathematical justification.

Theorem 2.2. Under the conditions of Theorem 2.1, it follows that, for every $0<T<\infty$,

$$c(T)=\left(k(0)+\int_0^T m(y)(1-F(y))\,dy\right)\bigg/\left(\int_0^T (1-F(y))\,dy\right).$$

Proof. This is an easy application of Theorem 2.1 and (1). □

Observe that, for the infinite-horizon discounted cost criterion with discount factor $\alpha>0$, under the same conditions on the discounted marginal cost function m_α, a similar result can be derived. In this case, the discounted cost function c_α has the representation

$$c_\alpha(T)=\left(k(0)+\int_0^T m_\alpha(y)(1-F(y))\,dy\right)\bigg/\left(\alpha\int_0^T \exp(-\alpha y)(1-F(y))\,dy\right).$$

The optimal policy $0<T^*<\infty$ (if it exists) is a solution of the optimization problem (P) given by

$$\inf\{c(T): 0<T<\infty\},$$

with c a continuous function on $(0, \infty)$ and $c(0)=\infty$. Denote now by $\mathscr{P}^*$ the possibly empty and closed set of optimal solutions of (P). To verify that, under certain conditions, the set $\mathscr{P}^*$ is nonempty, we will first compute the upper right Dini derivative of the objective function c. Since the denominator in (1) is positive and continuously differentiable on $(0, \infty)$, and since the numerator is a continuous function, we obtain by the calculus rules for upper Dini derivatives [see formula (2a) of Ref. 7] that, for every $T>0$, the upper Dini derivative equals

$$(D^+c)(T)=(1-F(T)(m(T)-c(T))/(\mathbb{E}(T\wedge X_1)).$$

Observe that this result holds irrespective of whether $m(T)$ is finite or not, and so we might encounter situations where the above formula holds, but the function c does not have the representation of Theorem 2.2. Using the above formula, it is relatively easy to derive sufficient conditions for $\mathscr{P}^*$ to be empty or not. These conditions are known under the additional assumption that the marginal cost function m is continuous on $(0, \infty)$; see Refs. 1, 10, 11.

Lemma 2.1. If $c(\infty):=\lim_{t\uparrow\infty} c(t)$ exists and $c(\infty)=\infty$, then $\mathscr{P}^*$ is a nonempty compact subset of $(0, \infty)$. Moreover, if $c(\infty)$ is finite and

$\liminf_{t\uparrow\infty} m(t) > c(\infty)$, then $\mathscr{P}^*$ is also a nonempty compact subset of $(0, \infty)$.

Proof. The first part is trivial to prove. To verify the second part, we observe that

$$\liminf_{t\uparrow\infty} m(t) > c(\infty)$$

implies the existence of some finite T_0 such that, for every $T \geq T_0$, it follows that $m(T) > c(T)$. This implies that

$$(D^+c)(T) > 0, \qquad \text{for every } T \geq T_0;$$

and since c is continuous, we may conclude from Theorem 1.14 of Ref. 7 that c is strictly increasing on (T_0, ∞). By this observation, the desired result follows. □

We will now introduce a class of marginal cost functions for which it is easy to compute the optimal age replacement policy.

Definition 2.3. See Ref. 12. A function $g: [0, \infty) \to [-\infty, \infty]$ is called unimodal with parameter $0 \leq b < \infty$ if the function g is decreasing on $[0, b]$ and increasing on (b, ∞).

In the sequel, it will be assumed that the marginal cost function m is finite-valued and unimodal with parameter $0 \leq b < \infty$. This implies by Theorem 2.2 that the representation for the function c holds. Moreover, by Theorem 4.1.2 of Ref. 13, the function m is continuous almost everywhere with respect to the Lebesgue measure, and so the integral in Theorem 2.1 can be interpreted as a Riemann integral. It is now possible to prove the following result.

Lemma 2.2. If the optimal solution set $\mathscr{P}^*$ is nonempty, and if m is finite-valued and unimodal with parameter $0 \leq b < \infty$, then $\mathscr{P}^*$ is contained in $[b, \infty)$.

Proof. Clearly the result follows for $b = 0$. Assume therefore that $0 < b < \infty$. Since m is decreasing on $[0, b]$, it follows that, for every $t < b$,

$$m(t) \int_0^t (1 - F(y))\, dy - \int_0^t m(y)(1 - F(y))\, dy - k(0) < 0.$$

This yields by Theorem 2.2 that

$$m(t) - c(t) < 0,$$

or equivalently,

$$(D^+c)(t) < 0, \qquad \text{for every } t < b.$$

Hence, by the continuity of c and the remark after Theorem 1.14 of Ref. 7, the function c is strictly decreasing on $(0, b)$, implying that $\mathscr{P}^* \subseteq [b, \infty)$. □

Under the additional assumption that the marginal cost function m is continuous, the same result is also shown in Ref. 11. Clearly, by the above observation, we can restrict our search for an optimal policy to the interval $[b, \infty)$. Finally, we will show under the assumption of unimodality of the marginal cost function that a simple bisection locates an optimal age replacement policy.

Theorem 2.3. If the optimal solution set $\mathscr{P}^*$ is nonempty, and if the marginal cost function m is finite-valued and unimodal with parameter $0 \leq b < \infty$, then it follows that $m(T) \geq c(T)$ if and only if the intersection of $\mathscr{P}^*$ and $[b, T]$ is nonempty.

Proof. If $m(T) \geq c(T)$, and if m is unimodal with parameter $0 \leq b < \infty$, it is shown in the proof of Lemma 2.2 that $T \geq b$. Since m is increasing on (b, ∞) and $T \geq b$, we obtain that, for every $t \geq T$,

$$\int_T^t (m(t) - m(y))(1 - F(y))\, dy + (m(t) - m(T)) \int_0^T (1 - F(y))\, dy \geq 0.$$

By Theorem 2.2, it follows that, for every $t \geq T$, the following identity holds:

$$\begin{aligned}(m(t) - c(t)) \int_0^t (1 - F(y))\, dy &= (m(T) - c(T)) \int_0^t (1 - F(y))\, dy \\ &\quad + \int_T^t (m(t) - m(y))(1 - F(y))\, dy \\ &\quad + (m(t) - m(T)) \int_0^T (1 - F(y))\, dy,\end{aligned}$$

and so

$$m(t) \geq c(t), \qquad \text{for every } t \geq T, \text{ if } m(T) \geq c(T).$$

This yields

$$(D^{+}c)(t) \geq 0, \qquad \text{for every } t \geq T;$$

and since c is a continuous function, we obtain that c is increasing on (T, ∞), implying the desired result. To prove the converse implication, we observe that a nonempty intersection of the sets $\mathscr{P}^*$ and $[b, T]$ implies the existence of an optimal solution T^* of (P) within the interval $[b, T]$. This yields

$$(D^{+}c)(T^*) \geq 0,$$

or equivalently,

$$m(T^*) \geq c(T^*).$$

Since m is increasing on (b, ∞) and $T^* \leq T$, this implies by the same arguments as in the first part that $m(T) \geq c(T)$, and this shows the reverse implication. □

By Theorem 2.3, it is clear that one has to compare the values $m(t)$ and $c(t)$ in order to determine which part of the positive axis will contain with certainty an optimal solution. By Theorem 2.3, observe that one has to assume that the marginal cost function is finite-valued and unimodal, and so continuity of the function m is not required. This result also covers unimodal finite-valued cost functions which are almost everywhere continuous. Under the above conditions, it is also not difficult to verify that $m(T) = c(T)$ implies that T is an optimal policy, but the reverse implication might not hold due to the absence of continuity everywhere of the function m.

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