The t-median function on graphs

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23 August 2005

ECONOMETRIC INSTITUTE REPORT EI 2005-31

Abstract

A median of a sequence $\pi = x_1, x_2, \ldots, x_k$ of elements of a finite metric space (X, d) is an element x for which $\sum_{i=1}^{k} d(x, x_i)$ is minimum. The function M with domain the set of all finite sequences on X and defined by $M(\pi) = \{x : x$ is a median of $\pi\}$ is called the median function on X, and is one of the most studied consensus functions. Based on previous characterizations of median sets $M(\pi)$, a generalization of the median function is introduced and studied on various graphs and ordered sets. In addition, new results are presented for median graphs.

Keywords: Consensus; Median function; Median graph; Median semilattice

MCS: primary 06A12; secondary 05C75

1 Introduction

The axiomatic approach to the study of consensus functions effectively began with the famous work of Kenneth Arrow in 1951, with the domain of interest being the set of preference rankings of a given set of alternatives. Since then domains have been extended to sets of phylogenetic trees, classifications, molecular sequences, etc., where the goal is to produce an output consensus object(s) for an input collection of objects in the domain. See [4] for many references and results in this growing research enterprise.

Often the domain of interest will admit one (or more) distance measures between pairs of objects so that a metric space results. The general setting in these cases is as follows. Let (X, d) be a finite metric space and $X^* = \bigcup_{k \geq 1} X^k$. One of the reasonable ways to produce a consensus of a sequence $\pi = x_1, x_2, \ldots, x_k$ of elements in X is to find elements x in X that are closest to π, and one way to do this is to find x that minimize $\sum_{i=1}^{k} d(x, x_i)$. The function $M : X^* \longrightarrow 2^X \backslash \{\emptyset\}$, where $M(\pi) = \{\ x \mid \sum_{i=1}^{k} d(x, x_i)$ is minimum$\}$ is called the *median function*, and has been the subject of extensive study. (see [4]) Usually X has additional graph theoretic or order theoretic structure such as median graph or distributive semilattice structure.

In the present paper we introduce a parametrized family of functions, M_t and m_t, where $\frac{1}{2} \leq t \leq 1$ where $M_{\frac{1}{2}}$ is the median function on graphs and $m_{\frac{1}{2}}$ is the median function on semilattices. Using natural generalizations of the axioms that characterize the median function, we study these *t-median functions* and observe strikingly different behavior. In Section 2, some new results on median graphs are presented and some of the basic axioms, such as faithfulness and consistency, are stated. The consensus functions M_t and m_t are defined and two versions of the t-Condorcet axiom are given in Section 3. Section 4 contains a somewhat surprising impossibility result and Section 5 focuses on the consistency of M_t.

A consequence of our work is that a consensus function c on a median graph G satisfies the axioms of faithfulness, consistency, and t-Condorcet for some t in $[1/2, 1)$ if and only if $t = 1/2$ and c is the median function. On the other hand, for any t in $[1/2, 1)$, we observe that the t-median function M_t is faithful, quasi-consistent, and t-Condorcet. Consequently, there is a subtle interplay between different types of consistency and the t-Condorcet axiom for various values of t. A complete characterization of the t-median function for $t > 1/2$ is still an open problem.

2 Preliminaries

In this section we give much of the required background and definitions. Because of the fairly large numbers of such items, we ask the reader for tolerance.

2.1 Basics

Throughout this paper $G = (V, E)$ is a finite connected graph with distance function d, where $d(u, v)$ is the length of a shortest u, v-path (geodesic), for any two vertices u and v of G. Clearly, (V, d) is a finite metric space. For any subset W of V the subgraph of G induced by W is denoted by $\langle W \rangle$.

A subset W of V is *isometric* if $\langle W \rangle$ contains a geodesic between u and v, for any u, v in W. An *isometric subgraph* is a subgraph induced by an isometric subset. The *interval* $I(u, v)$ between vertices u and v consists of all vertices on geodesics between u and v. A subset W of vertices of G is *convex* if $I(u, v) \subseteq W$ for any $u, v \in W$. Observe that the intersection of two convex sets is again convex. A *convex subgraph* is a subgraph induced by a convex set. Let W be a subset of vertices and z any vertex. A vertex $x \in W$ is a *gate* in W for z if $x \in I(z, w)$, for any $w \in W$. Note that, if z has a gate in W, then it is unique, and is the vertex in W closest to z. A subset W of V is *gated* if every vertex has a gate in W, so a *gated subgraph* is a subgraph induced by a gated set. It is easily seen that a gated subgraph is convex. Because of the uniqueness of gates, it is easily seen that a vertex z outside a gated set W has at most one neighbor in W, and if it has a neighbor in W, then this is the gate for z.

In the sequel we will not distinguish between a subset W of vertices of the graph G and the subgraph $\langle W \rangle$ of G induced by W. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs, then the *intersection* $G_1 \cap G_2$ of G_1 and G_2 is the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$.

2.2 Splits and Partial Cubes

For an edge uv of G, we use the following notation.

$$W_u^{uv} = \{ \ x \mid d(u, x) < d(v, x) \ \},$$
$$G_u^{uv} = \langle W_u^{uv} \rangle,$$
$$F_{uv} = \{ \ xy \mid xy \ \text{ is an edge with } x \in W_u^{uv} \text{ and } y \in W_v^{uv} \}.$$

If we have xy an edge in F_{uv}, then, by convention, we assume that x is in G_u^{uv} and y is in G_v^{uv}, that is, uv and xy are written in the "same order". If edge xy is an edge in F_{uv} distinct from uv, then in general it is possible that $G_x^{xy} \neq G_u^{uv}$.

If there are no vertices with equal distance to u and v, then connectivity of G implies that $V = W_u^{uv} \cup W_v^{uv}$, that is, G_u^{uv} and G_v^{uv} *cover* G. Evidently, G is bipartite if and only if G_u^{uv} and G_v^{uv} cover G, for all edges uv of G. We call the pair G_u^{uv}, G_v^{uv} a *split* if they cover G and we have $G_x^{xy} = G_u^{uv}$ and $G_y^{xy} = G_v^{uv}$, for any edge xy in F_{uv}. The subgraphs G_u^{uv} and G_v^{uv} are the *splithalves* of the split.

Lemma 1 *Let G be a connected graph, and let uv be an edge of G such that G_u^{uv}, G_v^{uv} is a split. Then G_u^{uv} and G_v^{uv} are convex, and $d(u, x) = d(v, y) = d(u, y) - 1 = d(x, v) - 1$, for any edge xy in F_{uv}.*

Proof. Let G_u^{uv}, G_v^{uv} be a split for the edge uv, and let xy be an edge in F_{uv}. Then x is closer to u than to v, and y is closer to v than to u. Hence we have

$$d(x,u) = d(x,v) - 1 \leq d(y,v) = d(y,u) - 1 \leq d(x,u).$$

So we have equality throughout.

To show G_u^{uv} is convex, choose any two vertices p, q in G_u^{uv}, and let P be a shortest p,q-path. We have to prove that P does not contain vertices in G_v^{uv}. Assume the contrary, and let rs be the first edge from F_{uv} on P in going from p to q along P. At some point we have to return to G_u^{uv}. Let xy be the next edge from F_{uv} on P, where this edge is traversed from y to x. Now G_x^{xy}, G_y^{xy} and G_r^{rs}, G_s^{rs} are the same split as G_u^{uv}, G_v^{uv}. So, by the first part of the proof, we have $d(r,x) = d(s,y) = d(r,y) - 1$. Replacing the part of P between r and x by a geodesic between r and x, we obtain a p,q-walk of length $d(p,q) - 2$, which is in conflict with P being a shortest path. From this we conclude the convexity of G_u^{uv}, and similarly that of G_v^{uv}. □

A *partial cube* is an isometric subgraph of a hypercube. Djokovic [5] was the first to characterize these graphs, with another characterization given by Winkler [16]. For a formulation of the Djokovic-Winkler characterization see Imrich-Klavžar [6]. In our terminology their result reads as follows.

Theorem A *Let G be a connected graph. Then G is a partial cube if and only if G_u^{uv}, G_v^{uv} is a split for every edge uv in G.*

Note that this implies that a connected graph is a partial cube if and only if G is bipartite and G_u^{uv} is convex, for any edge uv in G.

If we consider a split without specifying an edge between the splithalves, then, by convention, we denote the split just by G_1, G_2 with vertex sets W_1 and W_2 respectively. The set of edges between G_1 and G_2 is denoted by F_{12}.

2.3 Median Graphs

A *median graph* is a graph $G = (V, E)$ such that

$$|I(u,v) \cap I(v,w) \cap I(w,u)| = 1 \quad \text{for all } u, v, w \in V.$$

In other words, G is a median graph if there exists a unique vertex x lying on some geodesic between each pair out of u, v, w, for any three vertices u, v, w in V. This vertex x is called the *median* of u, v, w. For an extensive study of median graphs see [15], for a survey of characterizations and applications of median graphs see [7]. A connected graph G satisfies the *quadrangle property* if, for any four vertices u, v, w, z with $d(u,v) = d(u,w) = d(u,z) - 1$ and $d(v,w) = 2$ and z a common neighbor of v and w, there exists a common neighbor x of v and w with $d(u,x) = d(u,v) - 1 = d(u,w) - 1 = d(u,z) - 2$. Theorem B from [15] is needed in order to prove our Theorem 2, a new characterization of median graphs.

Theorem B *Let G be a connected triangle-free graph. Then G is a median graph if and only if G satisfies the quadrangle property and does not contain $K_{2,3}$ as a subgraph.*

Theorem 2 *Let G be a connected bipartite graph. Then G is a median graph if and only if G_u^{uv} is gated for all edges uv in G.*

Proof. Let G_u^{uv} be gated for every edge uv in G. We first show that G satisfies the quadrangle property. Let u, v, w, z be four vertices with $k = d(u,v) = d(u,w) = d(u,z) - 1$ and z a common neighbor of v and w. Since G is bipartite, we have $d(v,w) = 2$. Assume that there is no common neighbor x of v and w with $d(u,x) = d(u,v) - 1 = d(u,w) - 1 = d(u,z) - 2$. Under these circumstances we choose the vertices u, v, w, z such that k is as small as possible. Note that we have $k \geq 2$. By minimality of k, we have $I(u,v) \cap I(u,w) = \{u\}$. Let x be a neighbor of v in $I(u,v)$, and let y be a neighbor of u in $I(u,w)$. Then we have $d(u,x) = k - 1$ and $d(y,w) = k - 1 = d(y,z) - 1$. From $I(u,v) \cap I(u,w) = \{u\}$ it follows that $d(y,v) \geq d(u,v)$. Since G is bipartite, it follows that $d(y,v) = d(u,v) + 1$, so the edge uy is an edge in F_{vz}. By convexity of splithalves, we have $I(u,v) \subseteq G_v^{vz} = G_u^{uy}$ and $I(y,z) \subseteq G_z^{vz} = G_y^{uy}$. Since $d(y,v) = k + 1$, we have $d(y,x) = k = d(y,z)$. Thus z is not on a shortest x,y-path, which implies that z is not the gate for x in G_z^{vz}. Note that $d(x,z) = 2$, so the gate of x in G_z^{vz} must be a neighbor t of x in G_z^{vz}. Since x is not adjacent to w, we have $t \neq w$. By Lemma 1, we have $d(y,t) = k - 1 = d(y,w)$. By the minimality of k, we deduce the existence of a common neighbor s of t and w with $d(y,s) = d(y,t) - 1 = k - 2$. Now $d(u,x) = d(u,s) = k - 1 = d(u,t) - 1$. Again by minimality of k, we deduce the existence of a common neighbor r of x and s with $d(u,r) = d(u,x) - 1$. By the choice of u, v, w, z, being four vertices dissatisfying the quadrangle property, there is no common neighbor of r, v, and w. So r, v, w, s dissatisfy the quadrangle property. Hence, by minimality of k, we may assume that $r = u$. Now we have the situation that $d(w,v) = d(w,u) = 2$. So the gate of w must be a common neighbor of w, u, and v, which contradicts our choice of u, v, w, z. This concludes the proof of the quadrangle property for G.

Next assume that there is a $K_{2,3}$ in G. Since G is bipartite, this $K_{2,3}$ is an induced subgraph. Let u, y, z be the vertices of degree 2, and let x, v be the vertices of degree 3 in this $K_{2,3}$. Then u, x are in G_u^{uv} and v, y, z are in G_v^{uv}. But now x, being outside G_v^{uv} has two neighbors in the gated subgraph G_v^{uv}. Since this is impossible, there is no $K_{2,3}$ in G. Hence, by Theorem A, G is a median graph.

The converse is a well known consequence of the characterization of median graphs in [14] (also see [15]). □

A simple corollary to Theorem 2 is that median graphs are partial cubes. In [14] it was proved that they are precisely the graphs that can be isometrically embedded in a hypercube such that medians of triples are preserved.

2.4 Distributive and median semilattices

Some required order-theoretic preliminaries we now borrow from [9]. As before, all sets are finite. A *partially ordered set* is a nonempty set V together with a reflexive, antisymmetric, transitive relation $\leq$ defined on V. If $(V, \leq)$ is a partially ordered set and $x, y \in V$, then y *covers* x if $x \leq y$, and $x \leq z < y$ implies that $x = z$. The *covering graph* of V is the graph $G = (V, E)$ where $xy \in E$ if and only if x covers y or y covers x. The partially ordered set $(V, \leq)$ is a *meet semilattice* if and only if every two element set $\{x, y\}$ has an infimum, denoted $x \wedge y$, and is a *join semilattice* if and only if $\{x, y\}$ has a supremum, $x \vee y$. An element s in the meet semilattice V is *join irreducible* if $s = x \vee y$ implies that either $s = x$ or $s = y$. An *atom* of the meet semilattice V is an element that covers the universal lower bound of V. A *lattice* is a partially ordered set V for which $x \wedge y$ and $x \vee y$ exist for all $x, y \in V$. The lattice $(V, \leq)$ is *distributive* when $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in V$. A meet semilattice is *distributive* if for every $x \in V$, the set $\{t | t \leq x\}$ is a distributive lattice.

Now consider the following ordered version of median graphs. A meet semilattice $(V, \leq)$ is a *median semilattice* if and only if V is a distributive semilattice, and any three elements of V have an upper bound whenever each pair of them have an upper bound. The relationship between median graphs and median semilattices is well-known (see [1], [2], and [15]): If $G = (V, E)$ is a median graph and $z \in V$, then $(V, \leq_z)$ is a median semilattice where $\leq_z$ is defined by $x \leq_z y$ if and only if $x \in I(z, y)$. Conversely, the covering graph of a median semilattice is a median graph. Note that non-isomorphic median semilattices may have the same median graph as their covering graph.

A nice consequence of this close relationship between median graphs and median semilattices is that one can use both the graph perspective and the order perspective in proofs by going back and forth between these two appearances of median structures. An example if this feature is shown in the next theorem from [9], which we shall need below.

Theorem C *Let $G = (V, E)$ be a median graph and let z be any vertex of G. For any split G_1, G_2 of G with z in G_1, the gate s of z in G_2 is the unique join-irreducible in G_2 in the median semilattice $(V, \leq_z)$.*

This theorem provided us with the following surprising corollary, see [9].

Corollary D *Let $G = (V, E)$ be a median graph. Then all median semilattices $(V, \leq)$ having G as covering graph have the same number of join-irreducibles.*

For $(V, \leq)$ a median semilattice and $x \in V$, let $h(x)$ denote the length of a shortest path from x to the universal lower bound of V, in the covering graph of $(V, \leq)$. Finally recall that the usual lattice metric $d_{\leq}$ on $(V, \leq)$ defined by $d_{\leq}(u, v) = h(u) + h(v) - 2h(u \wedge v)$ coincides with the geodesic metric on the covering graph of $(V, \leq)$ (see [13] and [8]).

2.5 Consensus functions

A *profile* on a set V is a sequence $\pi = v_1, v_2, \ldots, v_k$ of elements in V, with $|\pi| = k$ the *length* of the profile. By V^* we denote the set of all profiles on V. A *consensus function* on a set V is a function $c : V^* \to 2^V - \{\emptyset\}$ that returns a nonempty subset for each profile. A standard problem in consensus theory is the study of the effects of various axioms on consensus functions. Here we present some relevant ones.

Anonymity (A) : for any profile $\pi = v_1, v_2, \ldots, v_k$ on V and any permutation σ of $\{1, 2, \ldots, k\}$, we have $c(\pi) = c(\pi^\sigma)$, where $\pi^\sigma = v_{\sigma(1)},\ v_{\sigma(2)},\ \ldots,\ v_{\sigma(p)}$.

Faithfulness (F) : $c(v) = \{v\}$, for all $v \in V$.

Unanimity (U) : $c(v, v, \ldots, v) = \{v\}$, for all $v \in V$.

Consistency (C) :
if $c(\pi) \cap c(\rho) \neq \emptyset$ for profiles π and ρ, then $c(\pi, \rho) = c(\pi) \cap c(\rho)$.

Let $\pi = v_1, v_2, \ldots, v_k$ be a profile on $G = (V, E)$. For a subset W of V, the *subprofile* π_W on W is the subsequence or π of vertices in W. Similarly, the subprofile π_H on a subgraph H is defined. In case the subgraph is G_u^{uv}, we write the subprofile on G_u^{uv} as π_u^{uv}. The next two consensus axioms involve the metric properties of the graph.

Betweenness (B) : $c(u, v) = I(u, v)$, for all $u, v \in V$.

$\frac{1}{2}$-Condorcet : $u \in c(\pi)$ if and only if $v \in c(\pi)$, for each profile π on G and for each split G_u^{uv}, G_v^{uv} of G with $|\pi_u^{uv}| = |\pi_v^{uv}|$.

It is easy to see on any graph if c satisfies axioms (B) and (C), then c satisfies axiom (F), and if c satisfies (C) and (F), then it satisfies (U) .

For a profile $\pi = v_1, v_2, \ldots, v_k$ and a vertex x, let

$$D(x, \pi) = \sum_{i=1}^{k} d(x, v_i).$$

A *median vertex* of π is a vertex x minimizing $D(x, \pi)$. The *median set* $M_G(\pi)$ of π is the set of all median vertices of π. The median function M_G on G is the consensus function that returns the median set for any profile on G. If no confusion arises, we delete the subscript G.

The median function is an important and well studied consensus function. It is easily verified that the median function M on a graph G satisfies the axioms (A), (B), and (C), and therefore also (U) and (F). It is an open problem to characterize the graphs on which the median function is the only consensus function satisfying (A), (B) and (C). A first, but far from trivial, result in this direction was proved in [11].

Theorem E *Let c be a consensus function on a cube-free median graph G. Then c satisfies (A), (B), and (C) if and only if $c = M$.*

In [11] the median function was characterized on arbitrary median graphs using an extra axiom. In [9] the following result was proved.

Theorem F *Let c be a consensus function on a median graph G. Then c satisfies (A), (B), (C), and $\frac{1}{2}$-Condorcet if and only if $c = M$.*

For other characterizations of the median function on median graphs see [7, 11].

Of course there are consensus functions that do not satisfy one or more of the above axioms. In that case a weaker condition might still be satisfied, such as one of the following axioms.

Subfaithfulness : $v \in c(v)$, for all $v \in V$.

Subunanimity : $v \in c(v, v, \ldots, v)$, for all $v \in V$.

In the case of consistency there are various sensible possibilities.

Subconsistency : $c(\pi) \cap c(\rho) \subseteq c(\pi, \rho)$, for any profiles π and ρ on V.

Quasi-consistency :
$c(\pi) = c(\rho) \Rightarrow c(\pi, \rho) = c(\pi) = c(\rho)$, for any profiles π and ρ on V.

Subquasi-consistency :
$c(\pi) = c(\rho) \Rightarrow c(\pi) = c(\rho) \subseteq c(\pi, \rho)$, for any profiles π and ρ on V.

3 The M_t consensus function

Based on a previous characterization of the median set of a profile as an intersection of splithalves we next introduce a generalization of the median function.

3.1 M_t on graphs

Throughout the rest of the paper t is a rational number with $\frac{1}{2} \leq t < 1$. Let $\pi = v_1, v_2, \ldots, v_k$ be a profile on the connected graph $G = (V, E)$. As recalled in Section 2.5 the median function M is $\frac{1}{2}$-Condorcet on median graphs. In the proof of Theorem F the following result was used, see [9].

Theorem G *Let G be a median graph, and let M be the median function on G. Then*

$$M(\pi) = \bigcap \{ \ G_u^{uv} \mid G_u^{uv} \textit{ is a splithalve with } |\pi_u^{uv}| > \tfrac{1}{2}|\pi| \ \},$$

for any profile π on V.

This is the motivation for considering the consensus function M_t defined by

$$M_t(\pi) = \bigcap\{\ G_u^{uv} \mid G_u^{uv} \text{ is a splithalve with } |\pi_u^{uv}| > t|\pi|\ \}$$

for any profile π on G. We call this function the *t-median function* on G. By Lemma 1, the set $M_t(\pi)$ is convex for any profile π.

If G_1, G_2 is a split of G with $|\pi_1| > t|\pi|$, then we call this split *t-distinguishing* with G_1 the *t-side* of the split and G_2 the *opposide* of the split.

The median function M is trivially faithful on any graph. But for M_t this is quite different, and leads to a new characterization of partial cubes.

Lemma 3 *Let G be a connected graph. Then M_t is faithful on G if and only if G is a partial cube.*

Proof. Assume that M_t is faithful. Take any edge uv in G. Then $M_t(u) = \{u\}$. Since v is not in $M_t(u)$, there exists an edge xy such that G_x^{xy}, G_y^{xy} is a split with u in G_x^{xy} and v in G_y^{xy}. Then uv is an edge in F_{xy}, and, by the definition of split, $G_u^{uv} = G_u^{xy}$ and $G_v^{uv} = G_y^{xy}$. So G_u^{uv}, G_v^{uv} is a split in G.

Conversely, let x be any vertex of G. Since G is a partial cube, every edge of G defines a split. So for each neighbor y of x, we have $x \in G_x^{xy}$ and $y \in G_y^{xy}$, where G_x^{xy}, G_y^{xy} is a split. Let w be any vertex in G different from x, and let y be a neighbor of x on some geodesic between x and w. Then w lies in G_y^{xy}, so w is not in $M_t(x)$. Hence we have $M_t(x) = \{x\}$. □

An analogue that we use of the $\frac{1}{2}$-Condorcet axiom for $\frac{1}{2} \leq t < 1$ reads as follows.

***t*-Condorcet** :
$u \in c(\pi) \iff v \in c(\pi)$ for each profile π and each split G_u^{uv}, G_v^{uv} with $|\pi_u^{uv}| = t|\pi|$.

Lemma 4 *Let $G = (V, E)$ be a partial cube. Then M_t is t-Condorcet on G.*

Proof. Consider a profile π on V, and let G_u^{uv}, G_v^{uv} be any split of G with $|\pi_u^{uv}| = t|\pi|$. Hence this split is not t-distinguishing. Assume that one end of uv is in $M_t(\pi)$ and the other end is not. Then there exists a t-distinguishing split G_1, G_2 such that one of u and v belongs to the t-side G_1 and the other belongs to the opposide G_2. Now uv is an edge between the splithalves of G_u^{uv}, G_v^{uv} as well as G_1, G_2, which means that these splits are the same. i.e., this split would be t-distinguishing and not t-distinguishing at the same time. This impossibility proves the lemma. □

3.2 M_t on distributive semilattices

First we introduce some notation and definitions from [3]. Let $(V, \leq)$ be a finite distributive semilattice, S be the set the join-irreducibles of $(V, \leq)$, and $\pi = v_1, v_2, \ldots, v_k$ be a profile on V. Then the *index* of an element $v \in V$ is

$$\gamma(v, \pi) = \frac{|\{\ i \mid v \leq v_i\ \}|}{k}.$$

For the profile π we define

$$\alpha_t(\pi) = \bigvee\{\ s \mid s \in S \text{ with } \gamma(s, \pi) > t\ \}.$$

The *t-median function*, m_t, on $(V, \leq)$ is defined by

$$\begin{aligned}&m_t(\pi) = \{\alpha_t(\pi)\}\cup\\ &\quad \cup\{\alpha_t(\pi) \vee s_1 \vee \ldots \vee s_k \mid \gamma(s_i, \pi) = t,\ i = 1, \ldots, k, \text{ provided the join exists }\}.\end{aligned}$$

The t-Condorcet axiom for a consensus function c on the semilattice V is phrased as follows.

(order) t-Condorcet :
if s is join-irreducible in $(V, \leq)$ covering w_s and $\gamma(s, \pi) = t$, then $x \vee s$ is in $c(\pi)$ if and only if $x \vee w_s$ is in $c(\pi)$, provided $x \vee s$ exists.

In [12] the following result was proved.

Theorem H *Let $(V, \leq)$ be a distributive meet semilattice in which all join-irreducibles are atoms, and let t be a rational number with $\frac{1}{2} \leq t < 1$. Let c be a consensus function on $(V, \leq)$. Then $c = m_t$ if and only if c satisfies F, C, and t-Condorcet.*

We are now able to improve Theorem H with the following result, which was proved in [9] for the special case of the median function. (i.e., when $t = \frac{1}{2}$.)

Theorem 5 *Let $(V, \leq)$ be a median semilattice , and let t be a rational number with $\frac{1}{2} \leq t < 1$. Let c be a consensus function on $(V, \leq)$. Then $c = m_t$ if and only if c satisfies F, C, and t-Condorcet.*

Proof. If $c = m_t$, then it is clear that c satisfies faithfulness. It follows from Lemma 25 in [3] that m_t is consistent. To show that m_t satisfies t-Condorcet, let π be a profile and s a join-irreducible covering w_s. Assume $\gamma(s, \pi) = t$. If j is a join-irreducible and $x \vee s$ exists, then

$$j \leq x \vee s\ \Leftrightarrow j \leq x \vee w_s \text{ or } j = s.$$

Since an element y belongs to $m_t(\pi)$ if and only if $\alpha_t(\pi) \leq y$ and $j \not\leq y$ for all join-irreducibles j such that $\gamma(j, \pi) < t$ it follows that

$$x \vee s \in m_t(\pi)\ \Leftrightarrow\ x \vee w_s \in m_t(\pi).$$

For the converse we need the following fact. For any nonzero element x in $(V, \leq)$, there exist join-irreducibles $s_1, \ldots, s_r$ such that $x = s_1 \vee \ldots \vee s_r$ and $s_i \leq s_j$ if and only if $i = j$. Therefore, for any z strictly less than x, there exists a join-irreducible s_i such that $s_i \leq x$, $s_i \not\leq z$, and $s_i \not\leq a$ where $a = \bigvee\{s \in S | s \leq x \text{ and } s \neq s_i\}$. The expression $s_1 \vee \ldots \vee s_r$ is called an *irredundant join* and so x can be represented as an irredundant join of join-irreducibles.

Assume c satisfies faithfulness, consistency and t-Condorcet. Let $\pi = x_1, \ldots, x_k$ be a profile. If $k = 1$, then, by unanimity, $c(\pi) = m_t(\pi) = \{x_1\}$. So we may assume $k \geq 2$. Now $t = m/n$ for some positive integers m and n such that $m < n$.

Claim 1: For any $x \in c(\pi)$ and for any $s \in S$, if $s \leq x$, then $\gamma(s, \pi) \geq t$.

Proof of Claim 1. Assume that there exists $x \in c(\pi)$ and a join-irreducible s' such that $s' \leq x$ and $\gamma(s', \pi) < t$. Since x can be represented as an irredundant join of join-irreducibles, there exists a join-irreducible j such that $s' \leq j \leq x$ and $j \not\leq a$ where $a = \bigvee\{s \in S | s \leq x \text{ and } s \neq j\}$. So $a < x$ and $x = a \vee j$. Let w_j be the element covered by j, then $a \vee j$ covers $a \vee w_j$. Note that $\gamma(j, \pi) \leq \gamma(s', \pi) < t$. So $\gamma(j, \pi) = \frac{u}{k} < \frac{m}{n}$ for some integer u such that $0 \leq u < k$. Consider the profile

$$\pi^* = \pi^{n-m}, x^{(km-nu)} \in V^{kn-nu}$$

consisting of $(n - m)$ copies of π followed by $(km - nu)$ copies of the profile x. It follows from unanimity and consistency that $c(\pi^*) = \{x\}$. It can be verified that $\gamma(j, \pi^*) = t$. Therefore, by t-Condorcet, $x = a \vee j \in c(\pi^*)$ implies that $a \vee w_j \in c(\pi^*)$ contrary to $c(\pi^*) = \{x\}$. This completes the proof of Claim 1.

Claim 2: For any $x \in c(\pi)$, the element $z = x \wedge \alpha_t(\pi)$ belongs to $c(\pi)$.

Proof of Claim 2. Assume that there exists $x \in c(\pi)$ such that $z = x \wedge \alpha_t(\pi) \notin c(\pi)$. Then $z < x$. Choose $y \in V$ such that $y \in c(\pi)$, $z < y < x$, and there does not exist $y' \in c(\pi)$ such that $z < y' < y$. Since y can be represented as an irredundant join of join-irreducibles, there exists a join-irreducible j such that $j \leq y$, $j \not\leq z$, and $j \not\leq a$ where $a = \bigvee\{s \in S | s \leq y \text{ and } s \neq j\}$. So $y = a \vee j$, $z \leq a$, and $a \vee j$ covers $a \vee w_j$ where w_j is the unique element covered by j. Since $j \leq x$ and $x \in c(\pi)$ it follows from Claim 1 that $\gamma(j, \pi) \geq t$. Since $j \not\leq z$ and $j \leq x$ it follows that $j \not\leq \alpha_t(\pi)$ and so $\gamma(j, \pi) = t$. By t-Condorcet, $y = a \vee j \in c(\pi)$ implies that $a \vee w_j \in c(\pi)$. This contradicts our choice of y since $z < a \vee w_j < y$. This completes the proof of Claim 2.

Claim 3: For any $x \in c(\pi)$ and for any $s \in S$ such that $\gamma(s, \pi) > t$, we have $s \leq x$.

Proof of Claim 3. Assume that there exist $x \in c(\pi)$ and $s' \in S$ such that $s' \not\leq x$ and $\gamma(s', \pi) > t$. By Claim 2, the element $z = x \wedge \alpha_t(\pi)$ belongs to $c(\pi)$. Since $s' \not\leq z$, there exists $j \in S$ such that j covers w_j, $j \leq s'$, $j \not\leq z$, and $w_j \leq z$. Observe that $\gamma(j, \pi) \geq \gamma(s', \pi) > t$ and so $j \leq \alpha_t(\pi)$. Since $z \leq \alpha_t(\pi)$ it follows that $z \vee j$ exists. Moreover, $z \vee j$ covers $z \vee w_j = z$. Now $\gamma(j, \pi) = \frac{u}{k}$ for some integer u such that $0 < u \leq k$ and $\frac{u}{k} > \frac{m}{n}$. Consider the profile

$$\pi^* = \pi^m, z^{(nu-mk)} \in V^{nu}$$

consisting of m copies of π followed by $(nu - mk)$ copies of the profile z. It follows from unanimity and consistency that $c(\pi^*) = \{z\}$. It can be verified that $\gamma(j, \pi^*) = t$. Therefore, by t-Condorcet, $z = z \vee w_j \in c(\pi^*)$ implies that $z \vee j \in c(\pi^*)$ contrary to $c(\pi^*) = \{z\}$. This completes the proof of Claim 3.

Claim 4: For any $x \in c(\pi)$ and for any $s \in S$, if $x \vee s$ exists and $\gamma(s, \pi) = t$, then $x \vee s \in c(\pi)$.

Proof of Claim 4. Assume $x \vee s \notin c(\pi)$ for some $x \in c(\pi)$ and $s \in S$ such that $\gamma(s, \pi) = t$. Choose $y \in c(\pi)$ such that $x \leq y < x \vee s$ and there does not exist $y' \in c(\pi)$ such that $y < y' < x \vee s$. There exists $j \in S$ such that $j \leq s$, $j \not\leq y$, and $w_j \leq y$ where w_j is the unique element in V covered by j. So $\gamma(j, \pi) \geq \gamma(s, \pi) = t$. On the other hand, $j \not\leq x$ and $x \in c(\pi)$ implies that $\gamma(j, \pi) \leq t$ by Claim 3. So $\gamma(j, \pi) = t$. Since $y \vee w_j = y \in c(\pi)$ it follows from t-Condorcet that $y \vee j \in c(\pi)$. Since $y < y \vee j < x \vee s$ we get a contradiction to the choice of y. This completes the proof of Claim 4.

It follows from Claims 1 and 3 that $c(\pi) \subseteq m_t(\pi)$. It follows from Claims 2 and 3 that $\alpha_t(\pi) \in c(\pi)$. Finally, by Claim 4, $m_t(\pi) \subseteq c(\pi)$ and the proof is complete. $\square$

4 Impossibility result

Does the t-median function on a graph, M_t with $t > \frac{1}{2}$, behave like the median function $M = M_{\frac{1}{2}}$? Apparently not, as is shown by the following impossibility result which shows when $t > \frac{1}{2}$ that there is **no** function satisying all the axioms that characterize M. Let $G = (V, E)$ be a median graph and a any vertex of G. Denote the t-median function of the median semilattice $(V, \leq_a)$ by m_t^a.

Theorem 6 *Let $G = (V, E)$ be a median graph with $|V| \geq 3$, and let t be a rational number with $\frac{1}{2} < t < 1$. Then there does not exist a consensus function $c : V^* \to 2^V - \{\emptyset\}$ on G satisfying (F), (C), and t-Condorcet.*

Proof. Assume to the contrary that such an c exists. First we prove that, for any vertex a in G, the function c is a consensus function on the median semilattice $(V, \leq_a)$ satisfying (F), (C), and order t-Condorcet.

Claim: c satisfies order t-Condorcet.
Proof of Claim. Take any profile π on V. Let s be a join-irreducible element with $\gamma(s, \pi) = t$, and let w_s be the element covered by s. Choose any element x in V.

First suppose that $x \geq_a s$. Then we have $x \vee s = x \vee w_s = x$. So we have $x \vee s \in c(\pi)$ if and only if $x \vee w_s \in c(\pi)$.

Next suppose that $x \not\geq_a s$. Then $x \vee s$ covers $x \vee w_s$, by the upper semi-modularity of a median semilattice. So sw_s is an edge in G. Let G_1, G_2 be the split in G of this edge with x in G_1. Let W_i be the vertex set of G_i, for $i = 1, 2$. Then we have

$$W_1 = \{ z \in V \mid z \geq_a s \},$$

and

$$W_2 = \{ z \in V \mid z \not\geq_a s \}.$$

Since $\gamma(s,\pi) = t$, we have $|\pi_1| = t|\pi|$. Now let $v_1 = x \vee s$, and $v_2 = x \vee w_s$. Then v_1 lies in G_1 and v_2 lies in G_2, so $v_1 v_2$ is an edge in F_{12}. Hence, c being t-Condorcet on G, we have

$$v_1 \in c(\pi) \text{ if and only if } v_2 \in c(\pi).$$

This implies that

$$x \vee s \in c(\pi) \text{ if and only if } x \vee w_s \in c(\pi).$$

Thus we may conclude that c is order t-Condorcet, that is, $c = m_t^a$ on the median semilattice $(V, \leq_a)$, for any a in V.

Since $|V| \geq 3$, we can find three vertices p, q, r in G such that pqr is an induced path of length 2 in G. Consider the profile $\pi = (p, r)$. First we take $(V, \leq_p)$. Then q covers p, and r covers q in $(V, \leq_p)$. Now we have $c(\pi) = m_t^p(\pi) = \{p\}$. Second take $(V, \leq_q)$. Now we have $c(\pi) = m_t^q(\pi) = \{q\}$. But this is impossible. This settles the impossibility of the existence of a consensus function c on G that satisfies (F), (C), as well as t-Condorcet. □

5 The consistency of M_t

On partial cubes, M_t satisfies (F) by Lemma 3, and is t-Condorcet by Lemma 4, so the impossibility result of the previous section tells us that consistency is not satisfied by the consensus function M_t. Thus a natural question is whether M_t satisfies any of the weaker consistency conditions.

Theorem 7 *Let $G = (V, E)$ be a connected graph. Then M_t satisfies subconsistency and subquasi-consistency on G.*

Proof. Let π and ρ be profiles on G. Let G_1, G_2 be a t-distinguishing split for the profile $\pi\rho$. Then we have

$$|\pi_1| + |\rho_1| = |(\pi\rho)_1| > t|(\pi\rho)| = t|\pi| + t|\rho|.$$

So we must have $|\pi_1| > t|\pi|$ and/or $|\rho_1| > t|\rho|$. Hence G_1, G_2 is t-distinguishing for π or ρ (or both). This implies that, if y is not in $M_t(\pi\rho)$, then y is not in $M_t(\pi)$ or not in $M_t(\rho)$, whence y is not in $M_t(\pi) \cap M_t(\rho)$. This settles that M_t is subconscious.

For subquasi-consistency, let π and ρ be profiles with $M_t(\pi) = M_t(\rho)$. As above, if a split G_1, G_2 is t-distinguishing for $\pi\rho$, then it is t-distinguishing for π and/or for ρ. So, if y is not in $M_t(\pi\rho)$, then y is not in $M_t(\pi) = M_t(\rho)$, by which we have the subquasi-consistency of M_t. □

In the case of quasi-consistency, we have the following problem. Let π and ρ be profiles with $M_t(\pi) = M_t(\rho)$. Then, unfortunately, we are not sure whether the same spits are involved in making the intersection for $M_t(\pi)$ as well as for $M_t(\rho)$. So a split G_1, G_2 might be t-distinguishing for π but not for ρ, and vice versa, whereas we still have $M_t(\pi) = M_t(\rho)$. But for a partial cube we have quasi-consistency because now every edge defines a split.

Lemma 8 *Let G be a partial cube. Then M_t on G is quasi-consistent.*

Proof. Since G is a partial cube, every edge in G defines a split. This has the following consequence. Let uv be any edge, and let π be any profile. Then we have one end of uv in $M_t(\pi)$ and the other end not if and only if the split of uv is t-distinguishing. Now let π and ρ be two profiles with $M_t(\pi) = M_t(\rho)$. Let y be any vertex not in $M_t(\pi) = M_t(\rho)$. Take any geodesic from y to a vertex u in $M_t(\pi)$ closest to y, and let v be the vertex on this geodesic right before u. Then v is not in $M_t(\pi) = M_t(\rho)$. So G_u^{uv}, G_v^{uv} is t-distinguishing with respect to π as well as ρ, whence also with respect to $\pi\rho$. By the definition of splits y is in G_v^{uv}. So y is not in $M_t(\pi\rho)$. Together with subquasi-consistency we conclude the quasi-consistency of M_t. $\square$

Lemmas 3, 4, and 8 provide us with the following Theorem.

Theorem 9 *Let G be a partial cube. Then M_t is faithful, quasi-consistent and t-Condorcet on G.*

The converse of this theorem is not true as is shown by the following example. Define $c: V^* \to 2^V - \{\emptyset\}$ on a partial cube G by

$$c(\pi) = \begin{cases} M_t(\pi) & \text{if } |M_t(\pi)| = 1, \\ V & \text{otherwise.} \end{cases}$$

with $\frac{1}{2} < t < 1$ and t small enough, since $c = M_{t'}$ for t' close enough to 1. This consensus function trivially is faithful, quasi-consistent and t-Condorcet. But, also trivially, it is not M_t on G as soon as G is not just a K_2.

Finally, we consider the other axioms subfaithful and subunanimous and prove the following easy result.

Proposition 10 *Let G be a connected graph. Then M_t is subfaithful and subunanimous.*

Proof. Let π be the profile consisting of a repetition of x of length k, with $k \geq 1$. Consider any split G_1, G_2. Then this split is t-distinguishing for π ($k \geq 1$) if and only if x lies in G_1. So $x \in M_t(\pi)$. $\square$

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