OPTIMAL RETIREMENT DECISION

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Optimal retirement decision

Optimale pensioneringsbeslissing

Thesis

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Grzegorz Kula

Rotterdam, November 2003

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Chapter 1

Introduction

1.1 Retirement systems - the problem we all face

In May 2003 huge demonstrations and strikes shook France. Teachers, air traffic controllers, public transport workers, tax collectors, hospital workers, postmen... in short almost all the public sector. The reason? The government wanted to reform the pension system by taking away the privileges of the public sector and making their situation identical to that of the private sector. Almost everybody agrees that a pension reform is necessary, but the big question is: how to do this?

In the second half of the last century the labor force participation of elderly decreased rapidly and substantially around the world. The graphs in Figure 1.1. are very familiar to everybody. This process, combined with an increased length of lives and a fall in fertility, especially when the retirement of the baby boom generation born after the second world war is approaching, is the reason for reforms of retirement systems: pay-as-you-go schemes cannot work much longer. There are simply not enough working people to pay for the retirement benefits of elderly. In Europe at the moment one pensioner is supported by four workers, in 2050 the dependency ratio of older people to those of working age will rise to one-to-two (*The Economist*, 15.03.2003). In Japan such a ratio will be achieved already by 2020. Even in Norway, which is building up a huge fund from oil revenues as a reserve, it is possible that without the reform of the retirement system public finances will be unsustainable (OECD, 2002). Of course the situation would be easier if people, while living longer, would agree to work longer and retire later.

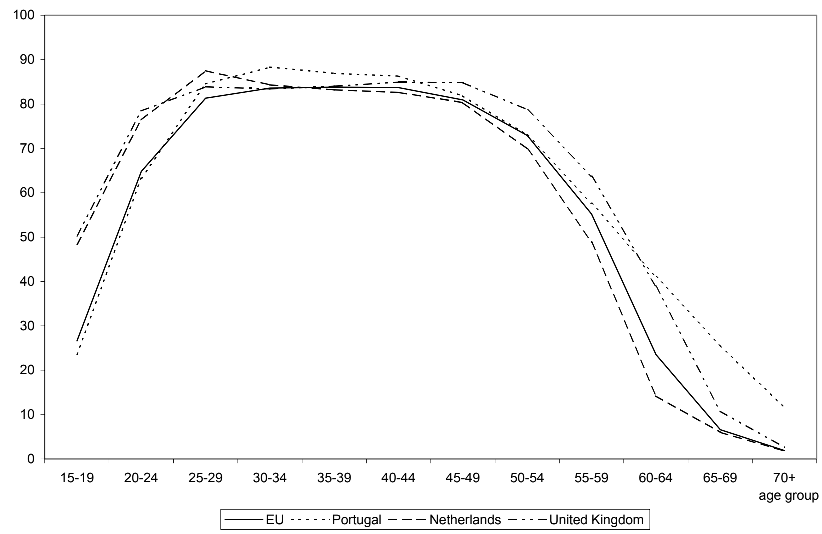


Figure 1.1: Economic activity rates by age groups

Source: *Labour Force Survey*, Eurostat (1997)

The most common suggestion is to raise the fixed retirement age, traditionally set at 65. Granted, it is not certain that an increase in the retirement age would bring only benefits, since with a shorter retirement period people may save less, they will have to be retrained and jobs will have to be adjusted for older workers' possibilities. On the other hand an increase in the retirement age may reduce government spending on pensions and other retirement expenditures, increase the level of production and tax revenues. So why not raise it? The main reason is not economical but political. People, especially those close to retirement, hate the idea of working longer and not enjoying their long awaited benefits. There are also other reasons. If the retirement age is higher, the supply of labor rises, but it does not guarantee employment. Thus the government, while increasing the retirement age, should at the same time somehow try to increase the labor demand. Labor demand may even fall if wages are not flexible enough to fall when productivity of older workers starts decreasing. But, if retirement benefits depend on the last wage, then nobody will want to work for less. It is also necessary to provide older workers with training. Companies may be reluctant to do so, since the returns from such an investment are low - even with a higher retirement age older workers may not remain working long enough to bring profits from training to their firms.

As we can see, even such a simple looking change as an increase of the retirement age is difficult and complex. Even in countries which decided to do it, like the United States, the whole process it distributed over forty years. However, there are voices that a fixed retirement age is an anachronism, since many workers retire as early as possible, much earlier than the retirement age. In France the effective retirement age is below 60, and the OECD countries average is generally between 60 and 63 (OECD, 2002). This it the main reason for the drop in the labor force participation of elderly shown in Figure 1.1. The prime suspect responsible for this process is the social security system, with increased benefits, guaranteed income, implicit tax on earnings and the option of early retirement. Other possibilities include those of employer and private pension plans and private savings. It is also possible to enter early retirement through the disability system, although, since the overall health level of older generations is improving, there should be less reasons for using it. A high number of workers enters early retirement through unemployment. Most of the countries have special programs for older unemployed, allowing them to receive unemployment benefits longer than usually, till the time they become eligible for early retirement. Finally we should mention the income effect: people are getting richer and their wealth is partly spent on leisure. Nowadays, with a number of countries reforming their social security and retirement systems or being on the verge of such reforms, the question of what causes people to stop working gains particular importance.

Another question is how to change state pension systems and what are the alternatives. An obvious way for governments to proceed is to push some of the retirement burden on employers and employees themselves. According to *The Economist* "...Encouraged by tax deductions and regulations, half of all American companies with pension plans and many British ones offered their staff "defined-benefits" schemes, which pay pensions linked to an employee's final salary..." (*The Economist*, 10.05.2003, p.9). However this was the case before the economic slowdown. Afterwards everything has changed. A large number of firms have now high deficits in their pension plans, in some cases reaching more than half of their market capitalization. The Pension Benefit Guaranty Corporation - an agency insuring companies' defined benefits pension plans in the US - closed the year 2002 with a $3.6 billion deficit. Quite a change from a $7.7 billion surplus in the previous year (*The Economist*, 15.02.2003). The problem was caused by falling stockmarkets and the level of the discount rate used by firms to calculate the present value of future pensions. Such a high deficit raises questions about defined benefits plans that are supposed to form an addition and/or alternative to public retirement systems. Obviously, if the market changes and stock prices start to rise, the deficits may disappear very fast, but the feeling of security is probably gone for good.

Due to the problems of defined benefit plans more and more firms switch to defined contribution plans, which are also used in savings plans like 401(k) in US. Defined contribution plans have many advantages over defined benefit. First of all, with workers' mobility increasing, they are easier to use and manage. Second, all the risks (together with inflation) are borne by the plan participants, and not by the companies. True, this also concerns rewards in case of good performance of the investment, but employers are ready to give them up in exchange for not bearing the risk. Third, in defined contribution it is known exactly how much and which assets belong to each participant. Nevertheless, defined contribution plans have their drawbacks. Since usually part of the contributions is paid by the employer, it may happen, and it indeed did happen, that firms cut or stop their contributions if their market situation is worsening. When a company is using its own stock as its contribution to pension accounts the consequences may be even more serious, like in the Enron case, when pension accounts became worthless.

Of course in practice the existing pension schemes are in between defined benefit and defined contribution, like plans setting a floor to protect workers from the effects of poor investments, or adjusting the contributions periodically in order to move closer to the planned level of benefits. Even public pay-as-you-go schemes in some countries are being transformed according to defined benefit or defined contribution lines by creating individual accounts, like in Sweden, Latvia, or Poland.

People have shown the obvious thing - the incentives do matter. Existing institutions provide a lot of incentives to retire earlier, what can be seen from the studies presented in the literature review below. As Wise (2003) wrote in his report on the NBER Program on the Economics of Aging: "...Over the years, the strong relationship between the economic incentives of retirement policies and the ages at which individuals retire from the labor force has been confirmed in multiple studies, using multiple data sources, and applying multiple research methodologies..."[1]. Some of the studies give very good results, which fit closely the actual data, let us just give examples of Gustman and Steinmeier (1986, 2002) or Rust and Phelan (1997)[2]. Their models show that the peaks in the retirement ages are

[1] Wise (2003), p. 4

[2] Although Gustman and Steinmeier (2002) claim that Rust and Phelan's methodology "...leads to an

due to the rules of the retirement system. Social security system establishes a number of rules which are responsible for the retirement patterns: the early eligibility age, the normal eligibility age, the earning test, the problem of the actuarial fairness of the system when retirement is postponed, etc. The same we can see with some pension plans; the incentives may be even stronger, since some companies for example offer special bonuses for retiring at a certain age. The purpose of this whole system is to allow older and less productive workers to stop working and "enjoy the fruits of their long, hard working life". However we cannot ignore the moral hazard: the majority of people just do not like to work, and will use any possibility to stop working as soon as possible, no matter if they are really old or less productive.

All papers treat institutions as given, although in many of them the effects of changes in institutions are considered, like shifting the early retirement age or increasing benefits. No one could deny importance of the studies which try to show what elements of existing systems have the strongest influence on the decision to retire and the timing of this decision, although the complexity of interacting effects leads sometimes to conflicting results, as we show in the review of the literature below. Nevertheless, despite the fact that we may say more and more about the working of current systems, we still do not know what the optimal system should look like.

This is the first question we ask in this book: are the institutions optimal? We want to know what are the optimal retirement institutions when labor incomes are uncertain. In order to do it, we do not impose any retirement ages or benefits' levels, but instead we allow an individual to retire in a moment which maximizes her/his utility. Thus we define optimal retirement institutions as those maximizing individual's lifetime utility. The second question of the thesis is: can improving institutions explain the fall in the retirement age? In particular - can improvements in insurance explain the fall in the labor force participation of elderly? Better insurance, neutralizing income risk, yields two opposite effects: first, with wages more certain it is more attractive to work and earnings become more valuable. Second, savings becomes more valuable, since returns on savings depend on wage, and with better insurance it is possible to plan both saving and dissaving better. A priori we cannot say which effect is stronger. However, when we compare perfect insurance studied in Chapter 2 with self-insurance from Chapter 4, it becomes clear that

overestimate of the effects of market imperfections on retirement..." (p. 6).

better insurance, i.e. insurance providing a more secure income, increases the probability of retirement.

1.2 Review of the literature

Ever since the downward trend in the labor force participation of elderly became visible, i.e. from the 1950's on, the number of studies of the problem is growing rapidly. It is simply not possible to give a detailed picture of the whole field, but we will try to present a brief overview, with a few examples of characteristic or important models shown in more details. Although the literature covers all, or almost all, aspects of the phenomenon, we are interested only in the models studying the retirement decision from the perspective of an individual worker. All the papers discussed below study the incentives to retire provided by existing retirement institutions. The main characteristics of these papers are presented in Table 1.1. In this table the uncertainty is defined as something unknown to an individual, about which it is possible to form beliefs, like for example the survival probability. The randomness is defined as something unknown to an economist, a process following or a random shock drawn from the assumed distribution. Most of the studies suggest that early retirement is caused by the incentives in the social security system, although some claim that firm and private pension plans are more important. It is impossible to decide which results are true, although with the increasing share of those covered by employer provided and private pension plans, and saving periods getting longer, the role of pension plans' incentives is becoming more and more important.

1.2.1 Static models

Even the early retirement literature, approaching the problem of retirement within the static framework, like in Boskin (1977), was able to show that incentives are important. He focuses on incentives which social security may provide to retire earlier and estimates the probability of retirement. There are three ways in which social security, included in the budget constraint, can affect retirement decisions: the benefits, the social insurance contributions and the earning test. The conclusion is that "...the overall impact of the social security system, through the income guarantee and the earnings test, is clearly to induce earlier retirement for a substantial fraction of the elderly population..."[3]. The effect of social security benefits is much more important than the effect of income from assets, because the former are guaranteed for the rest of worker's life and are indexed against inflation. A similar approach is presented in Boskin and Hurd (1978), where the probability of retirement is determined not only by social security but also by an individual's health. They too find that "...the change in the budget constraint caused by the social security system has a considerable impact on retirement..."[4] and that a poor health significantly increases the probability of retirement.

1.2.2 Dynamic life cycle models

The static models showed that social security influences retirement decisions. However their results were based on current incomes only and did not include uncertainty. Such an approach was far from reality, since people make the retirement decision taking into account possible future incomes, and trying to reduce uncertainty as much as possible. Thus researchers started to investigate the role of future earnings in the labor market participation of elderly. One of the first attempts to use a dynamic life cycle framework in the probability of retirement model was in given Burkhauser (1979). In this paper the retirement decision depends on the difference between the present value of pensions accepted immediately and the present value of pensions when the acceptance is delayed. When this difference is positive, a worker should accept pensions now. However such an approach ignores the value of waiting, since the only source of uncertainty in the model is the probability of living through the next period, which does not influence the results. As expected, the probability of accepting an early pension increases with the difference between the present value of the early pension over the normal one. Also as expected, the value of the earnings stream in the present job influences negatively the early pension acceptance. Surprisingly the age effect on pension acceptance is negative, although it could be explained by the fact that seniority affects wages. This paper considered pensions in the car industry in the US and compares them to social security only in the summary. For the study of the latter we may turn to another paper by Burkhauser (1980). The subject of this study is the decision to accept social security at the age of 62. The results

[3] Boskin (1977), p. 14.

[4] Boskin and Hurd (1978), p.371

Table 1.1. Literature review

Model	Paper	Main assumptions	Decision rule	Savings	Results
static	Boskin (1977)	- no uncertainty - no randomness	- none	- income from assets	- social security induces early retirement - effect of social security benefits bigger than effects of income from assets
static	Boskin and Hurd (1978)	- no uncertainty - no randomness	- none	- nonwage income	- social security and health influence significantly probability of retirement
life cycle	Burkhauser (1979)	- uncertainty – survival probability - no randomness	- differences between actual values of pension plans at different ages	- assets	- rise in the present value of early retirement benefits increases probability of early retirement - age effect on pension acceptance is negative
life cycle	Burkhauser (1980)	- no uncertainty - no randomness	- potential loss in market earnings when retired vs. change in total expected value of retirement benefits when retirement is postponed	- no	- social security system encourages early retirement
life cycle	Gordon and Blinder (1980)	- no uncertainty - randomness- individuals' tastes for leisure	- market wage vs. reservation wage, which takes retirement and retirement benefits into account	- no	- social security has little effect on retirement decisions - age effects and pension plans influence retirement
life cycle	Burtless and Moffitt (1985)	- uncertainty – survival probability - randomness – tastes for work heterogeneous	- optimal stopping rule: utility from retiring vs. utility from postponing retirement	- assets	- social security has rising and significant impact on the timing of retirement and hours of work after retirement up to age 62, then it falls
life cycle	Mitchell and Fields (1984)	- uncertainty – survival probability - randomness – errors have Weibull distribution	- utility from retiring now vs. utility of the next closest retirement ages	- no	- effects of changes in pension plans and social security on retirement are small
life cycle	Diamond and Hausman (1984)	- no uncertainty - randomness – unobserved individual factors	- reduced form specification	- wealth	- pensions and social security have positive influence on probability of retirement - bad health increases probability of retirement
life cycle	Hausman and Wise (1985)	- no uncertainty - randomness – unobserved individual factors	- reduced form specification	- liquid assets	- strong effect of social security benefits on probability of retirement

Table 1.1. Literature review (continued)

Model	Paper	Main assumptions	Decision rule	Savings	Results
life cycle	Burtless (1986)	- uncertainty – survival probability - randomness – heterogeneous preferences	- consumption financed from additional year of working vs. leisure from year of retirement	- wealth	- increase in social security benefits has small negative effect on retirement age
	Anderson et al. (1986)	- no uncertainty - randomness - unexpected change in social security wealth	- expected vs. actual values of explanatory variables	- wealth	- increase in social security wealth increases significantly probability of early retirement
structural life cycle	Gustman and Steinmeier (1985)	- no uncertainty - randomness – weight of leisure and individual preferences	- utility of leisure vs. utility of consumption	- no	- increase in the retirement age and more actuarially neutral social security delay retirement and increase work activity
	Gustman and Steinmeier (1986)	- no uncertainty - randomness – weight of leisure and individual preferences	- utility of leisure vs. utility of consumption	- no	- simulated distributions captures the peaks in actual retirement distribution
	French (2000)	- uncertainty – wage, health, survival - randomness - wages have an AR(1) component	- consumption and leisure now vs. consumption and leisure in the future	- control variable	- cuts in social security benefits increase labor supply
	Gustman and Steinmeier (2002)	- uncertainty – survival probability - randomness – weight of leisure and individual preferences, but not time preference rates	- utility of leisure vs. utility of consumption	- assets	- increase in the early retirement age causes substantial delay in retirement
option value	Stock and Wise (1990a) Stock and Wise (1990b)	- uncertainty – survival probability - randomness – unobserved determinants of retirement, can be correlated	- expected present value of continued work vs. expected present value of retirement	- no	- very strong influence of pension plans on the decision to retire
	Lumsdaine et al. (1997)	- uncertainty – survival probability - randomness – unobserved determinants of retirement, can be correlated	- expected present value of continued work vs. expected present value of retirement	- no	- pension plan provisions changes have strong effect on retirement - social security provisions changes have weak effect on retirement
	Samwick (1998)	- uncertainty – survival probability - randomness- heteroskedastic and serially dependent error term	- expected present value of continued work vs. expected present value of retirement	- assets	- accrual of retirement wealth has more influence on retirement that its level

Table 1.1. Literature review (continued)

Model	Paper	Main assumptions	Decision rule	Savings	Results
dynamic programming	Rust (1989) Rust (1990)	- uncertainty – survival, health, martial status, employment status - randomness – random components of state variables follow the parametric Markov transition density, uncorrelated	- comparison of expected discounted values of utility resulting from different retirement and consumption decisions	- wealth	- poorer and less healthy workers are more likely to retire earlier
	Berkovec and Stern (1991)	- uncertainty – survival probability - randomness – random components of wage follow the extreme value distribution, other errors - the normal distribution	- current and expected flows of wages, pensions and leisure while working vs. current and expected flows while retired	- no	- bad health, age, low education increase probability of retirement
	Lumsdaine et al. (1992) Lumsdaine et al. (1994)	- uncertainty – survival probability - randomness – errors drawn from normal distribution or from the extreme value distribution	- utility of retiring now vs. utility from working one more year and retaining the option to retire	- no	- dynamic programming and option value performed equally well, with slightly better fit for dynamic programming and slightly better forecast for option value - no significant difference between retirement behavior of men and women
	Blau (1994)	- no uncertainty - randomness – permanent and spell specific random effects, and NIID disturbance	- reduced form specification	- assets	- social security incentives cause people to retire exactly at age 65 - lagged endogenous variables influence probability of retirement
	Christensen and Gupta (1994)	- no uncertainty - randomness – utility shock following extreme value distribution	- utility from retiring now vs. utility from postponing retirement, given the work status of the spouse	- no	- husbands have at least as strong preference for leisure as wives - evidence of complementarity in leisure
	Daula and Moffitt (1995)	- uncertainty – probability of not leaving by a certain time - randomness - serially correlated error terms	- weighted sum of current and future wage differences when staying in the military or when leaving	- no	- military retirement benefits have the strongest effect on reenlistment decision
	Rust and Phelan (1996)	- uncertainty – survival, health, martial status, wage and social security benefits - randomness – unobservable state variables follow extreme value distribution	- expected, discounted utility from retiring vs. expected discounted utility from postponing retirement	- no	- social security incentives are especially strong for those with low incomes and no health insurance - model is able to capture peaks in retirement

show that the social security system encourages early retirement, even more so, since it is not actuarially fair.

Opposite result was found by Gordon and Blinder (1980), who claim that social security has little effect on retirement decisions and that the major causes of retirement are aging effects and pension plans. If the reservation wage exceeds the market wage, it means taking the retirement decision. There is no uncertainty from an individual point of view, like for example the survival probability in papers above. However, the model includes randomness: through the normally distributed error term in the wage equation and through the individual differences in tastes, influencing the weight of leisure in the reservation wage equation. This is an example of a frequent treatment of randomness in retirement models: if shocks in incomes are allowed (what is often not a case) they are usually transitory. Any other unexpected changes or persistence of the effects of shocks are attributed to some unobserved characteristics of preferences. The results of estimation show that age affects wages and changes labor-leisure preferences by increasing the reservation wage (the growth rate of the reservation wage is higher than the decline rate of the real wage). Pension plans cause people to retire at the age of the eligibility for the pension. Social security has some influence on the reservation wages after the age of 62, but this effect is economically unimportant. The reason why pension plans have a stronger effect than social security is explained by the mandatory retirement age in many pension plans and the reduction in the discounted present value of pensions if a worker continues working after passing the eligibility age.

This result was partially confirmed by Burtless and Moffitt (1985), who found that the impact of social security on the timing of retirement and hours of work after retirement is smaller after 62, however in the late 50's this impact is significant and rises with age. This paper is the first one we are aware of in which retirement is analyzed as an optimal stopping problem: an individual compares the utility from retiring now with the utility from postponing retirement. To maximize utility an individual must choose jointly a date of retirement and the number of working hours after retirement. The social security inclusion in the budget constraint causes its nonlinearity, because retirement benefits depend among other factors on the age of retirement. Since social security both increases wealth and imposes the earnings test, the overall impact of its inclusion into the budget constraint is to reduce postretirement hours of work. As in Gordon and Blinder (1980) preferences depend on age. In fact, if preferences were independent of age, there would be no retirement in the model. The hours of postretirement work are clustered around the social security exempt amount. Simulations show that a reduction of benefits or an increase in the normal retirement age would increase retirement ages and hours worked.

As in the paper above, usually in order to find the effects of changes in the retirement system it is necessary to simulate them. Nevertheless sometimes it is possible to avoid simulations. The influence of differences in income opportunities of retirement plans on retirement patterns is a subject of Mitchell and Fields (1984) paper, which compares 10 different pension plans. Since the plans are different, it is no longer necessary to include heterogenous preferences in the model, as was the case in studies presented above. Both models used to analyze the retirement decision show that workers with different pension plans answer differently to the incentives resulting from them. According to this paper, results of changes in pension plans and in social security are of similar magnitude. Moreover, the effects of changes in both of these systems on the retirement age are small. It is also suggested that workers sort themselves according to pension plans offered by firms.

The problem with the models mentioned above is that, although they give many useful and important information, they do not allow for unexpected events. The uncertainty and randomness appear in them as the probability of survival, a transitory shock in earnings or random components of heterogenous preferences, but their formulation prevented inclusion of uncertainty about future values of variables. However, it is important to know what happens if there is sudden change of pension plan's rules, a rise in wages, new job opportunities or a decline in health. As Diamond and Hausman (1984) said: "...These uncertainties imply that people are continuously reconsidering their plans for retirement and wealth accumulation as their economic and health positions develop..."[5]. In order to take it into account they build and estimate a hazard-rate model of retirement, where uncertainty is a part of the equation describing the hazard rate, as a random variable defining unobserved individual factors. The hazard model allows for variables that change over time, like health, wealth, etc. They found that both pensions and social security have a positive effect on the probability of retirement. Both permanent income and wealth reduce the probability of retirement, while a shift from good to bad health

[5]Diamond and Hausman (1984), p. 97.

increases it substantially.

However, there is one problem with the Diamond and Hausman model: for all the discussion of the importance of uncertainty, there is not much said about it in the results. There is a random variable, and the expected values of variables are taken into account, but we do not see any effect to the probability of the change, no reply to risk. A quite similar approach was taken in Hausman and Wise (1985) paper, in which one of the models considered was the proportional hazard model. Again the results show a strong effect of social security benefits on the probability of retirement, while about the uncertainty we only learn that the pattern of the hazard rates motivates the assumption that the unobserved random terms induce proportional shifts in the hazard rate.

The sudden changes of variables, in this case the unanticipated benefit increases, are the subject of the paper by Burtless (1986), in which he examines the effect of social security changes in the US in 1969 and 1972. However, Burtless is studying the effects of an actual change, and not the uncertainty that this change will take place. In contrast to Diamond and Hausman (1984) and Hausman and Wise (1985) the analysis is conducted by maximization of the lifetime utility function. The crucial component of the lifetime budget constraint is the discount rate of future wages and benefits, which determines the location and shape of this constraint. Contrary to previous models the discount rate is estimated, not assumed a priori. The retirement age is obtained by calculating the demand for retirement leisure consumption. The unanticipated increases in retirement benefits are represented as vertical shifts in the budget constraint. The estimation results suggest that increases in social security have a small influence on retirement ages causing them to decline, although it is somehow stronger at the kink points at ages 62 and 65.

Anderson et al. (1986) studied the same reforms, which were analyzed by Burtless (1986). Their aim was to check what was the effect of an unanticipated rise in social security benefits on the retirement plans. The model is much simpler than that of Burtless - there are two periods, in the first we observe the current labor supply and the planned future labor supply and in the second we compare the plans with actual decisions. Social security enters the model as part of the initial wealth in period one. The retirement status in period two relative to the earlier plans is defined by a trichotomous choice variable: retired earlier than planned, on time, later than planned. The estimation results show that increases in social security wealth increased the probability of retiring early and decreased the probability of retiring late. This result confirms that of Burtless, but in this case the change is significant: over 40 percent of the plans were inaccurate.

1.2.3 Structural life cycle models

As Burtless (1986) and Anderson et al. (1986) showed, the availability of information about future developments and incentives, and uncertainty about them, play an important role in the labor supply and retirement decisions. However the previous models were not able to capture the constant changes and adjustments of the retirement plans, which occur every time that the new information becomes available. Therefore it was necessary to build new models. One of the best examples of this structural representation of retirement and labor supply was the model presented by Gustman and Steinmeier in 1985 and developed further in their paper of 1986. Gustman and Steinmeier (1985) study the effect of a system change, which is introduced slowly and gradually over time. They analyze the long-run effects of the 1983 social security reforms in the US, which raised the normal retirement age, increased the delayed retirement credit and lowered the reduction rate for earnings over the test amount. It means that the aim of the paper is to check how changes in incentives affect the labor supply and the retirement decisions of elderly. The utility is optimized with respect not only to consumption and leisure, but also to work effort. As in Gordon and Blinder (1980), individuals' valuation of leisure increases with age. There are two time invariant stochastic terms: the weight of leisure and the curvature of the indifference curves, i.e. the substitution between consumption and leisure. When the first one is high, the individual quits full-time work. The choice between part-time work and full retirement depends then on the second stochastic term. If it is high the individual is more likely to choose complete retirement.

In their next paper Gustman and Steinmeier (1986) develop the model presented above to make it more suitable for analyzing retirement behavior and potential policy changes. Using detailed information on the life cycle compensation profiles, they are able to explain the peaks in U.S. retirement at age 62 and 65. However, the major drawback of this model is that it does not consider uncertainty, since it would make it computationally infeasible. This means, as the authors admit themselves, "...that the model is not appropriate for analyzing short term labor supply responses to unanticipated change in incentives.."[6]. It

[6] Gustman and Steinmeier (1986), p. 556.

is used to derive the optimal time paths for consumption and leisure. A function relating compensation to leisure can be nonlinear and even discontinuous.

In the model there are three components of compensation: wages, pension benefits and social security benefits. The wage component of the compensation is obtained by estimating the wage equations for full- and part-time jobs. The pension and social security components are calculated as the change in present value of benefits arising from working full-time another year. The estimation results suggest that an increase in compensation would reduce the percentage of individuals working full-time and increase the percentage of those fully retired, with mixed effects on those working part-time. Although there are no dummies for age 62 and 65 in the model, the patterns of observed and simulated distributions are very similar. Thus again we see that the social security rules, with eligibility for early retirement at 62, and for normal retirement at 65, have decisive impact on the timing of retirement.

1.2.4 Option value models

The models presented above showed that an ideal model to study retirement should include the structure of earnings and retirement wealth/benefits and uncertainty about future. These characteristics were with high degree of success combined in the dynamic programming approach. However, following the example of Lumsdaine and Mitchell (1999) we depart now from the chronological review, and instead of analyzing dynamic programming models we turn first to their simpler form - the option value models. This approach was developed by Stock and Wise (1990a, 1990b), who analyze the effect of a pension plan in a large firm on retirement decisions. They claim from the beginning that firm pension plans create much greater incentives for early retirement than social security, since the most advantageous conditions are offered if a worker leaves at a certain age, quite often the early retirement age. The term "option value" comes from the worker's decision not to retire and thus preserving the option to retire later. There are two key aspects of the model. First, a person continues to work for as long as the expected present value of continued work is greater than the expected present value of immediate retirement. Second, the decision is adjusted with the new information that come with age. The most important simplifying assumption is that retirement is based on the maximum of the expected present values of future utilities if retirement takes place now compared to each of the future ages.

Savings, and any assets other than the present value of the firm's pension and social security benefits, are not included into the model, since, as the authors claim, their effect on retirement is small relative to the social security wealth, and a large majority of elderly have little wealth, other than retirement benefits, pension and housing. Thus the retirement decision function is based on wage earnings and retirement benefits. The decision rule is: continue to work at year t if the expected gain from postponing retirement to the year yielding the highest expected value is positive. Otherwise retire.

In the model there are two individual-specific and time dependent random effects, which should capture the unobserved determinants of retirement, like preference for work versus leisure, health status, etc. According to Stock and Wise "...there should be considerable persistence in these random effect over time (...) Such persistence is captured by assuming that the random individual effects follow a Markovian or first order autoregressive process..."[7]. In particular in empirical work they treat random effects as evolving according to a random walk. However, the test of the estimated model rejects the assumption of a random walk for unobserved determinants of retirement. In our opinion the reason for this rejection is that the persistence of shocks is mainly due to the randomness in wages, which follow a random walk. We will return to this problem in the last section of the review.

The estimation results suggest that a dollar of income without work has more value than a dollar of income while working. Simulations show a very strong influence of pension plans on the decision to retire. An increase in the early retirement age would make many workers to postpone their retirement. On the other hand some of them would decide to retire even earlier, in order not to wait so long. Change from defined benefit to defined contribution plan would provoke even bigger effects, increasing the proportion of workers who would choose early retirement.

Let us give two other examples of the option value model. First, Lumsdaine et al. (1997) simulate the effects of changes in pensions and social security on retirement rates. Changes in pension schemes have a strong influence on the timing of the retirement decision, but those of social security relatively weak, since the pension plan provisions dominate those of social security. We cannot, however, ignore social security effects

[7]Stock and Wise (1990a), p. 1160

completely, since their elimination would have a substantial effect on retirement rates. The authors suggest coordinating changes in firm pension plans and social security provisions. Their simulations show that such a coordination would give substantial effects, reducing the departure rates. In the second example the relative importance of incentive effects of pensions and social security is studied by Samwick (1998). Due to a data set that combines representative household survey with detailed pension data he is able to check if Stock and Wise (1990) results are true on a general level and not only in a single large firm. He uses a reduced form alternative of their option value model, based on assumed parameter values. Both the option value and the accumulation of retirement wealth significantly reduce the probability of retirement. The important result is that first, the change in retirement wealth is more important than its level and second, pensions have much more influence on retirement wealth and retirement decisions than social security. The author claims that in the postwar period a rise in the pension coverage was responsible for up to one-fourth of the decrease in the labor force participation. When information about pensions was excluded from the model, the retirement wealth accruals and pension coverage appeared to be insignificant. Comparisons with Stock and Wise (1990) show that their estimates "...likely understate the full impact of pensions and Social Security on retirement from the labor force..."[8].

1.2.5 Dynamic programming models

In contrast to the option value models, in the dynamic programming models the retirement decision is taken not on the basis of the maximum of the expected present values of future versus current utilities, but on the basis of the expected value of the maximum of current versus future options. It allows us to avoid some problems that may arise in the option value model, as described in Lumsdaine et al.: (1992) "...The expected value of the maximum of a series of random variables will be greater than the maximum of the expected values. Thus to the extent that this difference is large, the Stock-Wise option value rule underestimates the value of postponing retirement. And to the extent that the dynamic programming rule is more consistent with individual decisions than the option value rule, the Stock-Wise rule may undervalue individual assessment of future

[8] Samwick (1998), p. 230.

retirement options..."[9]. Since dynamic programming is based on the expected value of the maximum of current versus future options, its evaluation requires an $n-1$ dimensional integration of n time periods. The solution of such a problem is very complex, unless the dynamic programming rule is approximated by simplifying the covariance structure to reduce the number of dimensions (like in Rust (1989) or Berkovec and Stern (1991)). In contrast, the option value model yields a much simpler separation property, especially with the assumption that the individual specific random effects follow a random walk. Finally, in the option value model random disturbances can be correlated, and in dynamic programming they are independent (although this is not an iron rule - see Daula and Moffitt (1995)).

In dynamic programming everything depends on the assumed error structure, since the decision rule evaluates the maximum of future disturbance terms. Since there is no single dynamic programming model we start with the work of Rust (1989, 1990). Rust's objective is to derive and estimate a model of the retirement behavior of older male workers. They optimize jointly over both the age of retirement and future consumption, which makes the model very complex. The retirement behavior is modeled as a discrete time Markovian decision problem. The solution to the decision problem is built from a sequence of decision rules maximizing the expected discounted utility over an infinite horizon. The expected utility contains current values of the state variables (among others accumulated wealth, total income from earnings and assets, and social security benefits), as well as the unobserved state variables. The decision rules belong to a choice set with a finite number of feasible values for each decision rule given the observed state variable. This value function is the unique solution of the Bellman equation maximizing the sum of current and expected utility with respect to the elements of the choice set.

The option value model was developed to avoid the complexity of dynamic programming models like the one presented above. There are, however, also simpler versions of dynamic programming. Berkovec and Stern (1991) estimate a discrete-time, discrete-state, dynamic programming model. The utility flows contain time-specific errors, of which the current values are known, but the future ones not. There is also a time preference component of the discount factor, which is unobserved by the econometrician. Thus the model includes uncertainty about future wages. The problem with this model

[9] Lumsdaine et al. (1992), p. 28

is that it incudes only the level of retirement benefits and not the features of the social security system. Thus it is not able to capture the peaks in retirement rates. According to Lumsdaine and Mitchell (1999) it is also not able to handle sharply nonlinear budget constraints, which is also a feature of the Rust (1989, 1990) model.

Lumsdaine et al. (1992, 1994) compare two dynamic programming models, one of which follows Berkovec and Stern (1991), with the option value approach. In their specification an individual has only two choices: retire now, or work for a year and retain the option to retire next year. Their basic mathematical model is a good example of the dependence of dynamic programming on the assumed error structure. The value function is given by

$$V_t = \max\left(\bar{V}_{1t} + \varepsilon_{1t}, \bar{V}_{2t} + \varepsilon_{2t}\right) \tag{1.1}$$

where

$$\begin{aligned}\bar{V}_{1t} &= U_W\left(Y_t\right) + \psi\pi\left(t+1 \mid t\right) E_t V_{t+1} \\ \bar{V}_{2t} &= \sum_{\tau=t}^{H} \psi^{\tau-t}\pi\left(\tau \mid t\right) U_R\left(B_\tau\left(t\right)\right)\end{aligned}$$

where $U_W\left(Y_t\right)$ is utility from earned income, $U_R\left(B_\tau\left(t\right)\right)$ utility from retirement benefits, $\pi\left(\tau \mid t\right)$ is the probability of surviving to year τ given that an individual was alive in year t, ψ is a discount factor, H is the maximum possible year of life. ε_{1t} and ε_{2t} are the error terms, which are assumed to be identically, independently distributed. A worker chooses to retire when $\bar{V}_{1t} + \varepsilon_{1t} < \bar{V}_{2t} + \varepsilon_{2t}$, otherwise she/he continues working.

If, as in Berkovec and Stern (1991), errors are drawn from an extreme value distribution with scale parameter σ, then

$$\frac{E_t V_{t+1}}{\sigma} \equiv \mu_{t+1} = \gamma_e + \ln\left[e^{\frac{U_W\left(Y_{t+1}\right)}{\sigma}} e^{\psi\pi(t+2|t+1)\mu_{t+2}} + e^{\frac{\bar{V}_{2t+1}}{\sigma}}\right] \tag{1.2}$$

where γ_e is Euler's constant. Equation (1.2) can be solved by backward recursion with the terminal value coming from the condition that $\mu_M = \bar{V}_{2M}$, where M is the age of mandatory retirement. The assumption about the extreme value distribution provides a closed form expression for the probability of retirement in year t:

$$\Pr\left[\text{retire in } t\right] = \Pr\left[\bar{V}_{1t} + \varepsilon_{1t} < \bar{V}_{2t} + \varepsilon_{2t}\right] = \frac{e^{\frac{\bar{V}_{2t}}{\sigma}}}{e^{\frac{\bar{V}_{1t}}{\sigma}} + e^{\frac{\bar{V}_{2t}}{\sigma}}} \tag{1.3}$$

If errors are independent draws from the normal distribution with $N(0,\sigma^2)$, then the probability of retirement is simple:

$$\Pr[\text{retire in } t] = \Pr\left[\frac{\varepsilon_{1t}-\varepsilon_{2t}}{\sqrt{2}\sigma} < \frac{\bar{V}_{2t}-\bar{V}_{1t}}{\sqrt{2}\sigma}\right] = \Phi(a_t) \tag{1.4}$$

where $a_t = \frac{\bar{V}_{2t}-\bar{V}_{1t}}{\sqrt{2}\sigma}$. Then the recursive equation becomes:

$$\frac{E_t V_{t+1}}{\sigma} \equiv \mu_{t+1} = \frac{\bar{V}_{1t+1}}{\sigma}\left[1-\Phi(a_{t+1})\right] + \frac{\bar{V}_{2t+1}}{\sigma}\Phi(a_{t+1}) + \sqrt{2}\phi(a_{t+1}) \tag{1.5}$$

where $\phi(\cdot)$ is the standard normal density and $\Phi(\cdot)$ the cumulative normal distribution function.

After the individual specific terms are included into the model as random terms, the utility functions $U_W(Y_t)$ and $U_R(B_\tau(t))$ also have different forms, depending on the assumed error structure. Thus we can see that within one basic framework, as in (1.1), the form of the model depends on the assumed error structure, and there is no unique dynamic programming model.

The main goal of Lumsdaine et al. is to compare the option value and the dynamic programming approach. Both models performed equally well, with a slightly better fit for the latter and a slightly better forecast for the former. The results of both models show that the retirement rates (both actual and predicted) show jumps corresponding to specific provisions of the pension plans or social security. However, they are not able to explain fully the peak in retirement at the age 65. There appears to be some effect unrelated to earnings or benefits.

Most of the dynamic programming models used in the literature are simplified as in Berkovec and Stern (1991). Nevertheless, there are some general and sophisticated models, like the one formulated in Rust and Phelan (1997) to estimate the joint labor supply and retirement decisions. They include in the analysis not only social security but also health insurance. Their model treats labor supply and application for social security benefits as separate decisions. Social security, Medicare, private pensions and health insurance plans are viewed as dynamic incentive schemes, while wealth is not taken into account. The decision rule and utility depend on the vector of the details of social security and Medicare, and on the vector of individual preferences and beliefs. The dynamic programming model gives a sequence of optimal decision rules which maximizes the individual's expected discounted utility. The estimation is conducted in two stages: in the first stage parameters characterizing individual's beliefs about uncertain events are estimated. In the second stage, using obtained estimates, the recursive value function of expected discounted utility is solved numerically. It is worth noticing that those who expected to receive private pensions were excluded from the data. The dynamic programming model appears to be able to capture peaks in retirement at ages 62 and 65 and the impacts of income, employment, health insurance and individual characteristics on behavior. It also predicts correctly who is more likely to apply for early retirement and shows that it is optimal to apply for social security at age 65, regardless of the later labor supply decisions. Expectably social security and Medicare have strong disincentive effects on labor supply. Exclusion of social security from the model causes the disappearance of peaks in hazard rates for labor force at ages 62 and 65. However, since the model takes into account only men with social security as pension plan, it cannot give a definite answer to the problem of early retirement.

1.2.6 Other retirement models

Lumsdaine et al. (1992), after comparing the option value with dynamic programming, conclude that both models fit the sample data equally well, with "...a slight advantage to the normal dynamic programming model..."[10]. Thus the only way to choose between them is on the basis of numerical complexity, and there is no doubt that the option value model is simpler. However, in the literature there are more examples of different forms of dynamic programming models than of option value models. A number of papers used some simplified version of dynamic programming. Blau (1994) derives from the dynamic programming framework of labor force in transition a discrete time hazard model of the quarterly transition rates among three labor force states (full-time work, part-time work, and retirement). The use of quarterly data allows to observe the dynamics which are otherwise invisible, like the fact that 23.9% of workers retire within three months of their 65th birthday - the age of full eligibility for social security benefits. The results suggest that, although social security benefits are very important in influencing retirement behavior, their changes may be not the main cause of decline in labor force participation of elderly. The results also show that the dynamic aspects of labor supply of elderly, through several lagged endogenous variables, have important effects on transition rates.

[10] Lumsdaine et al. (1992), p. 50.

A somehow similar approach, as far as the connection between dynamic programming and retirement hazard is used, is presented in Christensen and Gupta (1994) who consider the joint retirement decision of married couples. In their model the decision to retire or not is based on the husband's or wife's retirement status. Randomness enters the model through future utilities, as some utility shock, following the extreme value distribution of errors from Berkovec and Stern (1991). The results of the model show some signs of the complementarity in leisure times of the couples.

Another simplified model is used by Daula and Moffitt (1995) to estimate military reenlistment. This model is particularly simple because the choice of the military reenlistment is a simple leave-stay decision. Then "...the retention decision is a linear function of a simple weighted sum of current and future wage differences..."[11]. This simplicity allows for the introduction of serial correlation in the error terms through the assumption of unobserved heterogeneity in the form of a random individual effect. The fit of the model is not better than that of other models, but it is able to yield accurate out-of-sample predictions.

In the recent paper, Gustman and Steinmeier (2002) build a model of retirement and savings. It is a life cycle model, as in their previous papers. However, this time they relax the assumption made in all other models about the constant time preference rates across the population. Their estimates suggest that time preference rates are heterogenous, and in fact bimodal. Individuals may be heterogenous in the discount rate, the value of leisure relative to consumption and the utility value of leisure. Since part of the retirement benefits may be given as spouse benefits or survivor benefits, the death of the two spouses is modeled as stochastic. An individual is assumed to start without any assets, and to accumulate them later. Assets are non-negative, in order to prevent high time preference individuals from borrowing against future incomes and benefits. The model with heterogenous time preferences generates peaks in the retirement rates at ages 62 and 65, which are very similar to observed peaks. Simulations suggest that an increase in the early retirement age would shift much of the peak at age 62 into a peak at the new retirement age.

All the models presented in this survey concentrate on elderly around retirement age. French (2000) studies effects of changes in social security on the labor supply over the

[11] Daula and Moffitt (1995), p. 500.

whole life cycle. He does so by developing and estimating a model of optimal lifetime decision-making and simulating the effect of reducing social security benefits. Individuals maximize their lifetime utility by trading current consumption and leisure for future ones. The level of utility depends also on health condition. Wage uncertainty is treated as an innovation in wages, of which only the distribution is known in an autoregressive component of wages. There is also uncertainty about health and survival. The savings decision is an additional control variable. The level of savings results from the asset profile, the asset accumulation equation and the assumed rate of interest. The model predicts that individuals accumulate precautionary savings when young. Other results show that the labor force participation rates should not decline as fast as they do according to the data. Simulations of cuts in social security benefits, using estimated profile of preferences, show an increase in hours worked after age 50, because of reduction in lifetime wealth.

1.2.7 Randomness in wages

French (2000) is the only author studying retirement we are aware of in which wages are assumed to follow an AR(1) process, and the estimated parameters of this process show that it is almost a random walk. It is surprising that this assumption was not used in other retirement studies, which, if they allow for randomness in wages, assume often that wages have a white noise component, like in Rust and Phelan (1997). This is despite the fact that a number of job and earnings studies found that wages follow approximately an AR(1) process, usually a highly persistent one. This was shown among others in Abowd and Card (1987,1989), Carroll (1992), Topel and Ward (1992) or Farber and Gibbons (1996). For example French (2000) reports an AR(1) process with an autoregressive parameter equal to 0.977. In Topel and Ward the value is almost the same: 0.97, but they claim that the evolution in wages is approximately a random walk with drift. Farber and Gibbons' estimation yields a lower value of 0.91, but is still close enough to one to assume that income follows a random walk. In many cases the results of estimations show that changes in wages are composed from two elements: one transitory and other permanent. The results from different models are surprisingly consistent, with standard deviations of both permanent and transitory shocks in the range of 0.1 and 0.17. In the long run, and retirement models often consider the long time horizon, only permanent shocks play a role, while transitory shocks can be ignored. This suggest again that wages follow some kind of an AR process.

In contrast to retirement studies, consumption studies often consider the case in which wages do follow a random walk (or Brownian motion in the continuous time), for example Deaton (1989) or Caballero (1990). In some cases it is assumed that only the permanent income follows a random walk or a geometric random walk, like in Hall and Mishkin (1982) or Zeldes (1989), and the transitory part of income follows a different process (MA(2) in Hall and Mishkin). Carroll (1992) in turn assumes that it is the log of permanent income, which follows a random walk with drift. These assumptions allow for a much easier explanation of the sudden changes and shocks experienced by individuals and their persistence. Of course the assumptions made in the studies presented above make sense: individuals do have unobserved tastes for work and leisure, and they do have some other preferences which may influence their behavior. Also all the sudden shocks in consumption, employment, martial status, etc. may take place, thus some assumptions about their distributions may be necessary. However, the wage uncertainty is the most important source of uncertainty (together with survival probability) while retirement decisions are taken. Why the randomness in wages is considered only in some of the retirement studies, and why, if considered, despite the evidence mentioned above, wages are not assumed to follow a random walk, is something we do not understand. What is more, an assumption that wages follow a random walk makes both the analytical and empirical analysis easier. In this book the random walk assumption allows us to use the real options models to analyze retirement.

1.3 Methodology

In the thesis we study the retirement decisions of an individual worker, maximizing her/his lifetime utility subject to the lifetime budget constraint. Retirement is assumed to be irreversible. We are not concerned with the macroeconomic problems of aging populations, or the necessity to reform particular retirement systems. As the literature review above shows, the change of retirement institutions is crucial, but we would like to know if proposed changes are the right ones. Our objective is to compare the retirement behavior of an individual under two theoretical retirement systems: perfect insurance and pure private savings. Perfect insurance is defined as an insurance contract giving a constant level of income while working and constant benefits while retired, in exchange for the total lifetime output of the insured person. In this way an individual is completely insured against any risk of a fall in labor income or productivity. In contrast, private savings act as a self-insurance: an individual insures herself/himself by saving part of the labor income and using it first to smooth consumption when wages are falling and then to finance retirement.

We do study insurance - self-insurance in the case of private savings - but, despite of what we have said above, we do not take moral hazard into account, what may affect the retirement decision: I am insured, therefore I may retire earlier. Most likely it does happen in our models: since people usually dislike working they want to retire as soon as possible. However this effect is partially countered by the results of the postponement of retirement - the retirement benefits increase, what allows for a higher consumption. When people do not like to work, consumption while retired is more efficient than consumption while working, since utility from consumption is made stronger by the utility from leisure. Therefore the overall effect of omitting the moral hazard should not be too large.

In the thesis we use a dynamic programming model. As we have seen in the literature review such models are quite often used to study retirement behavior, since they provide "... a framework that is rich enough to accurately model the dynamic structure of Social Security rules and the uncertainties and sequential nature of individuals' decision making processes..."[12]. In our case it allows us to combine different retirement schemes with stochastic wages or productivity and individuals' decisions. The individuals' decision in dynamic programming studies is to work or to retire. The basis for a decision is the comparison of the expected discounted values of the utility over the remaining lifetime, resulting from the possible decisions. This is usually done through a value function, which is the solution of the Bellman equation combining current and expected utilities. According to Lumsdaine et al. (1992), since dynamic programming evaluates the maximum of future disturbance terms, the outcome of the model depends on the assumed error structure. In this book the equivalent assumption is the one about the worker's wage or productivity following geometric Brownian motion with drift[13]. The law of motion of wages is given

[12]Rust and Phelan (1997), p. 791.

[13]The empirical justification of the assumption that income follows Brownian motion with drift is in Topel and Ward (1992); see the discussion in the previous section.

by:

$$dW(t) = \mu W(t)\,dt + \sigma W(t)\,dv(t) \tag{1.6}$$

where parameter μ is the deterministic drift parameter, σ the instantaneous standard deviation, and dv is the increment of a Wiener process[14], with $E(dv) = 0$ and $V[dv] = E[(dv)^2] = dt$. The definition of geometric Brownian motion shows that the absolute changes in wages are lognormally distributed. It is necessary to assume that the discount rate is larger than the drift, otherwise the expected future values of wages would always be higher than its current value, rising even to infinity, hence waiting would be a better policy and there would be no optimum. The labor income risk is the only source of risk in the book, and since we use the infinite lifetime, there is no life insurance.

We do not use the dynamic programming in the same way as in the models presented above. Instead we use the dynamic programing framework in the real options models. These models are not widely used in labor and public economics. Nevertheless they can be very useful. One of the examples we are aware of is the Pfann (2001) paper on the worker's decision about the optimal separation time when offered an outside option but faced with the cost of quitting. The real options models are designed to analyze investment under uncertainty. The retirement decision may be treated as an investment process: first we collect capital/retirement wealth, and if we have accumulated enough we invest/retire, depending on the actual and expected prices/wages. The result of the model is a certain critical price/wage level, with investment/retirement optimal if price/wage is on one side of it and continuation of waiting/work on the other. The real options model is presented in detail in Chapter 2.

If the dynamic programming model has a finite horizon the last decision at its end has nothing to follow, and can be solved by standard optimization. Then working backwards it is possible to solve the initial problem. If the planning horizon is infinite, the problem is simple because of the recursiveness: each decision leads to an identical problem. Therefore it makes the numerical solution much easier, and sometimes allows even for an analytical

[14] If $v(t)$ is a Wiener process, then any change in v, Δv, in time interval Δt, satisfies:

1. $\Delta v = \epsilon_t \Delta t$, where ϵ_t is a random variable with distribution $N(0,1)$
2. ϵ_t is serially uncorrelated, i.e. $E[\epsilon_t, \epsilon_s] = 0$ for $t \neq s$. Therefore the values of increment Δv for any two different time intervals are independent.

solution. We conduct the analysis assuming the infinite lifetime with a constant discount rate, which is equivalent to the constant death rate - at each moment there is a certain small probability of dying. This notion is obviously an oversimplification, but it allows for the analytical analysis. Moreover, it is possible that at the beginning of life, or rather at the start of working career, people do not think about death or old age and treat them as something belonging to very distant future. In order to make the problem more realistic in Chapter 2 we consider also an increasing probability of death. In such a case the lifetime is theoretically infinite, but the probability of death is rising with time, slowly approaching one. Thus in practice an infinite lifetime is impossible. The drawback of this approach is that it is much more difficult to analyze and we are not able to get a final solution.

The retirement problems in this thesis are optimal stopping problems in continuous time. Models presented in the literature review are in discrete time, since they usually have a finite horizon and are used for econometric analysis of the real data. We are interested in theoretical analysis and use an infinite lifetime. With the assumption of Brownian motion as the underlying stochastic process it is possible to use Ito's lemma to transform the Bellman equation into a differential equation defining utility or the value function (depending on the specification). The problem is then reduced to finding a solution of the differential equation.

An important problem in all the models is the precaution. We use the constant relative risk aversion utility function, thus risk aversion is taken into account, but precaution depends on the third derivative of the utility. It is discussed in more details in the next section. The combination of the constant relative risk aversion utility with geometric Brownian motion means that everything is relative to income or wealth, but wealth itself depends on the path followed by wage. There is, however, the problem with the drift of wages or productivity. If the drift parameter is positive then in the infinite lifetime the wages should overall increase. Therefore there is no need to act precautionary - sooner or latter wages will be high and it will be possible to pay all the debts. One may even ask why an individual would like to retire in such a case. It is surely better to wait for wages to grow and even to postpone consumption until retirement - the retirement benefits should be so high that additional utility of consumption while retired should clearly offset the loss of not consuming while working. Therefore the analysis is conducted for both positive and negative drifts. Negative drift introduces a precautionary motive - if wages are expected to fall, we cannot wait very long for they increase before retiring.

In all chapters we use numerical simulations to test the analytical results or to solve the model when there is no analytical solution. We have chosen the values of the parameters, which are suggested by empirical studies of wages and consumption or are often used in the literature. The positive trend in wages is equal to 0.03, according to Carroll (1992,2001). The negative trend in wages is assumed to be -0.01, which allows us to consider a negative trend without moving too far from realistic values The standard deviation is 0.1. This value is within range of Topel and Ward (1992) results and it is equal to Carroll (1992) result for the standard deviation of the permanent and transitory shocks to income. Finally, the interest rate, which is assumed to be equal to the time preference rate, is equal 0.05. It is a compromise between the real interest rates, which are currently very low, and the individuals' time preference rates which are much higher, since people have usually quite a short time horizon.

1.4 Importance of the third derivative of utility

Since throughout this work we are looking for expected utility or trying to approximate the utility function, the properties of the utility function are of particular importance. Theoretical works on utility and consumption all underline the role of the third derivative of the utility function. As Caballero wrote in his 1990 paper: "...whenever the utility function is separable and has a positive third derivative (...) an increase in labor-income uncertainty, when insurance markets are not complete, will reduce current consumption and alter the slope of the consumption path..."[15]. This remark is useful for the study of consumption and private savings with and without retirement. However, we are more interested in the retirement threshold and how it depends on risk aversion and precaution. Also, in this case we cannot neglect the third derivative of the utility function. According to Deaton "...Risk-aversion is controlled by the degree of concavity of the utility function (...) but the degree of precaution is the degree of convexity of the *marginal* utility function. Risk-aversion depends on the second derivative of the utility function, and precaution on

[15] Caballero, 1990, p. 113.

the third derivative..."[16]. The detailed, formal discussion of these claims is presented in Kimball (1990). However, the role of the third derivative of the utility function can be shown by a simple example:

Suppose that life consists of two periods. The total lifetime consumption C is stochastic. The budget constraint is:

$$C = C_1 + C_2$$

Expectations of C are equal: $E(C) = 2C_0$. The problem is how to choose consumption in the first period in order to maximize the lifetime utility.

In the optimum, the marginal expected utility of consumption is set equal for both periods. Therefore

$$U'(C_1) = E\left[U'(C - C_1)\right] \tag{1.7}$$

Using Taylor expansion around C_0 on both sides of (1.7) yields:

$$\begin{aligned} & U'(C_0) + U''(C_0)(C_1 - C_0) + \frac{1}{2}U'''(C_0)(C_1 - C_0)^2 \\ = \; & E\left[U'(C_0) + U''(C_0)(C - C_1 - C_0) + \frac{1}{2}U'''(C_0)(C - C_1 - C_0)^2\right] \end{aligned}$$

This equation can be transformed to:

$$\begin{aligned} & U'(C_0) + U''(C_0)(C_1 - C_0) + \frac{1}{2}U'''(C_0)(C_1 - C_0)^2 \\ = \; & U'(C_0) + U''(C_0)(C_0 - C_1) + \frac{1}{2}U'''(C_0)\,E\left[(C - C_1 - C_0)^2\right] \end{aligned}$$

Then, using

$$E\left[C^2\right] - E^2[C] = V(C)$$

and simplifying, we get:

$$2U''(C_0)(C_1 - C_0) = \frac{1}{2}U'''(C_0)\,V(C) \tag{1.8}$$

We were interested in the consumption in the first period. Equation (1.8) gives:

$$C_1 = \frac{U'''(C_0)\,V(C)}{4U''(C_0)} + C_0$$

Therefore optimal consumption in the first period is defined by expectations of total consumption, its variance, and the second and third derivatives of the utility function.

[16]Deaton, 1992, p. 178.

Since the third derivative of a concave utility is positive and the second one negative, the first expression on the right hand side is negative, reducing consumption in the first period. According to Kimball (1990) precaution is measured by $-\frac{U'''}{U''}$. Thus the stronger the precautionary motive, the lower consumption in the first period. This confirms that the role of the third derivative of utility is crucial.

1.5 Structure of the thesis

This introduction is followed by four chapters. Chapters 2 and 4 study the retirement decision of an individual, or rather, in the case of Chapter 2, the decision of an insurer trying to maximize the lifetime utility of his client. In Chapter 3 we consider the optimal consumption path without retirement. Chapter 5 concludes and compares results of the models from the previous chapters.

In Chapter 2 we build a model of retirement under perfect insurance. The focus is on the relation between a worker's productivity and the retirement decision. Assuming that the value of the productivity is stochastic and follows geometric Brownian motion with drift, there is such a level of productivity for which it will be not attractive to work any longer. The worker buys an insurance, which gives a constant income while working and constant retirement benefits in exchange for the total lifetime output. To attract as many customers as possible an insurer sets the level of income and benefits as to maximize the lifetime utility of the worker. If productivity falls so low that the expected gains for the insurer are negligible, he allows the worker to retire. The model is solved for the infinite lifetime with a constant probability of death. The case of the increasing probability of death is also presented. It was however impossible to solve.

Chapter 3 analyzes the intertemporal planning of lifetime consumption when income follows geometric Brownian motion and the rate of return on savings is constant. This chapter is a first step towards the analysis of retirement if private savings are the only source of retirement wealth. In contrast to most of the existing literature on consumption, we use a continuous time framework and ignore transitory shocks. Following Deaton (1989) and Carroll (1997, 2001) and using elements of the real options approach we find a differential equation defining utility, and then transform it into the equation defining only the self-insurance component of utility. Using economic theory and empirical research we are able to formulate conditions for the utility function when either income or accumulated savings are close to zero. Then, with Taylor expansion, we can approximate the functional form of utility and find the optimal consumption path.

The question we ask in Chapter 4 is what is the retirement behavior if there is no retirement insurance, and people must save themselves for their retirement. In such a case at each moment an individual decides how much to consume and how much to save, and if enough has been saved to stop working and finance the retirement. Using the results from Chapter 3 we repeat the analysis, including now the possibility of retirement, to approximate the lifetime utility function and to find what is the ratio of expected incomes to accumulated savings for which it is optimal to retire. The model yields an equation defining this critical ratio and we can simulate the results. For comparison we conduct also numerical simulations of this threshold for different attitudes towards risk and taste for leisure. For one special case we simulate the distributions of consumption and savings at different stages of the labor career and calculate the probabilities of retirement.

Chapter 5 concludes. First, it summarizes the main results. Next, we compare the probabilities of retirement with perfect insurance and private savings. The results show without doubt that a less risky retirement scheme, i.e. the perfect insurance, motivates people to retire earlier than when retirement is financed from private savings. Finally we conclude and discuss the possible directions for further research.

Chapter 2

Retirement with perfect insurance

2.1 Introduction

As we wrote in Chapter 1 many papers tried to show which elements of existing retirement systems have the strongest influence on the decision to retire and its timing, and what would be the results of a reform of these systems. The model described in this chapter approaches the problem of retirement from a different perspective than that of expected wages and rules of social security system. We ask two questions: first, can the drop in the participation rate of elderly be explained by the introduction of better, more efficient insurance schemes? With the increased insurance coverage and the huge variety of insurance schemes it seems quite possible that workers, or their employers choose a scheme that gives higher individual utilities and offers more flexibility than older retirement systems (or, from the historical point of view, the lack of retirement system). In this chapter we consider perfect insurance, i.e. an insurance, which gives to an individual a constant income while working and constant benefits while retired in exchange for the total lifetime output. The level of income and benefits is chosen to maximize the lifetime utility.

The lifetime insurance scheme proposed above may seem unrealistic. The two most common questions it raises are: why should an insurer bother to maximize worker's utility and why is the worker allowed to retire, if, even with low productivity now, there is some possibility that it will increase in the future. Both these questions are answered when we assume that the insurer operates in a perfectly competitive market. If insurance firms are forced to compete for clients, they are forced to offer an insurance policy that maximizes worker's utility subject to the insurance company budget constraint. The worker is allowed to retire when her/his expected productivity is lower than her/his cost of continued work. The potential gain for the insurer from keeping her/him working is small in such a case and the possibility of retirement attracts clients. The same argument refers to the utility maximization - the worker buys insurance from the firm that offers the best conditions, and therefore insurance firms try to maximize utilities of their clients. The insurer has high potential gains anyway, since the insurance scheme is constructed in such a way that the most productive individuals will stay working till the end of their lives.

One may ask why such an insurance scheme in not used in practice, if it has so many advantages? The reason for this is that this scheme can work only in the absence of the principal-agent problem. The insurance does not include any control mechanism, which would give the insurer the guarantee that the insured does not shirk. On the other hand the motivation for shirking is very big: if the worker lowers her/his output on purpose, the insurer sees only a fall in productivity and may allow the worker to retire. In the model we assume that the principal-agent problem does not exist. This insurance scheme is obviously not used. It could be argued that it has some features of social security, like certain benefits not connected with the earlier contributions. However, social security is more like the saving scheme since both are last income related.

The second question of this chapter is: what is the relation between the worker productivity and her/his retirement decision? In many papers earnings are treated as more or less constant, with a sudden shock, which makes individuals retire. Worker's productivity is usually assumed to rise with age or experience, at least for a part of the labor career. However, we cannot neglect neither an occurrence of sudden falls in the productivity nor its possible decline for older agents. We also cannot neglect the presence of transitory shocks in earnings. Many of these elements may be included into the model with an assumption of stochastic productivity. If productivity is stochastic there will be such a level of productivity for which there would be no point in working any longer. Assuming the perfect insurance case in which the worker's lifetime income is equal to the expected value of her/his output from work, we will calculate the threshold value of productivity, i.e. the work-quitting level of productivity.

This chapter is organized as follows: Section 2 studies individual's utility and income when retirement is not allowed. Section 3 analyzes retirement in the infinite lifetime setup. In this case the model gives a critical, work-quitting threshold value of productivity as a function of starting productivity. In Section 4 we allow for an increasing probability of death, i.e. the lifetime can still be infinite but the probability of death is rising with time. With such a possibility included the retirement threshold is a function of both the starting probability and time. Section 5 concludes.

2.2 No retirement

2.2.1 Infinite lifetime

Before turning to the problems of the retirement decision we study utility and income of an individual living for an infinite time without a possibility of retirement. Her/his productivity, defined as the value of output, is given by the geometric Brownian motion with drift:

$$dP = \bar{\mu}Pdt + \sigma Pdv \tag{2.1}$$

where $\bar{\mu} \equiv \mu + \frac{\sigma^2}{2}$ is a drift parameter, σ a variance parameter, and dv is the increment of a Wiener process, with $E(dv) = 0$ and $V[dv] = E[(dv)^2] = dt$. The discount rate ρ is exogenous, and $\bar{\mu} < \rho$: otherwise, with the expected rise in productivity bigger than the discount rate, expected future values of productivity would always be higher than its current value, rising even to infinity. The total lifetime income of an individual is equal to the expected discounted value of lifetime productivity calculated at the beginning of the labor career, i.e. at $t = 0$. The level of income is chosen to maximize worker's lifetime utility and it is spread equally over the entire lifetime. Thus it is equivalent to the case in which worker buys an insurance giving constant benefits in exchange for her/his life-time output.

Assume that an individual lives from $t = 0$ to infinity. The lifetime utility can be written as:

$$U(Y) = \int_0^\infty \frac{1}{1-\alpha} Y^{1-\alpha} e^{-\rho t} dt \Leftrightarrow U(Y) = \frac{1}{1-\alpha} \frac{Y^{1-\alpha}}{\rho} \tag{2.2}$$

and for $\alpha = 1$ it is $U(Y) = \frac{1}{\rho} \ln Y$, where α is the coefficient of relative risk aversion and Y is the income/insurance benefit. The budget constraint is:

$$E \int_0^\infty P_t e^{-\rho t} dt = \int_0^\infty Y e^{-\rho t} dt \tag{2.3}$$

where the left hand side represents the expected total lifetime productivity and the right hand side the insurance wealth, i.e. lifetime income. Since the insurance wealth is equal to the expected discounted value of lifetime productivity calculated at $t = 0$, it must depend only on the starting productivity P_0 and the stochastic process defining productivity. We cannot directly calculate the value of $\int_0^\infty P_t e^{-\rho t} dt$. We can, however, calculate its expected value. The expected value of productivity at any period t is equal to $E(P_t) = P_0 e^{\bar{\mu} t}$. Discounting this value with ρ yields:

$$\int_0^\infty P_0 e^{\bar{\mu} t} e^{-\rho t} dt = \frac{P_0}{\rho - \bar{\mu}} \tag{2.4}$$

Without loss of generality we normalize the initial productivity P_0 to one.

Since the income level over the lifetime is constant, we can rewrite both the lifetime utility function and the budget constraint:

$$\rho U(Y) = \frac{1}{1-\alpha} Y^{1-\alpha} \tag{2.5}$$

$$Y - \frac{\rho}{\rho - \bar{\mu}} = 0 \tag{2.6}$$

Each individual wants to maximize her/his utility, subject to the budget constraint. Thus we must maximize (2.5) subject to (2.6). Therefore we can write the Lagrangian:

$$\rho U(Y, \lambda) = Y \left(\frac{Y^{-\alpha}}{1-\alpha} - \lambda \right) + \lambda \frac{\rho}{\rho - \bar{\mu}} \tag{2.7}$$

The first order conditions are:

$$\begin{aligned} \rho \frac{\partial U^*(P, \lambda)}{\partial Y} &= Y^{-\alpha} - \lambda = 0 \\ \rho \frac{\partial U^*(P, \lambda)}{\partial \lambda} &= -Y + \frac{\rho}{\rho - \bar{\mu}} = 0 \end{aligned}$$

This gives the optimal income based on the initial productivity and expectations of its future values for all values of the coefficient of relative risk aversion:

$$Y = \frac{\rho}{\rho - \bar{\mu}} \tag{2.8}$$

This optimal value of income, or consumption, is a natural consequence of the model: since the insurance wealth is equal to the expected discounted total productivity, the benefits through the infinite lifetime cannot be higher than the interests from the insurance wealth - otherwise the stock of wealth could be depleted.

In the optimum the Lagrange multiplier is equal to:

$$\lambda = \left(\frac{\rho}{\rho - \bar{\mu}}\right)^{-\alpha}$$

and the expected value of lifetime utility, when initial productivity is normalized to one, is equal:

$$\begin{aligned} U &= \text{for } \alpha \neq 1: \frac{1}{(1-\alpha)\rho}\left(\frac{\rho}{\rho - \bar{\mu}}\right)^{1-\alpha} \\ &= \text{for } \alpha = 1: \frac{1}{\rho}\ln\left(\frac{\rho}{\rho - \bar{\mu}}\right) \end{aligned} \tag{2.9}$$

Both income and lifetime utility are proportional to the initial productivity. This is the expected result, because the higher the initial productivity, the higher is the expected total productivity. And, since total income is equal to the expected discounted value of worker's output, it is straightforward that the level of income and hence lifetime utility grows with initial productivity. On the other hand, the Lagrange multiplier is a decreasing function of the initial productivity. If we interpret λ as the shadow price of income, the interpretation follows directly: if the starting value of productivity is low, a worker expects that her/his productivity over the whole lifetime will be relatively low, and thus income obtained from this productivity will be low. Therefore, when the level of income is low, its price is high.

Special case for $\alpha = 0$

When an individual is risk neutral, i.e. when $\alpha = 0$ the lifetime utility is:

$$U(Y) = \int_0^\infty Y e^{-\rho t} dt \Leftrightarrow U(Y) = \frac{Y}{\rho} \tag{2.10}$$

The budget constraint is the same as (2.3) and the Lagrangian is now given by:

$$\rho U(Y, \lambda) = Y(1 - \lambda) + \lambda \frac{\rho}{\rho - \bar{\mu}} \tag{2.11}$$

The first order conditions give the optimal income based on the initial productivity and expectations of its future values:

$$Y = \frac{\rho}{\rho - \bar{\mu}}$$

The Lagrange multiplier is equal

$$\lambda = 1$$

and the expected value of lifetime utility is:

$$U = \frac{1}{\rho - \bar{\mu}} \tag{2.12}$$

Since people are risk neutral, there is no risk aversion to influence their utility. Thus utility is multiplicative in the initial productivity, and the shadow price of income is equal 1.

As we can see from these results, in perfect insurance case with an infinite lifetime and without the possibility of retirement, independent from the value of risk aversion, the optimal consumption = benefit level is equal to the interests gained from the insurance wealth, which is equal to the expected total productivity.

2.2.2 Increasing probability of death

The notion of the constant probability of death, as in the infinite lifetime case, although tempting because of the analytical simplicity, is not very plausible. Therefore we introduce an increasing probability of death: an individual has a potential for the infinite life but the probability of death is rising with age. Let us assume that the probability of death is a linear function of time δt, where $0 < \delta < 1$ is a trend in the dying rate. In such a case the lifetime utility function without the possibility of retirement is equal:

$$U(Y) = \int_0^\infty \frac{1}{1-\alpha} Y^{1-\alpha} e^{-\left(\rho + \frac{\delta}{2} t\right)t} dt \tag{2.13}$$

with an equivalent of $U(Y) = \int_0^\infty \ln Y e^{-\left(\rho + \frac{\delta}{2} t\right)t} dt$ for $\alpha = 1$, where $\frac{\delta}{2}$ is necessary to make the solution of the differential equation in the case when retirement is allowed possible. The budget constraint is almost the same as in (2.3):

$$E \int_0^\infty P_t e^{-\left(\rho + \frac{\delta}{2} t\right)t} dt = \int_0^\infty Y e^{-\left(\rho + \frac{\delta}{2} t\right)t - \rho t} dt \tag{2.14}$$

The expected present value of lifetime productivity at the beginning of the labor career is now more complicated than in (2.4):

$$E \int_0^\infty P_t e^{-\left(\rho + \frac{\delta}{2} t\right)t} dt = \int_0^\infty e^{-\frac{\delta}{2} t^2 - (\rho - \bar{\mu})t} dt \tag{2.15}$$

with starting productivity normalized to one. Repeating the analysis above we optimize the Lagrangian. The first order conditions yield the optimal level of benefits given the starting value of productivity and its future expectations:

$$Y=\frac{\int_0^\infty e^{-\frac{\delta}{2}t^2-(\rho-\bar{\mu})t}dt}{\int_0^\infty e^{-\left(\rho+\frac{\delta}{2}t\right)t}dt} \tag{2.16}$$

This value is the same for all values of the coefficient of relative risk aversion. If we compare it with the result in (2.8) it is basically the same thing corrected for the increasing probability of death. Optimal income is equal to the interest payments from the insurance wealth, where the interest rate is increased by the probability of death - this follows from the denominator. The insurance wealth is equal to the expected discounted total productivity as shown in the budget constraint. The expected discounted total productivity is lower than in the infinite lifetime case, however, whether consumption is lower depends on the trend in productivity $\bar{\mu}$. If the expected trend is high, then the benefits level is higher in the increasing probability of death case - life is shorter and the long spells of low productivity are less probable, thus it is possible to consume more. If the expected trend is low, or even negative, then benefits level is lower in the increasing probability of death case than in the infinite lifetime case - once the productivity starts falling the possibility of its future rise is much lower than with the infinite lifetime to wait. Therefore consumption must be lower.

The optimal utility is equal:

$$\begin{aligned} U(Y) &= \text{for } \alpha\neq 1: \frac{1}{1-\alpha}\left(\frac{\int_0^\infty e^{-\frac{\delta}{2}t^2-(\rho-\bar{\mu})t}dt}{\int_0^\infty e^{-\left(\rho+\frac{\delta}{2}t\right)t}dt}\right)^{1-\alpha}\int_0^\infty e^{-(\rho+\delta t)t}dt \\ &= \text{for } \alpha=1: \ln\left(\frac{\int_0^\infty e^{-\frac{\delta}{2}t^2-(\rho-\bar{\mu})t}dt}{\int_0^\infty e^{-\left(\rho+\frac{\delta}{2}t\right)t}dt}\right)\int_0^\infty e^{-(\rho+\delta t)t}dt \\ &= \text{for } \alpha=0: \int_0^\infty e^{-\frac{\delta}{2}t^2-(\rho-\bar{\mu})t}dt \end{aligned} \tag{2.17}$$

As with consumption the optimal utility with an increasing probability of death can be either higher or lower than the utility in the infinite lifetime case. This depends on the trend in productivity and on risk aversion. Only for risk neutrality we may say that utility is always lower in the increasing probability of death case, since it is equal to the expected discounted value of productivity, and this value is lower when there is a probability of death than when an individual lives and produces forever.

2.3 Retirement with the infinite lifetime

In this section we allow the worker to retire. If productivity falls below a certain level P^* at some moment T, an individual quits work forever. An individual buys an insurance giving, in exchange for the lifetime productivity, benefits ensuring constant utility. As a result there are two levels of benefits: one for work period, which compensates an individual for the disutility of working, and the other for retirement. The insurance wealth is equal to the expected discounted value of the lifetime productivity calculated at the beginning of the labor career, i.e. at $t = 0$.

Let the insurance benefit while working be given by Y_W and the benefit while retired by Y_R. The lifetime utility is equal:

$$\begin{aligned} U(Y_W, Y_R) &= \text{for } \alpha \neq 1: E\int_0^T \frac{1}{1-\alpha} Y_W^{1-\alpha} e^{-\rho t} dt \\ &\quad + E\int_T^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-\rho(T+t)} dt \\ &= \text{for } \alpha = 1: E\int_0^T \ln Y_W e^{-\rho t} dt + E\int_T^\infty [\ln Y_R + \ln \varepsilon] e^{-\rho(T+t)} dt \end{aligned} \tag{2.18}$$

where ε is a cost of effort or the disutility from working - since people generally prefer leisure to work we assume that $\varepsilon > 1$. If people dislike working, then it is possible that their utility is actually higher than while working, even with lower benefits while retired. Thus the multiplier of the utility while retired must be bigger than one. The first element in (2.18) shows the discounted expected utility while working and the second the discounted expected utility while retired. The presence of the cost of effort in the utility while retired shows the gain of the utility due to not working.

The utility while working is just the utility while retired increased by the compensation for the disutility from working. Hence we can rewrite the lifetime utility[1]:

$$U(Y_W, Y_R) = \int_0^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-\rho t} dt + E\int_0^T \frac{1}{1-\alpha} \left(Y_W^{1-\alpha} - \varepsilon^{1-\alpha} Y_R^{1-\alpha}\right) e^{-\rho t} dt \tag{2.19}$$

The budget constraint is:

$$E\int_0^T P_t e^{-\rho t} dt = E\int_0^T Y_W e^{-\rho t} dt + E\int_T^\infty Y_R e^{-\rho(T+t)} dt \tag{2.20}$$

[1] All the subsequent formulas and results are for $\alpha \neq 1$. Their equivalents for $\alpha = 1$ are presented in Appendix 2.

The right hand side of the budget constraint represents the insurance wealth, composed of the sum of benefits paid while working and the sum of benefits paid while retired. Since the insurance wealth is equal to the expected discounted value of life-time work earnings calculated at $t = 0$, it must depend only on the initial productivity P_0 and the time spent working T. However, due to the randomness of P, which makes T also random, we can calculate the value of neither side of (2.20) directly. Transforming (2.20) we get:

$$E\int_0^T (P_t - Y_W + Y_R)\, e^{-\rho t}dt = \int_0^\infty Y_R e^{-\rho t}dt = \frac{Y_R}{\rho} \tag{2.21}$$

Each individual maximizes her/his utility subject to the budget constraint. Thus we must maximize (2.19) subject to (2.21). This gives the Lagrangian:

$$\rho U\left(Y_W, Y_R, \lambda\right) = \frac{1}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha} - \lambda Y_R + \rho G(P) \tag{2.22}$$

where

$$G(P) = E\int_0^T \left[\frac{1}{1-\alpha}\left(Y_W^{1-\alpha} - \varepsilon^{1-\alpha}Y_R^{1-\alpha}\right) + \lambda\left(P - Y_W + Y_R\right)\right] e^{-\rho t}dt$$

and where we ignore the time subscript of productivity. The first element of the Lagrangian is known, the other is stochastic, and we must find its expected value depending on productivity. To simplify the problem we first maximize the Lagrangian to find the values of income while working (Y_W) and the Lagrange multiplier (λ) in terms of income while retired (Y_R). From the first order conditions we have that:

$$\lambda = Y_W^{-\alpha} \tag{2.23}$$

$$\lambda = \varepsilon^{1-\alpha}Y_R^{-\alpha} \tag{2.24}$$

Together (2.23) and (2.24) give the relation between Y_R and Y_W:

$$Y_W = \varepsilon^{\frac{\alpha-1}{\alpha}}Y_R \tag{2.25}$$

As we can see, income while working does not have to be bigger than income while retired - it depends on the risk aversion being bigger or smaller than one. Optimality requires equal marginal utilities of consumption while working and while retired. Actual levels of consumption, and hence income and benefits, depend on two opposite effects: the substitution and income effects. The income effect causes marginal utility while retired to fall, reducing consumption after the retirement decision is taken. The substitution effect makes marginal utility of consumption to rise, increasing consumption while retired. The substitution effect dominates when $\alpha < 1$, hence for risk aversion smaller than one income while working is smaller than income while retired. The income effect is stronger when $\alpha > 1$ and then the opposite situation takes place. For $\alpha = 1$ the effects cancel each other, and the income while working is equal to the income while retired.

By substitution of (2.23), (2.24) and (2.25) into the Lagrangian we can rewrite it as a function only of income while retired:

$$\rho U\left(Y_R\right) = \frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha} + \rho G(P) \tag{2.26}$$

where

$$G(P) = E\int_0^T \left[\frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) + \varepsilon^{1-\alpha}Y_R^{-\alpha}P\right]e^{-\rho t}dt$$

Now we can look for the expected value of $G\left(P\right)$. First define the value of $G\left(P\right)$ at any moment t:

$$G\left(P\right) = \left(\frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) + \varepsilon^{1-\alpha}Y_R^{-\alpha}P\right)dt + E\left[G\left(P+dP\right)e^{-\rho dt}\right] \tag{2.27}$$

Using Ito's lemma and the definition of the stochastic process (2.1) in (2.27) yields the Bellman equation[2] defining $G\left(P\right)$:

$$0 = \frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) + \varepsilon^{1-\alpha}Y_R^{-\alpha}P - \rho G\left(P\right) + \bar{\mu}PG_P\left(P\right) + \frac{\sigma^2}{2}P^2G_{PP}\left(P\right) \tag{2.28}$$

The general solution of (2.28) is of the form:

$$G(P) = A_1P^{\beta_1} + A_2P^{\beta_2} + \frac{\varepsilon^{1-\alpha}Y_R^{-\alpha}P}{\rho-\bar{\mu}} + \frac{1}{\rho}\frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) \tag{2.29}$$

[2] Applying Ito's Lemma and (2.1) gives:

$$\begin{aligned}
&G\left(P\right) = \\
&= \left(\frac{\alpha\varepsilon^{1-\alpha}Y_R^{1-\alpha}}{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) + \varepsilon^{1-\alpha}Y_R^{-\alpha}P\right)dt + (1-\rho dt)\left[G\left(P\right) + G_P\left(P\right)dP + \frac{1}{2}G_{PP}\left(P\right)\left(dP\right)^2\right] \\
&= \left(\frac{\alpha\varepsilon^{1-\alpha}Y_R^{1-\alpha}}{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) + \varepsilon^{1-\alpha}Y_R^{-\alpha}P\right)dt + (1-\rho dt)\left[G\left(P\right) + \bar{\mu}PG_P\left(P\right)dt + \frac{\sigma^2}{2}P^2G_{PP}\left(P\right)dt\right] \\
&= \left(\frac{\alpha\varepsilon^{1-\alpha}Y_R^{1-\alpha}}{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) + \varepsilon^{1-\alpha}Y_R^{-\alpha}P\right)dt + (1-\rho dt)\,G\left(P\right) + \bar{\mu}PG_P\left(P\right)dt + \frac{\sigma^2}{2}P^2G_{PP}\left(P\right)dt
\end{aligned}$$

where the last transformation results from the fact that $(dt)^2$ goes to zero faster than other terms. Now we have:

$$0 = \left(\frac{\alpha\varepsilon^{1-\alpha}Y_R^{1-\alpha}}{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right) + \varepsilon^{1-\alpha}Y_R^{-\alpha}P\right)dt - \rho dtG\left(P\right) + \bar{\mu}PG_P\left(P\right)dt + \frac{\sigma^2}{2}P^2G_{PP}\left(P\right)dt$$

Dividing both sides by dt we obtain the differential equation (2.28).

where β_1 and β_2 are the roots of the equation:

$$\frac{1}{2}\sigma^2\beta(\beta-1)+\bar{\mu}\beta-\rho=0$$

which arise from substitution of (2.29) into (2.28) and then equalization of the sum of all the terms which include P to zero. The roots are:

$$\begin{aligned}\beta_1 &= \frac{1}{2}-\frac{\bar{\mu}}{\sigma^2}+\sqrt{\left(\frac{\bar{\mu}}{\sigma^2}-\frac{1}{2}\right)^2+2\frac{\rho}{\sigma^2}}>1 \qquad (2.30)\\ \beta_2 &= \frac{1}{2}-\frac{\bar{\mu}}{\sigma^2}-\sqrt{\left(\frac{\bar{\mu}}{\sigma^2}-\frac{1}{2}\right)^2+2\frac{\rho}{\sigma^2}}<0\end{aligned}$$

Since we are interested only in stopping work, we can simplify the solution. When the productivity P rises to ∞, the probability that an individual will quit work becomes very small. Therefore the value of the quitting option should go to zero as productivity increases. Hence, the coefficient A_1 corresponding to the positive root β_1 should be zero. Thus the value of $G(P)$ is:

$$G(P)=A_2P^{\beta_2}+\frac{\varepsilon^{1-\alpha}Y_R^{-\alpha}P}{\rho-\bar{\mu}}+\frac{1}{\rho}\frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right) \qquad (2.31)$$

where the first expression on the right hand side is the value of the option to stop working, while two other expressions constitute the value of continued work.

With $G(P)$ found we may start looking for P^*, i.e. the critical, work-quitting level of productivity. At such a critical threshold two conditions, value matching and smooth pasting, must be fulfilled:

$$G(P^*) = 0 \qquad (2.32)$$

$$\frac{\partial G(P^*)}{\partial P^*} = 0 \qquad (2.33)$$

Condition (2.32) is a value matching condition showing the net gain/loss from stopping work - an individual quits work and does not have to undertake effort but has to live from the fixed level of insurance. Since a worker cannot change the levels of benefits, she/he cannot gain more wealth by retiring earlier or working longer. Thus at the moment of retirement the gains and losses from stopping work should be equal. Condition (2.33) is a smooth pasting condition, ensuring continuity and smoothness at the threshold P^*.

Now we have two equations with two unknowns. Solving together (2.32) and (2.33) we are able to define P^* and A_2. Some transformations yield the appropriate formulas:

$$P^* = \frac{\beta_2}{1-\beta_2}\frac{\rho-\bar{\mu}}{\rho}\frac{\alpha}{1-\alpha}Y_R\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right) \tag{2.34}$$

$$A_2 = \frac{\varepsilon^{1-\alpha}Y_R^{-\alpha}\left[\frac{\alpha}{1-\alpha}Y_R\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right)\right]^{1-\beta_2}}{\rho\left(\beta_2-1\right)\left[\frac{\beta_2}{1-\beta_2}\frac{\rho-\bar{\mu}}{\rho}\right]^{\beta_2}} \tag{2.35}$$

(2.34) gives the optimal work-quitting threshold: whenever in the lifetime productivity level, while falling, hits P^*, an individual stops working forever. (2.35) yields a constant coefficient of the solution of the differential equation. Using (2.35) in (2.31) yields the full solution for $G(P)$. This solution may be substituted into the Lagrangian (2.22) but, since the Lagrangian maximizes the utility for the whole life and the insurance income depends only on the starting value of productivity and the expected time of work, productivity P in $G(P)$ is equal to the initial productivity P_0 normalized to one. Thus the final Lagrangian is given by:

$$\rho U\left(Y_R\right) = \varepsilon^{1-\alpha}Y_R^{-\alpha}\left[\frac{\left[\frac{\alpha}{1-\alpha}Y_R\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right)\right]^{1-\beta_2}}{\left(\beta_2-1\right)\left[\frac{\beta_2}{1-\beta_2}\frac{\rho-\bar{\mu}}{\rho}\right]^{\beta_2}}+\frac{\rho}{\rho-\bar{\mu}}+\frac{\alpha}{1-\alpha}Y_R\varepsilon^{\frac{\alpha-1}{\alpha}}\right] \tag{2.36}$$

The first order condition of (2.36) yields an equation defining benefits while retired:

$$\frac{\left[\frac{\alpha}{1-\alpha}Y_R\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right)\right]^{1-\beta_2}}{\left(\beta_2-1\right)\left[\frac{\beta_2}{1-\beta_2}\frac{\rho-\bar{\mu}}{\rho}\right]^{\beta_2}}\left(1-\beta_2-\alpha\right)+\alpha Y_R\varepsilon^{\frac{\alpha-1}{\alpha}}-\alpha\frac{\rho}{\rho-\bar{\mu}}=0 \tag{2.37}$$

This equation cannot be solved analytically. We can however solve it numerically assigning the values of the parameters in the model. This yields the value of Y_R and with it the values of Y_W and λ, which are then used to find the retirement threshold according to formula (2.34).

Some simulation results are presented in Table 2.1 for different values of the coefficient of relative risk aversion[3] and the values of parameters discussed in Section 1.3,

[3] $\alpha = 1.33$ was included into the set of analyzed coefficients since for the positive drift and the values of parameters given above condition (3.19) yields $\alpha < 1.33\,(33)$.

Table 2.1: $\bar{\mu} = -0.01$						
Parameters values: $\rho = 0.05$, $\sigma = 0.1$						
	$\varepsilon = 1.5$			$\varepsilon = 2$		
α	Y_W	Y_R	P^*	Y_W	Y_R	P^*
0.5	0.772	1.158	0.309	0.601	1.203	0.481
1	0.818	0.818	0.265	0.769	0.769	0.426
1.33	0.825	0.746	0.254	0.798	0.672	0.407
2	0.83	0.678	0.244	0.821	0.581	0.385

i.e. for $\rho = 0.05$, $\sigma = 0.1$, $\bar{\mu} = -0.01$ and 0.03. Since the cost of effort must be bigger than one for people to want to retire, we consider two values: $\varepsilon = 1.5$ and $\varepsilon = 2$. The results show that the work-quitting threshold of productivity is always positive and growing with the disutility from work - the more an individual dislikes work, the more she/he is likely to retire early. P^* is smaller than the starting productivity, i.e. smaller than one, because as long as productivity is higher than one the insurance company is gaining and it is not likely to let the worker retire. We can also see that the retirement threshold is decreasing with an increasing risk aversion. It is possible that more risk averse individuals are afraid to retire and thus retire later, but it does not explain this result, since in our model individual's income and benefits are certain. Therefore, risk aversion should not influence the retirement decision directly. The better answer follows from another result - income while working is increasing with risk aversion, while retirement benefits are decreasing. Thus a risk averse worker enjoys a high working income, but knows that her/his retirement benefits will be low. Therefore she/he retires as late as possible. In other words, income while working and retirement benefits are given by marginal utilities of income, but at the moment of retirement an individual is comparing average utilities.

The question is why income from working is an increasing function of α, while retirement benefits are a decreasing one? As we said above when talking about substitution and income effects, the answer is in the equation (2.25), where $\varepsilon^{\frac{\alpha-1}{\alpha}}$ is increasing with risk aversion. In order to keep marginal utilities of work and retirement equal at the moment of retirement, Y_W must increase and Y_R must decrease with rise of α. It shows that with the rise of risk aversion the income effect becomes stronger. Another result concerning incomes is that they are falling when the cost of effort is rising. One explanation is that since an individual who dislikes work is more likely to retire early, her/his salary must be

Table 2.2: $\bar{\mu} = 0.03$ Parameters values: $\rho = 0.05$, $\sigma = 0.1$						
	$\varepsilon = 1.5$			$\varepsilon = 2$		
α	Y_W	Y_R	P^*	Y_W	Y_R	P^*
0.5	2.49	3.736	0.432	2.064	4.129	0.716
1	2.499	2.499	0.351	2.453	2.453	0.59
1.33	2.5	2.26	0.334	2.481	2.089	0.548
2	2.5	2.041	0.318	2.494	1.763	0.507

lower in order to finance the retirement. The second argument is that it also arises from the relation between income and benefits in equation (2.25).

Comparing the results in Table 2.1 we may see that for a risk aversion smaller than one income while working is lower than the retirement benefits, while for $\alpha > 1$ the situation is opposite. This also results from the interaction of the substitution and income effects, since in the first case $\varepsilon^{\frac{\alpha-1}{\alpha}} < 1$, and in the second it is bigger than one. For $\alpha < 1$ it is also true that $\varepsilon^{\frac{\alpha-1}{\alpha}}$ is decreasing in ε (and increasing in ε for $\alpha > 1$), and that is why for such risk aversion the retirement benefits are increasing with disutility from work, and why for $\alpha > 1$ they are decreasing with ε. This last result is more intuitive - the more an individual dislikes work the less she/he needs to be paid to stay retired.

With the negative trend in productivity ($\bar{\mu} = -0.01$) each individual expects her/his productivity to fall, even if it is currently rising. That is why incomes are low relative to the initial productivity, and why the separation threshold grows relatively slow. In order to see what is happening when people do not expect any fall in productivity, in Table 2.2 we present the results of our model calculated for the positive trend ($\bar{\mu} = 0.03$). In such a case individuals know that their productivity is likely to rise sooner or later and that makes the insurer to behave less precautionary. Comparing the results in Tables 2.1 and 2.2 shows that there are some important differences between two cases. First, the retirement threshold of productivity grows faster with the initial productivity than in the negative trend case. This is a straightforward consequence of the positive trend: if productivity is on average rising the retirement threshold is higher than when it is on average falling. Second, incomes are higher - since an employer/insurer expects higher lifetime productivity. Because the initial productivity is the basis for calculating income levels -and on average productivity is rising from the initial level - incomes are higher than its value. Since incomes are high, the Lagrange multiplier, the shadow price of income, is much lower than in the negative trend case.

The last result in table 2.2 which we need to consider is $Y_W \simeq 2.5$ in all but one case. This is not some kind of magical number. It is simply the interest earned from the discounted expected lifetime productivity. With our values of parameters $\frac{\rho}{\rho-\bar{\mu}} = 2.5$. Hence, in almost all the cases income while working is equal to the "safe amount": faced with prospects of paying the benefits infinitely, the insurer does not want to pay more than the interests gained from the wealth she/he is expecting to earn, in order to not deplete this wealth when the productivity starts falling or when the worker retires.

Special case for $\alpha = 0$

As in Section 2 we examine the case of risk neutrality. The lifetime utility for $\alpha = 0$ is given by:

$$U(Y_W, Y_R) = E\int_0^T Y_W e^{-\rho t} dt + E\int_T^\infty \varepsilon Y_R e^{-\rho(T+t)} dt \tag{2.38}$$

Following the steps described above the results maximizing the Lagrangian are given by

$$Y_R = 0 \tag{2.39}$$

$$Y_W = \frac{\rho}{\rho - \bar{\mu}} \tag{2.40}$$

If we use these to compute the retirement threshold it appears that a risk-neutral individual never retires, since:

$$P^* = 0 \tag{2.41}$$

what follows also from formula (2.34) if we take the limit: $\lim_{\alpha \to 0^+} P^* = 0$.

The question is: why does a risk-neutral individual never retire? It is inconsistent with the results presented above for risk-averse individuals: with the fall in risk aversion the retirement threshold is rising, thus the less risk-averse individuals are more likely to retire earlier. However, since the levels of benefits are certain, the risk aversion should not influence the retirement decision directly, only through benefits. More risk averse individuals retire later because they have a high income while working, but a low income while retired, i.e. in their case the income effect dominates. Here the mechanism is opposite - only the substitution effect matters, thus an individual never retires. As can be seen from the definition of the utility (2.38), with disutility from working $\varepsilon > 1$, consumption after the retirement gives higher utility than consumption while working per unit of income. A risk-neutral individual is by definition indifferent to risk and thus she/he will consume everything when consumption is more efficient, i.e. after retirement. Therefore, with an infinite lifetime, she/he is willing to postpone retirement waiting for the possibility of higher consumption. However, in the conditions just described, with an infinite lifetime and risk neutrality, such decision is never taken, and thus the result (2.41) is true.

2.3.1 Probability of retirement

Knowing the retirement thresholds we would like to find out what is the probability of reaching them in a given time. From Harrison (1990) we know that the probability that X will remain below the certain level y for the time t, when X follows standard Brownian motion, is given by[4]:

$$P\left\{M_t < y\right\} = \Phi\left(\frac{y-\mu t}{\sigma t^{\frac{1}{2}}}\right) - e^{\frac{2\mu y}{\sigma^2}}\Phi\left(\frac{-y-\mu t}{\sigma t^{\frac{1}{2}}}\right) \tag{2.42}$$

where $M_t \equiv \max\left\{X_s, 0 \leq s \leq t\right\}$ and $\Phi\left(\cdot\right)$ is the $N\left(0,1\right)$ distribution function. However, we are not interested in the probability of remaining below the certain level, but in the probability of falling below the certain level. This can be done if we transform (2.42) into the formula defining the probability that X will remain above the certain level y, as in Corollary B.3.4 in Musiela and Rutkowski (1998):

$$P\left\{m_t > y\right\} = \Phi\left(\frac{-y+\mu t}{\sigma t^{\frac{1}{2}}}\right) - e^{\frac{2\mu y}{\sigma^2}}\Phi\left(\frac{y+\mu t}{\sigma t^{\frac{1}{2}}}\right) \tag{2.43}$$

where $m_t \equiv \min\left\{X_s, 0 \leq s \leq t\right\}$. In order to find the probability of falling below y, we need simply to subtract (2.43) from one:

$$P\left\{m_t \leq y\right\} = 1 - P\left\{m_t > y\right\} = \Phi\left(\frac{y-\mu t}{\sigma t^{\frac{1}{2}}}\right) + e^{\frac{2\mu y}{\sigma^2}}\Phi\left(\frac{y+\mu t}{\sigma t^{\frac{1}{2}}}\right) \tag{2.44}$$

To switch to our case we must change from the standard Brownian motion starting at zero into the geometric Brownian starting at $P_0 = 1$. The substitution of:

$$\begin{aligned} y &= \ln\frac{P^*}{P_0} = \ln P^* \\ \mu &= \bar{\mu} - \frac{\sigma^2}{2} \end{aligned}$$

[4] This is the formula (8.11) in Harrison (1990). For examples of its use check Sarkar (2000) or Pawlina and Kort (2002).

yields the final formula:

$$P\{m_t \leq P^*\} = \Phi\left(\frac{\ln P^* - \left(\bar{\mu} - \frac{\sigma^2}{2}\right)t}{\sigma t^{\frac{1}{2}}}\right) + P^{*\frac{2\bar{\mu}}{\sigma^2}-1}\Phi\left(\frac{\ln P^* + \left(\bar{\mu} - \frac{\sigma^2}{2}\right)t}{\sigma t^{\frac{1}{2}}}\right)$$

where $P\{m_t \leq P^*\}$ is the probability that the geometric Brownian motion with a starting value 1, drift $\bar{\mu}$ and standard deviation σ, will hit the lower boundary P^* before time t. The plots of this probability for the critical thresholds from the previous section are presented in Figures 2.1 and 2.2.

The plots in Figures 2.1 and 2.2 show the probability that the worker retires before time t. Most of these results are intuitive: the probability is an increasing function of time and the cost of effort ε. The older people are, i.e. the longer their tenure, the more likely they are to retire. Also if individuals dislike work, the possibility of retirement is increasing. It grows much faster with time for high ε and it seems to stabilize for high t - showing that for higher ages the probability of retirement does not change much with time. It can be explained when we notice that it appears only for a positive drift, suggesting that if an individual has not retired so far, the chance that the productivity falls to the retirement threshold is low, or rather dominated by its expected rise. The probability of retirement is falling with the increase of risk aversion, but this is explained by the fall of the retirement threshold, resulting from the fact that the income effect decreases benefits while retired. When we compare the probabilities for positive and negative drift, it is clear that, as expected, with a negative drift the probability of retirement is much bigger than for a positive drift.

When we compute the limits of probability at infinity the results show that the probability of retirement converges from below to a constant. For the positive drift these constants are in each case close to the values reached by the probabilities at $t = 50$. Therefore with the positive drift most of the workers do not retire. For the negative drift, for all values of the coefficient of relative risk aversion, the probability of retirement is converging to one, showing that when individuals expect a fall in wages sooner or latter everybody retires. These results contradict those of Teulings and van der Ende (2000), who show that the separation rate for workers is hump shaped: first the separation rate increases since a sufficient number of negative shocks of productivity have been accumulated. Later it declines, since the trajectories with a large number of negative shocks have already been eliminated.

Figure 2.1: Probability of retirement for $\bar{\mu} = -0.01$, parameters values: $\rho = 0.05$, $\sigma = 0.1$

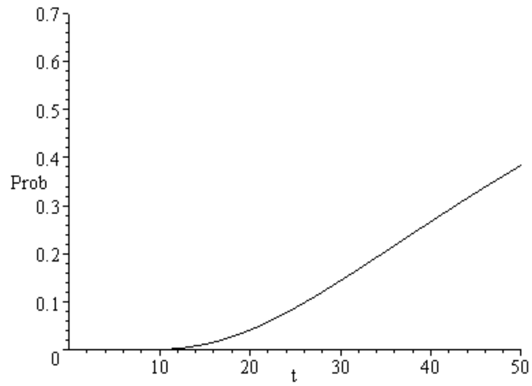
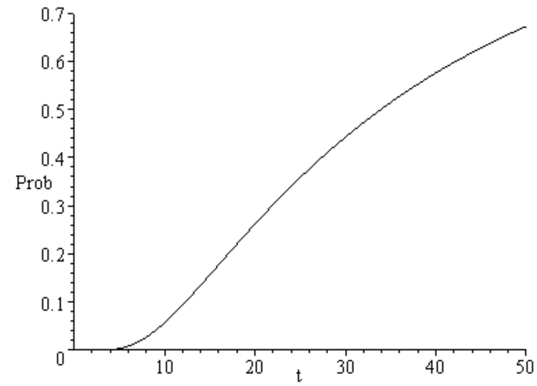
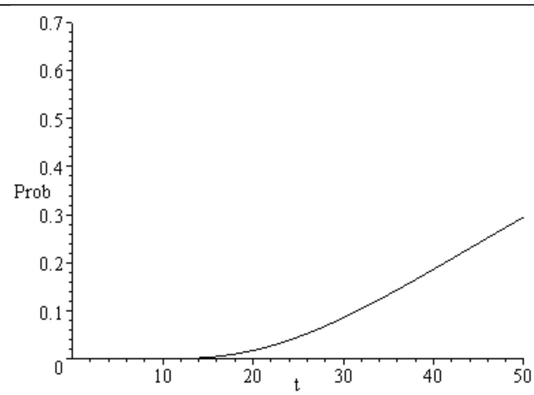
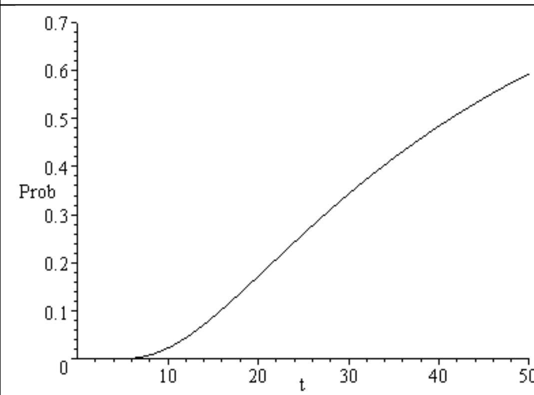
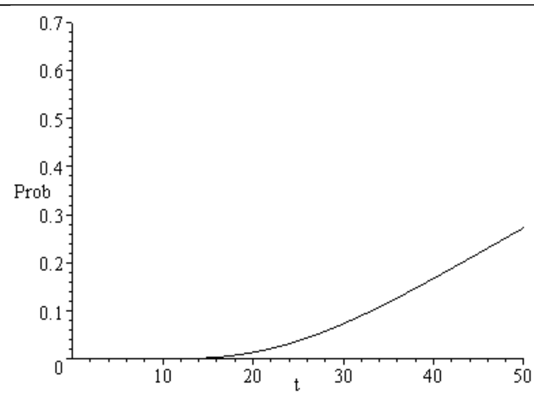
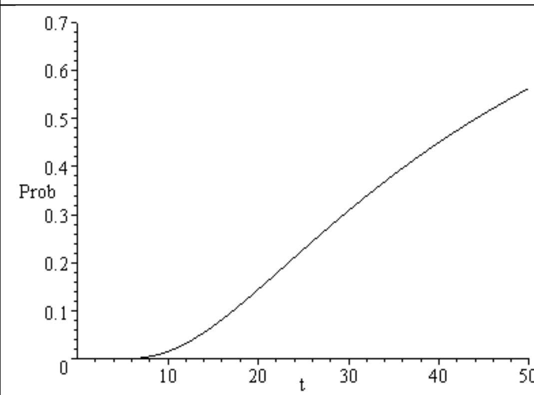
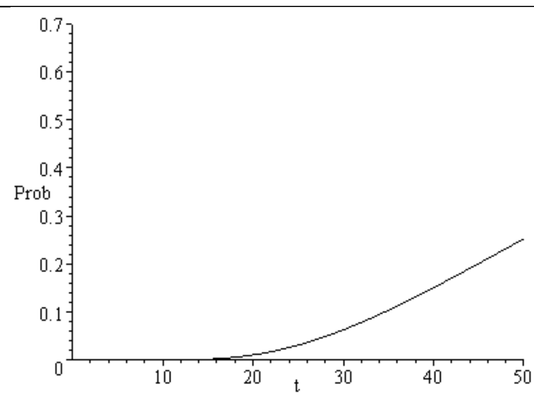
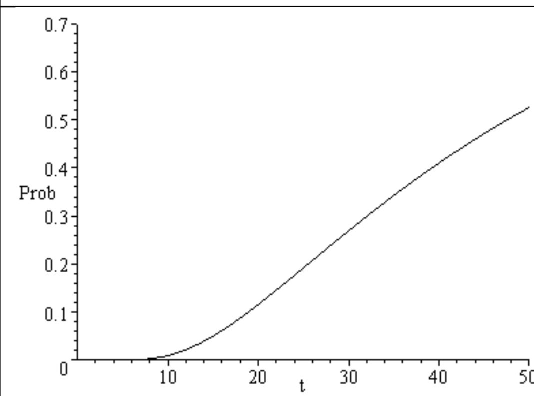
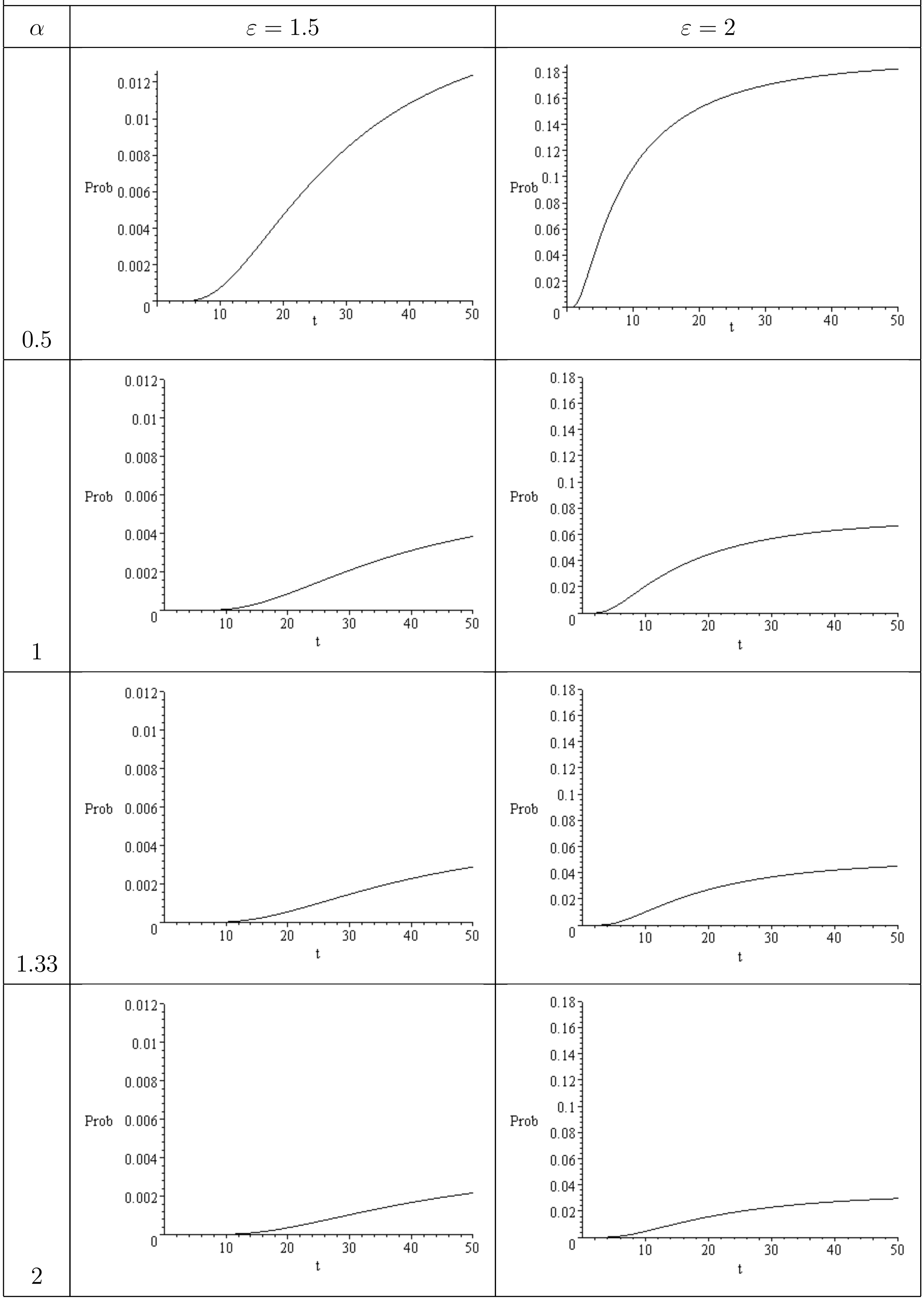
α	$\varepsilon = 1.5$	$\varepsilon = 2$
0.5		
1		
1.33		
2		

Figure 2.2: Probability of retirement for $\bar{\mu} = 0.03$, parameters values: $\rho = 0.05$, $\sigma = 0.1$



2.4 Retirement with increasing probability of death

As in section 2 we now turn to a much more interesting scenario than the one with the infinite lifetime. We introduce an increasing probability of death, as a linear function of time δt, where $0 < \delta < 1$. The retirement system is the same as in the previous section, with perfect insurance giving benefits while working and while retired in exchange for the lifetime output. In such a setup the lifetime utility is:

$$U(Y_W, Y_R) = E\int_0^T \frac{1}{1-\alpha} Y_W^{1-\alpha} e^{-\left(\rho+\frac{\delta}{2}t\right)t} dt + E\int_T^\infty \frac{1}{1-\alpha}\varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-\left(\rho+\frac{\delta}{2}t\right)(t-T)} dt \quad (2.45)$$

where the only difference in comparison with (2.18) is that the discount rate is increased by the probability of death. It can be rewritten as:

$$U(Y_W, Y_R) = E\int_0^T \frac{1}{1-\alpha}\left(Y_W^{1-\alpha} - \varepsilon^{1-\alpha} Y_R^{1-\alpha}\right) e^{-\left(\rho+\frac{\delta}{2}t\right)t} dt + \int_0^\infty \frac{1}{1-\alpha}\varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-\left(\rho+\frac{\delta}{2}t\right)t} dt \quad (2.46)$$

In the same manner the budget constraint looks as follows:

$$E\int_0^T (P_t - Y_W + Y_R) e^{-\left(\rho+\frac{\delta}{2}t\right)t} dt = \int_0^\infty Y_R e^{-\left(\rho+\frac{\delta}{2}t\right)t} dt \quad (2.47)$$

In order to maximize utility subject to budget constraint we form a Lagrangian:

$$U(Y_W, Y_R, \lambda) = \int_0^\infty \left[\frac{1}{1-\alpha}\varepsilon^{1-\alpha} Y_R^{1-\alpha} - \lambda Y_R\right] e^{-\left(\rho+\frac{\delta}{2}t\right)t} dt + G(P,t) \quad (2.48)$$

where

$$G(P,t) = E\int_0^T \left[\frac{1}{1-\alpha}\left(Y_W^{1-\alpha} - \varepsilon^{1-\alpha} Y_R^{1-\alpha}\right) + \lambda (P_t - Y_W + Y_R)\right] e^{-\left(\rho+\frac{\delta}{2}t\right)t} dt$$

is the stochastic part of the utility.

To simplify the problem we first maximize the Lagrangian to find the values of income while working (Y_W) and the Lagrange multiplier (λ) in terms of income while retired (Y_R). From the first order conditions we have:

$$Y_W^{-\alpha} = \lambda \quad (2.49)$$

$$\varepsilon^{1-\alpha} Y_R^{-\alpha} = \lambda \quad (2.50)$$

As we can see this is the same result as in the infinite lifetime case, and it gives an identical relation between Y_W and Y_R:

$$Y_W = \varepsilon^{\frac{\alpha-1}{\alpha}} Y_R \quad (2.51)$$

Thus the relation between the levels of incomes in the case of increasing probability of death is the same as in the case of the infinite lifetime, although the levels themselves are different. This results from the insurance setup we assumed.

Using the results from (2.49), (2.50) and (2.51) we can rewrite the Lagrangian as a function of only income while retired:

$$U\left(Y_R,t\right)=\int_t^{\infty}\frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}e^{-\left(\rho+\frac{\delta}{2}s\right)s}ds+G\left(P,t\right) \tag{2.52}$$

where

$$G\left(P,t\right)=E\int_t^{T}\left[\frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right)+\varepsilon^{1-\alpha}Y_R^{-\alpha}P\right]e^{-\left(\rho+\frac{\delta}{2}s\right)s}ds$$

We are looking for the expected value of $G\left(P,t\right)$ as a function of productivity and time. With Ito's lemma and the law of motion of productivity (2.1) we can write the differential equation defining $G\left(P,t\right)$:

$$0=A+\varepsilon^{1-\alpha}Y_R^{-\alpha}P-\left(\rho+\delta t\right)G\left(P,t\right)+\bar{\mu}PG_P\left(P,t\right)+\frac{\sigma^2}{2}P^2G_{PP}\left(P,t\right)+G_t\left(P,t\right) \tag{2.53}$$

where

$$A\equiv\frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right)$$

The method of solving equation (2.53) is presented in Appendix 1. The general solution is:

$$\begin{aligned}G\left(P,t\right) &= \int_{-\infty}^{\beta_2}B_0\left(x\right)P^x\exp\left[\frac{1}{2}\delta t^2+\left(\rho-\bar{\mu}x+\frac{1}{2}\sigma^2x\right)t-\frac{1}{2}\sigma^2x^2t\right]dx \\ &\quad+\varepsilon^{1-\alpha}Y_R^{-\alpha}Pe^{\frac{1}{2}\delta t^2+\left(\rho-\bar{\mu}\right)t}\int_t^{\infty}e^{-\frac{1}{2}\delta s^2-\left(\rho-\bar{\mu}\right)s}ds+Ae^{\frac{1}{2}\delta t^2+\rho t}\int_t^{\infty}e^{-\frac{1}{2}\delta s^2-\rho s}ds\end{aligned} \tag{2.54}$$

In order to find the work-quitting threshold of productivity P^* and the value of $B_0\left(x\right)$ we must ensure that the value matching and smooth pasting conditions hold. The value matching condition (equation (2.55) below) shows that at the moment of retirement the utility from continued work with the possibility of future retirement must be equal to the utility from retiring now. We cannot, as we did in the infinite lifetime with constant probability of death case, just equalize the stochastic part $G\left(P,t\right)$ of the Lagrangian to zero, since the increasing probability of death makes timing of the retirement decision important. The worker cannot change the level of benefits but she/he can influence the utility after retirement - the higher the probability of death, the lower the utility. The utility after retirement is given by:

$$U_R(Y_R,t)=\int_t^\infty \frac{1}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}e^{-\left(\rho+\frac{\delta}{2}s\right)(s-t)}dt$$

and it must be equalized to the Lagrangian (2.52), after the substitution of (2.54). The smooth pasting condition (equation (2.56) below) guarantees the continuity of the utility at the threshold P^*:

$$\begin{aligned}\int_t^\infty \frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}e^{-(\rho+\delta s)(s-t)}ds+\int_{-\infty}^{\beta_2}B_0(x)\,P^{*x}e^{\frac{1}{2}\delta t^2+\left(\rho-\bar{\mu}x+\frac{1}{2}\sigma^2x\right)t-\frac{1}{2}\sigma^2x^2t}dx\\ +\varepsilon^{1-\alpha}Y_R^{-\alpha}P^*e^{\frac{1}{2}\delta t^2+(\rho-\bar{\mu})t}\int_t^\infty e^{-\frac{1}{2}\delta s^2-(\rho-\bar{\mu})s}ds+Ae^{\frac{1}{2}\delta t^2+\rho t}\int_t^\infty e^{-\frac{1}{2}\delta s^2-\rho s}ds\\ =\int_t^\infty \frac{1}{1-\alpha}\varepsilon^{1-\alpha}Y_R^{1-\alpha}e^{-(\rho+\delta s)(s-t)}ds\end{aligned} \tag{2.55}$$

$$\begin{aligned}\int_{-\infty}^{\beta_2}B_0(x)\,xP^{*x-1}e^{\frac{1}{2}\delta t^2+\left(\rho-\bar{\mu}x+\frac{1}{2}\sigma^2x\right)t-\frac{1}{2}\sigma^2x^2t}dx\\ +\varepsilon^{1-\alpha}Y_R^{-\alpha}e^{\frac{1}{2}\delta t^2+(\rho-\bar{\mu})t}\int_t^\infty e^{-\frac{1}{2}\delta s^2-(\rho-\bar{\mu})s}ds=0\end{aligned} \tag{2.56}$$

Solution of these two equations would yield the retirement threshold P^* and function $B_0(x)$. However, since there is no analytical solution, it has to be solved numerically. In such a case we need the third equation, which would define income while retired (Y_R). We can get it from the Lagrangian (2.52), after substitution of the functional form of $G(P,t)$ from formula (2.54). Then the first order condition with respect to Y_R is given by:

$$\begin{aligned}\alpha\varepsilon^{1-\alpha}Y_R^{-\alpha}\int_t^\infty e^{-\frac{\delta}{2}s^2-\rho s}ds\left[1+\left(\varepsilon^{\frac{\alpha-1}{\alpha}}-1\right)e^{\frac{1}{2}\delta t^2+\rho t}\right]\\ +\int_{-\infty}^{\beta_2}\frac{\partial B_0(x)}{\partial Y_R}P^x\exp\left[\frac{1}{2}\delta t^2+\left(\rho-\bar{\mu}x+\frac{1}{2}\sigma^2x\right)t-\frac{1}{2}\sigma^2x^2t\right]dx\\ -\alpha\varepsilon^{1-\alpha}Y_R^{-\alpha-1}Pe^{\frac{1}{2}\delta t^2+(\rho-\bar{\mu})t}\int_t^\infty e^{-\frac{1}{2}\delta s^2-(\rho-\bar{\mu})s}ds=0\end{aligned} \tag{2.57}$$

In this equation everything is defined in terms of the known parameters and the three unknown variables. However, we do not know the value of the derivative $\frac{\partial B_0(x)}{\partial Y_R}$. It may well be that the function $B_0(x)$ depends on the income while retired, and thus we cannot just assume that $\frac{\partial B_0(x)}{\partial Y_R}=0$.

We now have a set of three equations (2.55), (2.56) and (2.57), and three unknowns P^*, $B_0(x)$ and Y_R. Nevertheless we are still unable to solve it, mainly because of the presence of $\frac{\partial B_0(x)}{\partial Y_R}$.

2.5 Conclusions

In this chapter we have found the critical retirement level of worker's productivity under the perfect insurance retirement scheme. The crucial assumption of the model was that productivity follows geometric Brownian motion with drift. In the perfect insurance scheme in exchange for the output of the whole lifetime the worker is paid constant benefits chosen to maximize her/his lifetime utility. After solving the differential equation defining the stochastic part of the lifetime utility and using value matching and smooth pasting conditions, we calculated the expected lifetime utility and the retirement threshold of productivity in the case of the constant probability of death for both positive and negative drifts in productivity.

The results show that the retirement threshold of productivity is increasing with the individual's dislike of work, but it is decreasing with the rise of risk aversion. However in the risk neutral case there is no retirement - a risk neutral individual is willing to postpone consumption until it is more efficient, i.e. for after retirement. With the infinite lifetime it means that the decision to retire is never taken and only the substitution effect matters. Although our model is theoretical the probabilities of retirement it yields are very reasonable. The probability is increasing with time and with the disutility from working, but falling with the rise of risk aversion. It is also much higher for the negative drift than for the positive one.

Unfortunately, we were not able to solve the model with the increasing probability of death. We found the functional form of the expected value of the stochastic part of lifetime utility, but it was impossible for us to find the retirement threshold as a function of time. Clearly this is the path for further research to follow. It may lead to interesting results and produce valuable tools for other research.

2.6 Appendix 1

The solution to the partial differential equation (2.53) is of the form:

$$G(P,t)=\int_{C_1}^{C_2} B(t,x)\,P^x dx + C(t)\,P + D(t)$$

The derivatives of this function are equal:

$$\begin{aligned} G_P &= \int_{C_1}^{C_2} xB(t,x)\,P^{x-1}dx + C \\ G_{PP} &= \int_{C_1}^{C_2} x(x-1)\,B(t,x)\,P^{x-2}dx \\ G_t &= \int_{C_1}^{C_2} B_t(t,x)\,P^{x}dx + C_tP + D_t \end{aligned}$$

Variable $x \in [C_1, C_2]$, where C_1 and C_2 are constants to be determined, is an equivalent of β in the infinite lifetime case. Since we want to study an option to quit it must be an equivalent of β_2. Thus x should be negative. When an individual has a very high probability of death the value of the option to retire is very high (what is the point in working if I am likely to die in a moment?). Therefore $C_1 = -\infty$. On the other hand, at the beginning of the life/working career the whole life is before the worker and to approximate it we can use the infinite lifetime case. Hence $C_2 = \beta_2$.

- P^x: For each x, after division by $B(t,x)$ and rearranging terms:

$$\begin{aligned} b(t,x) &\equiv \ln B(t,x) \Rightarrow b_t = \frac{B_t(t,x)}{B(t,x)} \\ b_t(t,x) &= \rho + \delta t - \bar{\mu}x - \frac{\sigma^2}{2}x(x-1) \\ b(t,x) &= \frac{1}{2}\delta t^2 + \left(\rho - \bar{\mu}x + \frac{\sigma^2}{2}x\right)t - \frac{\sigma^2}{2}x^2 t + b_0(x) \\ \int_{-\infty}^{\beta_2} B(t,x)\,P^x dx &= \int_{-\infty}^{\beta_2} B_0(x)\,e^{\frac{1}{2}\delta t^2 + \left(\rho - \bar{\mu}x + \frac{\sigma^2}{2}x\right)t - \frac{\sigma^2}{2}x^2 t + x\ln P}dx \end{aligned}$$

  where $b_0(x)$ is an integrating constant and for simplicity $B_0(x) \equiv e^{b_0(x)}$. Note that this integral can be written as a normal distribution function, since the log integrand is a parabola in x with a negative coefficient for the second order term.

- P:

$$\begin{aligned} C_t &= (\rho - \bar{\mu} + \delta t)\,C - \lambda \\ C(t) &= \lambda\Theta(t)\,e^{\frac{1}{2}\delta t^2 + (\rho-\bar{\mu})t} \\ \Theta(t) &\equiv \int_t^{C_3} e^{-\frac{1}{2}\delta s^2 - (\rho-\bar{\mu})s}ds \end{aligned}$$

  $C(t)$ is the net discounted value of a flow of income with value 1 with a discount rate $\rho - \bar{\mu}$ and a death rate of δs for $s > t$, measured a time t. Note that $\Theta(t)$ can be written as a normal distribution function again. C_3 is a constant to be determined. Economic interpretation suggest that it is equal to infinity, since this part of the solution together with the constant represent the value of continued work and, with the increasing probability of death and no definite end of life, there is a possibility of working for the infinite time.

- constant:

$$\begin{aligned} D_t &= (\rho + \delta t)\, D - A \\ D(t) &= A\Psi(t)\, e^{\frac{1}{2}\delta t^2 + \rho t} \\ \Psi(t) &\equiv \int_t^{C_3} e^{-\frac{1}{2}\delta s^2 - \rho s} ds \end{aligned}$$

$D(t)$ has a similar interpretation as $C(t)$, but now with a discount rate ρ.

These three elements form the general solution of equation (2.53) as shown in formula (2.54).

2.7 Appendix 2

Infinite lifetime:

Utility for $\alpha = 1$:

$$U(Y_W, Y_R) = \int_0^{\infty} [\ln Y_R + \ln \varepsilon]\, e^{-\rho t} dt + E \int_0^T [\ln Y_W - (\ln Y_R + \ln \varepsilon)]\, e^{-\rho t} dt \tag{2.19'}$$

The solutions to the set of three first order conditions are:

$$\lambda = \frac{1}{Y_R} \tag{2.24'}$$

$$Y_W = Y_R \tag{2.25'}$$

Lagrangian:

$$\rho U(Y_R) = \ln Y_R + \ln \varepsilon - 1 + \rho G(P) \tag{2.26'}$$

where

$$G(P) = E \int_0^T \left(\frac{1}{Y_R} P - \ln \varepsilon \right) e^{-\rho t} dt$$

The critical threshold of productivity P^* and constant A_2 are equal:

$$P^* = \frac{\rho - \bar{\mu}}{\rho} \frac{\beta_2}{\beta_2 - 1} Y_R \ln \varepsilon \quad (2.34')$$

$$A_2 = \frac{(\ln \varepsilon)^{1-\beta_2}}{\rho(\beta_2 - 1)\left[\frac{\rho-\bar{\mu}}{\rho}\frac{\beta_2}{\beta_2-1} Y_R\right]^{\beta_2}} \quad (2.35')$$

The equation, which can be solved numerically to find Y_R, has a form:

$$\frac{-\beta_2 Y_R^{1-\beta_2} \ln \varepsilon}{(\beta_2 - 1)\left[-\frac{\rho-\bar{\mu}}{\rho}\frac{\beta_2}{1-\beta_2} \ln \varepsilon\right]^{\beta_2}} P_0^{\beta_2} + \frac{\rho P_0}{\rho - \bar{\mu}} - Y_R = 0 \quad (2.37')$$

Increasing probability of death:

Utility for $\alpha = 1$:

$$U(Y_W, Y_R) = \int_0^\infty [\ln Y_R + \ln \varepsilon] e^{-\left(\rho + \frac{\delta}{2}t\right)t} dt + E \int_0^T [\ln Y_W - (\ln Y_R + \ln \varepsilon)] e^{-\left(\rho + \frac{\delta}{2}t\right)t} dt \quad (2.45')$$

Results from the optimization of utility:

$$\frac{1}{Y_W} = \lambda \quad (2.49')$$

$$\frac{1}{Y_R} = \lambda \quad (2.50')$$

and therefore for $\alpha = 1$ the income while working is equal to the income while retired, as in the infinite lifetime case.

The Lagrangian as a function of only income while retired:

$$U(Y_R) = \int_0^\infty [\ln \varepsilon + \ln Y_R - 1] e^{-\left(\rho + \frac{\delta}{2}t\right)t} dt + G(P, t) \quad (2.52')$$

where

$$G(P, t) = E \int_0^T \left[-\ln \varepsilon + \frac{1}{Y_R} P\right] e^{-\left(\rho + \frac{\delta}{2}t\right)t} dt$$

Chapter 3

Optimal consumption when income is geometric Brownian

3.1 Introduction

Since Milton Friedman formulated the "permanent income hypothesis" in 1957 many economists have studied optimal consumption and precautionary savings. Nevertheless, the general analytical solution for the optimal consumption in the presence of uncertainty has not been found. This chapter follows Deaton (1989) and Carroll (1997, 2001). They studied consumption and savings in a discrete time framework, in which, with finite lifetime, the utility maximization can be solved recursively, since in the last period of life there is no uncertainty. We conduct our analysis in a continuous time framework, when incomes follow geometric Brownian motion and the rate of return on savings is constant. In geometric Brownian motion, the current value of income depends on its previous value, therefore in practice all the shocks can be treated as permanent, although there is some possibility that income will fall to the level from before the shock. Thus the transitory shocks are ignored, even more so since they play a small role from the point of view of the entire lifetime. We assume that life is infinite and consumers have a constant relative risk aversion utility. There are no liquidity constraints, because the CRRA utility function allows for precaution, and, as shown by Carroll and Kimball (2001), both give the same effects. However, our results prove that in the infinite lifetime this is true only when we expect wages to fall.

We look for the optimal consumption path - how do people divide income between consumption and saving if they treat savings as an insurance against a fall in income. Since the lifetime is infinite, time does not influence the consumption decision, which depends only on current and expected incomes and on savings. The problem is what part of wage is saved and how individuals finance their consumption when there are "bad times". In order to answer this question we approximate the lifetime utility function and compare it with the numerical solution of the Bellman equation defining the optimal consumption path. This comparison shows that our approximation behaves well if risk aversion is not too big. For higher risk aversion the approximation is working when the expected income is smaller than accumulated savings. When the ratio of the net expected discounted value of wages to accumulated savings grows, the approximation becomes less reliable. This is also confirmed by the analysis of the optimal consumption path.

The optimal consumption paths produced by our model show that despite the risk aversion the precautionary motive is not strong enough to make people save, as long as they expect a rise in income. To give incentives for saving we would need to impose liquidity constraints, what, as we mentioned above, contradicts Carroll and Kimball results. However, when the expected change in income is very small or negative, the ratio of consumption to wage falls below one for high expected incomes or low savings and the precautionary motive is working. According to Deaton risk aversion depends on the second derivative of the utility function, and precaution on the third derivative. We have discussed this in Section 1.4, where we gave a simple example of the importance of the third derivative of the utility function. The results of this chapter show that in the self-insurance component of utility the coefficient of relative risk aversion starts to play a role in the third derivative. In the first and second derivatives it is only a scaling factor.

There are many other ways to study the problem of interactions between consumption and savings. An approach which is somehow similar to ours is presented in Caballero (1990) who is able to calculate the analytical formula of the optimal consumption function for constant absolute risk aversion utility and wages following a moving average process. However, most of the papers are empirical. Blundell et al. (2003) study consumption and income inequality to determine what is the degree of insurance to income shocks. They do so by analyzing the correlations between shifts in the distribution of income growth and consumption growth. Uncertainty enters the model through income, which is stochastic, since it contains a mean-reverting component. The model also includes observed and unobserved taste shifts. Somewhat similar approaches can be found in Attanasio and Davis (1996), who analyze how relative wage changes affect consumption in order to evaluate complete insurance hypothesis, or in Blundell and Preston (1998) with the research how consumption inequality can be used to identify and asses income uncertainty. The results of these studies are often quite general. Blundell and Preston (1998) are able to measure the precise path of growth in short term income risk over a specific period by comparing the growth paths of consumption and income inequalities. Attanasio and Davis (1996) reject the hypothesis of full consumption insurance, at least between groups. Blundell et al. (2003) present evidence against full insurance for permanent income shocks. It is, however, possible to insure against temporary shocks. Among other results they find also that partial insurance for permanent shocks is possible. When we compare these papers to our work it is clear that we go in a completely different direction: we are interested not in finding evidence of the existence of some relation or rejection of a hypothesis, but in finding and approximating a structural equation defining the relation between consumption, income and wealth.

This chapter is organized as follows: Section 2 describes the basic assumptions and the Bellman equation for the optimal consumption path. In Section 3 we analyze utility and its properties for very low wages and savings. We transform the partial differential equation from Section 2 into an ordinary differential equation with one variable, the ratio of wages and savings, and approximate the solution of this equation at zero and infinity. Section 4 proposes the approximated functional form of the utility and the optimal path of consumption as a share in income. Section 5 compares constant relative risk aversion utility with constant absolute risk aversion function. In Section 6 we check how reliable the analytical approximation is by comparing it with the numerical solution. Section 7 simulates the distributions of consumption and savings. Section 8 concludes.

3.2 Utility and stochastic process

Like in Chapter 2 suppose that workers have a CRRA utility function:

$$
\begin{aligned}
U_t &= \text{for } \alpha \neq 1 : \mathrm{E}\left[\int_t^{\infty} \frac{1}{1-\alpha} C_s^{1-\alpha} e^{-\psi t} ds\right] && (3.1) \\
&= \text{for } \alpha = 1 : \mathrm{E}\left[\int_t^{\infty} \ln C_s e^{-\psi s} ds\right] && (3.1')
\end{aligned}
$$

where C_t is consumption at time t, $\alpha \geq 0$ is the degree of risk aversion, and ψ is the rate of time preference. A constant relative risk aversion (or iso-elastic) utility function is both a priori plausible and easy to manipulate.

Let S_t be the worker's stock of savings and W_t the current wage. The laws of motion of S_t and W_t read, leaving out the arguments and subscripts:

$$
\begin{aligned}
dS &= W - C + \rho S \\
dw &\sim N(\mu dt, \sigma^2 dt)
\end{aligned}
$$

where dS is shorthand for $\frac{dS}{dt}$ and $w \equiv \ln W$. A change in savings dS is defined as the difference between current income, i.e. wage plus interests from the amount saved before, and the current consumption. The evolution of wages follows geometric Brownian motion with drift. Workers know their current wage W and their stock of savings S but they do not know the future evolution of wages. Hence the current state is fully determined by W and S and thus the optimal consumption and utility are also determined by these variables. Therefore worker's utility can be written as $U = U(W, S)$.

The future wages are not known, but since the laws of motion of savings and income are known, we can use Ito's lemma to derive a differential equation defining the expected utility:

$$
\begin{aligned}
\psi U &= \frac{1}{1-\alpha} C^{1-\alpha} + U_S dS + \bar{\mu} W U_W + \frac{1}{2}\sigma^2 W^2 U_{WW} && (3.2) \\
&= \frac{1}{1-\alpha} C^{1-\alpha} + U_S (W - C + \rho S) + \bar{\mu} W U_W + \frac{1}{2}\sigma^2 W^2 U_{WW}
\end{aligned}
$$

where $\bar{\mu} \equiv \mu + \frac{1}{2}\sigma^2$ is a drift in wages and where we leave out arguments and subscripts[1]. $\bar{\mu}$ must be smaller than the interest rate ρ - otherwise waiting would be better than making

[1] All the formulas and results presented in the chapter are for $\alpha \neq 1$. Their equivalents for $\alpha = 1$ are presented in Appendix 3.

any decision. Optimal consumption at time t maximizes the utility given by (3.2). Hence the first order condition is:

$$C^{-\alpha} = U_S \tag{3.3}$$

This means that in the optimum an additional unit of current consumption has the same value as an additional unit of savings to finance future consumption. Substituting it back into differential equation (3.2) gives:

$$\psi U = \frac{\alpha}{1-\alpha} U_S^{\frac{\alpha-1}{\alpha}} + U_S\left(W + \rho S\right) + \bar{\mu} W U_W + \frac{1}{2}\sigma^2 W^2 U_{WW} \tag{3.4}$$

This differential equation defines the optimal expected utility. Its solution yields the optimal path of lifetime consumption.

3.3 Analysis of utility

Equation (3.4) is a partial differential equation, which cannot be solved directly. However, it can be transformed into an ordinary differential equation. Given the fact that the utility function is CRRA and that income follows geometric Brownian motion, the optimal consumption path makes consumption relative to wages a function of savings relative to wages. This can be shown formally as follows:

Lemma 1 *Define*

$$z \equiv \frac{W}{\left(\rho - \bar{\mu}\right) S}$$

and

$$\bar{U}(W,z) \equiv U\left(W, \frac{W}{\left(\rho-\bar{\mu}\right) z}\right)$$

The optimal consumption path and utility satisfies:

$$\frac{C}{W} = \left[\left(\rho - \bar{\mu}\right) z^2 g'(z) g(z)^{\alpha-2}\right]^{-\frac{1}{\alpha}} \tag{3.5}$$

$$\bar{U}(W,z) = \frac{1}{1-\alpha} W^{1-\alpha} g(z)^{\alpha-1} \tag{3.6}$$

where $g(z)$ satisfies:

$$\begin{aligned}
\psi g(z)^2 \quad = \quad & \alpha\left(\rho-\bar{\mu}\right)^{\frac{\alpha-1}{\alpha}} z^{2\frac{\alpha-1}{\alpha}} g'(z)^{\frac{\alpha-1}{\alpha}} g(z)^{\frac{2}{\alpha}} + (1-\alpha)\left(\bar{\mu} - \frac{1}{2}\sigma^2\alpha\right) g(z)^2 \\
& + (1-\alpha)\left[(z+1)\left(\rho-\bar{\mu}\right) - \sigma^2(1-\alpha)\right] z g'(z) g(z) \\
& - \frac{1}{2}\sigma^2\left(1-\alpha\right)(\alpha-2) z^2 g'(z)^2 - \frac{1}{2}\sigma^2\left(1-\alpha\right) z^2 g''(z) g(z)
\end{aligned} \tag{3.7}$$

Proof. Equations (3.5) and (3.3) imply

$$C^{-\alpha} = (\rho - \bar{\mu})\, W^{-\alpha} z^2 g'(z) g(z)^{\alpha-2} = U_S$$

The equation (3.6) and the definitions of z and $\bar{U}(W,z)$ imply

$$\bar{U}(W,z) = \frac{1}{1-\alpha} W^{1-\alpha} g\left(\frac{W}{(\rho-\bar{\mu})\,S}\right)^{\alpha-1} = U(W,S)$$

Hence

$$\begin{aligned} U_S &= W^{-\alpha}\left(\frac{W}{(\rho-\bar{\mu})\,S}\right)^2 (\rho-\bar{\mu})\, g'\left(\frac{W}{(\rho-\bar{\mu})\,S}\right) g\left(\frac{W}{(\rho-\bar{\mu})\,S}\right)^{\alpha-2} \\ &= (\rho-\bar{\mu})\, W^{-\alpha} z^2 g'(z) g(z)^{\alpha-2} \end{aligned}$$

Substituting (3.6) into (3.4) yields equation (3.7).

In the same way we can prove Lemma 1 for $\alpha = 1$. ■

Given the current wage and given the fact that wages follow geometric Brownian motion, the expected lifetime income is equal $\frac{W}{\rho-\bar{\mu}}$. Hence, z is the ratio of the net expected discounted value of wages relative to accumulated savings. We managed to transform the partial differential equation into the ordinary differential equation using the redefined utility function in which we separated the relative utility derived from income, $W^{1-\alpha}$, and the self-insurance component, $g(z)^{\alpha-1}$, showing the relation between expected income and accumulated savings and its influence on the utility.

3.3.1 Utility when wages are low

Regrettably the ordinary differential equation (3.7) does not have an analytical solution. Hence we consider an approximation. For simplicity we assume $\psi = \rho$ in the subsequent analysis. For $\psi > \rho$ an individual would consume more early in life (or less for $\psi < \rho$). In the extreme case of $\alpha = 0$ all lifetime income is consumed at the beginning of life (for $\psi > \rho$) or consumption is postponed forever (for $\psi < \rho$).

We will approximate the solution close to $W = 0$, that is for $z = 0$.

Lemma 2 *Given* $\bar{U}(W,z)$*, for* $z = 0$ *we have*

$$g(0) = 0 \tag{3.8}$$

$$g'(0) = \rho^{\frac{\alpha}{1-\alpha}} (\rho - \bar{\mu}) \tag{3.9}$$

Proof. If $W = 0$, an individual must finance her/his consumption from savings[2]. Then there is no randomness and the optimal consumption path is $C = \rho S$, i.e. an individual consumes only interests earned from savings in fear of depleting the saved wealth. Hence, the lifetime utility satisfies:

$$\rho U(0, S) = \frac{1}{1-\alpha}(\rho S)^{1-\alpha} \tag{3.10}$$

From (3.6):

$$\lim_{W \to 0} \frac{1}{1-\alpha} W^{1-\alpha} g\left(\frac{W}{(\rho - \bar{\mu})\, S}\right)^{\alpha-1} = \frac{1}{1-\alpha}\rho^{-\alpha} S^{1-\alpha}$$
$$\Rightarrow \lim_{z \to 0} \frac{g(z)}{z} = \rho^{\frac{\alpha}{1-\alpha}} (\rho - \bar{\mu})$$

The proof for $\alpha = 1$ is similar. ■

Equation (3.9) can also be derived in a more intuitive way. The marginal utility of consumption is constant for z approaching zero, since if wages are zero then the consumption is equal to the interest payments from savings, which are nonstochastic. Hence the discounted marginal utility of consumption is equal at all t. Therefore the timing of this consumption is of a second order importance. The expected discounted monetary value of the random stream of wages is then a sufficient statistic for its effect on utility because the effect of variation of the marginal utility of consumption around the mean is of second order importance. Hence, for very small W a change in the utility caused by a small change in wage is equal to the discounted change in utility caused by a change in savings:

$$\lim_{W \to 0} U_W(0, S) = \frac{1}{\rho - \bar{\mu}} U_S(0, S) \tag{3.11}$$
$$\Rightarrow \lim_{z \to 0} z^{1-\alpha} g(z)^{\alpha-2} \left[\frac{g(z)}{z} - g'(z)\right] = \rho^{-\alpha}(\rho - \bar{\mu})^{\alpha-1}$$
$$\Rightarrow \lim_{z \to 0} \left(\frac{g(z)}{z}\right)^{\alpha-2} \left[\frac{g(z)}{z} - g'(z)\right] = 0$$
$$\Rightarrow \lim_{z \to 0} \frac{g(z)}{z} = g'(0)$$

and similarly for $\alpha = 1$. This confirms the results in the proof above.

---

[2] This is not the case at the beginning of the labor career, when an individual has not had time to accumulate any savings. However, in such conditions any wage is larger than savings and therefore z is increasing to infinity.

So far we were able to calculate the value of the function $g(z)$ and its first derivative at zero. In order to calculate higher order derivatives we must return to the differential equation (3.7). Rewriting it in such a way that it defines the second derivative, using l'Hospital's rule and taking the limit at zero we obtain the formula for $g''(0)$. For further derivatives we must differentiate the formula for $g''(z)$ and repeat the procedure. The calculations are presented in Appendix 1. The derivatives of $g(z)$ are as follows[3]:

$$\begin{aligned} g'(0) &= A \\ g''(0) &= -2A \qquad (3.12) \\ g'''(0) &= 6A\left[1-\frac{\alpha}{2\phi}\right] \qquad (3.13) \\ g^{(4)}(0) &= -24A\left[1-3\frac{\alpha}{2\phi}-\frac{\sigma^2\alpha\,(1+\alpha)}{(\rho-3\bar{\mu}-3\sigma^2)\,\phi}\right] \qquad (3.14) \end{aligned}$$

where

$$\begin{aligned} A &\equiv \begin{cases} \rho^{\frac{\alpha}{1-\alpha}}\left(\rho-\bar{\mu}\right) & \text{, for } \alpha\neq 1 \\ \frac{\rho-\bar{\mu}}{\rho} & \text{, for } \alpha = 1 \end{cases} \\ \phi &\equiv \frac{\sigma^2+2\bar{\mu}-\rho}{\sigma^2} \end{aligned}$$

ϕ is an effective discount rate of the variance, since from the distribution of Brownian motion we have $E\left[\int_0^\infty e^{-\rho t}W_t^2dt\right] = \frac{W_0^2}{\rho-2\bar{\mu}-\sigma^2}$.

An interesting feature of these derivatives is that risk aversion changes its role in the third derivative. In the first and second derivatives it only enters as a power of discount rate, thereby acting as a scaling factor. In the higher derivatives risk aversion affects not only the size but also the sign of the derivative. This shows the importance of the third derivative of function $g\left(z\right)$. However, since $g\left(z\right)$ is only the self-insurance component of the utility, this result is not enough to confirm the Deaton (1992) claim cited in the introduction that risk aversion depends on the second derivative, while precaution depends on the third derivative of the utility function.

[3] If there is no risk aversion, i.e. $\alpha = 0$, then we have a general expression for all the derivatives:

$$g^{(n)}(0) = (-1)^{n+1}\,n!\,(\rho-\bar{\mu})$$

3.3.2 Utility when savings are small

When savings fall to zero, z approaches infinity. This is the case at the beginning of the labor career - there was no time to accumulate any savings. Unfortunately we are not able to derive formally how $g(z)$ behaves for z approaching infinity, given the differential equation (3.7). We can however make a following proposition:

Proposition 1 *For all $z > 0$ the utility function satisfies:*

$$U_S(W,S) > 0$$

Proof. We know that the optimal utility is fully defined by wages and savings. Define

$$E(C_t) = C(W,S)$$

Therefore:

$$U_t = \max E[U(C_t)] = U(W,S)$$

If savings increase by an amount Δ, i.e. we have $U(W, S+\Delta)$, then it is possible to raise consumption in all future periods by $\rho\Delta$, since

$$E(C_t) = C(W,S) + \rho\Delta$$

Thus

$$U(W,S+\Delta) \geq \max E[U(C_t + \rho\Delta)]$$

Since the utility function is increasing with consumption, it means that the derivative of utility with respect to savings is positive. ■

The above proposition means simply that any increase in savings, which is not caused by the decrease in consumption, increases utility. Using Proposition 1 we can formulate a second proposition:

Proposition 2 *For all $z > 0$ the function $g(z)$ satisfies:*

1. $g'(z) > 0$

2. $g(z) > 0$

Proof. From Proposition 1: $U_S(W,S) > 0$. Using (3.6) the derivative of the utility function with respect to savings is equal:

$$\begin{aligned} U_S(W,S) &= -\bar{U}_z(W,z)\frac{W}{(\rho-\bar{\mu})S^2} \\ &\Rightarrow \bar{U}_z(W,z) < 0 \end{aligned}$$

The derivative of the utility with respect to z is:

$$\bar{U}_z(W,z) = -W^{1-\alpha}g(z)^{\alpha-2}g'(z) < 0$$

Since wages cannot be negative, the expression $g(z)^{\alpha-2}g'(z)$ must be positive. Hence, either both $g(z)$ and $g'(z)$ are positive or both are negative. However, we know from Lemma 2 and its proof, that the first derivative is positive in the neighborhood of zero. Therefore both $g(z)$ and $g'(z)$ must be positive for $z > 0$. ■

We need to know more about the function $g(z)$ in order to be able to say something about its functional form. Therefore we make the conjecture below:

Conjecture 1 *For all $z > 0$ the utility function satisfies:*

$$\lim_{z\to\infty}\bar{U}_z(W,z) = 0$$

Motivation For very high z, or equivalently for very high wages, a small change in savings does not influence utility. It just does not matter if we have a few cents more or less. It does not mean that Proposition 1 does not work any more. The changes are just so small that we can ignore them. Therefore the derivative of the utility function converges to zero.

From Proposition 2 we know that the function $g(z)$ is an increasing function. However we do not know how it behaves for large values of z. To answer this we use another proposition:

Proposition 3 *Given Conjecture 1, for all $z > 0$ the function $g(z)$ satisfies:*

$$\lim_{z\to\infty} g'(z) = 0$$

Proof. From Conjecture 1:

$$\lim_{z\to\infty} \bar{U}_z (W,z) = 0 \Leftrightarrow \lim_{z\to\infty} \left[-W^{1-\alpha} g(z)^{\alpha-2} g'(z)\right] = 0$$

The function $g(z)$ is an increasing function, rising from value zero for $z = 0$. Because of this

$$\lim_{z\to\infty} g(z) > 0$$

This yields:

$$\lim_{z\to\infty} g'(z) = 0$$

■

Thus given Conjecture 1 $g(z)$ is a concave, increasing function, and converges to a constant in infinity.

3.4 An approximation of function $g(z)$ and the optimal consumption path

According to Lemma 2 and Propositions 2 and 3 we are looking for a function which is increasing, concave, equal zero at zero and converging to a constant in infinity. We approximate $g(z)$ by:

$$g(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z}{(z+1)^n} \tag{3.15}$$

where n is the order of approximation. This function fulfills all the requirements[4].

Proposition 2 implies that there are two conditions that have to be imposed on this function:

$$\forall_z g'(z) > 0 \Rightarrow \forall_z \sum_{i=1}^{n} \left[i b_i z^{i-1} - (n-i) b_i z^i \right] > 0 \tag{3.16}$$

$$\lim_{z\to\infty} g(z) > 0 \Rightarrow b_n > 0 \tag{3.17}$$

The first condition is that the function $g(z)$ is always increasing, and the right hand side is a transformation of the first derivative of $g(z)$. In general we cannot guarantee that it is true for all values of b_n. It should be verified for each particular case. We can also consider two special cases: one for very small z:

$$\lim_{z\to 0} g'(z) = b_1 > 0$$

[4] An alternative approximation can be found in Appendix 2.

but this is always true by Lemma 2, and the other for very high z:

$$\lim_{z\to\infty} g'(z) > 0 \Rightarrow nb_n > b_{n-1}$$

The second condition must ensure that the function converges to a positive constant. When $z \to \infty$ the function $g(z)$ converges to:

$$\lim_{z\to\infty} g(z) = b_n$$

The second condition must be true whenever the first condition holds, since if $\underset{z}{\forall} g'(z) > 0$ and $g(0) = 0$ then $g(\infty) > 0$. Therefore when we are able to show that condition (3.16) holds there is no need to check condition (3.17).

To find the parameters of (3.15) we must calculate the derivatives of the functional form[5] when $z \to 0$. Then we only need to substitute the values of the derivatives of $g(z)$ which we have found before, and solve the set of equations to get the values of the parameters b_i.

Since we have four derivatives of $g(z)$ we are solving the problem for the second, third and fourth degree approximations, in order to find the best one. We need to check if conditions (3.16) and (3.17) are fulfilled. If we analyze the risk neutrality, i.e. when $\alpha = 0$, we can see that the exact solution is obtained already for the first degree approximation, and further approximations give the same result, namely:

$$g(z) = (\rho - \bar{\mu})\frac{z}{z+1} \tag{3.18}$$

In this case both conditions always hold since the interest rate must be higher than the trend in wages.

For $\alpha > 0$ we are not going to analyze the second degree approximation, since it is of not much interest to us - risk aversion influences only the scaling of the function. For the

[5] The kth derivative of $g(z)$ for $z \to 0$ is equal:

$$g^{(k)}(0) = k!b_k - k!nb_{k-1} + \frac{k!}{2!}n(n+1)b_{k-2} + \ldots + (-1)^{k+1}\frac{k!}{(k-1)!}n(n+1)\ldots(n+k-2)\,b_1$$

The kth parameter is given by:

$$\begin{aligned} b_k \;=\; & \frac{1}{k!}g^{(k)}(0) + \frac{n}{(k-1)!}g^{(k-1)}(0) + \frac{n(n-1)}{2!(k-2)!}g^{(k-2)}(0) + \frac{n(n-1)(n-2)}{3!(k-3)!}g^{(k-3)}(0) + \ldots + \\ & + \frac{n(n-1)(n-2)\ldots(n-d+1)}{d!(k-d)!}g^{(k-d)}(0) + \ldots + \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}g(0) \end{aligned}$$

fourth degree approximation the analytical solution of (3.16) is so complex, that there is no point in analyzing it. Hence we use the third degree approximation, for which condition (3.16) is a second order polynomial, which is always positive for

$$z \in (0, +\infty), \text{ for } \begin{cases} \alpha < \frac{2}{3}\phi \text{ if } \phi > 0 \\ \alpha > 0 \text{ if } \phi < 0 \end{cases} \tag{3.19}$$

For $\alpha > \frac{2}{3}\phi$ the polynomial changes from convex to concave and is positive only for some values of z. If $\phi < 0$ the polynomial is always convex and nonnegative. Thus the sign of ϕ is crucial - here it decides for what values of risk aversion the necessary condition (3.16) is fulfilled. This follows from the result that, depending on risk aversion, it may change the sign of the higher derivatives of $g(z)$ at zero. We already know that $\rho - \bar{\mu} > 0$ for the model to work. Now it is extended in the negative case to

$$\bar{\mu} + \sigma^2 > \rho - \bar{\mu} > 0$$

Thus not only the interest rate must be bigger than the drift in wages, but also the sum of possible changes in the growth of wages must be bigger than the difference between the discount rate and the drift. In the positive case the result is opposite:

$$\rho - \bar{\mu} > \bar{\mu} + \sigma^2$$

but, since the trend in wages may be negative, we do not know if the sum of the trend and the variance in wages is positive or not.

As for (3.17), the solution is similar to (3.19) and the parameter b_n is positive for

$$\begin{aligned} \alpha &\in (0, 2\phi), \text{ when } \phi > 0 \\ \alpha &\in (0, +\infty), \text{ when } \phi < 0 \end{aligned}$$

As we mentioned before, for both conditions to hold for all values of z we must only ensure that (3.19) is fulfilled.

When the conditions above are fulfilled the third degree approximation of function $g(z)$ is given by:

$$g(z) = A\frac{\left[1 - \frac{\alpha}{2\phi}\right] z^3 + 2z^2 + z}{(z+1)^3} \tag{3.20}$$

Now that we have the approximation of function $g(z)$, we can use it to find the optimal path of consumption. Substituting (3.20) into (3.5) from Lemma 1 gives the formula for the optimal consumption path as a share in income:

$$\frac{C}{W}=\frac{\rho}{\rho-\bar{\mu}}\frac{(z+1)^{3-\frac{2}{\alpha}}}{z}\left[\frac{\left(1-3\frac{\alpha}{2\phi}\right)z^2+2z+1}{\left[\left(1-\frac{\alpha}{2\phi}\right)z^2+2z+1\right]^{2-\alpha}}\right]^{-\frac{1}{\alpha}} \tag{3.21}$$

The optimal ratio of consumption to wage should fall below one for very high values of z, i.e. when expected incomes are high and accumulated savings low. In such a case precautionary individuals would not consume all their income but use part of it to increase savings, in order to insure themselves against a possible fall in income. The limit of $\frac{C}{W}$ is equal:

$$\lim_{z\to\infty}\frac{C}{W}=\frac{\rho}{\rho-\bar{\mu}}\left[\frac{1-3\frac{\alpha}{2\phi}}{\left(1-\frac{\alpha}{2\phi}\right)^{2-\alpha}}\right]^{-\frac{1}{\alpha}} \tag{3.22}$$

When the drift in wages is negative ($\bar{\mu}<0$) and $\phi<0$, the limit is lower than one for all the values of risk aversion. If one of these conditions does not hold the value of the limit depends on the values of α and the other parameters.

Let us consider the special case when $\alpha=2$. Then the function is of the form[6]:

$$\frac{C}{W}=\frac{\rho}{\rho-\bar{\mu}}\frac{(z+1)^2}{z\sqrt{\left(1-\frac{3}{\phi}\right)z^2+2z+1}} \tag{3.23}$$

and the limit in infinity is equal to:

$$\lim_{z\to\infty}\frac{C}{W}=\frac{\rho}{\rho-\bar{\mu}}\frac{1}{\sqrt{1-\frac{3}{\phi}}} \tag{3.24}$$

As in the general case for negative $\bar{\mu}$ and ϕ the limit is always smaller than one. If ϕ is positive then first of all the square root must be positive, i.e. $\phi>3$. Then everything depends on the value of the drift: if $\bar{\mu}>0$, then the limit is greater than one, if $\bar{\mu}=0$, then the limit is smaller than one when $\rho>\sigma^2$, and when $\bar{\mu}<0$, then everything depends on the values of the parameters.

In order to get some idea how the changes of the values of the parameters affect the limit (3.24) consider the comparative statics in Table 3.1. The analysis of this table shows that the limit is rising with an increase in the standard deviation, but this trend changes

[6] When we refer to the empirical papers estimating risk aversion, like Zeldes (1989), Lawrence (1991), Runkle (1991) or Attanasio and Weber (1995), we see that their results show usually α close to 2.

Table 3.1. Comparative statics for the limit of consumption path (3.24), $\alpha = 2$*

	$\rho = 0.03$						$\rho = 0.04$					
$\bar{\mu}$ / σ	-0.02	-0.01	0	0.01	0.02	0.03	-0.02	-0.01	0	0.01	0.02	0.03
0	0.6	0.75	1	1.5	3	·	0.67	0.8	1	1.33	·	4
0.05	0.57	0.7	0.89	1.06	4.74	·	0.64	0.75	0.91	1.12	·	4.9
0.1	0.49	0.57	0.63	·	·	·	0.56	0.63	0.71	0.67	·	·
0.15	0.39	0.4	0.32	·	·	·	0.45	0.48	0.45	·	·	·
0.2	0.27	0.21	·	·	·	·	0.33	0.3	·	·	·	·
0.25	0.12	·	·	·	·	·	0.19	·	·	·	·	·
0.3	·	·	·	·	·	·	·	·	·	·	·	·
	$\rho = 0.05$						$\rho = 0.06$					
$\bar{\mu}$ / σ	-0.02	-0.01	0	0.01	0.02	0.03	-0.02	-0.01	0	0.01	0.02	0.03
0	0.71	0.83	1	1.25	1.67	2.5	0.75	0.86	1	1.2	1.5	·
0.05	0.69	0.79	0.93	1.11	1.18	3.95	0.72	0.82	0.94	1.10	1.25	·
0.1	0.61	0.68	0.76	0.79	·	·	0.65	0.72	0.79	0.85	0.75	·
0.15	0.51	0.54	0.54	0.40	·	·	0.55	0.58	0.6	0.54	·	·
0.2	0.39	0.37	0.28	·	·	·	0.43	0.43	0.38	·	·	·
0.25	0.26	0.16	·	·	·	·	0.31	0.25	·	·	·	·
0.3	0	·	·	·	·	·	0.14	·	·	·	·	·

*Dots in the table signify the cases without solution or with solution in complex numbers.

for high values of the drift. For a low standard deviation an increase in $\bar{\mu}$ leads to an increase in the value of the limit, for a high standard deviation the situation is opposite, for $\sigma = 0.2$ or in some cases for $\sigma = 0.15$ the limit is first rising and then falling. When the drift is negative, the limit is always lower than one. For $\bar{\mu} > 0$ the picture is mixed, although in general the limit is greater than one for a low standard deviation and/or high values of the drift. Finally, with the rise of the interest rate ρ the chance that the limit is above one is falling. We can also see that in many cases, especially for a high standard deviation, the limit does not exist or the results are in complex numbers. For these values of parameters $\phi \in \langle 0; 3 \rangle$, and there is either a division by zero or a square root of negative number.

The approximated structural equation (3.21) defining the optimal consumption path is an interesting subject for empirical testing. Suppose we have data on the interest rate. Then we need two equations: one for wages, and the other for the ratio of consumption to wages. Wages can be estimated independent of consumption. Their estimation would give us estimates of the drift ($\bar{\mu}$) and the variance (σ^2), and show if wages are following geometric Brownian motion. That is the crucial test, since the consistency of the estimates of consumption depends on the correct assumption about the wage process. These parameters and the interest rate can be used to estimate the equation of the ratio of consumption to wages, or just a consumption equation. At each period consumption depends on current wage and the stock of savings. One method of estimating the consumption equation is to take the logarithm of equation (3.21):

$$\begin{aligned} \ln\left(\frac{C}{W}\right) &= \ln\left(\frac{\rho}{\rho-\bar{\mu}}\right)+\left(3-\frac{2}{\alpha}\right)\ln(z+1)-\ln z \\ &\quad -\frac{1}{\alpha}\ln\left[\left(1-3\frac{\alpha}{2\phi}\right)z^2+2z+1\right]+\frac{2-\alpha}{\alpha}\ln\left[\left(1-\frac{\alpha}{2\phi}\right)z^2+2z+1\right] \end{aligned}$$

This equation can be estimated by nonlinear least squares. It is a function of wages, savings and parameters ρ, $\bar{\mu}$ and σ, and it will yield the coefficient of relative risk aversion α. Another method is to approximate (3.21) around $\alpha = 2$. This allows us to verify if risk aversion is indeed equal to two, and if this is true, we can estimate the consumption equation in a simpler way, even with ordinary least squares.

If estimated, this model can give us more or different information on consumption, income, savings and their relations than many existing studies of consumption. For example Blundell et al. (2003) estimate the loglinear demand function. The inverted demand yields trends in the variance of consumption, which are the main object of the empirical analysis. Then the autocovariance estimates of consumption and income give the income process, which fits the data well, and offers some "... weak evidence that transitory shocks impact consumption growth or that liquidity constraints are empirically important..."[7]. Minimum distance estimation using data on household income and predicted consumption gives "...evidence for partial insurance with respect to the permanent shocks, full insurance with respect to transitory shocks..."[8]. This model, as well as that of Blundell and Preston (1998), focuses on marginal elasticities of consumption and can explain the

[7] Blundell et al. (2003), p. 22.

[8] *op. cit.*, p. 23.

marginal differences in consumption and incomes. Our model analyzes not only consumption and incomes but also savings and can explain the levels of variables and their nonlinearity.

3.5 Comparison of CRRA and CARA utility

Throughout the thesis we use the constant relative risk aversion (CRRA) utility. In this section we want to compare it with another utility function: the constant absolute risk aversion (CARA). If CRRA is given by the formula: $U = \frac{1}{1-\alpha}C^{1-\alpha}$, then the formula for CARA is $U = -\frac{1}{\theta}e^{-\theta C}$, where θ is the coefficient of absolute risk aversion. The basic difference between these two functions is that in CRRA the precautionary motive is stronger than in CARA. According to Kimball (1990) the risk aversion is measured by $-\frac{U''}{U'}$ while precaution is given by $-\frac{U'''}{U''}$. If for both functions the risk aversion is the same, then, with consumption normalized to one, we have that $\alpha = \theta$. When we compare precaution for both functions, CRRA is more precautionary if $\alpha + 1 > \alpha$. This is always true, thus constant relative risk aversion is always more precautionary than constant absolute risk aversion. Since risk aversion depends on the convexity of the marginal utility function, it shows that the first derivative of CRRA utility is more convex than the first derivative of the CARA function.

The consequences of using CARA for the consumption function can be found in Caballero (1990). As we have mentioned in the introduction to this chapter, Caballero is able to calculate the analytical formula of optimal consumption function for a constant absolute risk aversion utility and wages following a moving average process. His results show that with the standard Brownian motion all the shocks of wages are immediately transferred to consumption. Savings remain constant, not affected by changes in uncertainty, since saving plans are defined only by the initial status. This is the most important difference between this model and ours - in our model individuals change the savings plans as soon as there is a change of conditions: you earn more, thus you save more. This is more realistic that the results of CARA utility. Caballero recognizes this, saying that with CRRA "... consumers can reduce their relative importance of future labor-income riskiness by saving more today in order to lower the coefficient of absolute risk aversion faced in the future (by consuming more)..."[9]

In both models savings are at first mainly accumulating, only later people start to dissave to increase their consumption when wages fall. However, even while consuming some of the accumulated wealth, they do not spend enough of it to stop the accumulation. The extend to which consumption is financed by savings, i.e. the strength of consumption smoothing, is another difference between the models. According to Caballero the CARA function exhibits two effects of consumption changes in reaction to positive income innovations: the wealth effect, which is similar to the one in the certainty-equivalence model, and the "...offsetting precautionary-savings response to an increase in forecasted variance..."[10] The CRRA function has the same two effects, and two more: one which reduces the coefficient of absolute risk aversion and raises current consumption, and another which increases savings now to consume more later. When incomes follow geometric Brownian motion the second effect dominates the third and fourth, smoothing consumption more than with CARA function. The excess-smoothness effect is more important when the accumulated savings are lower.

If we redo Caballero's calculations in continuous time and with wages following standard Brownian motion, we can compare his consumption function with our optimal consumption path (3.21). The calculations are presented in Appendix 4. Assuming for simplicity that there is no drift in wages, i.e. $\mu = 0$, Caballero's consumption function is equal:

$$C_t = \rho S_t + Y_t - \frac{\theta}{2\rho}\sigma^2$$

If we start at zero, then there are no savings, $S_t = 0$. We can also normalize income to one. The result is:

$$C = 1 - \frac{\theta}{2\rho}\sigma^2$$

In our case, under the same conditions, i.e. savings equal to zero, wage normalized to unity, $\mu = 0$, the optimal consumption is given by (3.22), with $\bar{\mu} \equiv \frac{1}{2}\sigma^2$.

We must remember that the relation between the coefficient of relative risk aversion (α) and the coefficient of absolute risk aversion (θ) is: $\frac{\alpha}{C} = \theta$. Substituting the values of the parameters into both consumption functions allows us to compare them. The results are presented in Table 3.2.

[9] Caballero (1990), p. 133, footnote 29.
[10] *op. cit.*, p. 133.

Table 3.2: Consumption for $\rho = 0.05$, $\sigma = 0.1$			
α	$CRRA$	$\theta = \frac{\alpha}{C}$	$CARA$
0	0.941^{+}	0	1
0.5	0.904	0.553	0.945
1	0.864	1.157	0.884
1.33	0.838	1.588	0.841
2	0.786	2.546	0.745

$^{+}$ This value is not equal 1, since it is $\lim_{\alpha\to 0} C$

As we can see consumption under CRRA is smaller than under CARA for small values of risk aversion, but higher for high risk aversion. A stronger precautionary motive makes individuals under CRRA save more from the very beginning of their labor career, however precaution is working only when wages are moving downwards. Thus, with initial wage equal to one, and geometric Brownian motion compressing the downward move in comparison to standard Brownian, it is not surprising that for higher risk aversion consumption under CARA is lower than under CRRA.

3.6 Numerical simulations

We can check the results from Section 4 by assigning values to the parameters in the model. As in Chapter 2 we study two cases: one with positive and another with negative drift. First let us consider the positive drift. Assume that $\bar{\mu} = 0.03$, $\rho = 0.05$, $\sigma = 0.1$. For small values of the coefficient of relative risk aversion (for $\alpha < 1.333$) both conditions (3.16) and (3.17) are fulfilled. For higher values of α the interval of z for which the condition (3.16) holds is reduced and the condition (3.17) does not work at all. It seems then, that the analytical approximation of the function $g(z)$ is correct for low risk aversion and small values of z.

The graphical representations of functional form approximations, for $\rho = 0.05$, $\bar{\mu} = 0.03$, $\sigma = 0.1$, are shown in the Figure 3.1, where each line shows the appropriate approximation. The first and second degree approximations are identical. The reason why the differences between the approximations for α close to zero are so small is that for $\alpha = 0$, i.e. for the risk neutral case, all approximations give the same result, as shown in Section 4.

In order to verify the accuracy of our approximation we solve equation (3.7) numerically, with (3.8) and (3.9) as initial conditions and step $dz = 0.0002$. The results of this solution for $\rho = 0.05$, $\bar{\mu} = 0.03$, $\sigma = 0.1$ and different values of coefficient of relative risk aversion are also presented in the Figure 3.1. As we can see, in all the cases the function $g(z)$ satisfies Conjecture 1.

Comparison of the numerical solution with the analytical approximations shows that for low values of risk aversion and z the fourth degree approximation is the best one. However, for small z the differences between the third and the fourth degree approximations are very small, and with the rise in z it is the third order approximation that follows most closely the numerical solution. Even for the values of z and α higher than those suggested by conditions (3.16) and (3.17), the differences between the numerical solution and the third degree approximation are very small.

When we plot the optimal consumption path for different coefficients of relative risk aversion we get the graphs showing the relation between the share of consumption in income and z (relation between current wealth and expected labor income) as shown in Figure 3.2. For $z \leq 1$ (and in case of $\alpha < 1$ for all z) the pattern is more or less as expected: the ratio of consumption to income is falling as the expected income grows in relation to accumulated savings. However, when expected income is higher than savings, then consumption to wage ratio is increasing, what is counterintuitive.

This is also confirmed by the optimal consumption path calculated using the numerical solution. Our approximation is almost identical to the numerical solution for small z but the difference is increasing with z. The numerical solution shows that as the ratio of net expected discounted incomes to savings increases, the share of consumption in income falls, although very slowly.

The question to consider is why, as z grows, both in the approximated and the numerical versions of the optimal consumption paths the consumption to wage ratio does not fall to 1, or even below. That is what we would expect: for small wages consumption expenditures are high in relation to income, and even risk averse individuals may live on credit, spending on consumption more than they earn. However, as income (wage) grows, the share of consumption in income should fall, and for high wages risk averse people will spend less than they earn, saving part of their wages as self-insurance. Because of this precautionary motive, we ignored the liquidity constraints, since both are supposed

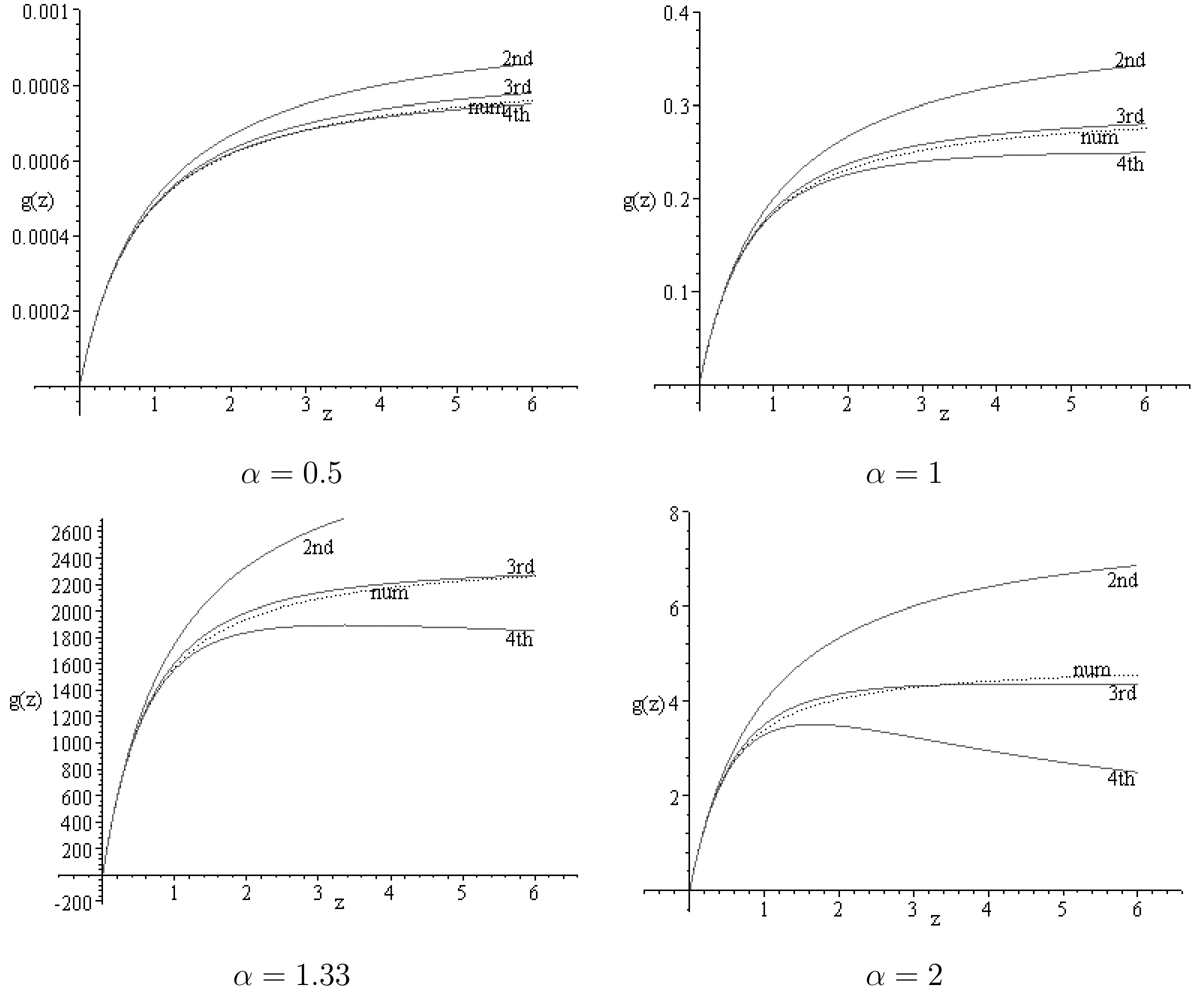


Figure 3.1: Numerical solutions of equations (3.7) and (7') and approximations of $g(z)$
Parameters' values: $\rho = 0.05$, $\bar{\mu} = 0.03$, $\sigma = 0.1$

to give similar effects (see Carroll and Kimball, 2001). Why is it not the case in Figure 3.2?

The answer to this is quite simple. Individuals do behave rationally, given the model and the values of the parameters we have chosen. If one has an infinite lifetime and a steady growth of income by 3 % each period, which is higher than the variance (1%), then it is natural that one consumes more than one earns - there will always be a possibility to repay debts. Because of this there is no need for precautionary savings. Therefore we need to check what happens if the trend in wages is smaller. Thus the variance plays a more important role and consumers have lower expected incomes. Let us consider the negative drift $\bar{\mu} = -0.01$, in which case an individual knows that wages are expected to decline over time, and thus tries to save when wages are high and the ratio of consumption

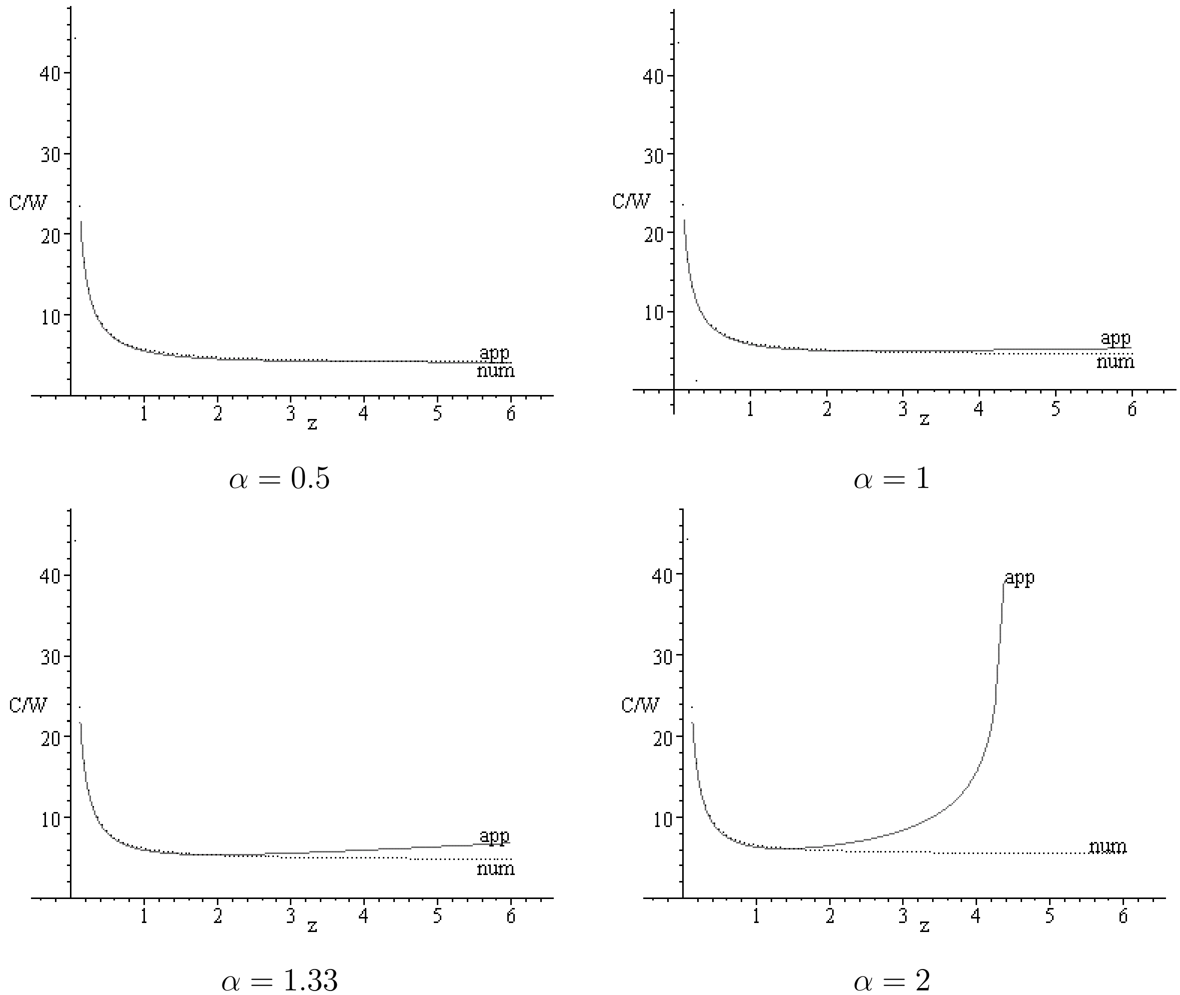
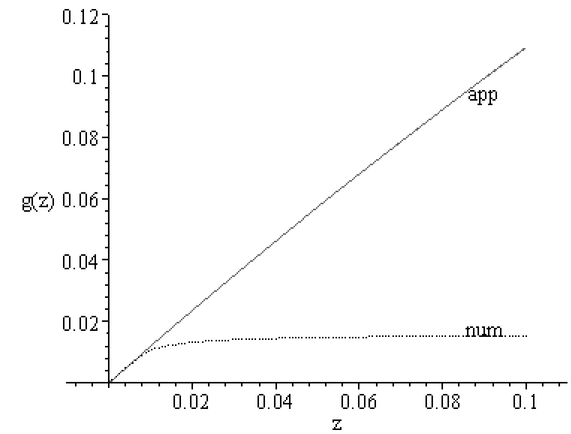


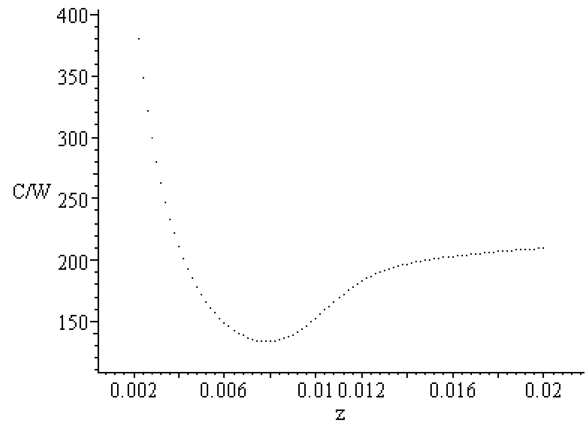
Figure 3.2: Optimal consumption paths for the third degree approximation of $g(z)$

Parameters' values: $\rho = 0.05$, $\bar{\mu} = 0.03$, $\sigma = 0.1$

to wage is smaller than 1. Unfortunately, with such a drift parameter, we quickly run into problems with the numerical solution. An example of these troubles for $\alpha = 1$ is presented in Figure 3.3. The numerical solution of equation (3.7') has values so much smaller than the analytical approximations of function $g(z)$ that there is no way to compare them. Hence, we cannot use the graphs in Figure 3.3.a to chose the best approximation, as we did in Figure 3.1. The consumption path calculated using the numerical solution does not behave as expected. Its fragment shown in Figure 3.3.b is first decreasing and then increasing. That would suggest that the ratio of consumption to wage is increasing with the rise in income, which is contrary to what we said above. Even more disturbing are the values of the consumption to wage ratio. For all we have said about why consumption can be much higher than income when wages are small and accumulated savings high,



a) Functional form of $g(z)^{+}$



b) Consumption path

Figure 3.3: Problems with numerical solution for $\bar{\mu} = -0.01$, $\alpha = 1$, $\rho = 0.05$, $\sigma = 0.1$

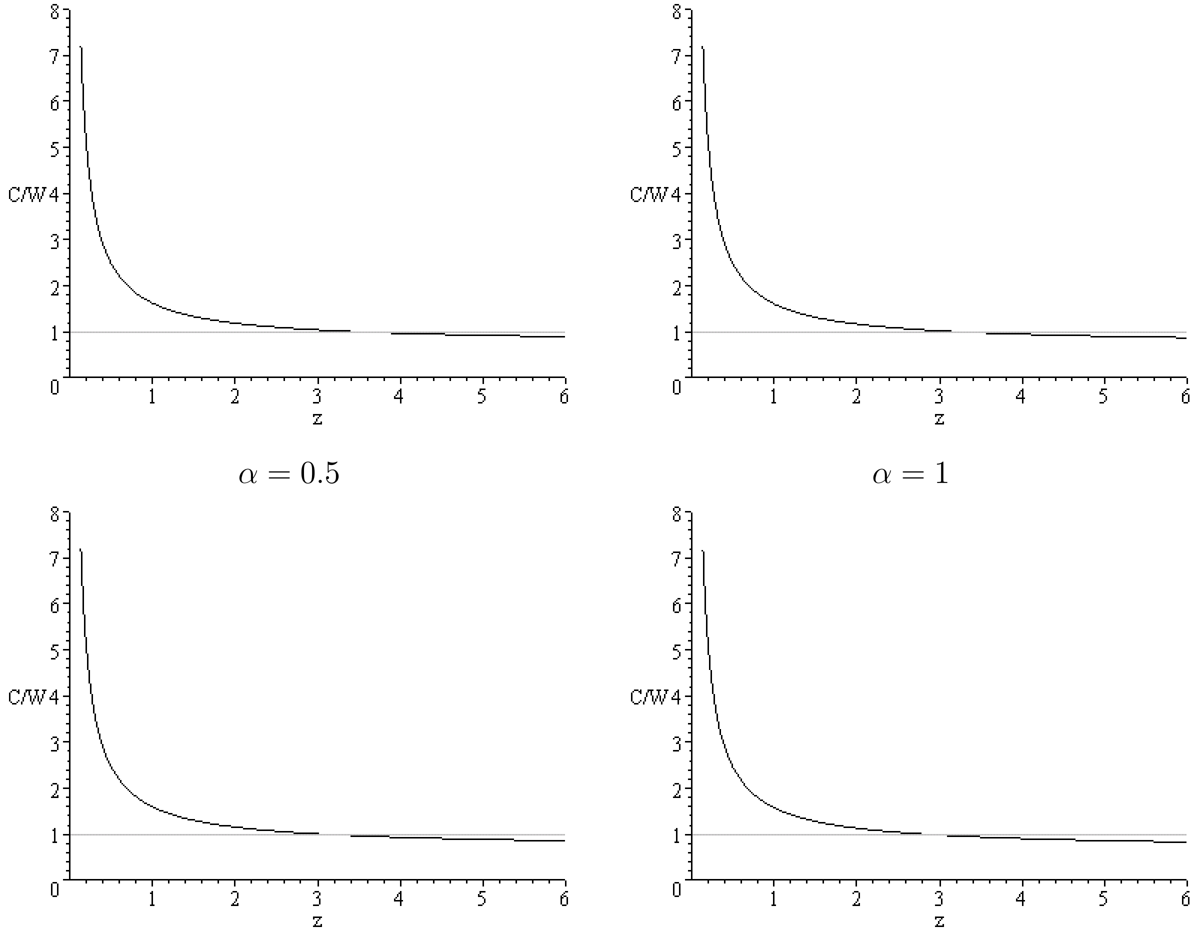
$^{+}$For such low values of z all approximations are identical

the notion that for all the combinations of expected incomes and accumulated savings consumption is more than a hundred times bigger than wage is absurd. Therefore the numerical solution does not work.

On the other hand, the analytical approximations of the function $g(z)$ for $\bar{\mu} = -0.01$ provide more insight. Conditions (3.16) and (3.17) hold for all values of z, since now $\phi < 0$. The results are presented in Figure 3.4. For all the values of the coefficient of relative risk aversion which we considered, the ratio of consumption to wage falls below one for high values of z. Indeed, as stated above, in all the cases considered in Figure 3.4 it is true that:

$$\lim_{z \to \infty} \frac{C}{W} < 1$$

Thus an individual is consuming more than she/he earns when her/his savings are rela-



$\alpha = 0.5$ $\alpha = 1$

$\alpha = 1.33$ $\alpha = 2$

Figure 3.4: Approximated optimal consumption path for the third degree approximation of $g(z)$ and $\bar{\mu} = -0.01$, with $\rho = 0.05$, $\sigma = 0.1$

tively high and expected income low. However, when expected income grows, or savings decrease, risk aversion combined with the negative trend in wages make the precautionary saving motive stronger and people consume less than they earn in order to insure themselves against the fall in incomes.

3.7 Distribution of consumption and savings

The simulations in Section 6 showed us the shape of the function $g(z)$ and the consumption path. In this section we want to find out what are the distributions of consumption and savings and how they change over time. In order to do this we simulate the development of wages, assuming that a coefficient of relative risk aversion $\alpha = 2$, and a drift in wages $\bar{\mu} =$

-0.01. For these simulations we must transform the continuous lognormal distribution of wages into a discrete process defined by $\bar{\mu}$ and σ. Starting from the initial wage of $W_0 = 1$, wages develop by a binomial tree, with the probability of moving up equal:

$$P_u = \frac{1}{2} + \frac{\bar{\mu}}{2\sigma}$$

At each node we calculate consumption from the formula for the consumption path (3.23), given the current wage and savings from the previous period. The current savings are calculated with the law of motion of savings, current consumption and wage. However, the process is path dependent, since the amount saved until moment t depends on the whole wage history up to t. Therefore, with each period the number of possibilities increases and we need to take into account 2^t paths at each moment. With many periods it creates too many paths to analyze, therefore the full sample is taken into account only for 10 and 20 periods. For a higher number of periods we choose randomly a number of paths[11], where the selection of paths is determined by the relation between the probabilities of moving up or down in the wage tree and a certain randomly chosen number between zero and one. Finally, to smooth the distributions we use the normal kernel density estimation.

The resulting distributions are presented in Figure 3.5. They are as expected: since the trend in wages is negative they are skewed to the left, with a long right tail, which gets longer with time. It is especially clear when we compare the differences between the 95 percentiles and the maximum values presented in Table 3.3, which presents some descriptive characteristics of the distributions. The plots of the distributions show that for a short time period, before substantial savings are accumulated, people consume less than they earn. For 10 periods the distribution of consumption is shifted to the left of the distribution of wages. Then gradually it shifts to the right, and from 30 periods onwards it is clear that consumers use their savings to increase consumption. It is confirmed by the characteristics of the distributions, shown in Table 3.3. For lower wages and consumption only for $t = 10$ is the fifth percentile of wages is higher than the fifth percentile of consumption. For high wages and consumption the 95th percentile of consumption is higher than that of wages only for $t = 40$ and $t = 50$. However, the maximum values of wages are always higher than consumption. Thus individuals with high earnings do not need to use their savings to increase consumption above the wage level.

[11]For $t = 30$ we have 2000000 paths, for $t = 40$: 3000000 and for $t = 50$: 4000000.

Figure 3.5: Distributions of consumption, wages and savings

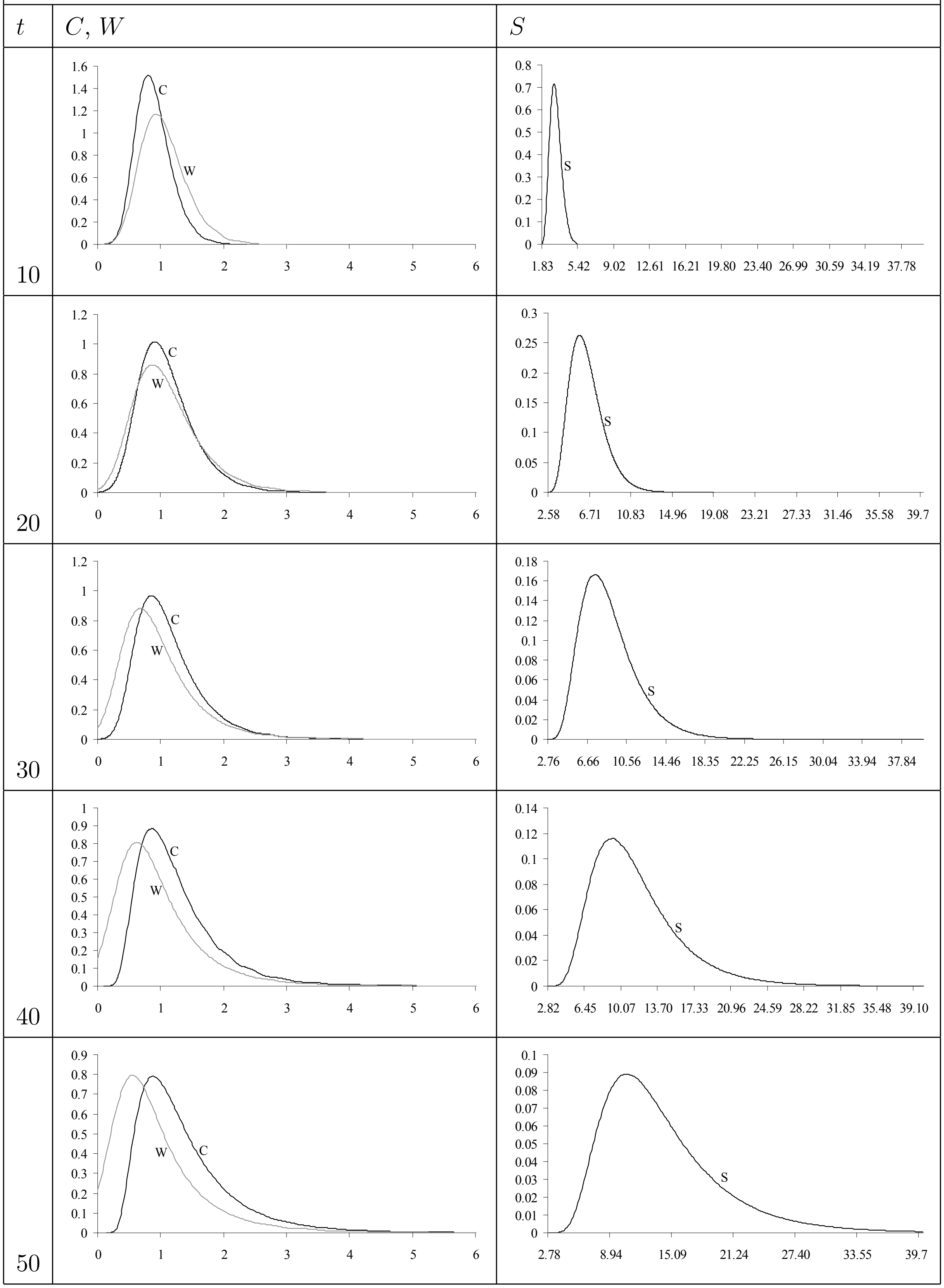


Table 3.3: Characteristics of the distributions of consumption, wage and savings

\	t	5%	95%	max	$mean$	$variance$	$skewness$	$kurtosis$
C	10	0.52	1.44	2.13	0.81	0.05	0.92	4.24
	20	0.56	1.97	6.05	0.93	0.13	1.35	6.22
	30	0.52	2.14	16.71	1.13	0.27	1.7	8.36
	40	0.54	2.51	45.68	1.26	0.44	2.03	11.04
	50	0.55	2.91	124.39	1.39	0.65	2.35	14.25
W	10	0.55	1.82	2.72	0.95	0.09	0.97	4.37
	20	0.45	2.23	7.39	0.9	0.18	1.48	6.84
	30	0.37	2.23	20.09	0.96	0.32	1.92	9.88
	40	0.3	2.23	54.6	0.95	0.44	2.39	14.21
	50	0.25	2.23	148.41	0.93	0.56	2.86	19.73
S	10	2.41	4.22	5.14	3.06	0.27	0.62	3.25
	20	4.17	9.55	18.88	5.85	2.2	0.95	4.4
	30	5.22	14.21	56.04	8.85	8.13	1.21	5.55
	40	6.21	19.78	156.85	11.52	19.12	1.46	6.93
	50	7.03	25.56	430.68	14.07	36.9	1.72	8.64

The distribution of savings changes more than the distributions of consumption or wages. The range of the distribution is broader and its mean is shifting significantly to the right with time. Also the values of accumulated savings are increasing faster than the values of consumption and wages. Despite the fact that people use their savings to increase consumption, they keep on accumulating and do not consume all they could have.

The value of the 95th percentile of wages may seem strange: for $t > 10$ it is always equal to 2.23. This is the result of the discretization of the process. The number of possible paths is much higher than the number of wage levels. Therefore with a very long right tail many paths pass through the same wage levels. Higher time periods add extreme values to the distribution of wages, but mainly increase the number of paths through already attained wage levels. In this way the values of the 95th percentile are identical for different periods.

3.8 Conclusions

The purpose of this chapter has been to find the optimal consumption when income follows geometric Brownian motion. We assumed that the rate of return on savings is constant, lifetime is infinite, utility is a constant relative risk aversion utility function, and transitory shocks are ignored. The problem was studied in a continuous time setting. With such a setup we tried to move further than studies like Deaton (1989) or Carroll (1997, 2001). As Carroll (2001) said, in presence of uncertainty about the future income, the derivation of an explicit solution for consumption in the form of an analytical function of the model parameters appears to be impossible. Therefore we formulated a partial differential equation defining the optimal utility, and found a way to transform it into an ordinary differential equation. We solved this equation numerically and approximated its analytical form. Analysis of this approximation shows that it works quite well for low values of the coefficient of relative risk aversion, and for high risk aversion when expected income is smaller than accumulated savings. The problems start for high z, i.e. when the expected income is higher than savings, and for high risk aversion. Finally, the optimal ratio of consumption to wage was calculated as a function of the ratio of expected incomes to accumulated savings (z). The approximated and numerical optimal consumption paths were similar for risk aversion smaller than 1 and, when $\alpha \geq 1$, for z smaller than 1, i.e. for the interval in which the approximation works the best. However, in order to have the consumption to wage ratio as expected, i.e. falling below 1 for high incomes, and for the precautionary effect to work, we had to reduce the trend in income. In such a case the approximation works well, but the numerical solution was not reliable. This shows that Carroll and Kimball (2001) result that liquidity constraints are not necessary when precaution is taken into account does not work for a positive drift in wages and an infinite lifetime.

The achievement of this chapter is the approximation of the functional form of the lifetime utility with stochastic wages, and the approximated structural equation defining optimal consumption as a function of current wage and accumulated savings. This gives us the analytical approximation of the relation between consumption, wages and assets, which may help in empirical studies of the problem.

The obvious subject for further research is to find a good approximation of the utility when the expected income is higher than accumulated savings. The case for which we were able to find an approximation may describe individuals close to the end of their labor career: savings accumulated over the lifetime are high and expected possible earnings are small. The opposite case may characterize individuals at the beginning of the labor career, when there was no time to accumulate any savings and the expected earnings in the whole lifetime are high.

3.9 Appendix 1

Calculating the derivatives:

$$g''(z) = \frac{\left[\begin{array}{c} \left[-\rho+(1-\alpha)\bar{\mu}-\frac{\sigma^2}{2}\alpha(1-\alpha)\right]\left(\frac{g(z)}{z}\right)^2+\alpha\left(\rho-\bar{\mu}\right)^{\frac{\alpha-1}{\alpha}}\left(g'(z)\right)^{\frac{\alpha-1}{\alpha}}\left(\frac{g(z)}{z}\right)^{\frac{2}{\alpha}} \\ +(1-\alpha)\left[(z+1)\left(\rho-\bar{\mu}\right)-\sigma^2(1-\alpha)\right]g'(z)\frac{g(z)}{z}-\frac{\sigma^2}{2}(1-\alpha)(\alpha-2)(g'(z))^2 \end{array}\right]}{\frac{\sigma^2}{2}(1-\alpha)g(z)} \tag{3.25}$$

We are looking for $g''(0)$. Since, as $z \to 0$, $g(z)$ approaches zero we must use l'Hospital's rule. However, first we have to check if the conditions allowing its use are fulfilled. It is clear that the denominator is zero at $z = 0$. The question is what about the numerator? We know from Lemma 2 and its proof that $g'(0) = \frac{g(z)}{z}$. Therefore

$$\lim_{z\to 0}\left[\begin{array}{c} \left[-\rho+(1-\alpha)\bar{\mu}-\frac{\sigma^2}{2}\alpha(1-\alpha)\right]\left(\frac{g(z)}{z}\right)^2+\alpha\left(\rho-\bar{\mu}\right)^{\frac{\alpha-1}{\alpha}}\left(g'(z)\right)^{\frac{\alpha-1}{\alpha}}\left(\frac{g(z)}{z}\right)^{\frac{2}{\alpha}} \\ +(1-\alpha)\left[(z+1)\left(\rho-\bar{\mu}\right)-\sigma^2(1-\alpha)\right]g'(z)\frac{g(z)}{z}-\frac{\sigma^2}{2}(1-\alpha)(\alpha-2)(g'(z))^2 \end{array}\right]$$
$$= -\alpha\rho g'(0)^2+\alpha\left(\rho-\bar{\mu}\right)^{\frac{\alpha-1}{\alpha}}g'(0)^{\frac{\alpha+1}{\alpha}} = \left[-\alpha\rho+\alpha\left(\rho-\bar{\mu}\right)^{\frac{\alpha-1}{\alpha}}g'(0)^{\frac{1-\alpha}{\alpha}}\right]g'(0)^2$$

Substituting (3.9) gives:

$$\left[-\alpha\rho+\alpha\left(\rho-\bar{\mu}\right)^{\frac{\alpha-1}{\alpha}}\left[\rho^{\frac{\alpha}{1-\alpha}}\left(\rho-\bar{\mu}\right)\right]^{\frac{1-\alpha}{\alpha}}\right](\rho^{\frac{\alpha}{1-\alpha}})^2 = \left[-\alpha\rho+\alpha\rho\right](\rho^{\frac{\alpha}{1-\alpha}})^2 = 0$$

Therefore the numerator at zero is equal to zero and we can compute the second derivative:

$$g''(0) =$$
$$= \lim_{z\to 0}\frac{\left[\begin{array}{c} \left[-\rho+(1-\alpha)\bar{\mu}-\frac{\sigma^2}{2}\alpha(1-\alpha)\right]\left(\frac{g(z)}{z}\right)^2+\alpha\left(\rho-\bar{\mu}\right)^{\frac{\alpha-1}{\alpha}}\left(g'(z)\right)^{\frac{\alpha-1}{\alpha}}\left(\frac{g(z)}{z}\right)^{\frac{2}{\alpha}} \\ +(1-\alpha)\left[(z+1)\left(\rho-\bar{\mu}\right)-\sigma^2(1-\alpha)\right]g'(z)\frac{g(z)}{z}-\frac{\sigma^2}{2}(1-\alpha)(\alpha-2)(g'(z))^2 \end{array}\right]}{\frac{\sigma^2}{2}(1-\alpha)g(z)}$$
$$= \frac{\frac{1}{2}(1-\alpha)(\rho-\bar{\mu}+\sigma^2)g''(0)+(1-\alpha)(\rho-\bar{\mu})^2\rho^{\frac{\alpha}{1-\alpha}}}{\frac{\sigma^2}{2}(1-\alpha)} \Leftrightarrow g''(0) = -2\rho^{\frac{\alpha}{1-\alpha}}\left(\rho-\bar{\mu}\right)$$

To calculate $g'''(0)$ we need to differentiate (3.25), and repeat the procedure used above: verify that both the numerator and the denominator are zero at zero and then use l'Hospital's rule to find a limit. Then we differentiate (3.25) once more and calculate $g^{(4)}(0)$. The same procedure is used for the case when $\alpha = 1$.

3.10 Appendix 2

An alternative approximation of function $g(z)$:

We are looking for a function which is increasing, concave, equal zero at zero and converging to a positive constant in infinity. We can approximate $g(z)$ by:

$$g(z) = A\frac{z}{z+1}\frac{z\left(\frac{(b_1+1)z+1}{z+1}\right)^{-a}+1}{z+1}\frac{z^2\left(\frac{(b_2+1)z+1}{z+1}\right)^{-a}+1}{z^2+1}$$

Such a functional form forces the first and second derivatives at $z = 0$ to be equal to the actual ones with $g'(0) = A$ and $g''(0) = -2A$. Parameters b_1 and b_2 can then be found from the third and fourth derivatives. There are different versions of this approximation possible, depending on whether $a = \alpha$ or not. However, if we assume equality, and find the parameters, the functional form is given by

$$g(z) = A\frac{z}{z+1}\frac{z\left(\frac{\left(\frac{1}{2}\frac{\sigma^2}{\sigma^2+2\bar{\mu}-\rho}+1\right)z+1}{z+1}\right)^{-a}+1}{z+1}\frac{z^2\left(\frac{\left(\frac{1}{8}\frac{\sigma^4}{(\sigma^2+2\bar{\mu}-\rho)^2}\frac{\sigma^2+5\bar{\mu}-3\rho}{3\sigma^2+3\bar{\mu}-\rho}(1+\alpha)+1\right)z+1}{z+1}\right)^{-a}+1}{z^2+1}$$

When we plotted the optimal consumption path for the parameters $\rho = 0.05$, $\sigma = 0.1$, and for two values of $\bar{\mu}$: 0.03 and -0.01, the results were very similar to the ones in the text above. For the positive trend, the consumption path was always above one, although the values of $\lim_{z\to\infty}\frac{C}{W}$ were smaller than for the approximation used in the chapter. For the negative trend, the consumption path always ended below one. The reason why we do not use this approximation, instead of

$$g(z) = \frac{b_n z^n + b_{n-1}z^{n-1} + \ldots + b_1 z}{(z+1)^n}$$

is that it gives only slightly better results in exchange for more computational complexity.

3.11 Appendix 3

Results for $\alpha = 1$:

The differential equation defining the expected utility:

$$\psi U = \ln C + U_S \left(W - C + \rho S\right) + \bar{\mu} W U_W + \frac{1}{2}\sigma^2 W^2 U_{WW} \tag{3.2'}$$

$$\psi U = -\ln U_S - 1 + U_S \left(W + \rho S\right) + \bar{\mu} W U_W + \frac{1}{2}\sigma^2 W^2 U_{WW} \tag{3.4'}$$

From Lemma 1: The optimal consumption path and utility satisfy:

$$\frac{C}{W} = [\frac{\rho - \bar{\mu}}{\rho} z^2 g'(z) g(z)^{-1}]^{-1} \tag{3.5'}$$

$$\bar{U}(W, z) = \frac{1}{\rho}\left[\ln W - \ln g(z)\right] \tag{3.6'}$$

and $g(z)$ satisfies:

$$\psi g(z)^2 = 2\rho g(z)^2 \ln \frac{g(z)}{z} - \rho g(z)^2 \ln\left(\frac{\rho - \bar{\mu}}{\rho}\right) - \rho g(z)^2 \ln g'(z) + \left(\bar{\mu} + \frac{\sigma^2}{2}\right) g(z)^2$$
$$+ \left(\rho - \bar{\mu}\right)\left(z + 1\right) z g'(z) g(z) - \frac{\sigma^2}{2} z^2 g''(z) g(z) + \frac{\sigma^2}{2} z^2 g'(z)^2 \tag{3.7'}$$

The formula for the optimal consumption path as a share in income arises from substitution of (3.20) into (3.5') from Lemma 1:

$$\frac{C}{W} = \frac{\rho}{\rho - \bar{\mu}} \frac{z+1}{z} \frac{\left[1 - \frac{1}{2\phi}\right] z^2 + 2z + 1}{\left[1 - 3\frac{1}{2\phi}\right] z^2 + 2z + 1} \tag{3.21'}$$

3.12 Appendix 4

This appendix presents the model from Caballero (1990) recalculated in continuous time and with wages following standard Brownian motion. To simplify the comparison equations are numbered as in the original model. The notation was changed to match our model, thus $r = \rho$, $A = S$ and $\sigma_w = \sigma$.

The solution to the utility maximization must satisfy the Euler equation:

$$E\left[e^{-\theta \dot{C}_t}\right] = 1 \tag{2}$$

Consumption is assumed to be linear in levels. Transforming it gives

$$\dot{C}_t = \Gamma_t + v_t \tag{3}$$

where v_t is a random innovation and Γ_t a slope of the consumption path. The functional form of Γ_t may be derived by substituting (3) into (2):

$$\Gamma_t = \frac{1}{\theta} \ln E\left[e^{-\theta v_t}\right] \tag{4}$$

To determine it fully we must know the distribution of v_t.

The intertemporal budget constraint shows that the total discounted difference between consumption and income must be equal to savings:

$$\int_t^{\infty} (C_i - Y_i)\, e^{-\rho(i-t)} di = S_t$$

It can be rewritten as:

$$\int_t^{\infty} C_i e^{-\rho(i-t)} di - \int_t^{\infty} (Y_i - E_t[Y_i])\, e^{-\rho(i-t)} di - \int_t^{\infty} E_t[Y_i]\, e^{-\rho(i-t)} di = S_t \tag{5}$$

We know that the income process follows MA with:

$$\dot{E}_t[Y_i] = \psi_i w_t$$

However for standard Brownian motion we have

$$\psi_0 = 1, \psi_{i>0} = 0$$

Thus (5) gives:

$$\int_t^{\infty} C_i e^{-\rho(i-t)} di - \int_t^{\infty} \frac{w_i}{\rho} e^{-\rho(i-t)} di - \int_t^{\infty} E_t[Y_i]\, e^{-\rho(i-t)} di = S_t$$

where $\frac{1}{\rho}$ in the second expression on the left hand side results from: $Y_i - E_t[Y_i] = \frac{w_i}{\rho}$.

This assumption is necessary to have the same innovation disturbances in wages and consumption, i.e. $v_t = w_t$ as in Caballero' equation (8) for wages following random walk, and not $v_t = \rho w_t$.

We want to substitute (3) into the transformed (5). Before we do this, it is necessary to use integration by parts on the first expression in (5):

$$\int_t^{\infty} C_i e^{-\rho(i-t)} di = -\frac{C_i}{\rho} e^{-\rho(i-t)} \big|_t^{\infty} + \frac{1}{\rho} \int_t^{\infty} \dot{C}_i e^{-\rho(i-t)} di$$

Substituting (3) and simplifying yields:

$$\int_t^\infty C_i e^{-\rho(i-t)} di = \frac{C_t}{\rho} + \frac{1}{\rho}\int_t^\infty (\Gamma_i + v_i)\, e^{-\rho(i-t)} di$$

Now we can transform (5):

$$\frac{C_t}{\rho} + \frac{1}{\rho}\int_t^\infty \Gamma_i e^{-\rho(i-t)} di + \frac{1}{\rho}\int_t^\infty v_i e^{-\rho(i-t)} di - \frac{1}{\rho}\int_t^\infty w_i e^{-\rho(i-t)} di - \int_t^\infty E_t [Y_i]\, e^{-\rho(i-t)} di = S_t \tag{6}$$

Taking expectations conditional on information available at time t yields the consumption function:

$$\begin{aligned} C_t &= \rho \left[\begin{array}{c} S_t + \int_t^\infty E_t [Y_i]\, e^{-\rho(i-t)} di - \frac{1}{\rho}\int_t^\infty \Gamma_i e^{-\rho(i-t)} di \\ \underbrace{-\frac{1}{\rho}\int_t^\infty v_i e^{-\rho(i-t)} di + \frac{1}{\rho}\int_t^\infty w_i e^{-\rho(i-t)} di}_{=0} \end{array} \right] \\ &= \rho \left[S_t + \int_t^\infty E_t [Y_i]\, e^{-\rho(i-t)} di - \frac{1}{\rho}\int_t^\infty \Gamma_i e^{-\rho(i-t)} di \right] \end{aligned}$$

The permanent income is given by:

$$Y_t^p = \rho \left[S_t + \int_t^\infty E_t [Y_i]\, e^{-\rho(i-t)} di \right]$$

If we assume for simplicity that there is no drift in income, then the expected value of income at time t is equal: $\int_t^\infty E_t [Y_i]\, e^{-\rho(i-t)} di = \frac{Y_t}{\rho}$. Therefore the permanent income is equal to:

$$Y_t^p = \rho S_t + Y_t$$

Then the consumption function can be presented as

$$C_t = \rho S_t + Y_t - \int_t^\infty \Gamma_i e^{-\rho(i-t)} di$$

When we replace C_t back into (6) we can identify the stochastic sequence $\{v_t\}$:

$$\int_t^\infty \left(\frac{1}{\rho} v_i - \frac{1}{\rho} w_i \right) e^{-\rho(i-t)} di = 0 \tag{7}$$

The result is:

$$v_t = w_t \tag{8}$$

It means that consumption is as volatile as wages, and consumption itself becomes a random walk with drift Γ_t and the variance equal to that of innovations.

Replacing (8) back into expressions (3) and (4) gives a full characterization of the consumption process:

$$\dot{C}_t = \Gamma_t + w_t$$

and of the slope of the consumption function:

$$\Gamma_t = \frac{1}{\theta} \ln E_t \left[e^{-\theta w_t} \right]$$

Since income follows Brownian motion, the innovation disturbances are NIID distributed. Therefore the slope of the consumption function is equal:

$$\Gamma = \frac{\theta}{2} \sigma^2$$

Then the consumption function given by:

$$C_t = \rho S_t + Y_t - \frac{\theta}{2\rho} \sigma^2$$

Chapter 4

Retirement and private savings

4.1 Introduction

In this chapter we study what happens if there is no retirement insurance and individuals have to save themselves for their retirement. In the previous chapter we have analyzed the optimal consumption when savings were used only as an insurance against unexpected decreases in income. Now we allow an individual to retire, if she/he saved enough to finance this retirement. By extending the problem in this way we have added an additional decision: an individual decides not only how much of income to consume and how much to save, she/he decides also if enough has been saved to stop working and to finance retirement. As in Chapter 2 the retirement decision is irreversible. The main question of this chapter is what is the optimal retirement decision, i.e. for what value of the ratio of expected incomes to accumulated savings an individual should retire. We find the retirement threshold by approximating the functional form of utility at this point and comparing it with the approximated utility function when no retirement is allowed. Then we compare the results of the approximation with the numerical simulations of the retirement threshold.

This chapter is organized as follows: Section 2 describes the basic assumptions, the utility before and after retirement and the Bellman equation defining the optimal consumption path. Section 3 formulates the value matching and the smooth pasting conditions which must be fulfilled at the moment of retirement, ensuring the continuity and smoothness of the utility function. In Section 4 we calculate the retirement threshold of the ratio of expected wages to accumulated savings. We also present numerical simulations of this threshold. Simulated distributions of consumption and savings and probability of retirement are in Section 5. Section 6 concludes.

4.2 Utility before and after retirement

As in Chapter 3 the utility while working is given by

$$U_{W,t} = \text{for } \alpha \neq 1 : \mathrm{E}\left[\int_t^T \frac{1}{1-\alpha} C_{W,s}^{1-\alpha} e^{-\psi(s-t)} ds\right] \quad (4.1)$$

$$= \text{for } \alpha = 1 : \mathrm{E}\left[\int_t^T \ln C_{W,s} e^{-\psi(s-t)} ds\right] \quad (4.1')$$

where $C_{W,t}$ is consumption at time $t \in (0,T)$, $\alpha \geq 0$ is the coefficient of relative risk aversion, and ψ is the rate of time preference. In the subsequent analysis we assume that the rate of time preference is equal to the discount rate ρ. T, i.e. the moment of retirement, is determined endogenously. Since it depends on the realizations of random variables prior to T, it is itself a random variable. The utility after retirement is different:

$$U_{R,t} = \text{for } \alpha \neq 1 : \int_t^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} C_R^{1-\alpha} e^{-\rho(s-t)} ds \quad (4.2)$$

$$= \text{for } \alpha = 1 : \int_t^\infty \left[\ln C_R + \ln \varepsilon\right] e^{-\rho(s-t)} ds \quad (2')$$

where $t \in (T, \infty)$ and ε is a multiplier of utility because of not working. In other words, ε is an equivalent of the cost of effort, a measure of the disutility from working. Since people usually prefer not working to working we assume that $\varepsilon > 1$.

Let S_t be the worker's stock of savings and W_t the current wage. As in Chapter 3 the laws of motion of S and W read, leaving out the arguments and subscripts:

$$dS = (W - C + \rho S)\, dt \quad (4.3)$$

$$dW = \bar{\mu} W dt + \sigma W dv \quad (4.4)$$

The change in savings is given by the amount saved at moment t, i.e. the difference between the current income and the current consumption, and the interest gained from the savings accumulated so far. The change in income is given by the increase due to the trend ($\bar{\mu}$) plus a possible deviation (with the standard deviation σ) from this trend, where dv is the increment of a Wiener process. Therefore income follows geometric Brownian motion with drift. Workers know their current wage W and their stock of savings S but they do not know the future evolution of wages. Hence the current state is fully determined by W and S and thus the optimal consumption and utility are also determined by these variables. Therefore worker's utility can be written as $U = U(W, S)$.

Utility after retirement is nonstochastic, since consumption after retirement does not depend on stochastic income but on the accumulated savings and the constant rate of return from savings. The decision how much to consume is straightforward: in the infinite lifetime, with $\psi = \rho$, a retired individual consumes only the interests earned from savings, from fear of depleting their stock. Thus the retirement consumption is constant over time.

The optimal utility while working is defined by the same differential equation (3.4) as in Chapter 3[1]:

$$\rho U = \frac{\alpha}{1-\alpha} U_S^{\frac{\alpha-1}{\alpha}} + U_S (W + \rho S) + \bar{\mu} W U_W + \frac{1}{2}\sigma^2 W^2 U_{WW} \tag{4.5}$$

We may again use Lemma 1 from Section 3.3 to redefine the utility and to transform the partial differential equation above into the ordinary differential equation. Thus, defining

$$\begin{aligned} z &\equiv \frac{W}{(\rho - \bar{\mu}) S} \\ \bar{U}(W, z) &\equiv U\left(W, \frac{W}{(\rho - \bar{\mu}) z}\right) \end{aligned}$$

the lifetime utility and optimal consumption path satisfy[2]:

$$\bar{U}(W, z) = \frac{1}{1-\alpha} W^{1-\alpha} \bar{g}(z)^{\alpha-1} \tag{4.6}$$

$$\frac{C}{W} = \left[(\rho - \bar{\mu}) z^2 \bar{g}'(z) \bar{g}(z)^{\alpha-2}\right]^{-\frac{1}{\alpha}} \tag{4.7}$$

where $\bar{g}(z)$ is defined by:

$$\begin{aligned} \rho \bar{g}(z)^2 = {} & \alpha (\rho - \bar{\mu})^{\frac{\alpha-1}{\alpha}} z^{2\frac{\alpha-1}{\alpha}} \bar{g}'(z)^{\frac{\alpha-1}{\alpha}} \bar{g}(z)^{\frac{2}{\alpha}} + (1-\alpha)\left(\bar{\mu} - \frac{1}{2}\sigma^2\alpha\right) \bar{g}(z)^2 \\ & + (1-\alpha)\left[(z+1)(\rho - \bar{\mu}) - \sigma^2(1-\alpha)\right] z \bar{g}'(z) \bar{g}(z) \\ & - \frac{1}{2}\sigma^2 (1-\alpha)(\alpha - 2) z^2 \bar{g}'(z)^2 - \frac{1}{2}\sigma^2 (1-\alpha) z^2 \bar{g}''(z) \bar{g}(z) \end{aligned} \tag{4.8}$$

[1] The differential equation for $\alpha = 1$ and the subsequent calculations for this case are presented in Appendix 3.

[2] To avoid any misunderstanding we use $\bar{g}$ for the function with the possibility of retirement and g for the function without this possibility.

Again the utility function is separated into the relative utility derived from income, $W^{1-\alpha}$, and the self-insurance (precautionary) component, $\bar{g}(z)^{\alpha-1}$, showing the ratio between the net expected discounted income and accumulated savings and its influence on utility.

4.3 Conditions of retirement

An individual retires when the expected utility from continued work and the utility of the immediate retirement are equal. Therefore the retirement decision is both forward and backward looking, since the utility from the retirement depends on the amount saved, and the expected utility of continued work depends on the expected incomes calculated on the basis of the current wage. An individual decides to retire for such a combination of wages (W) and savings (S) for which value matching and smooth pasting conditions hold:

$$\bar{U}(W,z) = \frac{1}{1-\alpha}\rho^{-\alpha}\varepsilon^{1-\alpha}S^{1-\alpha} \tag{4.9}$$

$$\bar{U}_W(W,z) = 0 \tag{4.10}$$

$$\bar{U}_S(W,z) = \rho^{-\alpha}\varepsilon^{1-\alpha}S^{-\alpha} \tag{4.11}$$

Condition (4.9) is a value matching condition, necessary to ensure continuity of the utility function, i.e. utility just before retirement is equal to the utility just after. The right hand side of condition (4.9) results from the fact that the utility after retirement is multiplicative in savings (S) and the not-working coefficient (ε), since, as explained above, consumption in retirement is equal to the interests earned from savings and the utility from consumption is enhanced by leisure. Thus, when we substitute $C = \rho S$ into the utility after retirement (4.2), the result is the right hand side of (4.9). Two smooth pasting conditions make certain that the utility is smooth with respect to both state variables, i.e. the marginal utilities just before and after the retirement are equal[3].

Using any of these conditions we may form the following proposition:

Proposition 1 *The decision to retire is fully defined by the ratio of two state variables* (W, S)*:*

$$z^* = \frac{W}{(\rho - \bar{\mu})S}$$

[3]For the detailed discussion of value matching and smooth pasting conditions see Dixit and Pindyck (1994).

Proof. The proposition may be proved with any of the three conditions above. Consider the value matching condition (4.9) and substitute (4.6) for $\bar{U}\,(W,z)$. Then:

$$\frac{1}{1-\alpha}W^{1-\alpha}\bar{g}(z)^{\alpha-1} = \frac{1}{1-\alpha}\rho^{-\alpha}\varepsilon^{1-\alpha}S^{1-\alpha}$$
$$\Rightarrow \bar{g}(z)^{\alpha-1} = \rho^{-\alpha}\varepsilon^{1-\alpha}\left(\frac{S}{W}\right)^{1-\alpha}$$

Since $z \equiv \frac{W}{(\rho-\bar{\mu})S}$ then in the last transformation $\frac{S}{W} = \frac{1}{z(\rho-\bar{\mu})}$, and thus the condition necessary for retirement is expressed in terms of z. Therefore there exist a particular value of the ratio of expected incomes to accumulated savings z^* for which it is optimal to retire. In the same way we may prove this proposition with either of the smooth pasting conditions. ■

Usually the value matching and smooth pasting conditions are used to determine the critical retirement threshold. However, in our case we use them to characterize the value of the function $\bar{g}(z)$ at the moment of retirement:

Lemma 1 *Given $\bar{U}(W,z) \equiv U(W,S)$, at the moment of retirement we have*

$$\bar{g}(z^*) = A\varepsilon^{-1}z^* \tag{4.12}$$
$$\bar{g}'(z^*) = A\varepsilon^{-1} \tag{4.13}$$

where

$$A \equiv \rho^{\frac{\alpha}{1-\alpha}}\,(\rho-\bar{\mu})$$

Proof. In this proof we follow the proof of Proposition 1: substituting (4.6) into the value matching condition gives:

$$U\,(W,z^*) = \frac{1}{1-\alpha}\rho^{-\alpha}\varepsilon^{1-\alpha}S^{1-\alpha}$$
$$\Rightarrow \frac{1}{1-\alpha}W^{1-\alpha}\bar{g}(z^*)^{\alpha-1} = \frac{1}{1-\alpha}\rho^{-\alpha}\varepsilon^{1-\alpha}S^{1-\alpha}$$
$$\Rightarrow \bar{g}(z^*) = \frac{A}{\varepsilon}z^*$$

In the same way we can prove Lemma 1 for $\alpha = 1$. ■

We can also obtain this result from the smooth pasting conditions. When we substitute

(4.6) into the first smooth pasting condition (4.10), we get:

$$U_W(W, z^*) = 0$$
$$\Rightarrow W^{-\alpha}\bar{g}(z^*)^{\alpha-1} - W^{-\alpha}z^*\bar{g}'(z^*)\bar{g}(z^*)^{\alpha-2} = 0$$
$$\Rightarrow \bar{g}(z^*) - z^*\bar{g}'(z^*) = 0$$
$$\Rightarrow \bar{g}'(z^*) = \frac{A}{\varepsilon}$$

The same result can be obtained from the second smooth pasting condition (4.11).

For $\varepsilon = 1$ the value of $\bar{g}'(z^*)$ is the same as the value of $g'(0)$ in the non-retirement case. When $\varepsilon = 1$ an individual is indifferent between work and leisure/retirement. Therefore there is no reason for retiring. Since in such a case the values of the derivatives are equal, it shows that $z^* = 0$.

The value matching and smooth pasting conditions yield another proposition:

Proposition 2 *Consumption directly before retirement is higher than the consumption after retirement for $\alpha > 1$. For $\alpha < 1$, consumption after retirement must be higher than that before, and for $\alpha = 1$ both consumptions are equal.*

Proof. By reformulating the second smooth pasting condition (4.11), using the first order condition that $C = U_S^{-\frac{1}{\alpha}}$ and the fact that consumption during retirement equals the interests from savings:

$$\bar{U}_S(W, z^*) = \rho^{-\alpha}\varepsilon^{1-\alpha}S^{-\alpha} = C_W^{-\alpha}$$
$$\Rightarrow C_W = \varepsilon^{\frac{\alpha-1}{\alpha}}\rho S^* = \varepsilon^{\frac{\alpha-1}{\alpha}}C_R$$

Therefore, when $\alpha > 1$, consumption before retirement is higher than the consumption after retirement and for $\alpha < 1$, i.e. for very low risk aversion, the situation must be opposite. For $\alpha = 1$ the case is much simpler:

$$U_S(W, z^*) = \frac{1}{\rho S}$$
$$\Rightarrow C_W = C_R$$

Thus for risk neutral individuals consumption directly before retirement is equal to consumption after retirement. ■

The optimal retirement decision influences the marginal utility of consumption. Therefore consumption after the retirement decision is taken can either rise, if $\alpha < 1$, fall, if

$\alpha > 1$, or remain at the same level, for $\alpha = 1$. However, empirically, risk aversion is larger than one, and thus in practice consumption after retirement is lower than consumption just before the retirement decision is taken. While retired an individual may maintain even higher utility than before retiring since her/his utility is increased by not working. This is the same result as the relation between income while working and retirement benefits in Chapter 2.

4.4 Retirement threshold

We want to find the critical retirement threshold of z^*. So far we know the value of function $\bar{g}(z)$ and its first derivative at z^* but we were not able to calculate what is the separation threshold z^* itself. We can find it by making a following proposition:

Proposition 3

$$g(\infty) = \bar{g}(\infty)$$

Proof. As wages are increasing the value of the option to retire is converging to zero, since there are no incentives to retire when the expected gains from staying at work are growing. Therefore, as wages go to infinity, the utility with the possibility of retirement is converging to the utility without the possibility of retirement. ■

We have seen this in Chapter 2 in equation (2.31), which was composed of the option to retire and the value of continued work. When the option to retire is equal to zero, the only element left is the value of continued work. Therefore the retirement threshold z^* is a value of z for which both functions converge to the same constant.

First we must approximate function $\bar{g}(z)$ at z^*. Using this function and the values of its derivatives at z^* allows us to find the parameters of $\bar{g}(z)$ in the neighborhood of z^*. Then, the comparison of the utility with the possibility of retirement with the utility without this possibility, i.e. with the utility we studied in consumption case, allows us to form an equation defining the retirement threshold of the ratio of expected discounted wages to accumulated savings.

To proceed further we must calculate higher derivatives of the function $\bar{g}(z)$ at the moment of retirement. The second derivative is obtained from the differential equation

(4.8). After substitution of (4.12) and (4.13), $\bar{g}''(z^*)$ becomes:

$$\bar{g}''(z^*) = 2\frac{[\alpha B + (1-\alpha)(\rho - \bar{\mu}) z^*]}{\sigma^2 (1-\alpha) z^*}\frac{A}{\varepsilon} \tag{4.14}$$

where for simplicity

$$B \equiv \rho\left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1\right)$$

In order to compare this case with the one without retirement we need more derivatives. By differentiating the formula for $\bar{g}''(z)$ and substituting (4.12), (4.13) and (4.14) we obtain[4]:

$$\bar{g}'''(z^*) = 4\frac{[z^*(\rho - \bar{\mu}) - B - \bar{\mu} + \sigma^2][\alpha B + (1-\alpha)(\rho - \bar{\mu}) z^*] - \frac{\sigma^2}{2}\alpha B}{\sigma^4 (1-\alpha) z^{*2}}\frac{A}{\varepsilon} \tag{4.15}$$

The value of $\bar{g}(z^*)$ and its three derivatives can be used to approximate the function using the same functional form as in the consumption case. We are not calculating higher order derivatives, since in Chapter 3 we have chosen the third order approximation of function $g(z)$. We know that the function is of the type:

$$g(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z}{(z+1)^n}$$

However in the retirement case it must be slightly changed, since for this function we cannot guarantee that it will fulfill the condition arising from Lemma 1, that $\bar{g}(z^*) = z^*\bar{g}'(z^*)$. Therefore we need to modify it, while maintaining all the basic properties: a positive first derivative, concavity and convergence to a positive constant in infinity. The new functional form is given by (4.16):

$$\bar{g}(z) = \frac{\bar{b}_n (z - z^*)^n + \bar{b}_{n-1}(z - z^*)^{n-1} + \ldots + \bar{b}_1 (z - z^*)}{(z - z^* + 1)^n} + \bar{b}_1 z^* \tag{4.16}$$

Using the third degree approximation, as shown in the non-retirement case, i.e. for $n = 3$, and knowing the value of the function and its derivatives at z^* we have a set of four equations with four unknowns: $\bar{b}_1$, $\bar{b}_2$, $\bar{b}_3$, and z^*:

$$\begin{cases} \bar{g}'(z^*) = \bar{b}_1 \\ \bar{g}''(z^*) = 2\bar{b}_2 - 6\bar{b}_1 \\ \bar{g}'''(z^*) = 6\bar{b}_3 - 18\bar{b}_2 + 36\bar{b}_1 \\ \bar{b}_3 + \bar{b}_1 z^* = b_3 \end{cases} \tag{4.17}$$

[4]The derivatives of $\bar{g}(z^*)$ for $\alpha = 1$ are presented in Appendix 2.

The last equation results from the convergence of both functions, with and without retirement, to the same constant. This constant is the highest parameter of the functional form increased by $\bar{b}_1 z^*$ for the function with retirement, as shown by:

$$\begin{aligned}\lim_{z\to\infty} g(z) &= b_n \\ \lim_{z\to\infty} \bar{g}(z) &= \bar{b}_n + \bar{b}_1 z^*\end{aligned}$$

In our case the highest parameter is b_3, since we are working with the third order approximation. Thus, in order to find the retirement threshold z^*, we have to compare the coefficient b_3 calculated when function $g(z)$ was approximated at zero with $\left(\bar{b}_3 + \bar{b}_1 z^*\right)$ calculated when function $\bar{g}(z)$ was at z^*. In the former case the coefficient was equal:

$$b_3 = \left[1 - \frac{\alpha}{2\phi}\right] A \tag{4.18}$$

where

$$\phi \equiv \frac{\sigma^2 + 2\bar{\mu} - \rho}{\sigma^2}$$

Since we want $\bar{g}(z)$ to converge to a positive constant we must check if $\bar{b}_3 + \bar{b}_1 z^* > 0$. This expression is a third order polynomial, thus it is possible to find an analytical solution of the inequality. Since the analytical solution is complex, we analyze a graphical solution for $\rho = 0.05$, $\bar{\mu} = 0.03$ and $\bar{\mu} = -0.01$, $\sigma = 0.1$ and the different values of the coefficient of the relative risk aversion. These graphs are presented in figures 4.7 and 4.8 in the Appendix 2. As we can see for most of the combinations of z^* and ε we have $\bar{b}_3 + \bar{b}_1 z^* > 0$. However there are some pairs of the cost of effort and the ratio of expected wages to savings for which this condition does not hold.

The reason for solving the problem for two different values of trend in wages is the same as in the previous chapter - with the positive trend there is no precautionary motive. In the case of retirement one may even ask why people with a positive trend in wages should retire, and exchange constantly growing incomes for the interests from savings. With the negative trend the precautionary motive is working, and individuals prefer to retire, because their incomes are more likely to decline than rise.

After substitution of (4.12), (4.13), (4.14), (4.15) and (4.18) into the set (4.17), we can find the values of the parameters and use them in the last equation, which yields the formula defining z^*:

$$\begin{aligned}0 \;=\;& \sigma^4(1-\alpha)z^{*3}+(1-\alpha)\left[\frac{2}{3}(\rho-\bar{\mu})^2+3(\rho-\bar{\mu})\sigma^2+3\sigma^4-\left(1-\frac{\alpha}{2\phi}\right)\sigma^4\varepsilon\right]z^{*2}\\&+\left[\left(\frac{2}{3}(\rho-\bar{\mu})+3\sigma^2\right)\alpha B+\frac{2}{3}\left(\sigma^2-B-\bar{\mu}\right)(1-\alpha)(\rho-\bar{\mu})\right]z^*\\&+\frac{2}{3}\left(\frac{\sigma^2}{2}-B-\bar{\mu}\right)\alpha B\end{aligned} \tag{4.19}$$

The equation (4.19) is a third order polynomial. Again we solve it graphically, because of the complexity of the analytical solution. However first we must verify if one result is true: when $\varepsilon = 1$ individuals are indifferent between work and leisure, and they will not retire as long as the retirement threshold is bigger than zero. When we substitute $\varepsilon = 1$ into (4.19) the solution is indeed $z^* = 0$. Therefore the basic condition is fulfilled.

Special case for $\alpha = 0$

Before proceeding with any further analysis we want to know what happens with the retirement threshold in the risk neutral case. The utility while working is then given by

$$U_{W,t} = E\left[\int_t^T C_{W,t}e^{-\rho(s-t)}ds\right]$$

while the utility after the retirement is equal:

$$U_{R,t} = E\left[\int_t^\infty \varepsilon C_R e^{-\rho(s-t)}ds\right]$$

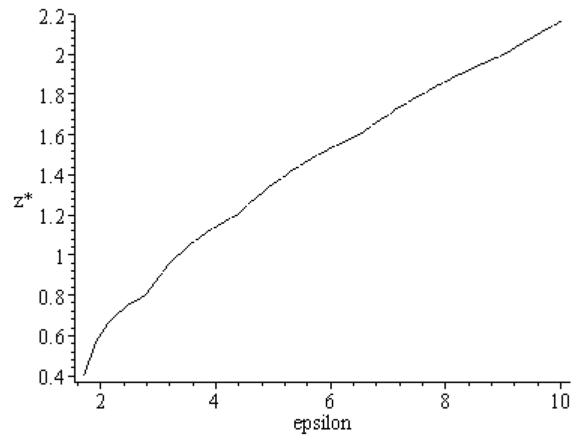
Then the first order condition with respect to consumption shows that the marginal utility of savings is constant and equal to one. This transforms the differential equation into:

$$\rho U = W + \rho S + \bar{\mu}WU_W + \frac{1}{2}\sigma^2W^2U_{WW}$$

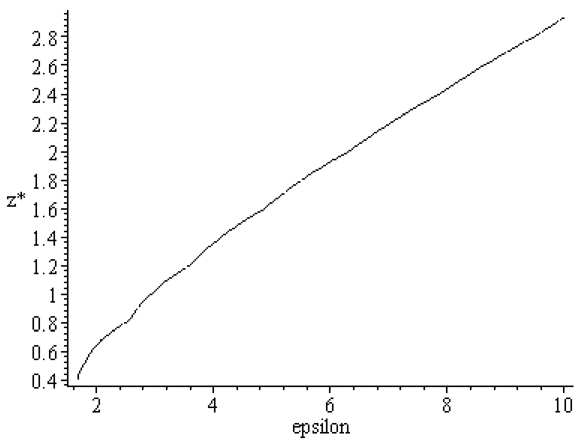
As we can see neither consumption nor marginal utility of savings influence the utility in the optimum. In the risk neutral case an individual is indifferent between consumption at any point of time. However, consumption is more efficient after retirement, since we assume $\varepsilon > 1$ and utility is defined as above. Therefore an indifferent individual consumes everything after retirement. It means that in the infinite lifetime an individual never retires, always waiting for more wealth to consume after the retirement.

We can now plot the solutions for $\alpha > 0$. The graphs of the function $z^*(\varepsilon)$ for $\rho = 0.05$, $\sigma = 0.1$, $\bar{\mu} = -0.01$ and $\bar{\mu} = 0.03$, and different values of the coefficient of relative risk

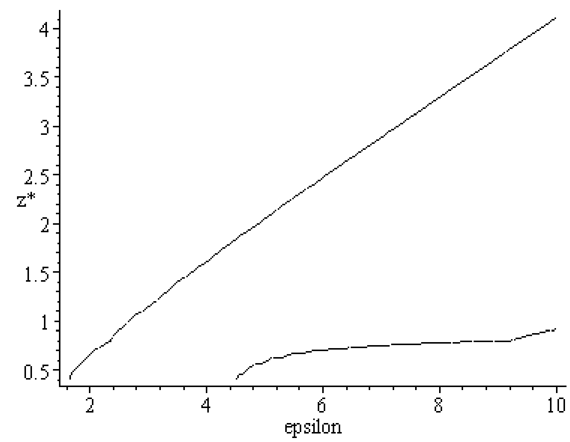
$\alpha = 0.5$



$\alpha = 1$



$\alpha = 1.33$

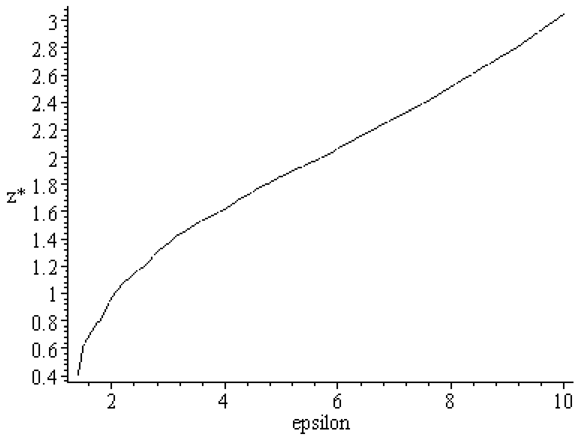


$\alpha = 2$

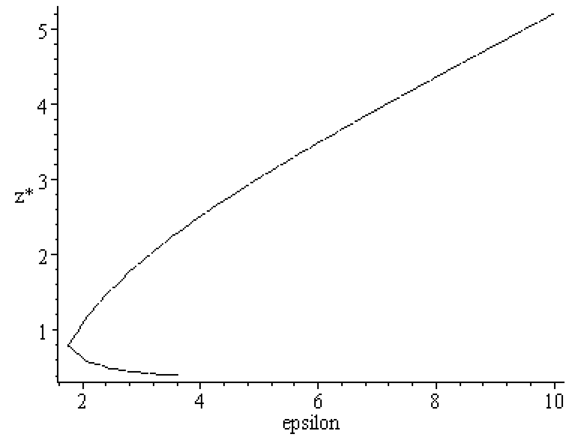
Figure 4.1: $z^*(\varepsilon)$, when $\bar{\mu} = -0.01$

aversion are presented in Figures 4.1 and 4.2. As expected $z^*(\varepsilon)$ is an increasing function of the cost of effort (ε) for all values of the coefficient of relative risk aversion (α), i.e. the more an individual dislikes work the earlier she/he retires. The critical threshold is rising with risk aversion - the more risk averse a worker is the sooner she/he retires, since waiting longer means accepting the risk of a fall in wages. Another result confirming the intuition is that in all the cases the values of z^* for the equivalent values of ε and α are higher for the positive drift. This is logical - with the negative drift in wages individuals retire later, i.e. when their savings are higher and expected incomes lower than when they expect an increase in wages.

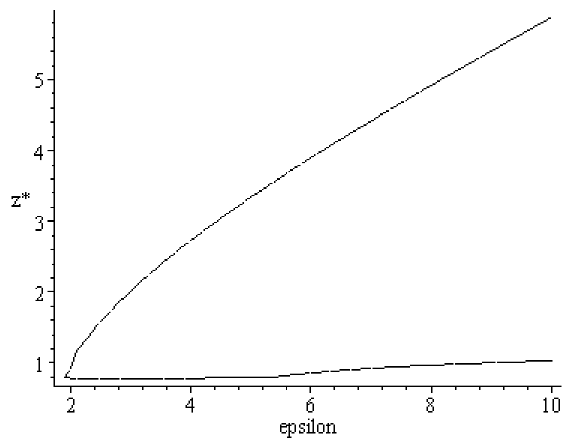
There are two problems with the plots in Figures 4.1 and 4.2. First, although $z^* = 0$ for $\varepsilon = 1$, in some cases there are other possible solutions. Second, there is the discontinu-



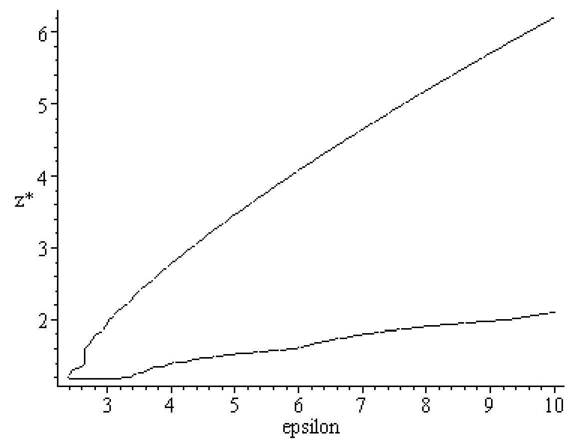
$\alpha = 0.5$



$\alpha = 1$



$\alpha = 1.33$



$\alpha = 2$

Figure 4.2: $z^* (\varepsilon)$, when $\bar{\mu} = 0.03$

ity of $z^* (\varepsilon)$ when we move outside the neighborhood of $\varepsilon = 1$. The problem with the multiple solutions is that an individual retires when the falling wage hits the critical threshold, thus it will never fall to the level of lower solutions. However, since the theory predicts that there is no retirement when $\varepsilon = 1$, $z^* (1)$ should be only zero and other solutions should be ignored. As far as the discontinuity of $z^* (\varepsilon)$ is concerned, we know that, since $z^* (1) = 0$ and $z^* (\varepsilon)$ is an increasing function of the disutility from working, it means that if the functions were continuous in the neighborhood of $\varepsilon = 1$, the values of z^* in this neighborhood would be very low. Therefore either the accumulated savings would be very high, or the expected discounted incomes would be zero, or close to zero. It is possible that, at least for the positive trend in wages, when individuals expect a rise in incomes, those with disutility of working close to one would still prefer not to retire. Therefore a relation between the retirement decision and z^* would be interrupted, since

Table 4.1. Comparative statics of the threshold z^* given by eq. (4.19), $\alpha = 2$, $\varepsilon = 2$*

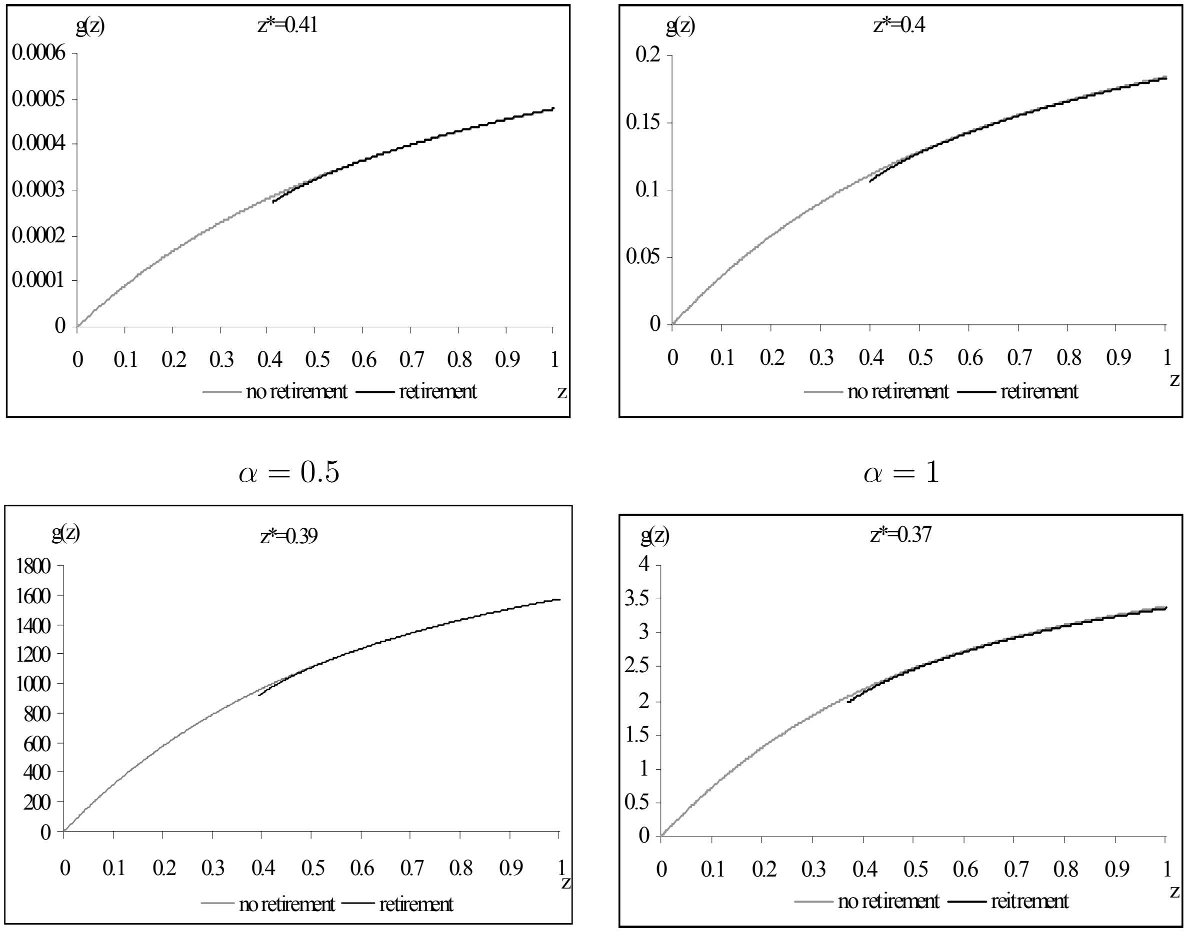
	$\rho = 0.03$						$\rho = 0.04$					
$\bar{\mu}$ / σ	-0.02	-0.01	0	0.01	0.02	0.03	-0.02	-0.01	0	0.01	0.02	0.03
0	0.50	0.62	0.83	1.24	3.24	·	0.55	0.66	0.83	1.10	·	·
0.1	0.44	0.54	0.71	·	·	·	0.50	0.59	0.71	1.01	·	·
0.2	0.53	·	·	·	·	·	0.52	0.97	·	·	·	·
	$\rho = 0.05$						$\rho = 0.06$					
$\bar{\mu}$ / σ	-0.02	-0.01	0	0.01	0.02	0.03	-0.02	-0.01	0	0.01	0.02	0.03
0	0.59	0.69	0.83	1.04	1.38	2.54	0.62	0.71	0.83	0.99	1.24	·
0.1	0.54	0.62	0.72	0.89	·	·	0.57	0.64	0.73	0.86	1.13	·
0.2	0.53	0.72	·	·	·	·	0.54	0.66	·	·	·	·

*Dots in the table signify the cases in which eq. (4.19) has no solution or the solution is not reliable due to the numerical problems.

z is a relation of both savings and expected incomes. This may explain the discontinuity for the positive drift, but we do not know what is an explanation when the drift is negative. This problem shows a potential weakness of our approximation.

To find out how the parameters influence the result we do the comparative statics for equation (4.19). The results are presented in Table 4.1. The exercise was conducted assuming $\alpha = 2$ and $\varepsilon = 2$. As we can see the retirement threshold is in general rising with the drift of wages and falling with the increase of the standard deviation. These rules are not always true, since for example when we plot z^* as a function of the standard deviation, then for higher values of σ there are discontinuities in the functions. As far as the influence of the interest rate on the threshold is concerned, we can say that for low values of the drift ($\bar{\mu} \leq 0$) and the standard deviation ($\sigma = 0$ and 0.1) z^* is an increasing function of ρ. For a positive drift and a high standard deviation z^* usually is a decreasing function of ρ, but it is not always the case.

We want to know if the characteristics of the solutions showed by the graphs in Figures 4.1 and 4.2 arise from the way in which we approximated the function, or if they are the results of the model. Therefore we solve the differential equation (4.8) in the same way as in Chapter 3. The difference is that this time we do not start from zero, but from z^*. We are



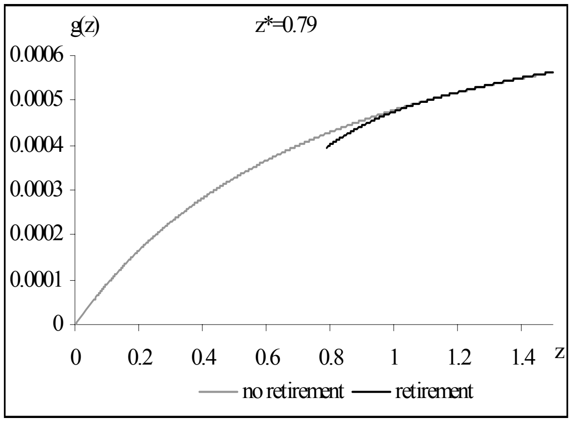
$\alpha = 0.5$ $\alpha = 1$

$\alpha = 1.33$ $\alpha = 2$

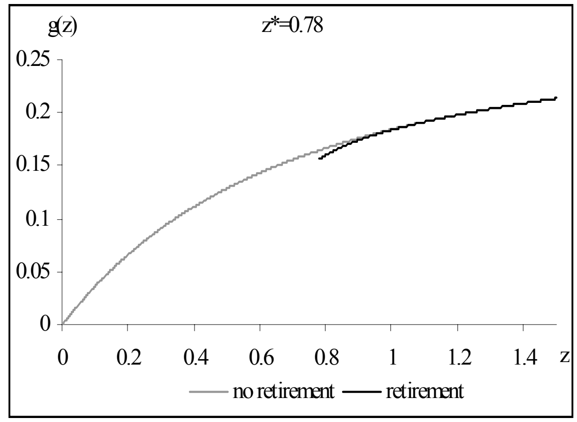
Figure 4.3: Retirement threshold for $\varepsilon = 1.5$, when $\bar{\mu} = 0.03$

looking for such z^* that, given the disutility from working ε, the function $\bar{g}(z)$ converges with the function $g(z)$ for high values of z, according to Proposition 3. The results are presented in Figures 4.3 and 4.4. They are only for the positive trend, since, as we have seen in the consumption case, the numerical solution did not work for the negative trend. In all graphs the gray line represents the graph of the function $g(z)$ starting at $z = 0$ (utility without the possibility of retirement), while the black line represents the graph of the function $\bar{g}(z)$ starting at a given value z^* (utility with the possibility of retirement).

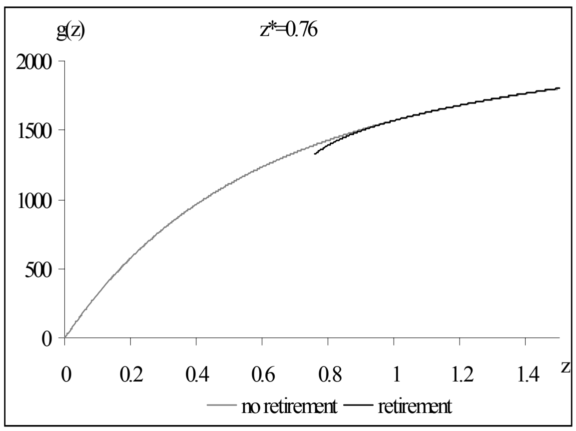
Although utility with the possibility of retirement is composed from the utility from working and the option to retire, and thus should be higher than utility without the possibility of retirement, the graph of $\bar{g}(z)$ is always below the graph of $g(z)$. This is because



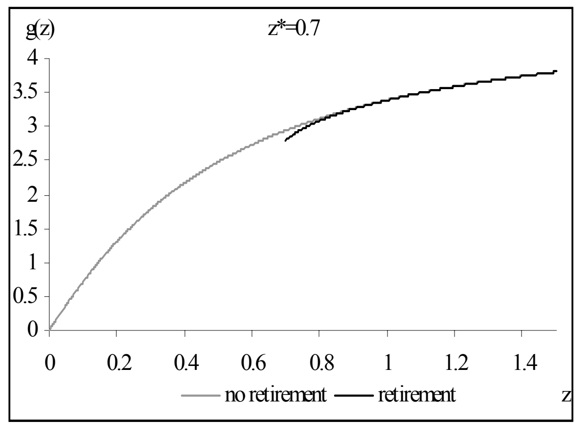
$\alpha = 0.5$



$\alpha = 1$



$\alpha = 1.33$



$\alpha = 2$

Figure 4.4: Retirement threshold for $\varepsilon = 2$, when $\bar{\mu} = 0.03$

utility is a decreasing function[5] of $g(z)$. As z increases, utility with the possibility of retirement is converging to the utility without it. The retirement threshold z^* is decreasing with the rise of risk aversion - the more risk averse people are, the less eager they are to save, since saving is risky, and its return depends on future values of z. Therefore with lower savings they retire later. The results also confirm the intuition: comparison of the retirement thresholds for $\varepsilon = 1.5$ and $\varepsilon = 2$ shows that the higher the disutility from working the higher is the retirement threshold, thus people who dislike work retire earlier.

The first thing visible when we compare the graphs from Figures 4.1 and 4.2 with those from Figures 4.3 and 4.4 is that in the case of the analytical approximations the

[5] This follows from (4.6):

$$\bar{U}_g(W, z) = -W^{1-\alpha} g(z)^{\alpha-2}$$

Table 4.2: Comparison of analytical and numerical values of the retirement threshold

	$\bar{\mu} = 0.03$				$\bar{\mu} = -0.01$	
	$\varepsilon = 1.5$		$\varepsilon = 2$		$\varepsilon = 1.5$	$\varepsilon = 2$
α	analytical eq. (4.19)	numerical	analytical eq. (4.19)	numerical	analytical eq. (4.19)	
0.5	$z^* = 0.55$	$z^* = 0.41$	$z^* = 0.94$	$z^* = 0.79$	$z^* = 0.25$	$z^* = 0.39$
1	$z^* < 0$	$z^* = 0.40$	$z^* = 1.11$	$z^* = 0.78$	$z^* = 0.30$	$z^* = 0.53$
1.33	$z^* < 0$	$z^* = 0.39$	$z^* = 1.07$	$z^* = 0.76$	$z^* = 0.31$	$z^* = 0.57$
2	$z^* < 0$	$z^* = 0.37$	$z^* < 0$	$z^* = 0.70$	$z^* = 0.32$	$z^* = 0.62$

retirement threshold rises with risk aversion, while in the numerical simulations it falls with the rise in risk aversion. It is difficult to decide which version is true, since there are arguments in favor of each of them. On the one hand the more risk-averse an individual is, the more likely she/he is to retire at the first possibility, being afraid that in case of postponing the retirement decision future incomes may be lower and it would be necessary to use savings, reducing in this way the retirement wealth - what supports the analytical results. On the other hand the more risk-averse an individual is, the less she/he is willing to save, thus delaying in retirement, as we have explained above - what in turn supports the numerical results. In the perfect insurance model with an infinite lifetime the result was that the retirement threshold falls with the rise of risk aversion and it was explained by the possibility of enjoying high incomes while working (rising with α) and the perspective of low benefits while retired, which are decreasing with rise of α. This is an argument in favor of the results of numerical simulations, but is it decisive? We do not know and we cannot decide which results are correct. These conflicting results show that there is a problem with our solutions, most probably with the analytical approximation.

The comparison of the graphs of the function $z^*(\varepsilon)$ from Figure 4.2 with the pairs (ε, z^*) from the numerical simulations shows that our approximation is not a good one. The results are presented in Table 4.2. Of course with only one or two pairs (ε, z^*) for each investigated value of the coefficient of relative risk aversion we cannot be sure if our approximation works for all (ε, z^*), for some of them or not at all, but even the fact that these eight pairs are or are not on the graphs gives some insight. Unfortunately, only in four cases, for ($\alpha = 0.5$, $\varepsilon = 1.5$), ($\alpha = 0.5$, $\varepsilon = 2$), ($\alpha = 1$, $\varepsilon = 2$), and ($\alpha = 1.33$, $\varepsilon = 2$), we can compare the results of the approximation obtained by comparison of the convergence constants with the numerical solution. The conclusion from this is clear: our approximation does not work very well. The probable reason is that the approximated utility in the non-retirement case in the previous chapter worked well for small values of z, and was not reliable for high ones. In this section we are dealing with this "not very reliable" part, since to calculate z^* we have to use the constant to which function $g(z)$ converges when $z \to \infty$, i.e. the parameter b_3.

Just for comparison in Table 4.2 we have also included the pairs (ε, z^*) for the negative trend from the analytical approximation. The most striking result is that the values of retirement threshold in this case are in the same range as the numerical values from the positive trend case. We also have the thresholds for all investigated values of risk aversion, and, as for the analytical approximation with the positive drift, they are increasing with risk aversion.

4.5 Simulated distribution and probability of retirement

The simulations in this section are conducted in the same way as the simulations in Section 3.6. We simulate the development of wages, assuming that the coefficient of relative risk aversion $\alpha = 2$, the cost of effort $\varepsilon = 2$, and the drift in wages $\bar{\mu} = -0.01$, at five different times during the potential labor career. We assume that 50 years is the maximum length of working life. The continuous lognormal distribution of wages is transformed into a discrete process defined by $\bar{\mu}$ and σ. The initial wage is equal $W_0 = 1$. Savings are calculated with the law of motion of savings, current consumption and wage. To obtain current consumption we use formula (4.7) into which we substitute the functional form of $\bar{g}(z)$ given by (4.16) with $z^* = 0.62$, which is the solution of equation (4.19) for this set of parameters. At each node we compare the current value of z with $z^* = 0.62$. If $z \leq z^*$ an individual retires and follows a single path with wage zero and consumption equal to the interests earned from savings accumulated at the moment of retirement. Thus the number of paths is smaller than in the non-retirement case in Chapter 3. However, for a higher number of periods we have chosen random samples with the same number of paths as before, i.e. 2000000, 3000000 and 4000000.

Figure 4.5. Distributions of consumption, wages and savings with retirement

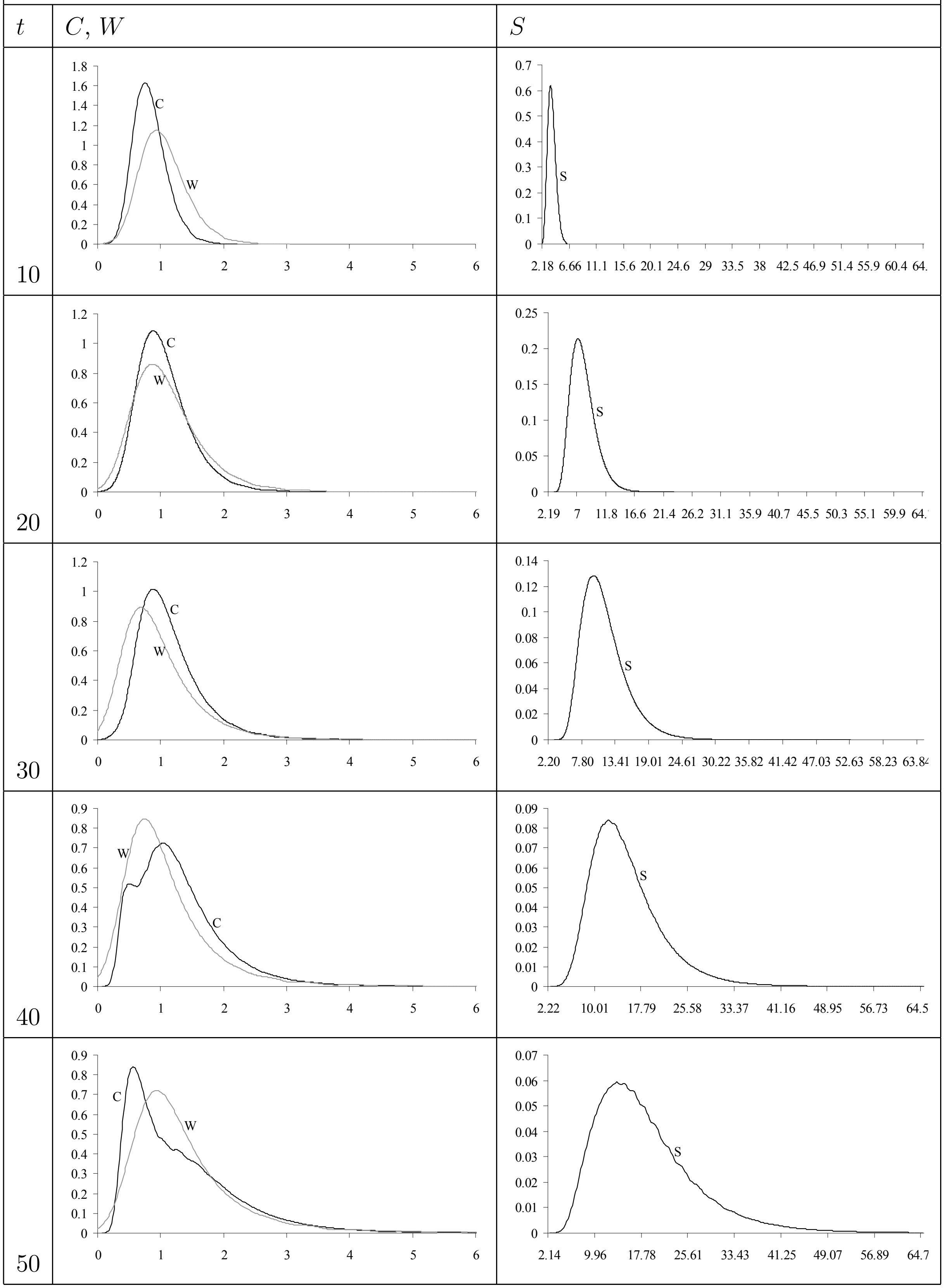


Table 4.3. Characteristics of the distributions of consumption, wage and savings with retirement

╲	t	5%	95%	*max*	*mean*	*variance*	*skewness*	*kurtosis*
C	10	0.49	1.35	2	0.76	0.05	0.93	4.26
	20	0.56	1.87	5.7	0.9	0.11	1.35	6.22
	30	0.54	2.1	15.8	1.14	0.25	1.6	7.91
	40	0.42	2.56	43.29	1.28	0.48	1.62	8.6
	50	0.41	3.04	118.02	1.33	0.82	1.93	10.16
W	10	0.55	1.82	2.72	0.95	0.09	0.97	4.37
	20	0.45	2.23	7.39	0.9	0.18	1.48	6.84
	30	0.37	2.23	20.09	0.97	0.32	1.94	9.89
	40	0.45	2.23	54.6	1.08	0.44	2.45	14.47
	50	0.55	2.72	148.41	1.34	0.68	2.78	18.64
S	10	2.85	4.95	6	3.61	0.37	0.61	3.23
	20	5.24	11.81	22.88	7.31	3.31	0.92	4.32
	30	6.79	18.35	69.32	11.49	13.35	1.17	5.38
	40	7.96	26.31	196.09	15.26	34.46	1.35	6.43
	50	8.18	34.63	541.27	18.48	73.62	1.55	7.62

The distributions are presented in Figure 4.5. The first thing we notice is a change in the distribution for consumption and wages for $t = 40$ and $t = 50$. This shows that retirement plays a role. The retired individuals have no wages, hence the shift of the distribution of wages visible for $t = 50$, and consumption is equal to ρS, what adds another hump in the distribution of consumption. This is much more pronounced for $t = 50$, suggesting that by this time more people are retired than working. Otherwise the distributions are as expected: since the trend in wages is negative they are skewed to the left, with a long right tail. Table 4.3, which presents descriptive characteristics of the distributions, confirms these results. The plots of the distributions show that, as in the non-retirement case, for short time periods people consume less than they earn. For 10 periods the distribution of consumption is shifted to the left of the distribution of wages, then for 30 periods consumers use their savings to increase consumption. Table 4.3 shows that for lower wages and consumption for $t = 10$ the fifth percentile of wages is higher than the fifth percentile of consumption. The situation is repeated for $t = 40$ and $t = 50$, with some people retired. It shows that some of those retired and some of the low income individuals have quite a low level of consumption. For high wages and consumption the 95th percentile of consumption is higher than that of wages only for $t = 40$ and $t = 50$.

Looking at the distributions of consumption for $t = 40$ and $t = 50$ we may ask: who is retiring? An individual retires if savings are high enough to finance retirement consumption and wage is small. Obviously those who see their wages permanently increasing keep on working - their wages are too high to make retirement valuable. Those who always had low wages also continue working, since they were not able to save enough to finance their retirement. Therefore the only people to retire are those who had wages high enough to accumulate initially some savings and then experienced a fall in wages. It would suggest that under self-insurance mainly the middle income individuals retire.

The distribution of savings is wider than those of consumption and wages and its mean is significantly shifting to the right with time. Also the values of accumulated savings are increasing faster than the values of consumption and wages. When we compare these distributions and their characteristics with those from Figure 3.5 and Table 3.2, we can see that when retirement is possible people save more than without this possibility, for both ends of the distributions, and distributions of savings shift right.

We can also compare the distributions of consumption with and without the possibility of retirement. Individuals with low incomes (5th percentile) consume more, when retirement is not possible. It shows that they want to save more for retirement and therefore reduce their consumption. Those with high incomes (95th percentile) for the first part of their career, up to $t = 30$ also reduce their consumption, and it is smaller than in the non-retirement case. However for $t \geq 40$ they know that they are less likely to retire due to high incomes, and they start to consume more than when retirement was not possible, making use of the high level of accumulated savings. We can also notice that the distributions presented in Figure 4.5 are less skewed than those in Chapter 3.

The same program which computes the simulations calculates the probability of retirement as a function of time. At each node of every path it verifies if $z \leq z^*$, and checks what is the probability of reaching this node. The sum of probabilities of all nodes for the same moment in time gives the probability of retiring. We have used more paths than in the simulated distributions to have a better approximation of the actual probability. The plot in Figure 4.6 shows the probability function. As we see, probability starts to play a

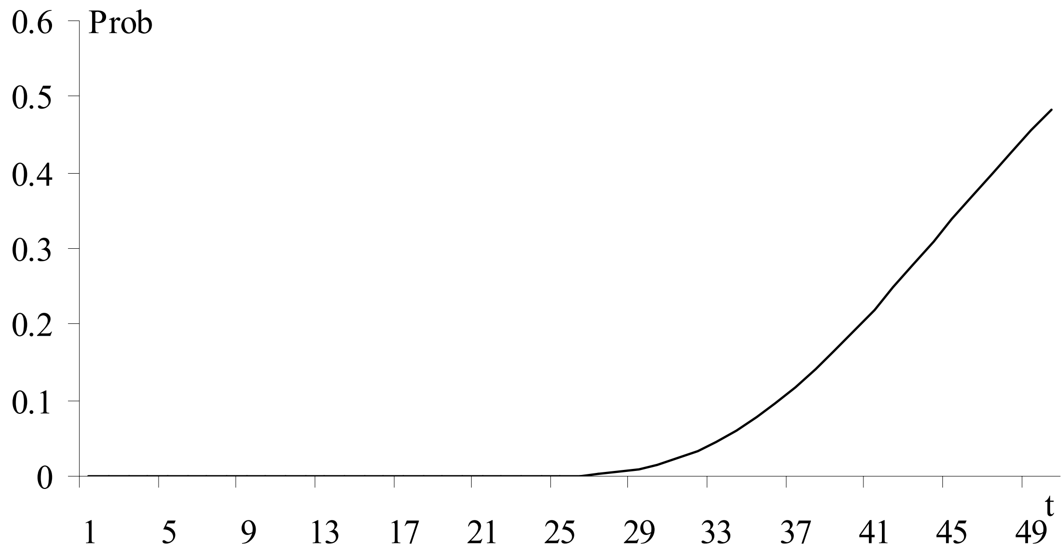


Figure 4.6. Probability of retirement

10^7 random paths, $\mu = -0.01$, $\alpha = 2$, $\varepsilon = 2$, $\rho = 0.05$, $\sigma = 0.1$

role after 25 years of labor career. It is an increasing function of time - by the time it reaches $t = 50$ more than 50% of workers have retired.

4.6 Conclusions

In this chapter we studied the retirement decision when retirement is financed from private savings. The main object was to approximate the utility function and with it to find the critical retirement threshold of the ratio of net expected discounted value of wages to accumulated savings subject to the coefficient of relative risk aversion and the parameter defining disutility from working. Our results, especially the simulated distributions of wages, consumption and savings, suggest that under self-insurance retirement is a middle income class phenomenon. The second important achievement of the chapter is the equation (4.19) defining the retirement threshold. This equation helps to predict the retirement decisions, although only in the infinite lifetime setup. Using it we were able to plot the probability of retirement as a function of time.

The comparison of the results of the approximation and of the numerical simulation showed that our approximation is not very good. Nevertheless it gave some interesting results. First of all it confirmed the intuition that the more an individual dislikes to work, the earlier, i.e. for the higher value of the threshold of the rate of expected discounted incomes to the accumulated savings, she/he retires. Second, we proved that if the risk aversion is smaller than 1, the disutility from working (ε) must be also smaller than one; otherwise the utility after retirement is lower then before it, and for $\alpha > 1$ we must have $\varepsilon > 1$. From the numerical simulations it is clear that the more risk averse an individual is, the lower must be the critical threshold for which she/he retires, i.e. the latter she/he retires.

Obviously this study does not close the subject. We were not able to find a mechanism giving the critical retirement threshold for any value of ε and for those we did find we still have problems with small values of the coefficient of relative risk aversion. Our search for the analytical approximation of the retirement threshold as a function of ε has not given straightforward results. We do not know how to solve the model in the finite lifetime or with increasing probability of death. Therefore there are many problems worth studying as possible extensions of this chapter.

4.7 Appendix 1

Calculations for $\alpha = 1$:

Utility can be given by:

$$\bar{U}(W,z) = \frac{1}{\rho}\left[\ln W - \ln g(z)\right] \tag{4.6'}$$

and the differential equation is:

$$\begin{aligned} \rho g(z)^2 &= 2\rho g(z)^2 \ln \frac{g(z)}{z} - \rho g(z)^2 \ln \frac{\rho - \bar{\mu}}{\rho} - \rho g(z)^2 \ln g'(z) \\ &+ \left(\bar{\mu} - \frac{\sigma^2}{2}\right) g(z)^2 + (\rho - \bar{\mu})(z+1) z g'(z) g(z) \\ &- \frac{\sigma^2}{2} z^2 g''(z) g(z) + \frac{\sigma^2}{2} z^2 g'(z)^2 \end{aligned} \tag{4.8'}$$

The value matching and smooth pasting conditions:

$$\bar{U}(W,z) = \frac{1}{\rho}\left[\ln(\rho S) + \ln \varepsilon\right] \tag{4.9'}$$

$$\bar{U}_W(W,z) = 0 \tag{4.10'}$$

$$\bar{U}_S(W,z) = \frac{1}{\rho S} \tag{4.11'}$$

From Lemma 1: Given $\bar{U}(W,z) \equiv U(W,S)$ we have

$$\bar{g}(z^*) = \frac{\rho-\bar{\mu}}{\rho\varepsilon} z^* \tag{4.12'}$$
$$\bar{g}'(z^*) = \frac{\rho-\bar{\mu}}{\rho\varepsilon} \tag{4.13'}$$

From the proof of Lemma 1: substituting (4.6') into the value matching condition

$$\begin{aligned} \frac{1}{\rho}\left[\ln W - \ln \bar{g}(z^*)\right] &= \frac{1}{\rho}\left[\ln(\rho S) + \ln \varepsilon\right] \\ \Rightarrow \ln \bar{g}(z^*) &= \ln W - \ln(\rho\varepsilon S) \\ \Rightarrow \bar{g}(z^*) &= \frac{\rho-\bar{\mu}}{\rho\varepsilon} z^* \end{aligned}$$

Substituting (4.6') into the first smooth pasting condition:

$$\begin{aligned} \frac{1}{\rho W} - \frac{1}{\rho}\frac{\bar{g}'(z^*)}{\bar{g}(z^*) S(\rho-\bar{\mu})} &= 0 \\ \Rightarrow 1 - \frac{\bar{g}'(z^*) z^*}{\bar{g}(z^*)} &= 0 \\ \Leftrightarrow \bar{g}'(z^*) = \frac{\bar{g}(z^*)}{z^*} &= \frac{\rho-\bar{\mu}}{\rho\varepsilon} \end{aligned}$$

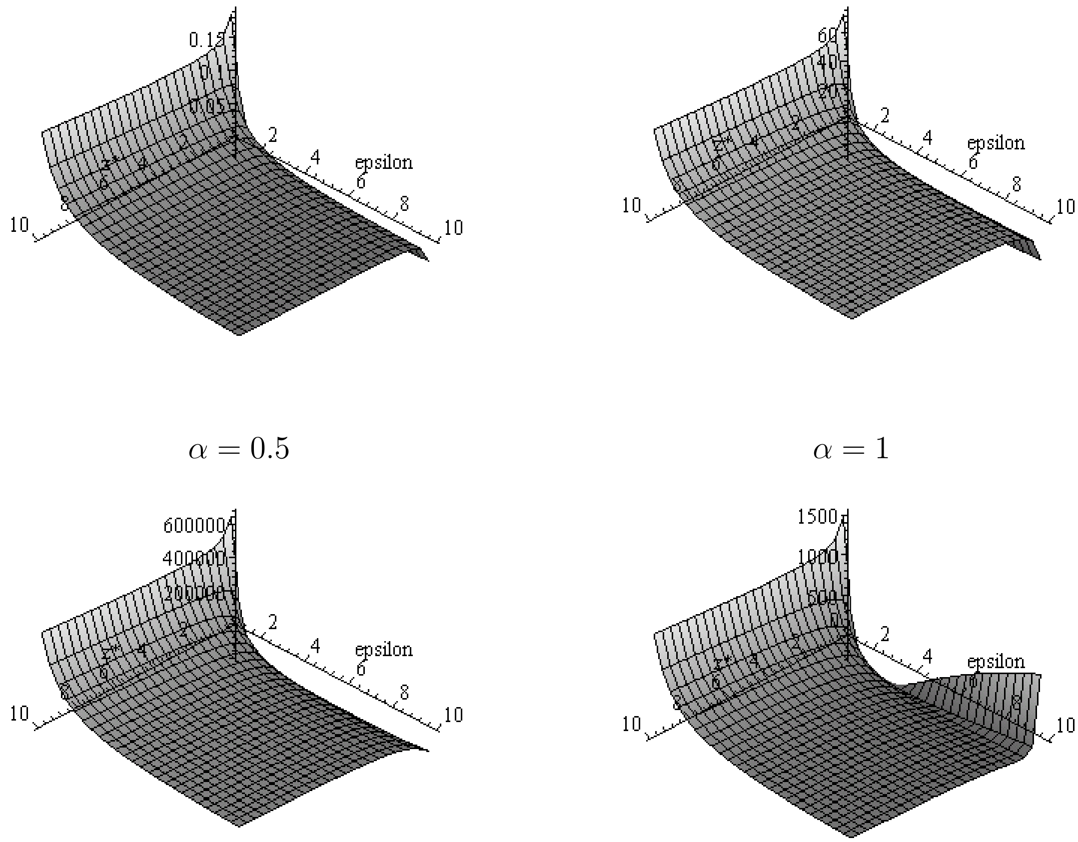
The derivatives of $\bar{g}(z^*)$

$$\begin{aligned} \bar{g}'(z^*) &= \frac{\rho-\bar{\mu}}{\rho\varepsilon} \\ \bar{g}''(z^*) &= \frac{\left[-\rho\ln\varepsilon + (\rho-\bar{\mu}) z^*\right]}{\frac{1}{2}\sigma^2 z^*}\frac{\rho-\bar{\mu}}{\rho\varepsilon} \qquad (4.14') \\ \bar{g}'''(z^*) &= \frac{\left[(\rho-\bar{\mu})(z^*+1) - \rho + \sigma^2\right]\left[-\rho\ln\varepsilon + (\rho-\bar{\mu}) z^*\right] + \frac{\sigma^2}{2}\rho\ln\varepsilon}{\frac{\sigma^4}{4} z^{*2}}\frac{\rho-\bar{\mu}}{\rho\varepsilon} \qquad (4.15') \end{aligned}$$

The equation defining z^*:

$$\begin{aligned} 0 &= \sigma^4 z^{*3} + \left[\left(\frac{2(\rho-\bar{\mu})}{3} + 3\sigma^2\right)(\rho-\bar{\mu}) + 3\sigma^4 - \left(1 + \frac{\sigma^2}{2}\frac{1}{\rho - 2\bar{\mu} - \sigma^2}\right)\varepsilon\sigma^4\right] z^{*2} \\ &\quad - \left[\left(\frac{2(\rho-\bar{\mu})}{3} + 3\sigma^2\right)\rho\ln\varepsilon + \frac{2(\rho-\bar{\mu})}{3}\left(\bar{\mu}-\sigma^2\right)\right] z^* + \frac{2}{3}\left(\bar{\mu} - \frac{\sigma^2}{2}\right)\rho\ln\varepsilon \qquad (4.19') \end{aligned}$$

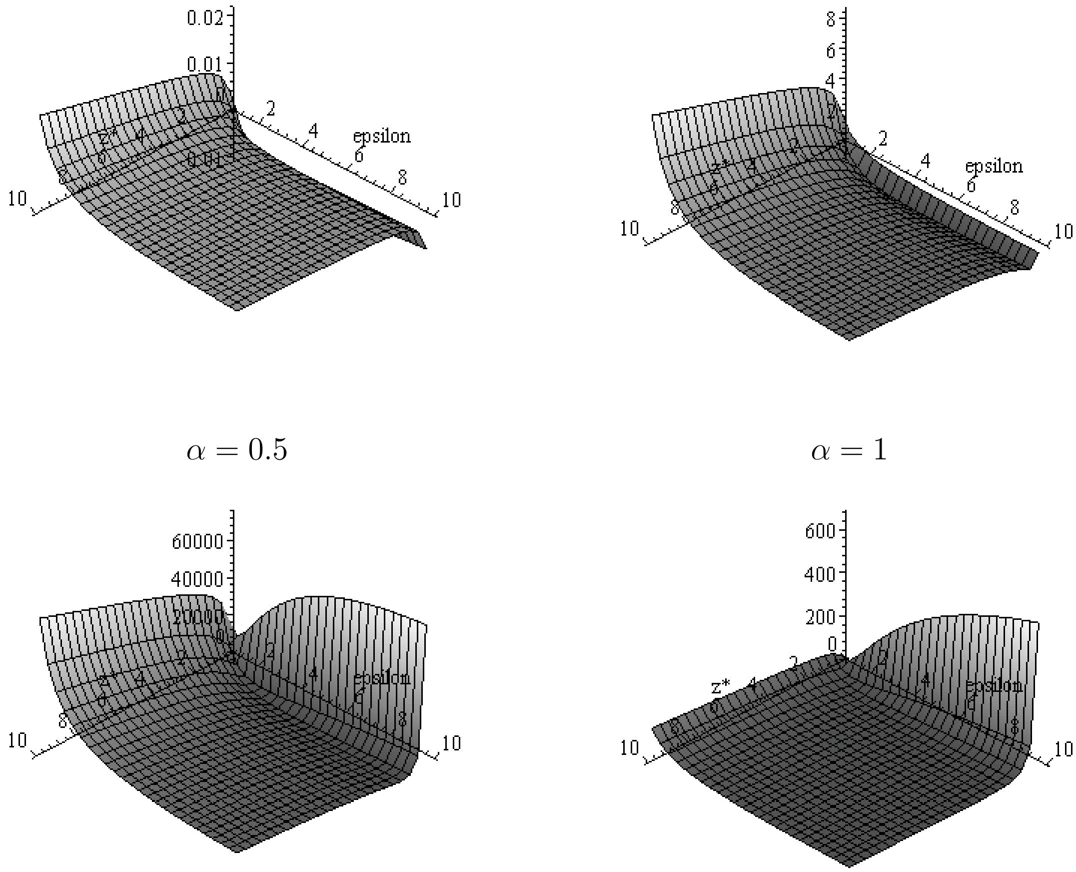
4.8 Appendix 2



$\alpha = 0.5$ $\alpha = 1$

$\alpha = 1.33$ $\alpha = 2$

Figure 4.7: parameter $\bar{b}_3$ as a function of z and ε, for $\bar{\mu} = -0.01$



$\alpha = 0.5$ $\alpha = 1$

$\alpha = 1.33$ $\alpha = 2$

Figure 4.8: parameter $\bar{b}_3$ as a function of z and ε, for $= 0.03$

Chapter 5

Conclusions

In the present day world aging, retirement and decreasing labor supply of elderly have become one of the main problems faced not only by economists, but also by governments and society as a whole. Many scientists and politicians try to solve the resulting political, sociological and cultural questions and challenges. In this thesis we address the most important economic question related to retirement: what makes people to retire. We answer two questions: what are the optimal retirement institutions when labor incomes are uncertain, and can improvements in insurance explain the fall in the labor force participation of elderly? In practice it means that we are imposing neither any retirement eligibility date nor rules on to what level of benefits an individual is entitled. Instead we look for the optimal retirement decision, which maximizes lifetime utility in two different retirement schemes: perfect insurance and private savings.

The main assumption of the thesis is that incomes follow geometric Brownian motion with drift. The lifetime is infinite, the real interest rate is constant and equal to the time preference rate, retirement is irreversible, and utility is a constant relative risk aversion function. Since retirement is defined as the optimal stopping problem in continuous time we are using a dynamic programming framework used in real options models.

Perfect insurance is defined as a system, in which an individual receives constant income and retirement benefits in exchange for total lifetime output. In such a system, as shown by the results of Chapter 2, the retirement threshold of productivity is increasing with the individual's dislike of work, but it is decreasing with the rise of risk aversion. In the risk neutral case there is no retirement - a risk neutral individual is willing to postpone consumption till it is more efficient, i.e. for after retirement. With the infinite lifetime it means that the decision to retire is never taken and only the substitution effect matters. The probability of retirement is increasing with time and the disutility from working, but falling with the rise of risk aversion. It is also much higher for the negative drift in productivity than for the positive one.

With private savings we do not allow for any external insurance. Individuals use their savings as self-insurance; in Chapter 3 to smooth consumption in case of sudden fall in incomes, and in Chapter 4 to finance retirement. The main achievement of Chapter 3 is the approximated structural equation defining optimal consumption as a function of current wage and accumulated savings. This gives us the analytical approximation of the relation between consumption, wages and assets, which may help in the empirical studies of the problem. However, for the consumption to wage ratio to fall below one for high incomes, and for the precautionary effect to work, the trend in wages must be very small or negative. This shows that Carroll and Kimball (2001) result that liquidity constraints are not necessary when precaution is taken into account does not work for a positive drift in wages and an infinite lifetime.

In Chapter 4 retirement is financed from private savings. By approximating the utility function we solve the model to get an equation defining the retirement threshold of the ratio of net expected discounted value of wages to accumulated savings, subject to the coefficient of relative risk aversion and the parameter defining disutility from working. With this equation we can simulate the distributions of consumption and savings and plot the probability of retirement as a function of time. The problem is that our approximation may be not very reliable: there is a discontinuity in the retirement threshold for low values of the coefficient of disutility from working, and the comparison of the approximation and the numerical simulation showed that results are similar only in some cases. Nevertheless the results show that the probability of retirement is rising in time, and the retirement threshold is falling when risk aversion increases.

Perfect insurance is the most extreme case of the welfare state. The introduction of the welfare state has made retirement possible for the low income class, since it offered them an insurance against the fall of productivity due to aging. When we look at the results of Chapter 2 this is clearly the case: those with low productivity, and thus low income, have a higher probability of retirement than those with high productivity. In contrast, results from Chapter 4 suggest that with self-insurance the middle class is most

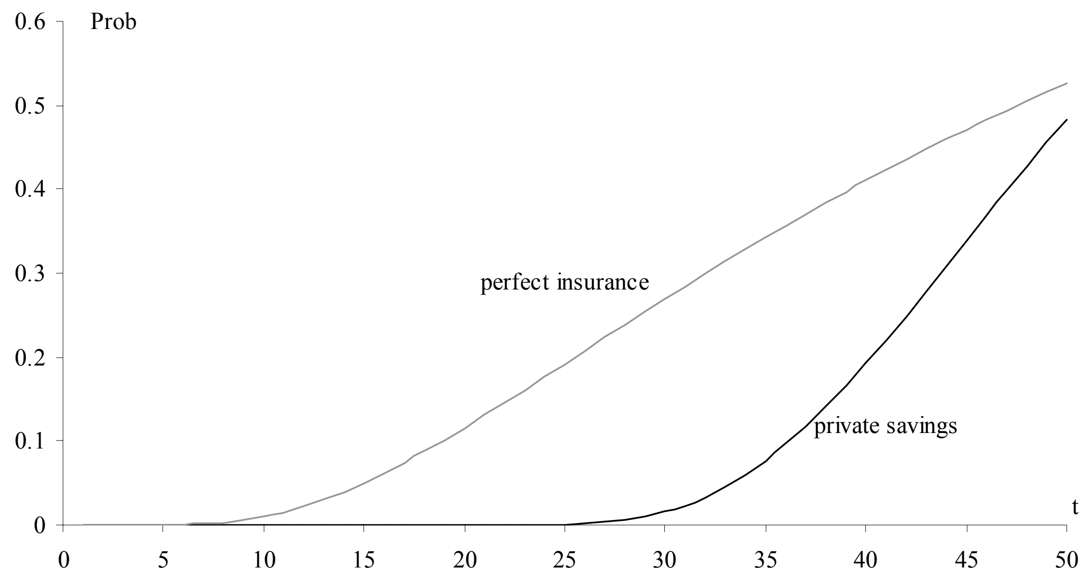


Figure 5.1: Probabilities of retirement for $\bar{\mu} = -0.01$, $\alpha = 2$, $\varepsilon = 2$, $\rho = 0.05$, $\sigma = 0.1$

likely to retire, since the only people to retire are those who have wages high enough to accumulate initially some savings but not high enough to make them continue working. However the question is which system is better in general.

Part of the answer and the most important conclusion of the thesis is given by the comparison of the probability of retirement with perfect insurance studied in Chapter 2 with the probability of retirement with self-insurance from Chapter 4, presented in Figure 5.1. Better insurance, i.e. insurance providing more secure income and reducing risk, increases the probability of retirement, and motivates people to retire earlier. The fall in labor force participation of elderly may indeed be attributed to more efficient insurance schemes.

Under perfect insurance an individual retires at the fixed retirement threshold, and once it is crossed all randomness disappears. The retirement rule can be summarized as follows: working is a risky investment, so keep working as long as you are productive. Once you are not productive - retire. Under self insurance an individual retires when savings are high and she/he has experienced a negative shock in wages. The main difference is that the retirement threshold is not fixed, but depends on the history, i.e. on the path followed by wages. Thus there are two conflicting effects: one is motivating a risk averse individual to retire early, since it is impossible to insure fully against labor income risk. The other effect is delaying retirement, since savings are risky - their returns depend on future values of the ratio of expected discounted wages to accumulated savings - and therefore a risk averse individual prefers to save less. The retirement decision depends on the relative strength of these effects. As a result people end up retiring late and saving too much.

These results can suggest that if we are interested in delaying retirement, the solution is to introduce more risk into retirement plans. Thus personal accounts and a move towards defined contributions would be a step in the right direction. The question is: do we really want to do this? The social security system with pay-as-you-go schemes must be changed, since it is too expensive. However, introducing more risk into the system and some kind of self-insurance is not necessarily the right way. Why? Simply perfect insurance is more efficient than self-insurance. The results of the simulated distributions of consumption and savings from Chapters 3 and 4 prove that, as we have said above, when people save to insure against the drop in incomes they save too much. The problem is that savings keep on accumulating, no matter if incomes are high or low. When we allow for retirement (Chapter 4) individuals save even more, and their consumption is lower than in the case without retirement. At least this is the result for the low end of the distribution. For the high end of the distribution the situation is different: people with high incomes first save for retirement and their consumption is also lower than without it. Then, when their tenure is long, and their incomes remain high, they realize that the probability of a substantial fall in their incomes, and thus retirement, is getting smaller. Therefore they suddenly start consuming more than they would do when retirement is not allowed, trying to make use of the excess savings. However, also their savings are constantly growing. Thus private savings are not efficient, and we must continue our search for optimal institutions.

In the introduction we pointed out that moral hazard is important while we discuss the retirement insurance. However, throughout the book we have assumed that there is no moral hazard. If we relax this assumption it is possible that the best retirement system is between perfect insurance and private savings. This points again to the introduction of more risk into retirement plans. Obviously further research of the problem is necessary.

If private savings are not efficient, an individual would be better off under perfect insurance. Using our results it is possible to calculate by how much we should increase the initial wage/productivity of the self-insured individual to offset the disadvantage of the absence of the perfect insurance scheme. In the case of perfect insurance the lifetime utility is given by equation (2.36). For self-insurance the lifetime utility at the beginning of the labor career is given by formula (4.6) into which we substitute the limit of (4.16) for z rising to infinity (z rises to infinity, since at the beginning of the labor career savings are zero). Assuming that the initial productivity and the initial wage are both equal to one, the compensation[1] for $\bar{\mu} = -0.01$, $\alpha = 2$, $\varepsilon = 2$, $\rho = 0.05$, $\sigma = 0.1$, i.e. the case presented in Figure 5.1, must be equal 0.29. This means that, in order to ensure that a self-insured individual has the same initial utility as the one covered by perfect insurance, we must raise the starting wage by 29%. The results for $\varepsilon = 2$ and other values of risk aversion also give high and positive compensation, although it is not clear if the compensation increases with risk aversion, probably due to the numerical problems. However, there is no doubt that the level of necessary compensation is surprisingly high.

As we can see, this thesis leaves many questions unanswered. Our last result shows again that further research is needed to find a good approximation of the utility when expected incomes are higher than accumulated savings. Our approximation is well defined for the ratio of expected discounted wages to accumulated savings close to zero, what may describe individuals close to the end of their labor career: savings accumulated over the lifetime are high and expected possible earnings are small. The description of the other end of utility, for high values of the ratio, is partially based on conjectures. High values of the ratio may characterize individuals at the beginning of the labor career, when there was no time to accumulate any savings and the expected earnings in the whole lifetime are high. If we could prove what the characteristics of utility are in this case, we would be able to tell much more about the full lifetime utility function. Such a utility function may also yield a much better mechanism giving the critical retirement threshold, working for any value of the coefficient of cost of effort (ε) and the coefficient of relative risk aversion (α). All our models are in the infinite lifetime framework, and this also affects the results. The model with the finite lifetime must be solved numerically and in discrete time. We tried to solve the model for an increasing probability of death. We found the functional

[1] The compensation formula is given by:

$$x = \frac{A\left[\frac{2}{3}\frac{\left[z^*(\rho-\bar{\mu})-B-\bar{\mu}+\sigma^2\right]\left[\alpha B+(1-\alpha)(\rho-\bar{\mu})z^*\right]-\frac{\sigma^2}{2}\alpha B}{\sigma^4(1-\alpha)z^{*2}}+3\frac{\left[\alpha B+(1-\alpha)(\rho-\bar{\mu})z^*\right]}{\sigma^2(1-\alpha)z^*}+3+z^*\right]}{\left[(1-\alpha)\,Y_R^{-\alpha}\left[\frac{\left[\frac{\alpha}{1-\alpha}Y_R B\right]^{1-\beta_2}\rho^{2(\beta_2-1)}}{(\beta_2-1)\left[\frac{\beta_2}{1-\beta_2}(\rho-\bar{\mu})\right]^{\beta_2}}+\frac{1}{\rho-\bar{\mu}}+\frac{\alpha}{1-\alpha}\frac{Y_R}{\rho}\varepsilon^{\frac{\alpha-1}{\alpha}}\right]\right]^{\frac{1}{\alpha-1}}}-1$$

form of the expected value of the stochastic part of lifetime utility, but it was impossible for us to find the retirement threshold as a function of time. This may be an interesting path for further research to follow.

Finally it would be very interesting to test the models with real data. In Chapter 3 we have offered some suggestions for the estimation of the optimal consumption path. If the empirical research can confirm that our approximated structural equation of consumption works, it would offer an important tool for studying consumption, incomes and savings.

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Samenvatting (Summary in Dutch)

Vergrijzing, pensionering en de verminderde arbeidsparticipatie van ouderen zijn tegenwoordig een van de voornaamste problemen waar niet alleen economen, maar ook regeringen en hele samenlevingen voor staan. Menig wetenschapper en politicus probeert een antwoord te vinden op de daaruit volgende politieke, sociologische en culturele vragen en uitdagingen. In dit proefschrift concentreren we ons op het belangrijkste probleem dat met pensionering te maken heeft, namelijk: wat zijn de redenen dat mensen met pensioen gaan? Wij geven antwoord op twee vragen: wat zijn de optimale pensioenschema's als arbeidsinkomen onzeker is, en kunnen verbeterde inkomensverzekeringen de sterk verminderde arbeidsmarktparticipatie van ouderen verklaren? In de praktijk betekent het dat we geen rekening houden met mogelijke wetgeving wat betreft de pensioengerechtigde leeftijd of de hoogte van een pensioen. In plaats daarvan vragen we welke beslissing om met pensioen te gaan het levenslange nut maximaliseert onder twee verschillende pensioenschema's: volledige verzekering en particuliere spaargelden.

De belangrijkste aanname in het proefschrift is dat de inkomens een geometrische Browniaanse beweging met trend volgen. De levensduur is oneindig, de reële rente constant, de beslissing om met pensioen te gaan onomkeerbaar, de nutsfunctie is een functie met constante relatieve risico-aversiteit en er is geen 'moral hazard'. Omdat het met pensioen gaan wordt gedefinieerd als een optimaal stopprobleem in continue tijd, gebruiken we een dynamisch programmerings raamwork zoals dat ook in reële optiemodellen gebruikt wordt.

In Hoofdstuk 2 bouwen we een pensioenmodel in een situatie waarin volledige inkomensverzekering mogelijk is. Deze verzekering geeft een werknemer een constant inkomen in de periode waarin hij werkt en een pensioen nadat hij met werken stopt, in ruil voor zijn levenslange productie (zijn totale productie tijdens zijn leven). Om zoveel mogelijk klanten te trekken kiest de verzekeraar het niveau van inkomens en uitkeringen zodanig dat het levenslange nut van de werknemers gemaximaliseertd wordt. We onderzoeken de relatie tussen de productiviteit van de werknemer en de beslissing om met pensioen te gaan. We zoeken een niveau van productiviteit waarop het niet meer aantrekkelijk is om te blijven werken. Als productiviteit zo veel daalt dat de verwachte winsten voor de verzekeraar verwaarloosbaar worden, laat hij toe dat de werknemer met pensioen gaat. Het model is opgelost voor een constante sterftekans. Het geval van een stijgende sterftekans is ook beschreven, het bleek echter onmogelijk om voor dit geval een oplossing te vinden.

De resultaten van Hoofdstuk 2 laten zien dat onder volledige verzekering de productiviteitsdrempel waarop er met pensioen gegaan wordt stijgt met de afkeer tegen werken, maar daalt naarmate de risico-aversiteit groter wordt. Als de werknemer risico-neutraal is gaat hij nooit met pensioen omdat een risico-neutrale persoon zijn consumptie wil uitstellen tot het moment waarop deze efficiënt is, d.w.z. tot na pensionering. Voor een oneindige levenduur betekent het dat de beslissing om te stoppen met werken nooit genomen wordt. De kans op met pensioen gaan wordt groter met de tijd en de afkeer tegen werken, maar daalt als de risico-aversiteit groter wordt. Het is ook veel hoger voor een negatieve dan voor een positieve trend in productiviteit.

Hoofdstuk 3 bestudeert de intertemporale planning van levenslange consumptie wanneer het alleen mogelijk is om zich door middel van individueel sparen te verzekeren en het niet mogelijk is om met pensioen te gaan. Deze zelfverzekering (self-insurance) maakt het mogelijk om consumptie vlakker te maken in het geval van een abrupte daling van inkomen. Dit hoofdstuk is een eerste stap naar een analyse van een pensioensysteem waarin het gespaard vermogen van de gepensioneerde zijn enige bron van vermogen tijdens het pensioen vormen. Voortbordurend op het werk van Deaton (1989) en Carroll (1997,2001) en gebruik makend van sommige elementen van de theorie van reële opties, vinden we een differentiaal vergelijking die het verwachte levenslange nut definieert. Deze transformeren we naar een vergelijking die alleen de zelfverzekeringcomponent van het nut definieert. Gebruik makend van de economische theorie en empirische bevindingen zijn we in staat om de begincondities voor de nutsfunctie te formuleren wanneer of inkomen, of het spaargeld vrijwel nul is. Met behulp van een Taylor expansie kunnen we de oplossing van de differentiaal vergelijking, d.w.z. de functionele vorm van het verwachte nut, benaderen.

De belangrijkste verworvenheid van Hoofdstuk 3 is een benadering van de structurele vergelijking die de optimale consumptie als een functie van huidig salaris en het gespaard vermogen definieert. Dat geeft ons een analytische uitdrukking voor de relatie tussen consumptie, salaris en vermogen, die bij empirische studies van deze vraagstukken nuttig kan zijn. Echter, de ratio van consumptie tot salaris voor hoge inkomens is alleen kleiner dan één, en het voorzorgeffect ('precautionary effect') werkt alleen, als de trend in salarissen negatief of heel zwak positief is. Dus, het resultaat van Carroll en Kimball (2001) dat aantoont dat liquiditeitsbeperkingen niet noodzakelijk zijn als er met voorzorg rekening gehouden wordt, geldt niet als salarissen een positieve trend vertonen en de levensduur oneindig is.

In Hoofdstuk 4 stellen we de vraag wat het pensioneringsgedrag zal zijn als er geen pensioenverzekering is en mensen zelf voor hun oude dag moeten sparen. In dat geval moet een individu niet alleen op elk moment beslissen hoeveel hij wil consumeren dan wel sparen, maar ook of hij al genoeg gespaard heeft om te stoppen met werken. Met gebruik van de resultaten van Hoofdstuk 3 herhalen we de analyse, waarin nu ook de mogelijkheid om met pensioen gaan meegenomen is, om de levenslange nutsfunctie te benaderen en de ratio van verwachte inkomens tot gespaard vermogen te berekenen waarbij het optimaal is om te stoppen met werken. Het model levert een vergelijking op die deze ratio definieert en die afhankelijk is van de coëfficiënt van de relatieve risico-aversiteit en de parameter die de afkeer tegen werken beschrijft. Het is echter mogelijk dat onze benadering niet volledig betrouwbaar is: de relatie tussen inkomen en vermogen die het moment van pensionering bepaalt is niet continu, en een vergelijking van de benadering en de numerieke oplossing toont aan dat deze resultaten alleen in bepaalde gevallen dicht bij elkaar liggen.

Met behulp van de vergelijking die de relatie definieert waarvan de beslissing om met pensioen gaan afhangt, kunnen we simulaties uitvoeren van de verdeling van consumptie en spaargelden in verschillende fases van de carrière en een grafische voorstelling geven van de kans om met pensioen te gaan als een functie van tijd. De resultaten laten zien dat de kans op pensionering stijgt met de tijd en dat de kritische relatie tussen inkomen en vermogen daalt wanneer de risico-aversiteit stijgt. Wanneer we de distributies van consumptie en vermogen in Hoofdstukken 3 en 4 vergelijken, kunnen we zien dat in Hoofdstuk 4, als het mogelijk is om met pensioen te gaan, mensen minder gaan consumeren en meer sparen. Het blijkt dat de risico-averse mensen te veel sparen, met name als zij een hoog inkomen hebben.

Hoofdstuk 5 vergelijkt de modellen. Deze vergelijking van kansen op pensionering onder perfecte verzekering en particulier sparen toont zonder twijfel aan dat een perfecte verzekering prikkels geeft om eerder met pensioen te gaan. De daling van arbeidsparticipatie van ouderen kan dus inderdaad worden veroorzaakt door betere inkomensverzekering.

Perfecte verzekering is een extreem geval van een welvaartsstaat. De welvaartsstaat heeft het ook voor groepen met lagere inkomens mogelijk gemaakt om te stoppen met werken omdat het hen verzekerde tegen de productiviteitsdaling als gevolg van het ouder worden. Wanneer we naar de resultaten in Hoofdstuk 2 kijken zien we het duidelijk: degenen met de lage productiviteit, en dus een laag inkomen, hebben een grotere kans om met pensioen te gaan dan degenen met een hoge productiviteit. Aan de andere kant, de resultaten in Hoofdstuk 4 suggereren dat in het geval van zelfverzekering de mensen met een middeninkomen de grootste kans hebben om met pensioen te gaan, omdat de enige mensen die met pensioen gaan diegenen zijn die in het begin van hun carriere genoeg verdienen om een bepaald vermogen op te bouwen, maar die op een gegeven moment niet meer genoeg verdienen om door te willen gaan met werken. De vraag is echter welk systeem in het algemeen beter is.

Gezien vanuit het standpunt van het individu is de volledige verzekering beter, omdat die een hoger verwacht nut oplevert. Wanneer mensen zelf sparen om zich tegen een inkomensdaling te verzekeren en om te kunnen stoppen met werken, sparen ze te veel en dus is zelfverzekering niet efficiënt. We laten zien dat om de afwezigheid van een perfecte verzekering te compenseren, het oorspronkelijke salaris/productiviteit van een zelfverzekerde persoon substantieel verhoogd zou moeten worden (29% in ons voorbeeld).

Het probleem is dat wat voor een individu goed is, niet noodzakelijk de beste oplossing voor de gehele samenleving is. Als de overheid de vermindering van arbeidsparticipatie tegen wil gaan, impliceren onze resultaten dat men dat zou kunnen bereiken door een bepaalde mate van risico in het pensioensysteem te introduceren. Het invoeren van persoonlijke rekeningen en 'defined contribution schemes' lijkt een stap in deze richting. Tegelijkertijd moet een optimaal pensioensysteem de inefficiëntie van de zelfverzekering vermijden. Het definiëren van zulk een systeem is een onderwerp voor verder onderzoek.

Streszczenie (Summary in Polish)

W obecnych czasach starzenie się społeczeństw, emerytury i zmniejszająca się podaż pracy wśród osób w podeszłym wieku stały się bardzo istotnym problemem nie tylko dla ekonomistów, ale i dla rządów, a nawet całych społeczeństw. Wielu naukowców i polityków stara się znaleźć rozwiązanie powstałych w związku z tym pytań i wyzwań z dziedziny polityki, socjologii, a także kultury. W tej pracy poruszamy najważniejszy ekonomiczny problem związany z emeryturami: co sprawia, że ludzie przechodzą na emeryturę? Odpowiadamy na dwa pytania: jakie są optymalne instytucje emerytalne, gdy dochody z pracy są niepewne, oraz czy zmiany w ubezpieczeniach mogą wyjaśnić spadek podaży pracy wśród osób w podeszłym wieku? W praktyce oznacza to, że nie bierzemy pod uwagę ani istnienia określonego wieku emerytalnego, ani przepisów określających poziom świadczeń emerytalnych do jakich dana osoba jest uprawniona. W zamian za to szukamy takiej decyzji o przejściu na emeryturę, która maksymalizuje życiową użyteczność w dwóch różnych systemach emerytalnych: całkowitym ubezpieczeniu i prywatnych oszczędnościach.

Głównym założeniem pracy jest, że dochód jest stochastyczny i da się opisać geometrycznym ruchem Browna z dryfem. Przyjmujemy też, że długość życia jest nieskończona, realna stopa procentowa stała, decyzja o przejściu na emeryturę nieodwracalna, funkcja użyteczności charakteryzuje się stałą względną awersją do ryzyka, oraz że nie ma hazardu moralnego. Ponieważ przejście na emeryturę jest zdefiniowane jako problem optymalnego zatrzymania, używamy programowania dynamicznego, podobnego do metodologii wykorzystywanej w modelach opcji rzeczowych.

W rozdziale 2 budujemy model emerytury w warunkach całkowitego ubezpieczenia. Polega ono na tym, że pracownik kupuje ubezpieczenie, które daje stały dochód w czasie pracy i świadczenia emerytalne w zamian za całkowity produkt wytworzony w ciągu życia. Aby przyciągnąć jak najwięcej klientów, ubezpieczyciel oferuje taki poziom dochodów i

świadczeń, który maksymalizuje życiową użyteczność ubezpieczonego pracownika. W takich warunkach badamy zależność pomiędzy produktywnością pracownika a decyzją o przejściu na emeryturę. Szukamy takiego poziomu produktywności, przy którym dalsza praca jest nieopłacalna. Jeżeli produktywność spadnie tak nisko, że oczekiwane zyski ubezpieczyciela są bliskie zera, pozwala on ubezpieczonemu przejść na emeryturę. Krytyczny poziom produktywności możemy znaleźć przy założeniu stałego prawdopodobieństwa śmierci. W pracy przedstawiamy też model z rosnącym prawdopodobieństwem śmierci; niestety ten przypadek okazał sie niemożliwy do rozwiązania.

Wyniki rozdziału 2 pokazują, że przy całkowitym ubezpieczeniu krytyczny poziom produktywności, przy którym przechodzi się na emeryturę, rośnie wraz z niechęcią do pracy, ale maleje ze wzrostem awersji do ryzyka. W przypadku neutralności wobec ryzyka nie ma emerytury – osoba o neutralnym stosunku do ryzyka jest skłonna odsunąć konsumpcję aż do czasu, gdy będzie ona bardziej efektywna (da wyższą użyteczność), czyli do czasu emerytury. Przy założeniu nieskończonego życia oznacza to, że decyzja o przejściu na emeryturę nigdy nie jest podjęta. Prawdopodobieństwo przejścia na emeryturę rośnie w czasie i z niechęcią do pracy, a spada ze wzrostem awersji do ryzyka. Jest również znacznie wyższe przy ujemnym dryfie produktywności niż przy pozytywnym.

Rozdział 3 analizuje międzyokresowe planowanie życiowej konsumpcji, gdy ludzie mogą się ubezpieczać tylko poprzez własne oszczędności i nie ma możliwości przejścia na emeryturę. Takie „samo-ubezpieczenie" pozwala na wygładzenie konsumpcji w przypadku nagłego spadku dochodów. Ten rozdział jest pierwszym krokiem ku analizie emerytury, jeżeli prywatne oszczędności są jedynym źródłem kapitału emerytalnego. Bazując na pracach Deatona (1989) i Carrolla (1997, 2001), i wykorzystując elementy modelowania opcji rzeczowych, znajdujemy równanie różniczkowe definiujące oczekiwaną życiową użyteczność, a następnie modyfikujemy je tak, by definiowało tylko składnik funkcji użyteczności odnoszący się do „samo-ubezpieczenia". Odwołując się do teorii ekonomii i badań empirycznych jesteśmy w stanie sformuować warunki początkowe spełniane przez funkcję użyteczności, kiedy albo dochód, albo zebrane oszczędności są bliskie zera. Następnie, używając rozwinięcia Taylora, znajdujemy przybliżone rozwiązanie równania różniczkowego, czyli przybliżoną postać funkcji użyteczności.

Głównym osiągnięciem rozdziału 3 jest przybliżone równanie strukturalne, definiujące optymalną konsumpcję jako funkcję obecnej płacy i zebranych oszczędności. To daje analityczne przybliżenie zależności między konsumpcją, płacą i majątkiem, które może pomóc w empirycznych badaniach tego zagadnienia. Jednak aby stosunek konsumpcji do płacy spadał poniżej jednego dla wysokich dochodów i aby zadziałał efekt przezorności, trend płac musi być bardzo mały albo ujemny. To pokazuje, że wynik uzyskany przez Carrolla i Kimballa (2001) o równoważności między ograniczeniami płynności i efektem przezorności w modelu nie działa dla dodatniego trendu płac i nieskończonej długości życia.

Rozdział 4 odpowiada na pytanie, jak na decyzję o przejściu na emeryturę wpływa fakt, że nie ma ubezpieczenia emerytalnego i ludzie muszą sami oszczędzać na emeryturę. W takich warunkach w każdej chwili trzeba zdecydować nie tylko jaką część dochodu można skonsumować, a jaką oszczędzić, ale też czy oszczędności są wystarczająco wysokie by przestać pracować i sfinansować emeryturę. Używając wyników z rozdziału 3, powtarzamy analizę dopuszczając teraz możliwość przejścia na emeryturę, by uzyskać przybliżoną życiową funkcję użyteczności i znaleźć taką relację między oczekiwanymi dochodami i zgromadzonymi oszczędnościami, dla której decyzja o przejściu na emeryturę jest optymalna. Z modelu uzyskujemy równanie definiujące tą krytyczną relację, jako funkcję współczynnika relatywnej awersji do ryzyka i parametru określającego niechęć do pracy. Problem polega na tym, że nasze przybliżenie może nie być bardzo wiarygodne: relacja między dochodami i oszczędnościami określająca moment przejścia na emeryturę jest nieciągła dla niskich wartości parametru określającego niechęć do pracy, a porównanie analitycznego przybliżenia rozwiązania i rozwiązania numerycznego pokazuje, że wyniki są podobne tylko w niektórych przypadkach.

Mając równanie definiujące zależność opisującą decyzję o przejściu na emeryturę, możemy przeprowadzić symulacje dystrybucji konsumpcji i oszczędności na różnych etapach kariery zawodowej i znaleźć graficzną postać prawdopodobieństwa przejścia na emeryturę jako funkcji czasu. Wyniki pokazują, że prawdopodobieństwo przejścia na emeryturę rośnie w czasie, a krytyczna zależność między dochodami a oszczędnościami spada, gdy rośnie awersja do ryzyka. Porównanie dystrybucji konsumpcji i oszczędności w rozdziale 3 i 4 pokazuje, że kiedy w rozdziale 4 dopuszczamy możliwość przejścia na emeryturę, ludzie konsumują mniej i oszczędzają wiecej. Okazuje się, że osoby przejawiające awersję do ryzyka oszczędzają za dużo, zwłaszcza w grupie o wysokich dochodach.

Rozdział 5 porównuje modele. Porównanie prawdopodobieństwa przejścia na emeryturę w warunkach całkowitego ubezpieczenia i przy finansowaniu emerytury z prywatnych oszczędności pokazuje wyraźnie, że całkowite ubezpieczenie motywuje ludzi do wcześniejszego przechodzenia na emeryturę, niż gdy muszą na nią sami oszczędzać. Wynika z tego, że lepsze ubezpieczenie, czyli takie, które daje bezpieczniejszy dochód i redukuje ryzyko, zwiększa prawdopodobieństwo przejścia na emeryturę i zachęca do wybrania wczesnej emerytury. Świadczy to o tym, że spadek podaży pracy wśród osób w podeszlym wieku może rzeczywiście być wyjaśniony bardziej efektywnymi ubezpieczeniami.

Całkowite ubezpieczenie jest skrajnym przykładem państwa opiekuńczego. Państwo opiekuńcze otworzyło możliwość przejścia na emeryturę osób z niskim poziomem dochodów, ponieważ oferowało im ubezpieczenie od spadku produktywności na starość. Kiedy spojrzymy na wyniki rozdziału 2 widać to dokładnie: ci, którzy są mało produktywni, czyli dysponują niskim dochodem, mają wyższe prawdopodobieństwo przejścia na emeryturę niż osoby o wysokiej produktywności. W przeciwieństwie do całkowitego ubezpieczenia, wyniki rozdziału 4 pokazują, że przy „samo-ubezpieczeniu" najbardziej prawdopodobne jest przejście na emeryturę osób należących do klasy średniej, ponieważ na emeryturę w tym przypadku idą tylko ci, którzy mieli wystarczająco wysokie pensje, by zebrać początkowo pewne oszczędności, ale nie były one tak wysokie, aby zachęcić ich do kontynuowania pracy. Pozostaje jednak pytanie, który system jest ogólnie lepszy.

Z punktu widzenia jednostki całkowite ubezpieczenie jest lepsze niż „samo-ubezpieczenie", ponieważ to pierwsze daje wyższą oczekiwaną użyteczność. Gdy ludzie sami oszczędzają, żeby ubezpieczyć się od spadku dochodów i na emeryturę, zaczynają oni oszczędzać za dużo i „samo-ubezpieczenie" jest nieefektywne. W pracy pokazujemy, że aby zrekompensować brak całkowitego ubezpieczenia, początkowa płaca/produktywność tych, którzy sami oszczędzają, musi być znacząco wyższa (w przedstawionym przykładzie konieczny jest 29% wzrost).

Problem polega na tym, że to, co jest dobre dla jednostki, może nie być najlepszym rozwiązaniem dla całego społeczeństwa. Jeżeli celem rządu jest ograniczenie spadku podaży pracy osób w podeszłym wieku, nasze wyniki sugerują, że można tego dokonać przez wprowadzenie pewnego stopnia ryzyka do systemu emerytalnego. Wprowadzenie indywidualnych rachunków i systemu zdefiniowanych kontrybucji wydaje się być krokiem właśnie w tym kierunku. Jednak optymalne instytucje emerytalne, wprowadzając elementy ryzyka by zwiększyć podaż pracy osób w podeszłym wieku, powinne w tym samym czasie starać sie uniknąć nieefektywności „samo-ubezpieczenia". Jak taki system powinien wyglądać jest tematem do dalszych badań.

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