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THE STABILITY OF SUBDIVISION OPERATOR

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Abstract[2]

We consider the univariate two-scale refinement equation $\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x-k)$, where $c_0, \cdots, c_N$ are complex values and $\sum c_k = 2$. The paper analysis the correlation between the existence of smooth compactly supported solutions of this equation and the convergence of the corresponding cascade algorithm/subdivision scheme. We introduce a criterion that expresses this correlation in terms of mask of the equation. We show that the convergence of subdivision scheme depends on values that the mask takes at the points of its *generalized cycles*. This means in particular that the stability of shifts of refinable function is not necessary for the convergence of the subdivision process. This also leads to some results on the degree of convergence of subdivision processes and on factorizations of refinable functions.

Key words. refinement equations, cascade algorithm, subdivision process, degree of convergence, stability, cycles, tree.

AMS subject classification. 26C10, 39B32, 42A05, 42A38

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I. Introduction.

Refinement equations have been studied by many authors in great detail in connection with their role in the study of wavelets and of subdivision schemes in approximation theory and design of curves and surfaces (see References). In this paper we study the correlation between the existence of smooth solutions of refinement equations and the convergence of the corresponding subdivision schemes. We restrict ourselves to univariate equations having compactly supported mask. We obtain a criterion for the convergence of subdivision process under the condition that the associated refinement equation has a smooth solution.

Throughout the paper we denote by $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the unit circle, by $\mathcal{H}$ the space of entire functions on $\mathbb{C}$, by $\mathcal{C}^l$ the space of l times continuously differentiable functions on $\mathbb{R}$, by $\mathcal{C}^0 = \mathcal{C}$ the space of continuous functions, by $\mathcal{C}_0^l$ the space of compactly supported functions from $\mathcal{C}^l$, and by $\mathcal{C}_0$ the space of compactly supported continuous functions on $\mathbb{R}$. A sequence $\{f_k\}$ converges to zero in $\mathcal{C}_0^l$ if it converges to zero in $\mathcal{C}^l$ and the supports of $f_k,\ k \in \mathbb{N}$ are uniformly bounded.

Consider a refinement equation

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x - k), \tag{1}$$

where $c_k \in \mathbb{C},\ \sum_k c_k = 2$. It is well-known that a $\mathcal{C}_0$-solution of this equation (*refinable function*), if it exists at all, is unique up to normalization, has its support on the segment $[0, N]$, and can be represented in frequency domain by the formula

$$\widehat{\varphi}(\xi) = \widehat{\varphi}(0) \prod_{r=1}^{\infty} m\left(\frac{\xi}{2^r}\right), \tag{2}$$

where $m(\xi) = \frac{1}{2}\sum_{k=0}^{N} c_k e^{-ik\xi}$ is the *mask* of equation (1) (as usually we denote $\hat{f}(\xi) = \int f(x) e^{-i\xi x} dx$). For a given mask $a(\xi)$ let us denote by $[a]$ the corresponding refinement equation. Let us also define the following subspaces of the space $\mathcal{C}_0$:

$$\mathcal{M}^l = \{f \in \mathcal{C}_0 \mid \widehat{f}(\xi)(1 - e^{-i\xi})^{-l-1} \in \mathcal{H}\},\ l \geq 0, \tag{3}$$

and the subspaces of $\mathcal{C}_0^l$:

$$\mathcal{L}^l = \{f \in \mathcal{C}_0^l \mid \widehat{f^{(l)}} \in \mathcal{M}^l\},\ l \geq 0.$$

In other words the Fourier transform of a function from M^l has zeros of order $\geq l+1$ at all the points $2\pi k,\ k \in \mathbb{Z}$. The Fourier transform of a function from $\mathcal{L}^l$ has zero at the point $\xi = 0$ and has zeros of order $\geq l+1$ at all the points $2\pi k,\ k \in \mathbb{Z} \setminus \{0\}$. Let us also denote $\mathcal{L} = \mathcal{L}^0 = \mathcal{M}^0$. By Poisson summation formula we have:

$$f \in \mathcal{L} \ \Leftrightarrow\ f \in \mathcal{C}_0,\ \sum_k f(x-k) \equiv 0.$$

The *cascade algorithm* for refinement equations was introduced in [D]. A single iteration of that algorithm is $f_n = Tf_{n-1}$, where f_0 is an initial function from $\mathcal{C}_0$, $Tf(x) = \sum_k c_k f(2x-k)$ is the *subdivision operator* associated to equation (1). This operator is defined on the space $\mathcal{C}_0$ and has the form

$$\widehat{Tf}(\xi) = m(\xi/2)\widehat{f}(\xi/2) \tag{4}$$

in frequency domain. If f_n converges in the space $\mathcal{C}^l$ to a function $\varphi \in \mathcal{C}_0^l$ ($l \geq 0$), then obviously it converges in $\mathcal{C}_0^l$ and φ is the solution of (1). Moreover, in that case the function $g = f_0 - \varphi$ necessarily belongs to $\mathcal{L}^l$ (see [CDM], [Du1]). The cascade algorithm *converges in* $\mathcal{C}^l$ if $T^n g \to 0,\ n \to \infty$ for any $g \in \mathcal{L}^l$. Properties of the cascade algorithms have been studied by many authors in various contexts. This algorithm gives a simple way for approximation of refinable functions. In particular this was put to good use in the study of wavelets ([D],[DL1], [Du2]). On the other hand the convergence of the cascade algorithm is equivalent to the convergence of the corresponding *subdivision scheme* (see [RS] for many references). For a given mask $m(\xi)$ we say that the *subdivision process* $\{m\}$ *converges in* $\mathcal{C}^l$ if the corresponding cascade algorithm or the corresponding subdivision scheme converges in that space.

It is clear that the convergence of subdivision process in $\mathcal{C}^l$ implies that the corresponding refinement equation has a $\mathcal{C}_0^l$-solution. In general the converse is not true (see [DL2] and [CDM] for many examples. See also [CH], [W], [RS] for general discussions of this aspect). In this paper we analyze the correlation between the existence of smooth solutions of refinement equations and the convergence of the corresponding subdivision process. In other words we study stability of subdivision operator at its fixed point. Let us first formulate several previously known results on this problem.

II. Preliminary results.

Necessary conditions for the convergence of subdivision processes were first introduced in the work [DGL2].

If a subdivision process $\{m\}$ *converges in* $\mathcal{C}^l$*, then its mask can be factored as*

$$m(\xi) = \Bigl(\frac{1+e^{-i\xi}}{2}\Bigr)^{l+1} a(\xi) \tag{5}$$

for some trigonometric polynomial $a(\xi)$*. In particular the condition*

$$m(\xi) = \Bigl(\frac{1+e^{-i\xi}}{2}\Bigr) a(\xi) \tag{6}$$

is necessary for the convergence of the subdivision process in $\mathcal{C}$ ([DGL2]).

For a given mask m denote by $\mathbf{l}(m)$ the maximal integer l such that condition (5) is satisfied. So if a subdivision process $\{m\}$ converges in $\mathcal{C}^k$, then $k \le \mathbf{l}(m)$. Let us remark that condition (5) is not necessary for the existence of $\mathcal{C}_0^l$-solutions of refinement equation ([DL2], [P2]).

Sufficient conditions for the convergence of subdivision process in the space $\mathcal{C}$ (i.e. in the case $l = 0$) were introduced in [CDM].

If a refinement equation $[m]$ *has a* $\mathcal{C}_0$*-solution and that solution is stable in the space* $L^\infty(\mathbb{R})$ *(i.e. its integer translates possess Riesz basis property in that space), then the subdivision process* $\{m\}$ *converges in* $\mathcal{C}$ ([CDM]).

This condition is simplified by the criterion of stability of refinable functions proved in [JW] and [Z] and introduced independently in [He1]. To formulate it we need some notation. Let $p(\xi)$ be a trigonometric polynomial. If for some $\alpha \in \mathbb{T}$ we have $p(\alpha/2) = p(\pi + \alpha/2) = 0$, then the pair $\{\alpha/2, \pi + \alpha/2\}$ is a *pair of symmetric roots* for $p(\xi)$. In order to be defined we set that for any $\alpha \in \mathbb{T}$ the value $\alpha/2 \in \mathbb{T}$ has the corresponding real value from the half-interval $[0, \pi)$. Further, a given set $\mathbf{b} = \{\beta_1, \cdots, \beta_n\} \subset \mathbb{T}$, where $n \ge 2$, is called a *cycle* of the polynomial $p(\xi)$ if $2\beta_j = \beta_{j+1}$ for $j = 1, \cdots, n$ (we set $\beta_{n+1} = \beta_1$) and $p(\beta_j + \pi) = 0$ for all $j = 1, \cdots, n$. We consider only irreducible cycles, i.e. we suppose everywhere that all elements of a cycle are different. Now let us remember the criterion of stability of refinable functions.

The $\mathcal{C}_0$-solution of a refinement equation is stable in L^∞ if and only if its mask has neither symmetric roots nor cycles. ([JW], [Z], [He1]).

Those two results can be summarized in the following Theorem.

Theorem 1. *(*[CDM],[JW],[Z],[He1]*). Suppose a mask m satisfying (6) has neither symmetric roots nor cycles; then if the equation $[m]$ has a $\mathcal{C}_0$-solution, then the process $\{m\}$ converges in $\mathcal{C}$.*

Remark 1. The statement of Theorem 1 can also be formulated in terms of *Cohen's criterion* (see [D]). Namely, it was shown in [V, proposition 2.4] that a mask satisfies Cohen's criterion if and only if it has neither symmetric roots nor cycles.

III. Statement of the fundamental theorems.

In this paper we give a criterion of stability of subdivision operator at its fixed point (Theorem 2). We will see that symmetric roots of mask do not influence the convergence of subdivision process (Corollary 3). It means in particular that the stability of solutions is not necessary for the convergence of subdivision process. The convergence depends on values of the mask at the points of cycles.

To formulate the criterion we need some further notation. Everywhere below we consider trigonometric polynomials without positive powers, i.e. polynomials of the form $p(\xi) = \sum_{k=0}^{N} a_k e^{-ik\xi}$. Us usually we set deg $p = N$ (assuming $a_N \neq 0$).

To an arbitrary trigonometric polynomial p we associate a polynomial $R[p]$ as follows: suppose $r(\xi)$ is the polynomial of smallest degree such that the function $\frac{p(\xi)r(\xi)}{r(2\xi)}$ is a polynomial without symmetric roots; then we set $R[p](\xi) = \frac{p(\xi)r(\xi)}{r(2\xi)}$. The reader will have no difficulty in showing that the mapping $p \mapsto R[p]$ is well-defined. For given p the polynomial $R[p]$ can by easily found algorithmically. If p has no symmetric roots, then $R[p] = p$. If $\{\alpha/2, \pi + \alpha/2\}$ is a pair of symmetric roots of p, then we pass from $p(\xi)$ to the polynomial $p_\alpha(\xi) = \frac{p(\xi)(1-e^{i(\alpha-\xi)})}{1-e^{i(\alpha-2\xi)}}$. After several steps we obtain a polynomial $\tilde{p}(\xi)$ that has no symmetric roots. In general there exist several different ways to realize each step of this algorithm: if there exist several pairs of symmetric roots, we can choose any of them to pass to the next polynomial. Nevertheless *the result (i.e. the polynomial $\tilde{p}(\xi)$) does not depend on that choice and coincides with the polynomial $R[p]$.* The proof of this fact is left to the reader.

For any trigonometric polynomial p and any finite subset $Y = \{\alpha_1, \cdots, \alpha_n\} \subset \mathbb{T}$ we denote $\rho_p(Y) = (\prod_{q=1}^{n} |p(\alpha_q)|)^{1/n}$. If the set Y is cyclic (i.e, $\alpha_{q+1} = 2\alpha_q,\ q = 1, \cdots, n$, where $\alpha_{n+1} = \alpha_1$), then $\rho_p(Y) = \rho_{R[p]}(Y)$ (the proof is trivial).

Now let us formulate the criterion of stability of subdivision process.

Theorem 2. *Suppose a refinement equation $[m]$ has a $\mathcal{C}_0^l$-solution, $l \geq 0$; then the process $\{m\}$ converges in $\mathcal{C}^l$ if and only if the mask m satisfies (5) and for any cycle $\mathbf{b}$ of the polynomial $R[m]$ we have $\rho_m(\mathbf{b}) < 2^{-l}$.*

The simplest corollary of this Theorem is the following generalization of Theorem 1 from the case $l = 0$ to an arbitrary integer factor $l \geq 0$.

Corollary 1. *Suppose a mask m satisfying (5) has neither symmetric roots nor cycles; then if the equation $[m]$ has a $\mathcal{C}_0^l$-solution, then the process $\{m\}$ converges in $\mathcal{C}^l$.*

Another problem is to explore the degree of convergence of subdivision processes. For a given integer $l \ge 0$, a mask m, and a function $f \in \mathcal{L}^l$ denote

$$\nu_l(m,f) = -\lim_{n\to\infty} \frac{\log_2 \|T^n[f^{(l)}]\|_{\mathcal{C}}}{n},$$

where T is the subdivision operator associated to m (we set $\log_2 0 = -\infty$). Also for a subspace $\mathcal{V} \subset \mathcal{L}^l$ we denote $\nu_l(m,\mathcal{V}) = \inf_{f\in\mathcal{V}} \nu_l(m,f)$. The value $\nu_l(m) = \nu_l(m,\mathcal{L}^l)$ is the *degree of convergence of the process* $\{m\}$ *in the space* $\mathcal{C}^l$.

For any mask m we have: $\nu_l(m) \le l+1$ (see [DL1]). Furthermore, it was shown in [DL1] and [HC] that a process $\{m\}$ converges in $\mathcal{C}^l$ if and only if $\nu_l(m) > l$. In particular, the inequality $\nu_0(m) > 0$ means that $\{m\}$ converges in $\mathcal{C}$. Let L be the maximal integer such that $\{m\}$ converges in $\mathcal{C}^L$ (if the process $\{m\}$ does not converge in $\mathcal{C}$, then we set $L = 0$). The values $\nu_l(m),\ l = 0,1,\cdots$ are connected as follows:

$$\nu_l(m) = l+1 \quad \text{for} \quad l < L; \quad \nu_l(m) = \nu_L(m) \quad \text{for} \quad l \ge L. \tag{7}$$

The proof can be found in [DL2]. The value $\nu_L(m)$ is said to be the *degree of convergence of the process* $\{m\}$ and denoted in the sequel by $\nu(m)$. Thus, if $\nu(m_1) = \nu(m_2)$, then $\nu_l(m_1) = \nu_l(m_2)$ for any $l \ge 0$.

The degree of convergence of subdivision processes in various functional spaces was studied in [CDM], [W], [Du1], [Du2], [R3], [RS]. The following Theorem reduces this problem (in the space $\mathcal{C}^l$) from general refinement equations to the case of refinement equations having stable solutions.

Theorem 3. *For a given mask m satisfying (5) for some integer $l \ge 0$ denote $m_1(\xi) = R[m](\xi)/\prod_{k=1}^{q}\prod_{\beta\in\mathbf{b}_k}(1+e^{i(\beta-\xi)})$, where $\{\mathbf{b}_1,\cdots,\mathbf{b}_q\}$ is the set of cycles of the polynomial $R[m]$ (counting with multiplicity). Then we have:*

the equation $[m]$ has a $\mathcal{C}_0^l$-solution if and only if $[m_1]$ does; furthermore,

$$\nu_l(m) = \min\{\nu_l(m_1), -\log_2\rho_m(\mathbf{b}_1),\cdots,-\log_2\rho_m(\mathbf{b}_q)\}.$$

Corollary 2. *Under the conditions of Theorem 3 we have:*

$$\nu_k(m) = \min\{\nu_k(m_1), -\log_2\rho_m(\mathbf{b}_1),\cdots,-\log_2\rho_m(\mathbf{b}_q)\} \qquad \textit{for any } k \le l.$$

Moreover, if $\mathbf{l}(m) = \mathbf{l}(m_1)$, *then*

$$\nu(m) = \min\{\nu(m_1), -\log_2\rho_m(\mathbf{b}_1),\cdots,-\log_2\rho_m(\mathbf{b}_q)\}.$$

Remark 2. Since the mask m_1 has neither symmetric roots no cycles, it follows that the $\mathcal{C}_0^l$-solution of the equation $[m_1]$ is stable. Some previously known results on subdivision processes deal with the stable case (see for instance [CDM]). Theorem 3 makes it possible to extend those results to the case of general refinement equations.

Corollary 3. *For an arbitrary mask m satisfying (5) we have*

$$\nu_l(m) = \nu_l(R[m]).$$

Moreover, in the case $\mathbf{l}(m) = \mathbf{l}(R[m])$ *we have* $\nu(m) = \nu(R[m])$.

To prove this it is sufficient to apply Theorem 3 to the masks m and $R[m]$ and note that $\rho_m(\mathbf{b}_i) = \rho_{R[m]}(\mathbf{b}_i)$.

Thus symmetric roots of mask do not have influence on the degree of convergence of subdivision process. So the sufficient conditions from Corollary 1 are not necessary for the convergence.

Remark 3. It is easily can be shown that $\mathbf{l}(m) \le \mathbf{l}(R[m])$ for any mask m. There are masks, such that $\mathbf{l}(m) < \mathbf{l}(R[m])$ and moreover $\nu(m) < \nu(R[m])$. That is why the condition $\mathbf{l}(m) = \mathbf{l}(R[m])$ is essential in the statement of Corollary 3 (see [P2]).

Remark 4. *(The degree of convergence in various subspaces of $\mathcal{C}_0$).*

Consider the family of embedded subspaces $\{\mathcal{M}^l\}$ defined from (3). It was shown in [DL2],[Du1] that $f \in \mathcal{M}^l$ whenever $\nu_0(m, f) > l$. So the subspaces $\{\mathcal{M}^l\}$ can be considered as spaces of fast convergence of subdivision processes. Moreover, if $\nu_0(m, \mathcal{M}^l) > l$, then the mask m satisfies (5) and hence all the subspaces $\mathcal{M}^k,\ k = 0, \cdots, l$ are invariant with respect to the corresponding subdivision operator. So it is natural to restrict a subdivision operator to suitable subspace $\mathcal{M}^l$ and consider the value $\nu_0(m, \mathcal{M}^l)$ instead of $\nu_l(m)$ (see for instance [CDM], [Du1], [Du2]). Theorems 2 and 3 of this paper can be reformulated in that terms without any change.

Theorems 2 and 3 will be proved in the next section. Then, in section V, we introduce the notion of *generalized cycles* and establish a correlation between zeros of mask m and cycles of the polynomial $R[m]$. As a corollary we shall formulate the criterion of Theorem 2 in terms of zeros of the mask m (without the transfer to the polynomial $R[m]$).

IV. Proof of the main results.

To prove Theorems 2 and 3 let us first consider the case $l = 0$. The proof will be split into several lemmas and propositions.

For a finite family of real values $\Delta = \{\delta_1, \cdots, \delta_n\}$ (that may coincide) let

$$\mathcal{C}_0\{\Delta\} = \mathcal{C}_0\{\delta_1, \cdots, \delta_n\} = \{f \in \mathcal{C}_0 \mid \widehat{f}(\xi) / \prod_{q=1}^{n} (1 - e^{i(\delta_q - \xi)}) \in \mathcal{H}\}.$$

It is clear that $\mathcal{M}^l = \mathcal{C}_0\{0, \cdots, 0\}$ ($l+1$ zeros). From Poisson summation formula it follows that for any $f \in \mathcal{C}_0\{\Delta\}$ we have

$$\sum_{k \in \mathbb{Z}} e^{ik\delta_q} f(x-k) = 0, \quad q = 1, \cdots, n. \tag{8}$$

Let us also denote

$$\mathcal{L}_\Delta = \mathcal{C}_0\{0, \Delta\} = \mathcal{C}_0\{0, \delta_1, \cdots, \delta_n\} \text{ and } \mathcal{L}_\Delta[0, N] = \{f \in \mathcal{L}_\Delta \mid \operatorname{supp} f \in [0, N]\}.$$

For given $\delta \in \mathbb{R}$ consider the difference operator S_δ acting from the space $\mathcal{C}_0\{\Delta\}$ into the space $\mathcal{C}_0\{\Delta, \delta\} = \mathcal{C}_0\{\delta_1, \cdots, \delta_n, \delta\}$ and defined by the formula
$S_\delta \psi(x) = \psi(x) - e^{i\delta}\psi(x-1)$.

Lemma 1. *For any $\delta \in \mathbb{R}$ the operator S_δ is a homeomorphism of the spaces $\mathcal{C}_0\{\Delta\}$ and $\mathcal{C}_0\{\Delta, \delta\}$.*

Proof. For arbitrary $\varphi \in \mathcal{C}_0\{\Delta, \delta\}$ denote $\psi(x) = S_\delta^{-1}\varphi(x) = \sum_{k=0}^{+\infty} e^{ik\delta}\varphi(x-k)$. If $\operatorname{supp}\varphi \subset [a, b]$ for some integers a, b, then by (8) we have: $\operatorname{supp}\psi \subset [a, b-1]$. Thus, $\psi \in \mathcal{C}_0$. It now follows that $\psi \in \mathcal{C}_0\{\Delta\}$. It remains to note that $S_\delta\psi = \varphi$ and the operators S_δ and S_δ^{-1} are obviously continuous. $\square$

The following Proposition is the first step in the proof of Theorems 2 and 3.

Proposition 1. *Suppose a mask $m(\xi)$ satisfying (6) possesses a pair of symmetric roots $\alpha/2$ and $\pi + \alpha/2$. Let $m_\alpha(\xi) = \frac{m(\xi)(1-e^{i(\alpha-\xi)})}{1-e^{i(\alpha-2\xi)}}$. Then the equation $[m]$ has a $\mathcal{C}_0$-solution if and only if $[m_\alpha]$ does. Furthermore, $\nu_0(m) = \nu_0(m_\alpha)$.*

Proof. Let T and T_α be the subdivision operators associated to the masks m and m_α respectively.

Consider the operator $(P\psi)(x) = \sum_{k=0}^{N-2} p_k \psi(2x-k)$, where $p_0, \cdots, p_{N-2}$ are the coefficients of the polynomial

$$p(\xi) = \sum_{k=0}^{N-2} p_k e^{-ik\xi} = \frac{m(\xi)}{1-e^{i(\alpha-2\xi)}}.$$

That is to say in the frequency domain $\widehat{P\psi}(\xi) = \widehat{\psi}(\xi/2)p(\xi/2)$. It is clear that P is a continuous operator on $\mathcal{C}_0$. Furthermore, it preserves the subspace $\mathcal{L}$. Indeed, for any $\psi \in \mathcal{L}$ and $n \in \mathbb{Z}$ we have $\widehat{P\psi}(2\pi n) = \widehat{\psi}(\pi n)p(\pi n) = 0$ (if n is even, then $\widehat{\psi}(\pi n) = 0$; if n is odd, then $p(\pi n) = 0$, since the mask m satisfies (6)). Now observe that

$$PS_\alpha = T_\alpha, \quad S_\alpha P = T. \tag{9}$$

To prove this we apply (4) and get consequently

$$\widehat{PS_\alpha\psi}(\xi) = p(\xi/2)(1-e^{i(\alpha-\xi/2)})\widehat{\psi}(\xi/2) = m_\alpha(\xi/2)\widehat{\psi}(\xi/2) = \widehat{T_\alpha\psi}(\xi).$$

The equality $S_\alpha P = T$ can be proved in the same way.

Let $\psi \in \mathcal{C}_0$ be a solution of the equation $[m_\alpha]$. Since $T(S_\alpha\psi) = S_\alpha P S_\alpha \psi = S_\alpha T_\alpha \psi = S_\alpha \psi$, we see that the function $S_\alpha\psi$ is a solution of the equation $[m]$. Conversely, if a function $\varphi \in \mathcal{C}_0$ satisfies $T\varphi = \varphi$, then by (9) we have $\varphi \in \mathcal{C}_0\{\alpha\}$. Hence, by Lemma 1, the function $\psi = S_\alpha^{-1}\varphi$ is well-defined and belongs to $\mathcal{C}_0$. Now arguing as above we obtain $T_\alpha\psi = \psi$.

¿From (9) it follows that $T^k = S_\alpha T_\alpha^{k-1} P$ for every $k \geq 1$. Therefore, since P and S_α are continuous and preserve the subspace $\mathcal{L}$, we see that $\nu_0(m) \geq \nu_0(m_\alpha)$. Conversely, from the equality $T_\alpha^k = PT^{k-1}S_\alpha$ it follows that $\nu_0(m_\alpha) \geq \nu_0(m)$. Proposition is proved. $\square$

So using Proposition 1 we can consequently eliminate all symmetric roots and pass from the refinement equation with mask m to one with mask $R[m]$. The next step is to eliminate all cycles of the polynomial $R[m]$. In order to realize it we use the *matrix technique*, which was successfully applied in the study of subdivision processes ([MP],[CDM], [DL1],[W],[E]). For a given refinement equation $[m]$ consider the two linear operators B_0 and B_1 acting on $\mathbb{C}^N$ and defined by $N \times N$ matrices as follows:

$$(B_0)_{ks} = c_{2k-s-1}, \;\; (B_1)_{ks} = c_{2k-s}, \tag{10}$$

where c_j is the coefficient of equation (1) if $j \in \{0, 1, \cdots, N\}$, and $c_j = 0$ otherwise. As usually we denote by *span* (M) the linear span of a given set M in $\mathbb{C}^N$, by A^* the conjugate operator for a given operator A, by $V^\perp$ the orthogonal complement of a subspace V in Euclidean space. Let us recall the notion of the joint spectral radius of finite-dimensional linear operators:

$$\hat{\rho}(A_1, A_2) = \lim_{n\to\infty} \max_{(d_1,\cdots,d_n)\in\{0,1\}^n} \|A_{d_1}\cdots A_{d_n}\|^{1/n}$$

See [RoS], [BW], [CH], [LW], [P1] for more details about the joint spectral radius.

We need the following two lemmas. The first one is a direct corollary of results of the works [DL2] and [CH]. The proof of the second one can be found in [HC] or [P1].

Lemma 2. *(*[DL2], [CH]*). Let Δ be a finite family of real values such that the space $\mathcal{L}_\Delta$ is invariant with respect to the subdivision operator T; then*

$$\nu_0(m, \mathcal{L}_\Delta) = -\log_2 \hat{\rho}(B_0|_V, B_1|_V),$$

where

$$V = span\ \{(f(x), \cdots, f(x+N-1))^T \in \mathbb{C}^N \ \mid \ f \in \mathcal{L}_\Delta[0,N],\ x \in [0,1]\}.$$

In particular,

$$\nu_0(m) = \hat{\rho}(B_0|_W, B_1|_W),\ \text{where}\ W = \{(x_1, \cdots, x_N)^T \in \mathbb{C}^N \ \mid \ \sum x_j = 0\}.$$

Lemma 3. *(*[HC], [P1]*). Let A_0 and A_1 be linear operators acting on a finite-dimensional Euclidean space E. Suppose E_0 is a nontrivial common invariant subspace of these operators; then*

$$\hat{\rho}(A_0, A_1) = \max\Big\{\hat{\rho}(A_0|_{E_0}, A_1|_{E_0}), \hat{\rho}(A_0^*|_{E_0^\perp}, A_1^*|_{E_0^\perp})\Big\}.$$

Now we are able to realize the second step of the proof of Theorems 2 and 3.

Proposition 2. *Suppose a mask $m(\xi)$ possesses a cycle $\mathbf{b} = \{\beta_1, \cdots, \beta_n\}$. Denote by $\tilde{m}(\xi)$ the polynomial $m(\xi)/\prod_{k=1}^n (1 + e^{i(\beta_k - \xi)})$. Then the equation $[m]$ has a $\mathcal{C}_0$-solution if and only if $[\tilde{m}]$ does. Furthermore, $\nu_0(m) = min\{\nu_0(\tilde{m}), -\log_2 \rho_m(\mathbf{b})\}$.*

Proof. Consider the polynomial $q(\xi) = \prod_{k=1}^n (1 - e^{i(\beta_k - \xi)})$ and the corresponding operator $Q = S_{\beta_1} \circ \cdots \circ S_{\beta_n}$, which has the form $\widehat{Q\psi}(\xi) = \widehat{\psi}(\xi) q(\xi)$ in the frequency domain. It follows from Lemma 1 that Q maps the space $\mathcal{C}_0$ one-to-one into $\mathcal{C}_0\{\mathbf{b}\}$ and Q^{-1} is well-defined and continuous on $\mathcal{C}_0\{\mathbf{b}\}$. Let T and $\tilde{T}$ be the subdivision operators associated to the masks m and $\tilde{m}$ respectively. For an arbitrary function $f \in \mathcal{C}_0\{\mathbf{b}\}$ we have

$$\widehat{Tf}(\xi)/q(\xi) = m(\xi/2)\hat{f}(\xi/2)/q(\xi) = \tilde{m}(\xi/2)\hat{f}(\xi/2)/q(\xi/2) \in \mathcal{H}.$$

Consequently Tf is in $\mathcal{C}_0\{\mathbf{b}\}$ whenever $f \in \mathcal{C}_0\{\mathbf{b}\}$. This yields that the operator equality

$$\tilde{T} = Q^{-1} T Q \tag{11}$$

holds on the space $\mathcal{C}_0$. If a function $\psi \in \mathcal{C}_0$ satisfies the equality $\tilde{T}\psi = \psi$, then $\varphi = Q\psi$ satisfies $T\varphi = \varphi$. Conversely, assume that a function $\varphi \in \mathcal{C}_0$ satisfies $T\varphi = \varphi$. First let us show that φ belongs to $\mathcal{C}_0\{\mathbf{b}\}$. Using (2) we get

$$\widehat{\varphi}(\xi) = \widehat{\varphi}(0) \prod_{r=1}^{\infty} m\left(\frac{\xi}{2^r}\right) = \widehat{\varphi}(0) \prod_{r=1}^{\infty} \frac{q(\xi/2^{r-1})}{q(\xi/2^r)} \tilde{m}\left(\frac{\xi}{2^r}\right) = \frac{q(\xi)\widehat{\varphi}(0)}{q(0)} \prod_{r=1}^{\infty} \tilde{m}\left(\frac{\xi}{2^r}\right).$$

Since the function $\prod_{r=1}^{\infty} \tilde{m}\left(\frac{\xi}{2^r}\right)$ is entire, it follows that $\varphi \in \mathcal{C}_0\{\mathbf{b}\}$. Whence the function $\psi = Q^{-1}\varphi$ is well-defined and obviously satisfies $\tilde{T}\psi = \psi$.

Now in order to prove the equality $\nu_0(m) = \min\{\nu_0(\tilde{m}), -\log_2 \rho_m(\mathbf{b})\}$ we are going to use Lemmas 2 and 3. Let B_0 and B_1 be the linear operators acting in $\mathbb{C}^N$ and defined from (10). For arbitrary $t \in \mathbb{T}$ let us denote the vector $u(t) = (1, e^{it}, e^{2it}, \cdots, e^{i(N-1)t})^T \in \mathbb{C}^N$. Further, define the following subspaces:

$$U = span\ \{u(\beta_1), \cdots, u(\beta_n)\}, \quad W = u(0)^\perp = \{(x_1, \cdots, x_N) \in \mathbb{C}^N \ \mid \sum x_k = 0\}$$

and

$$\tilde{W} = \{u(0), u(\beta_1), \cdots, u(\beta_n)\}^\perp.$$

Finally denote $A_i = B_i|_W,\ \tilde{A}_i = B_i|_{\tilde{W}},\ \ i = 0,1$.

¿From (11) it follows that the equality $T^k = Q\tilde{T}^k Q^{-1}$ holds on the space $\mathcal{L}_{\mathbf{b}}$ for any $k \geq 1$. This yields that $\nu_0(\tilde{m}) = \nu_0(m, \mathcal{L}_{\mathbf{b}})$. If we combine this with Lemma 2, we get $\nu_0(\tilde{m}) = -\log_2 \hat{\rho}(\tilde{A}_0, \tilde{A}_1)$. Now it remains to proof the equality

$$\hat{\rho}(A_0, A_1) = \max\{\hat{\rho}(\tilde{A}_0, \tilde{A}_1), \rho_m(\mathbf{b})\}. \tag{12}$$

To do this observe the following property of operators B_0 and B_1:

$$B_0^* u(t) = \overline{m\Big(\frac{t}{2}\Big)} u\Big(\frac{t}{2}\Big) + \overline{m\Big(\frac{t}{2} + \pi\Big)} u\Big(\frac{t}{2} + \pi\Big),\ t \in \mathbb{T}$$

$$B_1^* u(t) = e^{-\frac{it}{2}} \overline{m\Big(\frac{t}{2}\Big)} u\Big(\frac{t}{2}\Big) + e^{-i(\frac{t}{2} + \pi)} \overline{m\Big(\frac{t}{2} + \pi\Big)} u\Big(\frac{t}{2} + \pi\Big),\ t \in \mathbb{T}. \tag{13}$$

(This can be easily shown by a direct calculation, see also [P1] or [CD2]). Whence for arbitrary $\beta_k \in \mathbf{b}$ the following hold:

$$B_0^* u(\beta_k) = \overline{m(\beta_{k-1})} u(\beta_{k-1}), \quad B_1^* u(\beta_k) = e^{-i\beta_{k-1}} \overline{m(\beta_{k-1})} u(\beta_{k-1}).$$

Therefore for any $\beta_k \in \mathbf{b}$ and any set of indices $\{d_1, \cdots, d_n\} \in \{0,1\}^n$ we have

$$B_{d_1}^* \cdots B_{d_n}^* u(\beta_k) = e^{i\mu} \Big(\prod_{j=1}^{n} \overline{m(\beta_j)}\Big) u(\beta_k),$$

where $\mu \in \mathbb{T}$ depends on β and $d_1, \cdots, d_n$. Since the vectors $\{u(\beta_k)\}_{k=1}^{n}$ form a basis of the space U, it follows that the operator $B_{d_1}^* \cdots B_{d_n}^*|_U$ is expressed in that basis by a diagonal matrix and moreover, the modulus of each diagonal entry of that matrix is equal to $|\prod_{j=1}^{n} m(\beta_j)| = (\rho_m(\mathbf{b}))^n$. This implies immediately that

$$\hat{\rho}(B_0^*|_U, B_1^*|_U) = \rho_m(\mathbf{b}). \tag{14}$$

If we apply Lemma 3 to the space W, its subspace $\tilde{W}$, and operators A_0, A_1 defined above, we obtain

$$\hat{\rho}(A_0, A_1) = \max\Big\{\hat{\rho}(\tilde{A}_0, \tilde{A}_1), \hat{\rho}(A_0^*|_H, A_1^*|_H)\Big\},$$

where H is the orthogonal complement of the subspace $\tilde{W}$ in the space W. Let us finally note that $A_i^*|_H = P_H B_i^*|_U P_H^{-1},\ \ i = 0,1$, where P_H is the operator of orthogonal projection from U to H (since the vectors $u(0), u(\beta_1), \cdots, u(\beta_n)$ are linearly independent, it follows that P_H^{-1} is well-defined on the space H). Combining this with (14) we get:

$$\hat{\rho}(A_0^*|_H, A_1^*|_H) = \hat{\rho}(B_0^*|_U, B_1^*|_U) = \rho_m(\mathbf{b}),$$

that completes the proof of Proposition 2. □

Suppose we have a subdivision process $\{m_0\}$; then we pass to the process $\{R[m_0]\}$ and using Proposition 2 consequently eliminate all cycles of the mask $R[m_0]$. As a result we obtain the mask m_1 that has neither symmetric roots nor cycles. So we prove the following statement, which is a weaker version of Theorem 3.

Proposition 3. *For a given mask m_0 satisfying (6) let us denote*

$$m_1(\xi) = R[m_0](\xi) / \prod_{k=1}^{q} \prod_{\beta \in \mathbf{b}_k} (1 + e^{i(\beta - \xi)}),$$

where $\{\mathbf{b}_1, \cdots, \mathbf{b}_q\}$ is the set of cycles of the polynomial $R[m_0]$ (counting with multiplicity). Then we have:

the equation $[m_0]$ *has a* $\mathcal{C}_0$*-solution if and only if* $[m_1]$ *does; furthermore,*

$$\nu_0(m_0) = \min\{\nu_0(m_1), -\log_2 \rho_{m_0}(\mathbf{b}_1), \cdots, -\log_2 \rho_{m_0}(\mathbf{b}_q)\}.$$

Thus Theorem 3 is proved for the case $l = 0$. Combining this with Theorem 1 we obtain Theorem 2 for the case $l = 0$.

Now it remains to realize the third step of the proof, i.e. to extend the statements of Theorems 2 and 3 from the case $l = 0$ to general integer factor $l \geq 0$. To do this we introduce Proposition 4, which gives a method of factorization of refinement equations. That Proposition reduces the study of refinadle functions and subdivision processes from the space $\mathcal{C}^l$ to $\mathcal{C}$.

Let us first remember the definition of the *cardinal B-spline*:

$$B_0(x) = \chi_{[0,1]}(x); \quad B_k(x) = [\chi_{[0,1]} * \cdots * \chi_{[0,1]}](x) \quad (k \text{ convolutions}).$$

For any $k \geq 0$ the cardinal B-spline B_k is a solution of the refinement equation with mask $(\frac{1+e^{-i\xi}}{2})^{k+1}$ (see for instance [Sc] or [DL2]).

Proposition 4. *Suppose* m *and* m_0 *are masks of refinement equations such that* $m(\xi) = (\frac{1+e^{-i\xi}}{2})^l m_0(\xi)$, $l \geq 1$*; then*

a. *The equation* $[m]$ *has a* $\mathcal{C}_0^l$*-solution if and only if* $[m_0]$ *has a* $\mathcal{C}_0$*- solution. Moreover,* $\psi = S_0^{-l}\varphi^{(l)}$ *and* $\varphi = B_{l-1} * \psi$*, where* φ *and* ψ *are solutions of* $[m]$ *and* $[m_0]$ *respectively;* S_0 *is the difference operator:* $S_0 f(x) = f(x) - f(x-1)$.

b. *The subdivision process* $\{m\}$ *converges in* $\mathcal{C}^l$*, if and only if* $\{m_0\}$ *converges in* $\mathcal{C}$*. Moreover,* $\nu(m) = \nu(m_0) + l$.

Proof. It follows from Lemma 1 that the mapping $S_0^l : \mathcal{C}_0 \to \mathcal{M}^{l-1}$ is a homeomorphism. Furthermore, for any $k \geq 0$ the mapping $S_0^l : \mathcal{M}^k \to \mathcal{M}^{k+l}$ is a homeomorphism. Now observe that for any $f \in \mathcal{C}_0$ and $g \in \mathcal{M}^{l-1}$ we have

$$T_0 f = 2^l S_0^{-l} T S_0^l f, \quad f \in \mathcal{C}_0;$$

$$Tg = 2^{-l} S_0^l T_0 S_0^{-l} g, \quad g \in \mathcal{M}^{l-1}, \tag{15}$$

where T and T_0 are the subdivision operators associated to the masks m and m_0 respectively. This immediately implies item a). Further, from (15) it follows that $\nu_0(m, \mathcal{M}^{k+l}) = \nu_0(m_0, \mathcal{M}^k) + l$ for any admissible $k \geq 0$, i.e. whenever $k \leq \mathbf{l}(m_0)$. Therefore, $\nu_{k+l}(m) = \nu_k(m_0) + l$. Combining this with (7) we obtain item b), that completes the proof of Proposition 4. $\square$

Now to extend Theorems 2 and 3 from the case $l = 0$ it is sufficient to pass from the mask m to m_0 (applying Proposition 4) and note that $\rho_m(\mathbf{b}) = 2^{-l}\rho_{m_0}(\mathbf{b})$ for any cycle $\mathbf{b}$. This concludes the proof of the main theorems.

$\square$

Remark 5. The statement of item **a)** of Proposition 4 generalizes the result [E, theorem 2.2], which was obtained for refinement equations satisfying Cohen's criterion (see Remark 1).

Remark 6. It follows from results of the work [P2] that the statement of item **a)** of Proposition 4 can be extended to general refinement equations, i.e. equations without condition (5). Namely, the following hold:

If an equation $[m]$ *has a* $\mathcal{C}_0^l$*-solution* $\varphi(x)$, $(l \geq 1)$*, then there exist dyadic rational values* $\gamma_1, \cdots, \gamma_r$ *(perhaps coinciding) such that* $\varphi = B_{l-1} * (S_{\gamma_1} \circ \cdots \circ S_{\gamma_r}\psi)$ *(and* *correspondingly* $\psi = S_0^{-l} \circ S_{\gamma_1}^{-1} \circ \cdots \circ S_{\gamma_r}^{-1}\varphi^{(l)}$*), where* ψ *is the* $\mathcal{C}_0$*-solution of the equation having the mask*

$$m_0(\xi) = \frac{m(\xi)}{[(1+e^{-i\xi})/2]^l} \prod_{k=1}^{r} \frac{1-e^{i(2\pi\gamma_k - \xi)}}{1-e^{i(2\pi\gamma_k - 2\xi)}}.$$

So the study of smooth refinable functions can be reduced to the study of continuous refinable functions (see [P2] for more details; see also [R1] and [C] for similar factorization theorems).

V. Generalized cycles.

Theorems 2 and 3 are formulated in terms of cycles of the polynomial $R[m]$. It is easy to see that in general the sets of cycles of the polynomials m and $R[m]$ are different. The question arises how can cycles of $R[m]$ be characterized by roots of m? In other words we are going to reformulate the criterion of stability of subdivision operator in terms of zeros of its mask.

Let $p(\xi)$ be a given trigonometric polynomial (let us remember that we consider polynomials without positive powers). Assume that p possesses a pair of symmetric roots $\{\alpha/2, \pi+\alpha/2\}$. The transfer from $p(\xi)$ to the polynomial $p_\alpha(\xi) = \frac{p(\xi)(1-e^{i(\alpha-\xi)})}{1-e^{i(\alpha-2\xi)}}$ is said to be a *transfer to the previous level.* The inverse transfer from p_α to p is a *transfer to the next level.* So the polynomial $R[p]$ is obtained from p by a sequence of transfers to the previous level.

To a given value $\alpha \in \mathbb{T}$ we assign a binary tree denoted in the sequel by $\mathcal{T}_\alpha$. To every vertex of this tree we associate a value from $\mathbb{T}$ as follows: put α at the root, then put $\alpha/2$ and $\pi+\alpha/2$ at the vertices of the first level (the *level* of the vertex is the distance from this vertex to the root. The root has level 0). If a value γ is associated to a vertex on the n-th level, then the values $\gamma/2$ and $\pi+\gamma/2$ are associated to its neighbors on the $(n+1)$-st level. Thus there are the values $\frac{\alpha}{2^n} + \frac{2k\pi}{2^n},\ k = 0, \cdots, 2^n - 1$ on the n-th level of the tree $\mathcal{T}_\alpha$. A set of vertices $\mathcal{A}$ of the tree $\mathcal{T}_\alpha$ is called a *minimal cut set* if every infinite path (all the paths are without backtracking) starting at the root includes exactly one element of $\mathcal{A}$. For instance the one-element set $\mathcal{A} = \{root\}$ is a minimal cut set.

Definition 1. *A set* $\{\beta_1, \cdots, \beta_n\} \subset \mathbb{T}$ *is called a generalized cycle of the polynomial* $p(\xi)$ *if the following hold:*

- a. *This set is cyclic, i.e.* $\beta_{j+1} = 2\beta_j$ *for all* $j = 1, \cdots, n$ *(we set* $\beta_{n+1} = \beta_1$*);*
- b. *for any* $j = 1, \cdots, n$ *the tree* $\mathcal{T}_{\beta_j+\pi}$ *possesses a minimal cut set that consists of roots of the polynomial* p.

Any (regular) cycle of $p(\xi)$ is also a generalized cycle. Indeed, in this case each minimal cut set A_j is the root of the corresponding tree $\mathcal{T}_{\beta_j+\pi}$. Now we establish a correlation between generalized cycles of the polynomial $p(\xi)$ and (regular) cycles of $R[p]$.

Proposition 5. ***a)***. *Every cycle of the polynomial* $R[p]$ *is a generalized cycle of* p. ***b)***. *Every generalized cycle* $\mathbf{b}$ *of the polynomial* p *such that* $\rho_p(\mathbf{b}) \neq 0$ *is a cycle of* $R[p]$.

Proof. **(a).** Let $\mathbf{b} = \{\beta_1, \cdots, \beta_n\}$ be a cycle of the polynomial $R[p]$. The polynomial p is obtained from $R[p]$ by a sequence of transfers to the next level. That sequence takes the root of the tree $\mathcal{T}_{\beta_j+\pi}$ to some minimal cut set $\mathcal{A}_j$ of this tree. Since $\beta_j + \pi$ is a root of $R[p]$, it follows that all elements of $\mathcal{A}_j$ are roots of p. So the set $\mathbf{b}$ is a generalized cycle for $p(\xi)$.

(b). Let $\mathbf{b} = \{\beta_1, \cdots, \beta_n\}$ be a generalized cycle of the polynomial $p(\xi)$. Applying a suitable sequence of transfers to the previous level we pass from the minimal cut sets $\mathcal{A}_1, \cdots, \mathcal{A}_n$ to the roots $\beta_1 + \pi, \cdots, \beta_n + \pi$ of the corresponding trees. Then we continue applying transfers to the previous level until we obtain the polynomial $R[p]$. If at some step we involve an element $\beta_j + \pi$ in this process, then the polynomial $p_1(\xi)$, which is obtained from the polynomial $p(\xi)$ by this step, has the pair of symmetric roots $\{\beta_j, \beta_j + \pi\}$. This implies that $\rho_{p_1}(\mathbf{b}) = 0$ and hence $\rho_p(\mathbf{b}) = 0$. Consider the opposite case. If the elements $\beta_1 + \pi, \cdots, \beta_n + \pi$ are not involved, then each of them is a root of $R[p]$. Therefore $\mathbf{b}$ is a cycle of $R[p]$. This completes the proof. $\square$

Corollary 4. *If a polynomial $p(\xi)$ has no symmetric roots, then the set of its generalized cycles coincides with the set of its (regular) cycles.*

Corollary 5. *The set of all generalized cycles of a polynomial $p(\xi)$ is a union of the following two sets: the first one is the set of all cycles of $R[p]$, the second one consists of generalized cycles $\mathbf{b}$ such that $\rho_p(\mathbf{b}) = 0$.*

It follows from Propositions 2 and 4 that any cycle $\mathbf{b}$ such that $\rho_m(\mathbf{b}) = 0$ does not have influence on the convergence of the subdivision process $\{m\}$, i.e. $\nu(m) = \nu(\tilde{m})$ in terms of Proposition 2. Hence the criterion of convergence for subdivision processes can be formulated in terms of generalized cycles of mask. As a corollary we obtain the main result of this section:

Corollary 6. *The statement of Theorem 2 remains true if the notion "a cycle of the polynomial $R[m]$" is replaced by "a generalized cycle of the mask m".*

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