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# ADOPTION OF SUPERIOR TECHNOLOGY IN MARKETS WITH HETEROGENEOUS NETWORK EXTERNALITIES AND PRICE COMPETITION

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**Abstract.** In this paper we investigate whether markets with heterogeneous network externalities can be locked-in by old technologies even if superior technologies are available. Heterogeneous network externalities are present when some consumers care more about the size of the market share of a good than others. Interestingly, the answer depends on the quality difference between the old and the new technology and on whether firms compete in prices. Without price competition, a partial lock-in occurs if (and only if) the quality difference is small. In the presence of price competition, lock-in in the traditional sense completely disappears, although the old technology may keep some market share in some periods as the new technology is priced higher in equilibrium.

JEL codes: L1, L13, D43

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## 1. Introduction

The value individuals attach to consuming many technological products (like telephones, software and hardware) or products that require maintenance depends on how many others are using these goods. This phenomenon is known as network externality. In the literature it is well-known that network externalities may create barriers to entry, preventing adoption of new goods, possibly of a higher quality. This can lead to the society being "locked-in" with an inefficient technology. A classical – although according to some (e.g. Liebowitz and Margolis, 1990), mistaken – example is the QWERTY standard commonly used in type - writers and computer keyboards (see David, 1986). Another example are nuclear power reactors in Europe – the dominant technology is light water, although many scientists consider it to be inferior to heavy water of gas graphite technology (Cowan, 1990).

In many situations, the relative importance of network externalities for the individual adoption decision will differ between different consumers. An important reason for this differentiation is that people use the same technology in a variety of ways, and some require more coordination than others. In the QWERTY example, a large company with many typists and a high rotation of personnel will care more about the network externality than a free–lance journalist, who uses her keyboard herself and for whom typing speed is important. In the existing literature, however, consumer heterogeneity with respect to network externalities is usually not modelled. Typically, in models with horizontally non-differentiated good, consumers are either homogenous (as, e.g., in Katz and Shapiro, 1992), or they have identical preferences with respect to network externalities (as, e.g., in Katz and Shapiro 1985).

In this paper, we want to study the classical question concerning the possibility of "lock-in" in markets where the consumers' valuation of network externalities is heterogeneous across the population.[1] Specific questions that we ask are: is it possible that the new technology is not adopted by anyone, despite its higher quality? If it attracts some users, under what conditions will the new technology take over the market? Does there exist an equilibrium with both technologies present? We also investigate the market structure of the market when the technology is sponsored by two strategically acting sellers, who choose prices to maximize long - run profits.

To this end, we study a model with two products, an old, inferior good and a new, superior good. We assume that quality can be objectively measured and that consumers differ in their relative valuation of quality vis-à-vis the network externality. This means that if A is of a higher quality than B, then everyone regards A to be better than B, but for some people the quality difference is relatively more important than for others. Consumers decide in every period which good to buy solely on the basis of the present (net) utility.[2] We study the questions outlined above in two different environments. First, in Section 2, we address the pure demand side effect by considering technology adoption in a world where firms are passive. Second, in Section 3, firms play a game with infinitely many periods and simultaneously choose prices in every period to maximize long-run profits. In every period market shares adjust to their equilibrium values given the prices that are chosen in that period and the market share at the beginning of that period. Since prices influence future profits only through their impact on market share, we can use the concept of Markov equilibrium. Markov strategies specify optimal actions for each value of a state variable. In our case, they specify optimal prices for each value of the market share at the beginning of a period.

Our basic results are as follows. When firms are passive (technologies are not sponsored) two equilibrium market shares may emerge: if the difference in qualities is larger than a certain threshold value, the new technology will be the unique technology in the market. If the quality advantage is lower than this threshold value, the two technologies co-exist in the market and the entrant will have the smallest market share. It is easy to see why the new technology has to have the whole market if it is to be the dominant technology: if both quality and market share of one technology are higher, all consumers derive more utility from this product than from the other and hence, will switch to this technology. The possible emergence of two equilibrium market shares and the discontinuous jump of market shares at the threshold value are the main results of this section.

Section 3 examines whether these results continue to hold when technologies are provided by a price-setting incumbent and entrant. Like in the basic model the market share of the entrant positively depends on his quality advantage. However, some qualitatively new phenomena may arise. When the quality difference is large, the entrant will initially set a low price and take over the market. In the following periods, he will set the highest price that will ensure that the incumbent will stay out of the market forever.[3] When the quality difference takes intermediate values, the entrant will still take over the market in the first period, but then he will not try to keep the whole market, since that would require setting an unprofitably low price. Therefore, both technologies coexist, and the new technology dominates.

When the quality advantage is low, a pure strategy Markov equilibrium does not exist. It may still be optimal for the entrant to take over the market, but then it will not be optimal to set a price that would allow him to keep a large market share forever. Similarly, the

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[1] de Palma and Leruth (1996) also model heterogeneous network externalities, but as quality differences don't play a role in their analysis, they cannot address the issues we are interested in.

[2] Hence, in our model the consumers are not "locked –in" by their past purchases directly, but instead by the present choice of other consumers. This is in contrast to some of the existing literature (see for instance Katz and Shapiro 1986, 1992).

incumbent will not keep a large market share if he has one. Only mixed strategy equilibria exist, in which sometimes the incumbent, and sometimes the entrant, will have a large market share.

Interestingly, with price competition the traditional lock-in result—namely that an inefficient technology will continue to dominate the market despite the existence of a superior one—disappears as an equilibrium phenomenon. The entrant will always, no matter what the quality difference—have a large market share at some point, but he may not find it optimal to keep it, especially when the quality difference is low. This means that the old technology may re-appear in the market and be bought by those consumers who consider the quality difference to be too small to warrant the price difference.

We also analyze welfare properties of the different equilibrium configurations. Social welfare is maximized when the new technology has the whole market. For high values of quality difference this is the outcome both with and without price competition. When quality difference takes on intermediate values, some of the market is given away to the old technology in the presence of price competition, which implies a lower welfare. For low values of $q$ the welfare implications are not clear.

The rest of the paper is organized as follows. Section 2 considers the pure demand effects due to heterogeneous network externalities. Section 3 introduces price competition and section 4 examines the welfare implications. Section 5 concludes. Most proofs are given in the appendix.

## 2. A demand-driven model

In this section we describe the demand side in detail and show whether and if so to what extent a new technology will be adopted in markets with heterogeneous network externalities. We explicitly show how equilibrium market shares depend on the quality of both technologies.

As explained in the introduction, consumers care about quality and about network externalities, but the relative importance of these factors in the individual adoption decision varies across consumers. Let $u_\theta^1(t)$ and $u_\theta^0(t)$ denote the utilities that consumer $\theta$ derives from consuming the new and the old technology, respectively, in period $t$. These utilities are given by

$$u_\theta^1(t) = (1-\theta)q_1 + \theta x_t ,$$

[3] For other models of strategic pricing in markets with network externalities, see, for example, Katz and Shapiro (1992), Bental and Spiegel (1995) and, more recently, Cabral et al. (1999).

$$u_\theta^0(t) = (1-\theta)q_0 + \theta(1-x_t),$$

where $q_1 > q_0$ are qualities of new and old technologies, respectively, $x_t \in \langle 0,1 \rangle$ is the market share of the new technology in period *t*, $1-x_t$ is the market share of the old technology, and $\theta$ is the relative weight that consumer $\theta$ places on the popularity, compared to quality. We assume that $\theta$ is uniformly distributed on the interval [0, 1].

At the moment the new technology enters the market, the whole market belongs to the old technology and consumers decide which technology to choose. We define *an equilibrium* to be a situation in which every consumer makes her preferred choice given the choice of the other consumers, i.e., no consumer would individually like to switch to another technology. For certain parameter values, multiple equilibrium market shares are possible. We define a *stable equilibrium given an initial markets share of* $x_{t-1}$ as an equilibrium that would result as the outcome of a (loosely formulated) dynamic process according to which consumers who are not satisfied with their choice given $x_{t-1}$ switch. The final outcome of the adjustment process depends on the quality advantage of the new technology, $q = q_1 - q_0 \geq 0$, and on the initial market share $x_{t-1}$. For simplicity, we assume that this consumer adjustment process takes place instantaneously. Figure 2.1. below illustrates the process for low values of *q*.

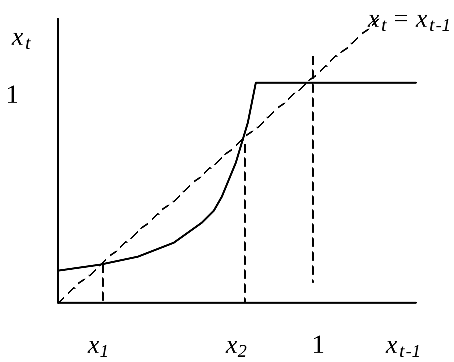


Figure 2.1.

There are three equilibria here, out of which $x_t = x_1$ and $x_t = 1$ are stable. If *q* is low and $x_{t-1} < x_2$, the process will result in an equilibrium market share $x_t = x_1$. Otherwise, $x_t = 1$. When $q$ is large, $x_1$ and $x_2$ disappear and the only equilibrium is $x_t = 1$. Since the new technology has initially no market share, in our case $x_{t-1} = x_0 = 0 < x_2$.

Proposition 1 describes equilibrium market shares of the new technology.

**Proposition 1** *The stable equilibrium market shares given* $x_{t-1}$*=0 are*

$$x_t = 1 \qquad \textit{if} \qquad q \geq 3 - 2\sqrt{2}$$

$$x_t = \frac{q + 1 - \sqrt{(q+1)^2 - 8q}}{4} \qquad \textit{if} \qquad q < 3 - 2\sqrt{2}.$$

*Proof.* In order to have an equilibrium coexistence of technologies, it must be true that there exist a $0 < q^* < 1$ such that

$$q^*(1 - 2x_t + q) = q \tag{1}$$

and all consumers with $q < q^*$ (and only those) choose the new technology. Since *è* is uniformly distributed on [0,1], $x_t$ must be equal to the $q$ of the indifferent consumer. Substituting for *è* in (1) we get a quadratic equation which we can solve for $x_t$.

$$-2x_t^{\ 2} + (q+1)x_t - q = 0 \tag{2}$$

Basically, two possibilities emerge:

*a*) If $(q+1)^2 - 8q \leq 0$, or $q \geq 3 - 2\sqrt{2}$, (2) has no stable solution. In this case, the only equilibrium is $x_t$ = 1.

*b*) If $(q+1)^2 - 8q > 0$, or $q < 3 - 2\sqrt{2}$, (2) has two solutions:

$$x_1 = \frac{q + 1 - \sqrt{(q+1)^2 - 8q}}{4} \text{ and } x_2 = \frac{q + 1 + \sqrt{(q+1)^2 - 8q}}{4},$$

where $x_t = x_2$ is an unstable equilibrium. If initially, $x_{t-1} < x_2$, the equilibrium that emerges is $x_t = x_1$. ///

Figure 2.2. shows the equilibrium market share of the new technology as a function of its quality advantage.

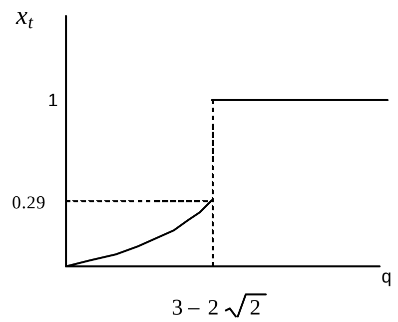


Figure 2.2

The equilibrium market share of the new technology will be either 1, or less then 0.29. Which value of the equilibrium market share will emerge depends on the quality difference.

A larger quality advantage leads to a larger market share, which is quite intuitive. However, the market share does not increase continuously with the quality difference: there is a critical value, $\overline{q} = 3 - 2\sqrt{2}$, such that if $q < \overline{q}$ the equilibrium market share is less than 0.29, and if $q > \overline{q}$, the new technology gains the whole market. This is a consequence of network externalities, which cause a critical mass effect: once the new technology has a sufficiently large market, it will be preferred by all types of consumers, not only those with a taste for quality, but also those with a preference for popularity.

Moreover, as $q > 0$ we see that the better technology always has some market share. When the quality advantage is small, the market will still be dominated by the old technology. If the quality advantage is large enough, however, there will be no technological lock-in: the new technology will drive out the old one.

### 3. Price competition

In this section we analyze the case where each technology is put on the market by one seller. Initially, there is only one seller in the market, the incumbent, who provides the old technology. Our analysis begins in the period in which an entrant appears, offering for sale a higher quality technology. This implies that $x_0 = 0$. A consumer $\theta$ will choose the good that will give him or her higher net utility, in period *t*, given by

$$u_\theta^1(t) - p_1^t = (1-\theta)q_1 + \theta x_t - p_1^t ,$$

$$u_\theta^0(t) - p_0^t = (1-\theta)q_0 + \theta(1-x_t) - p_0^t ,$$

where prices of the entrant and the incumbent are denoted by $p_1^t$ respectively $p_0^t$.

In every period sellers set a price in order to maximize their total discounted profits, given the pricing strategy of the competitor. That is, a seller *i* maximizes

$$\Pi_i = \sum_{t=1}^{\infty} \delta^{t-1} \Pi_i^t ,$$

where $\delta$ is a discount factor and $\Pi_i^t$ is profit in period *t*. Assuming that there are no costs of producing technology, per-period profits are

$$\Pi_1^t = p_1^t x_t \qquad \text{for the entrant,}$$

$$\Pi_0^t = p_0^t (1-x_t) \qquad \text{for the incumbent.}$$

Our notation equates demand to a market share, which implies that the market is covered. This is obviously true if prices are zero, since in that case everyone derives at least nonnegative utility from any technology. However, if prices are positive, it may happen that for some consumers prices exceed the gross utility from using a technology. It may also happen that sellers will find it optimal to serve only a part of the market. Since our interest lies in the structure of the market, that is its division between consumers of different technologies, and not in its size, we want to keep the total amount of consuming individuals constant. Hence, to make sure that the market will be covered in every period, we put restrictions on the qualities of both technologies. We assume that they are both contained in the interval [0.5, 3]. We show in the proofs of subsequent propositions that when this restriction is satisfied, the market is always covered in equilibrium.

To understand why this restrictions is needed, consider a seller who does not face any competition. Note that consumers who derive all utility from network externalities cannot value any technology at more than one. On the other hand, consumers who put all weight on quality value any technology only by its quality. If quality is very low or very large, the valuations of these two extreme types of consumers differ significantly. Then, it is optimal to sell only to consumers with highest valuation. Hence, the restriction on qualities makes sure that valuations of different consumers are not too far apart, and that it is not optimal to leave part of the market uncovered in exchange for a higher price.

We begin our analysis with obtaining per-period demand functions in subsection 3.1. We assume now that the market is covered, that is that seller can only loose some of the market to the other seller. This implies that having derived the demand for the new technology, we immediately know the demand for the old technology. In subsection 3.2 we describe equilibrium pricing strategies of sellers.

### 3.1. *Entrant's demand function*

Here, we show how the market share, or demand, of the entrant depends on prices and initial market share. To calculate demand, we use the same definitions about stable equilibrium market shares as in Section 2. Since $0.5 \le q_1, q_0 \le 3$, and since by assumption the new technology is of higher quality, we can restrict attention to $0 \le q \le 3$.

The following lemma characterizes the demand function of the entrant.

**Lemma 1** *Suppose that* $0 \le q \le 3$. *The demand function of the entrant is*

*(i)* *If* $p_1^t \le p_0^t + q$, *then*

- $$x_t = \frac{q+1-\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{4}$$

  $$\text{if } x_{t-1} < \frac{q+1}{4} \quad \text{and} \quad p_1^t > p_0^t + q - \frac{(q+1)^2}{8}, \text{ or}$$

*if* $\dfrac{q+1}{4} \le x_{t-1} < \dfrac{q+1}{2}$ *and* $p_1^t > p_0^t + 2x_{t-1}^2 - (q+1)x_{t-1} + q$

- $x_t = 1$ *otherwise.*

*(ii)* *If* $p_1^t \ge p_0^t + q$, *then*

- $x_t = \dfrac{q+3+\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}}{4}$

  *if* $q < 1,\ x_{t-1} \ge \dfrac{q+3}{4}$ *and* $p_1^t \le p_0^t + \dfrac{(q+3)^2}{8} - 1$ , *or*

  *if* $\dfrac{q+1}{2} \le x_{t-1} < \dfrac{q+3}{4}$ *and* $p_1^t \le p_0^t - 2x_{t-1}^2 + (q+3)x_{t-1} - 1$

- $x_t = 0$ *otherwise.*

*Proof*. The proof is analogous to the proof of Proposition 1 and can be found in the appendix. ///

Figure 3.1 illustrates lemma 1. It shows the market share of the entrant as a function of the price of his technology.

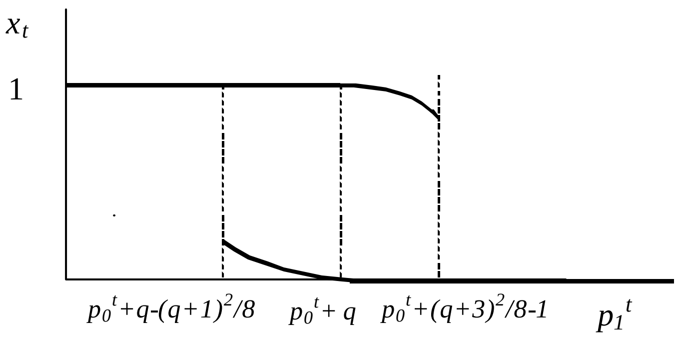


Figure 3.1

When $p_1^t$ is relatively small, the only equilibrium market share of the entrant is $x_t = 1$. When $p_1^t$ is relatively large, the only equilibrium is $x_t = 0$. At intermediate values of $p_1^t$ two demand values are possible depending on the initial market share. Note that there is no $p_1^t$ such that both solutions are interior, that is both demand values are between 0 and 1: if $p_1^t < p_0^t + q$, then $x_t < (q+1)/4$ or $x_t = 1$, and if $p_1^t > p_0^t + q$, $x_t > (q+3)/4$ or $x_t = 0$.

From figure 3.1 we can obtain the demand function for a given $x_{t-1}$. It consists of the upper curve up to a point $\tilde{p}_1$ and the lower curve to the right of this point. The value of $\tilde{p}_1$ at which demand is discontinuous depends positively on the initial market share and is given by:

$$\tilde{p}_1 = \begin{cases} p_0^t + q - \dfrac{(q+1)^2}{8} & \text{if} \quad x_{t-1} < \dfrac{q+1}{4} \\ p_0^t + 2x_{t-1}^2 - (q+1)x_{t-1} + q & \text{if} \quad \dfrac{q+1}{4} < x_{t-1} < \dfrac{q+1}{2} \\ p_0^t - 2x_{t-1}^2 + (q+3)x_{t-1} - 1 & \text{if} \quad \dfrac{q+1}{2} < x_{t-1} < \dfrac{q+3}{4} \\ p_0^t + \dfrac{(q+3)^2}{8} - 1 & \text{if} \quad \dfrac{q+3}{4} < x_{t-1} \end{cases}$$

Figure 3.2. shows an example of the demand function for a given $x_{t-1}$:

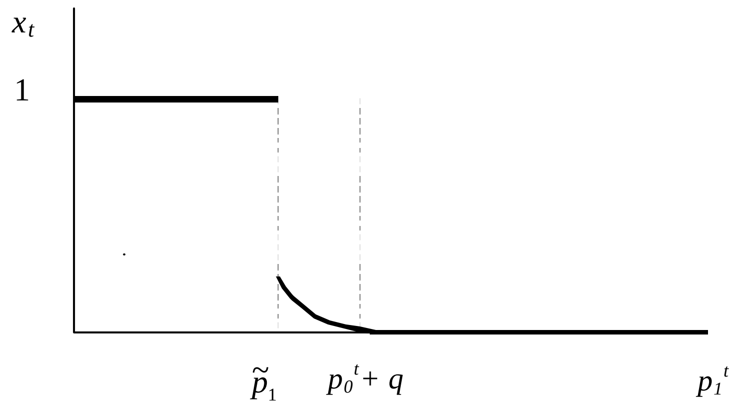


Figure 3.2.

Note that in comparison to Section 2, two additional stable equilibrium market shares are possible: no entry of the new superior good, $x_t = 0$, and a large market share of the entrant that is smaller than 1. Case (*i*) of lemma 1 is a generalization of Section 2: the entrant's market share is either small or equal to 1, depending on the initial market share and prices that are chosen. Both new types of equilibria occur in case (*ii*) of lemma 1 when the price of the new good exceeds its quality advantage, which is obviously not possible when prices are not included in the model and $q > 0$. It is no longer true that a higher quality good cannot have a market share larger than 0.5 and lower than 1. Its price may be so much higher than that of the other good that potential customers will not find it worthwhile to buy it.

Clearly, a large initial market share is an advantage: at given prices in the two – equilibrium range, a larger initial market share results in a larger eventual demand. This is a consequence of network externalities: a large market share increases the value of the good to consumers, who are therefore willing to pay a higher price.

3.2. *Markov Equilibrium prices and outcomes.*

In this subsection we describe equilibrium pricing strategies for both sellers. As firms set prices to maximize total profits, $\Pi_i = \sum_{t=1}^{\infty} d^{t-1} \Pi_i^t$, and as (from lemma 1) profits in period $t$ are equal to $\Pi_1^t = p_1^t x_t(p_1^t, p_0^t, x_{t-1})$ and $\Pi_0^t = p_0^t (1 - x_t(p_1^t, p_0^t, x_{t-1}))$, the impact of today's prices on future pay-offs is only through the impact on today's market share. Combined with the fact that the number of periods is infinite, this allows us to use the notions of Markov strategies and Markov equilibrium. In our context, a Markov strategy specifies, for each seller, a pricing strategy such that price in any period depends only on the market share at the beginning of that period in a way that is independent of time.

Propositions 2 to 5 describe Markov equilibrium strategies and outcomes for different values of the quality difference: $q \geq 1$, $0.5 \leq q < 1$, $\hat{q} < q < 0.5$ and $0 \leq q < \hat{q}$, where $\hat{q} \approx 0.3477$. Together these propositions give a full characterization for the case where $0.5 < q_1, q_0 < 3$. Interestingly, the propositions show that lock-in, in the sense of the old technology being able to keep a large market share forever, cannot be an equilibrium phenomenon under price competition for any value of the quality difference. For high levels of $q$ the entrant will be able to take over the whole market forever. For smaller values of $q$, the entrant will take over a large market share after some time, but gives the incumbent at least for some periods at least a small market share back.

Proposition 2 characterizes Markov equilibria for $q \geq 1$. When $q \geq 1$, Lemma 1 tells us that two types of market share can arise in any period: $x_{t-1} < (q+1)/4$ or $x_{t-1} = 1$. Hence, these components of the strategies are most relevant.

**Proposition 2.** *Suppose* $0.5 \leq q_1, q_0 \leq 3$ *and* $q \geq 1$. *Then, the strategies*

$$p_1^t = \begin{cases} q - \dfrac{(q+1)^2}{8} & \text{if} \quad x_{t-1} < 1 \\ 1 & \text{if} \quad x_{t-1} = 1 \end{cases}$$

$$p_0^t = 0 \qquad \textit{for all} \quad x_{t-1}$$

*form a Markov equilibrium. The equilibrium outcome is then* $x_T = 1$ *for all* $T \geq t$. *Moreover, this is the unique equilibrium outcome.*

Proposition 3 describes Markov equilibria for $0.5 \leq q < 1$. When $q < 1$, two types of market share can arise in any period: $x_{t-1} < (q+1)/4$ or $x_{t-1} \geq (q+3)/4$. This will be therefore the initial market share in most periods.

**Proposition 3.** *Suppose* $0.5 \leq q_1, q_0 \leq 3$ *and* $0.5 \leq q < 1$. *Then, the strategies*

$$p_1^t = \begin{cases} q - \dfrac{(q+1)^2}{8} & \text{if } \quad x_{t-1} < (q+3)/4 \\ q & \text{if } \quad x_{t-1} \geq \dfrac{(q+3)}{4} \end{cases}$$

$$p_0^t = 0 \qquad \textit{for each } x_{t-1}$$

*form a Markov equilibrium. The equilibrium outcome is then* $x_T = 1$ *for all* $T \geq t$. *Moreover, this is the unique outcome in all Markov equilibria.*

Propositions 2 and 3 show that when the quality of the new technology is much higher, it is optimal for the entrant to take over the market and then set the maximum price at which he will keep it. The incumbent will be driven completely out of the market already in the first period, and he will not be able to come back. Surprisingly, this result does not depend on the discount factor. With a large quality advantage, the entrant is able to set a relatively high price and still take over the market immediately, hence taking over is optimal even if only the current profits matter. If the entrant cares also about future profits, there is an additional gain from taking the whole market early on and enjoying the benefits of an installed consumer base.

The difference between the two propositions mainly lies in the price the entrant can ask in order to keep the whole market and can be understood by looking at Figure 3.3.

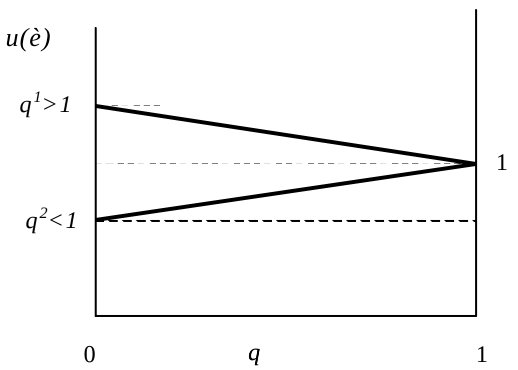


Figure 3.3.

Here, $u(q)$ is defined as the difference in gross utilities that consumer *q* can derive in period *t* from both technologies if $x_t = 1$, that is

$$u(q) = (1-q)q_1 + q - (1-q)q_0 = (1-q)q + q \,.$$

In other words, *u(è)* says how much more consumer *è* is willing to pay for the new technology, in comparison to the old one, if everyone else buys the new technology. The bold lines in the figure represent *u(è)* for two different values of $q$ , $q^1 > 1$ and $q^2 < 1$.

In equilibrium, the entrant cannot ask, from any consumer, a price which exceeds the price of the incumbent by more than the additional gross utility that the new technology offers in comparison to the old one. That is, the difference in prices cannot be larger than the lowest gross utility difference. Hence, $p_1^t - p_0^t \le \min_q u(q)$. If $q \ge 1$, *u(è)* reaches a minimum for consumers with $q = 1$ and it equals 1. Hence, the entrant's price cannot exceed the incumbent's price by more than 1. If $q < 1$, the gross utility difference reaches a minimum for consumers with $q = 0$ and equals $q$. Therefore, the price of the entrant cannot exceed the price of the incumbent by more than $q$.

We will now turn to an analysis of the case where the quality advantage of the entrant is smaller than 0.5 and we will show that in this case both sellers will coexist in the market. Propositions 4 and 5 deal with these cases. As in Proposition 3, $x_{t-1} < (q+1)/4$ and $x_{t-1} > (q+3)/4$ are the most relevant initial market shares. Let

$$\hat{q} = -\frac{7}{2} + \frac{1}{2}\left[\frac{1}{3}\left(41 + 2\sqrt{238}\right)^{1/3} + \frac{3}{\left(41 + 2\sqrt{238}\right)^{1/3}} + \frac{2}{3}\right]^2 \approx 0.3477 \,.$$

**Proposition 4.** *Suppose that* $0.5 \le q_1, q_0 \le 3$ *and* $\hat{q} \le q < 0.5$. *Then, the strategies*

$$p_1^t = \begin{cases} q - \dfrac{(q+1)^2}{8} & , \quad \text{if} \quad x_{t-1} < \dfrac{(q+3)}{4} \\ p_1 = \dfrac{3q^2 + 6q - 17 + (11+q)\sqrt{9q^2 + 38q + 9}}{100} & , \quad \text{if} \quad x_{t-1} \ge \dfrac{(q+3)}{4} \end{cases}$$

*and*

$$p_0^t = \begin{cases} 0 & , \quad \text{if} \quad x_{t-1} < \dfrac{(q+3)}{4} \\ \hat{p}_0 = \dfrac{-3q^2 - 46q - 23 + (9-q)\sqrt{9q^2 + 38q + 9}}{100}, & \quad \text{if} \quad x_{t-1} \geq \dfrac{(q+3)}{4} \end{cases}$$

*form a Markov equilibrium. The equilibrium outcome is then* $x_t = \hat{x}$ *for all* $t > T$ *, where* $\frac{q+3}{4} < \hat{x} < 1$. *Moreover, this is the unique equilibrium outcome.*

When the quality difference takes on intermediate values, as in Proposition 4, the entrant will still take over the market in the first period, but will not keep it. The two technologies will coexist, with more than three quarters of consumers buying the new technology. The reason is that when the quality difference is not so high, the entrant would have to set a low price to keep the whole market. It is optimal to set a somewhat higher price at the expense of giving away part of the market.

For $q$ closer to 0.5, the equilibrium prices are such that the incumbent is not able get back a large market share from the entrant. When $q$ is closer to $\hat{q}$, he is able to do it, but does not find it optimal. He would have to set a price much below the equilibrium price, and a large market share would not compensate for it. Moreover, in the following period the entrant would get a large market share again. This makes it optimal for the incumbent to accept a low market share in exchange for a higher price and the absence of a price war.

Note that in spite of the fact that the entrant has both a higher quality and a higher market share, some consumers buy the old good. This is not possible without price competition. More surprisingly, the few consumers who buy the old, low quality technology are precisely those who care most about quality. To understand this, observe that when $q < 1$, and $\hat{x} > (q+3)/4$, the market share advantage of the entrant is higher than the quality advantage, that is $\hat{x} - (1-\hat{x}) > q$.[4] As a consequence, "popularity loving" consumers are willing to accept a higher price difference (and still buy the new good) than "quality-loving" consumers. The consumers with a high valuation for quality play therefore the role of catalyst. Initially, the entrant sets a low price, and all consumers buy the new technology: the "quality lovers" because its quality is higher, and "popularity lovers", because the "quality lovers" provide benefits in terms of network externalities. After getting a high market share, the entrant sets a high price to exploit his established base. Then, for the "quality lovers" the quality difference does not justify the difference in prices. Therefore, they switch back to consuming the old good. On the other hand, there are now enough "popularity lovers" consuming the new technology, who can provide network benefits for each other. Thus, they are willing to accept a higher price difference.

Finally, we consider the case when $q$ is small.

**Proposition 5.** *If* $0.5 \le q_1, q_0 \le 3$ *and* $0 \le q < \hat{q}$, *there does not exist a pure strategy Markov equilibrium.*

When the quality difference is small, there exists a mixed strategy equilibrium, where both technologies will coexist over time, but neither seller will be able to keep a large market share in the long run. In a mixed strategy equilibrium sometimes the entrant, and sometimes the incumbent will dominate the market.

## 4. Social welfare

We now compare social welfare, measured in terms of social surplus, in situations with and without price competition. The social welfare consists of two components: consumer surplus and sellers' profits. In the case of unsponsored technology prices and profits are zero, so the welfare equals consumer surplus. With price competition, net consumer surplus is equal to gross consumer surplus minus the amount paid to sellers. Since the sellers' revenues are their profits, total social welfare is equal to the gross consumer surplus and depends only on the equilibrium market shares.

In subsections 4.1 and 4.2 we analyze the social welfare in the cases without and without price competition, respectively, and in subsection 4.3 we compare the two.

### 4.1. *Social welfare without price competition*

When everyone consumes the old technology, the total surplus is the area below the line connecting $q_0$ and 1 (see figure 4.1). If everyone consumes the new technology, the total surplus is the area below the line connecting $q_1$ and 1. If the market share of the new technology is $0 < x < 1$, total surplus is given by the area below the bold lines. To see why, note that the line connecting $q_1$ and $x$ shows, for every consumer type, the utility derived from consuming the new technology if its market share is $x$. Similarly, the line connecting

[4] This is easy to see as $\hat{x} - (1 - \hat{x}) = 2\hat{x} - 1 > (q + 1)/2 > q$ .

$q_0$ and $1-x$ shows utility derived from the old technology. We see now that consumers with $q < x$ prefer the new technology, and their total surplus is the area below the left segment of the bold line, while the remaining consumers prefer the old technology, and achieve total surplus equal to the area below the right segment of the bold line.

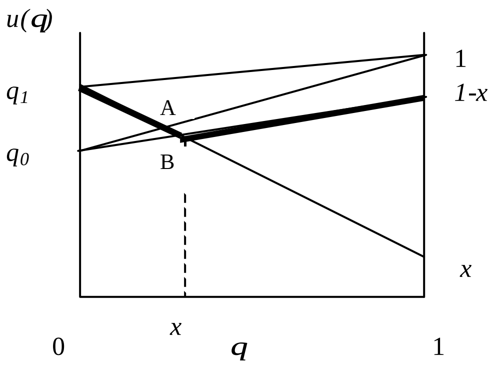


Figure 4.1

Clearly, the social welfare is maximized when everyone uses the new technology. Hence, if in equilibrium the new technology is the only one, society benefits from its introduction. However, when in the equilibrium the new technology has only a small market share, it is not so obvious that welfare has increased. The consumers who use the new technology in the equilibrium derive now more utility from its quality, but they lose a large part of the network externality. The remaining consumers lose a part of network externality and do not gain anything.

The net effect can be found by analyzing figure 4.1. The welfare gain due to higher quality equals to the area of the triangle $q_1q_0A$, and the welfare loss from decreased network externality is the area between 1, *A*, *B* and $1-x$. Hence, the welfare change is

$$S_{q_1q_0A} - S_{1AB(1-x)} = S_{q_1q_0B} - S_{1q_0(1-x)} = 0.5xq - 0.5x = 0.5x(q-1) .$$

Since the new technology can only have a small market share if $q < 3-2\sqrt{2}$, the welfare change is negative. Figure 4.2 shows how social welfare in equilibrium depends on the quality difference, given that the quality of the old technology, $q_0$, is fixed.

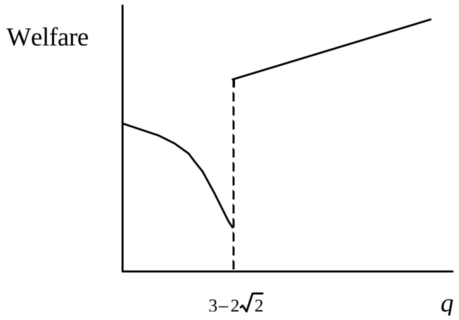


Figure 4.2.

As $q$ increases, social welfare changes discontinuously. Initially it decreases, because the market share of the new technology increases, which leads to decrease of network benefits. When $q$ reaches its critical value, social optimum is achieved in which the new technology gets the whole market.

4.2 *Social welfare with price competition.*

In this case social welfare equals gross consumer surplus, which depends on the equilibrium market shares. We know from section 3 that in the presence of price competition two types of pure strategy equilibrium outcomes can arise: $x = 1$ if $q \geq 0.5$ or $(q+3)/4 < x < 1$ if $0.34 < q < 0.5$. For lower $q$ no pure strategy equilibrium exists, which makes the welfare analysis in that case very difficult.

Figure 4.3. illustrates social welfare in the two possible pure strategy equilibria:

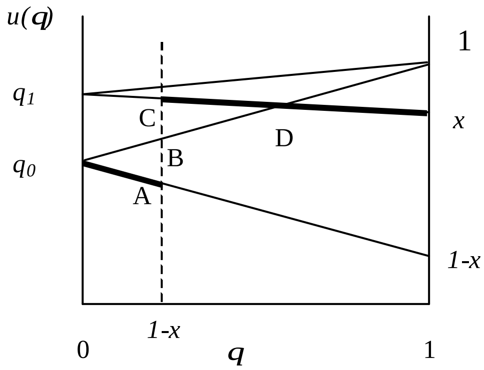


Figure 4.3.

As in figure 4.1., a line connecting a quality value on the left side of the diagram and a market share value on the right side shows, for each consumer type, the utility derived from consuming the technology of that quality and market share. The area below the bold line represents social welfare when $x$ is large. In this case, the new technology is being used by consumers with high $q$, and since its market share is $x$, the indifferent consumer has $q = 1 - x$. This consumer gets the same net utility from both technologies, but since the prices are not equal, gross utilities differ, which causes the jump in the bold line.

Similarly to the previous subsection, total surplus is maximized when $x = 1$. In an equilibrium with $(q+3)/4 < x < 1$ some consumers have a higher, and some a lower gross surplus in comparison with the situation before the introduction of the new technology. Those who in the equilibrium use the old technology lose a large part of the network benefits, equal to the area of the triangle $q_0 BA$. The remaining consumers benefit from higher quality, but lose a part of the network externality. The net change in their gross surplus equals the difference between the areas of triangles CBD and D1$x$. Hence, the total welfare change is equal to

$$S_{CBD} - S_{Dx1} - S_{q_0AB} = S_{q_1q_0(1-x)x} - S_{q_0(1-x)1} - S_{q_1q_0AC} =$$
$$= \frac{q+2x-1}{2} - \frac{x}{2} - \frac{q + xq_1 + (1-x)x - xq_0 - (1-x)^2}{2}(1-x) = -x^3 + 0.5(q+5)x^2 - 1.5x.$$

Since the lowest $x \approx 0.94$,[5] the welfare change is positive.

Figure 4.4. shows how welfare in the equilibrium depends on the quality difference, if $q_0$ is fixed, for $q > 0.34$.

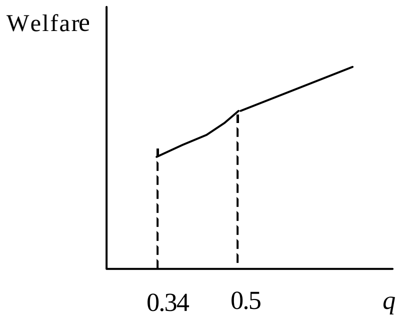


Figure 4.4.

The market share of the new technology increases as the quality difference increases. Since for high $x$ social welfare increases with $x$, it implies that welfare depends positively on $q$. When $q$ reaches 0.5, the new technology has the whole market in the equilibrium, which maximizes social welfare.

4.3. *Welfare comparison*

We can now compare the social welfare that is achieved in equilibrium with and without price competition, given the quality difference.

i) If $q \geq 0.5$, the equilibrium outcome is the same in these two cases. Everyone buys the new technology, and the social welfare is maximized.

ii) If $3 - 2\sqrt{2} < q < 0.5$, the new technology is the only one if there is no price competition, but it has less than the whole market with price competition. Since $x = 1$ is socially optimal, price competition leads to a lower welfare.

iii) If $q < 3 - 2\sqrt{2}$, in the equilibrium without price competition the new technology has a small market share. With price competition, there is no pure strategy equilibrium, and the

[5] This will be the market share of the entrant if $q$=0.34

new technology has sometimes a small, and sometimes a large market share. It is difficult to say in which case the welfare will be higher.

## 5. Conclusions

We have examined the adoption of technology in a market with heterogeneous network externalities where consumers have different relative valuations of both network size and quality. In section 2 we considered the case of unsponsored technology, and in section 3 we analyzed the implication of each of the two technologies being supplied by a strategic seller. We found that the market outcomes depend on the quality advantage of the new technology. In general, the higher the quality difference, the higher the market share of the new technology, both in the case of sponsored technology and in the case of unsponsored technology. However, when firms are passive the results are more clear-cut. There is a critical value such that if the quality difference is larger, the new superior technology will be used by the whole market, and if the quality difference is lower, the old technology remains dominant. Interestingly, with price competition it is not possible that the old technology stays dominant in every period, not even when the quality advantage is zero. That is, the traditional lock-in results does not hold under price competition.

This is not to say that it is always better for society to have price competition. There are values of the quality difference such that without price competition, the whole market will be served by the superior technology, whereas under price competition, the old and the new technology co-exist as the price difference between the two technologies is so large that some consumers prefer to buy the old technology. This means that under price competition social welfare may be lower.

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**Appendix**

*Proof of Lemma 1*. In order to have an equilibrium with coexistence of technologies, there must exist a $0 < \theta^* < 1$ such that

$$\theta^*(2x_t - 1 - q) = p_1^t - p_0^t - q\,. \tag{A.1}$$

There are two cases to be distinguished. If $p_1^t < p_0^t + q$, there may be an interior equilibrium $\theta^*$ such that all consumers with $\theta < \theta^*$ (and only those) consume the new technology. If, on the other hand, $p_1^t > p_0^t + q$, there may be an interior equilibrium $\theta^*$ such that all consumers with $\theta < \theta^*$ (and only those) consume the old technology. We consider the two cases in turn.

*Case I*: $p_1^t \le p_0^t + q$. Since è is uniformly distributed on [0,1], $x_t$ must be equal to the $\theta$ of the indifferent consumer. Substituting for è in (A.1) we get a quadratic equation which we can solve for $x_t$.

$$-2x_t^{\,2} + (q+1)x_t - (q - p_1^t + p_0^t) = 0. \tag{A.2}$$

Basically, two possibilities emerge:

*a*) If $p_1^t \le p_0^t + q - \dfrac{(q+1)^2}{8}$, (A.2) has no stable solution. In this case, the only equilibrium is $x_t = 1$.

*b*) If $p_1^t > p_0^t + q - \dfrac{(q+1)^2}{8}$, (A.2) has two solutions:

$$x_1 = \frac{q + 1 - \sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{4},$$

$$x_2 = \frac{q + 1 + \sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{4}.$$

If initially $x_{t-1} < x_2$, the equilibrium that emerges is $x_t = x_1$; if $x_{t-1} \ge x_2$, $x_t = 1$. The value of $x_2$ depends on $p_1^t$. From the expression for $x_2$ it follows that $x_{t-1} > x_2$, if and only if, $x_{t-1} > (q+1)/4$ and $p_1^t \le p_0^t + 2x_{t-1}^2 - (q+1)x_{t-1} + q$ (when $x_{t-1} > (q+1)/2$, the latter condition follows from $p_1^t \le p_0^t + q$). Otherwise, $x_{t-1} < x_2$.

*Case II:* $p_1^t > p_0^t + q$. Since è is uniformly distributed on [0,1], $x_t$ must be equal to the 1-$\theta$ of the indifferent consumer. Substituting for è in (A.1) we get a quadratic equation which we can solve for $x_t$:

$$-2x_t^{\,2} + (q+3)x_t - (1 + p_1^t - p_0^t) = 0\,. \tag{A.3}$$

Following a similar procedure as in Case I we have the following two possibilities:

*a*) If $p_1^t \geq p_0^t + \frac{(q+3)^2}{8} - 1$, (A.3) has no solution and the solution is unstable. In this case, the only stable equilibrium is $x_t = 0$.

*b*) When $p_1^t < p_0^t + \frac{(q+3)^2}{8} - 1$, two equilibria are possible depending on $x_{t-1}$: if $x_{t-1} < \frac{q+3-\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}}{4}$, which is true when $x_{t-1} < (q+3)/4$ and $p_1^t > p_0^t - 2x_{t-1}^2 + (q+3)x_{t-1} - 1$, then $x_t = 0$. This is always the case when $q \geq 1$.

In the reverse case and if $q < 1$, $x_t = \frac{q+3+\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}}{4} > \frac{q+3}{4}$. ///

*Proof of Proposition 2*. From Lemma 1 we know that the entrant's market share can never be $(q+1)/4 < x_t < 1$, so we do not consider these initial market shares. For the remaining initial market shares we show that it is possible and optimal for the entrant to take over the whole market in the first period and to keep it subsequently, no matter what the price of the incumbent. Hence, this will be the unique equilibrium outcome. From that it will follow that the entrant's strategy described in the proposition is the optimal response to the incumbent's equilibrium strategy. Demonstrating that the incumbent's strategy is an optimal response to the entrant's equilibrium strategy will complete the proof.

First, we prove that it is always possible and optimal for the entrant to take over and keep the whole market. That is, it is neither optimal to give some market share away to the incumbent, nor to leave some part of the market uncovered. We first show that taking over (or keeping) the market maximizes the entrant's current profits, or total profits if $d = 0$, for any $p_0^t$, and then we extend the argument to total discounted profits when $d > 0$.

We consider relevant initial market shares in turn:

*i*) $d = 0$ and $x_{t-1} = 1$. If in equilibrium $x_t = 1$, then no consumer prefers the old good to the new good, that is $p_1^t \leq p_0^t + 1$, and all consumers derive a positive utility from consumption, that is $(1-q)q_1 + qx_t - p_1^t \geq 0$ for all $q$. This requires $p_1^t = 1$, which implies $\Pi_1^t = 1$. We show now that $p_1^t = 1$ is optimal.

If $p_1^t \leq 1$, then $x_t = 1$ and $\Pi_1^t < 1$. If $p_1^t > 1$, only consumers with $q \leq \frac{q_1 - p_1^t}{q_1 - x_t}$ get nonnegative utility. As $x_t$ cannot be larger than the market, $x_t \leq \frac{q_1 - \sqrt{q_1^2 - 4(q_1 - p_1^t)}}{2}$, and profits cannot be larger than $p_1^t \frac{q_1 - \sqrt{q_1^2 - 4(q_1 - p_1^t)}}{2}$. When $p_1^t > 1$, this expression is always smaller than 1, provided that $q_1 \leq 3$.[6] Hence, $p_1^t = 1$ is optimal.

ii) $d = 0$ and $x_{t-1} < (q+1)/4$. To take over the market, the entrant must set $p_1^t \leq p_0^t + q - (q+1)^2/8$, and in addition $p_1^t \leq 1$ so that everyone gets positive utility in the equilibrium. The profit is then $\Pi_1^t = \min\{p_0^t + q - (q+1)^2/8, 1\}$. We show that $p_1^t = \min\{p_0^t + q - (q+1)^2/8, 1\}$ maximizes current profits. If $p_1^t$ is lower than this minimum, the market share is also 1, and the profits are lower, hence this is not optimal. Suppose $p_1^t$ is higher. We consider now two cases:

*a)* $\min\{p_0^t + q - (q+1)^2/8, 1\} = p_0^t + q - (q+1)^2/8$. Then, the profit is at most[7]

$$\Pi_1^t = p_1^t \frac{q+1-\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{2} < \left(p_0^t + q - \frac{(q+1)^2}{8}\right)\left(\frac{q+1}{4}\right),$$ [8]

which is lower than $\Pi_1^t = p_0^t + q - (q+1)^2/8$.

*b)* $\min\{p_0^t + q - (q+1)^2/8, 1\} = 1$. Now, the profit is $\Pi_1^t < p_1^t \frac{q_1 - \sqrt{q_1^2 - 4(q_1 - p_1^t)}}{2}$, which is lower than $\Pi_1^t = 1$.

This shows that when $x_{t-1} < (q+1)/4$, it is optimal to take over the market.

We have shown that when current profits are being maximized, $x_t = 1$ for all relevant $x_{t-1}$. If $x_{t-1} < (q+1)/4$, the entrant's total discounted profits are

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[6] If $0.5 \leq q_1 < 2$, $p_1^t \frac{q_1 - \sqrt{q_1^2 - 4(q_1 - p_1^t)}}{2}$ achieves a maximum, equal to $\left(q_1 - q_1^2/4\right)\left(q_1/2\right) < 1$, at $p_1^t = q_1 - q_1^2/4$. If $2 \leq q_1 \leq 3$, the maximum is achieved at $p_1^t = 1$ and equals 1. For other values of $q_1$ other $p_1$ may be optimal, and the market may be not covered.

[7] This is the share of consumers who prefer the new good to the old good. An additional restriction is that all consumers who buy the new good should get positive utility from it. Hence, the market share may be even smaller.

[8] The per-period profit function is decreasing when $p_0^t > (q+1)^2/6 - q$ (which is always true if $q > 2 - \sqrt{3} \approx 0.26$, since in that case $(q+1)^2/6 - q < 0$), and $p_1^t > p_0^t + q - (q+1)^2/8$. As $p_1^t$ approaches $p_0^t + q - (q+1)^2/8$, the market share converges to $(q+1)/4$.

$\Pi_1 = \min\{ p'_0 + q - (q+1)^2/8, 1\} + d/(1-d)$, where $p'_0$ denotes the equilibrium price of the incumbent when $x_{t-1} < (q+1)/4$. No other strategy can give higher total profits. To see this, note that under the proposed strategy current profits are maximized, and the future per-period profits equal 1. Since per-period profits can never be higher than 1, any other strategy will result in lower profits. This shows that taking over the market in the first period and keeping $x_t = 1$ forever is the entrant's optimal strategy, independently of the strategy of the incumbent and discount factor. Thus, this must be the outcome in any equilibrium.

As $\Pi_0^t = 0$ for any $p_0^t$ it is easy to see that $p_0^t = 0$ is (one of the optimal) incumbent's strategies. Hence, both players play an optimal response. ///

*Proof of Proposition 3*. From Lemma 1 we know that the entrant's market share will never be $(q+1)/4 < x_t < (q+3)/4$, hence we do not consider these cases. For remaining initial market shares we first show that if $d = 0$ it is always optimal for the entrant to take over the market and keep it, and we use this result to show that the proposed strategies form an equilibrium if $d = 0$. Next, we show that they also form an equilibrium if $d > 0$, and that the outcome in this equilibrium is the only outcome that can arise.

Suppose $d = 0$. We consider relevant initial market shares in turn:

i) $d = 0$ and $x_{t-1} > (q+3)/4$. The entrant must set $p_1^t \le p_0^t + q$ to have $x_t = 1$, to make sure that all consumers prefer the new good to the old one, and to ensure that all consumers must have positive utility, $p_1^t \le (1-q)q_1 + qx_t$ for each $q$. When $q_1 \ge 1$ (which is the case if $q > 0.5$ and $q_0 \ge 0.5$) and $x_t = 1$, this requires $p_1 \le 1$. Therefore, to keep the whole market, the entrant must set $p_1^t = \min\{ p_0^t + q, 1\}$. The profit is then $\Pi_1^t = \min\{ p_0^t + q, 1\}$. Lower $p_1^t$ will not increase the market share beyond 1, thus it is not optimal. We check now if it may be optimal to set a higher $p_1^t$. We have to consider two cases.

*a)* $\min\{ p_0^t + q, 1\} = p_0^t + q$. Then, the current profit is at most[9]

$$\Pi_1^t = p_1^t \frac{q+3+\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}}{4} \qquad \text{if} \quad p_1^t \le p_0^t + \frac{(q+3)^2}{8} - 1.$$

Taking first and second order derivatives we see that if $p_0^t \ge 1 - 2q$, which is always the case if $q \ge 0.5$ and $p_0^t \ge 0$, this profit is maximized when $p_1^t = p_0^t + q$. Then, $x_t = 1$ and $\Pi_1^t = p_0^t + q$.

[9] see footnote 7.

*b)* $\min\{p_0^t + q, 1\} = 1$ Then, if $q_1 < 3$, the profit is less than 1. [10] This shows that when $x_{t-1} = 1$, $p_1^t = \min\{p_0^t + q, 1\}$ and keeping the whole market is optimal.

ii) $d = 0$ and $x_{t-1} < (q+1)/4$. This proof is similar to the case $x_{t-1} < (q+1)/4$ and $d = 0$ of Proposition 2.

The above argument shows that for any $x_{t-1}$, the strategy at which the entrant takes over and keeps the market, is a best response to any incumbent's strategy if $d = 0$. As the incumbent always earns 0 independently of his strategy, $p_0^t = 0$ is an optimal response for him. Hence, the proposed strategies form an equilibrium if $d = 0$. The argument showing this is also an equilibrium if $d > 0$ is similar as in proposition 2 and therefore omitted.

We now show that no other equilibrium outcome can arise. (So far, we have only shown that there exists at least one equilibrium). In every period $x_t < (q+1)/4$ or $x_t > (q+3)/4$, and moreover $(q+3)/4 < x_t < 1$ is never optimal, because it gives lower current and the same future profits as $x_t = 1$. Hence, possible outcomes are

i) $x_t < (q+1)/4$ for every $x_{t-1}$

ii) $x_t = 1$ for every $x_{t-1}$

iii) $x_t < (q+1)/4$ if $x_{t-1} > (q+3)/4$ and $x_t = 1$ if $x_{t-1} < (q+1)/4$

iv) $x_t < (q+1)/4$ if $x_{t-1} < (q+1)/4$ and $x_t = 1$ if $x_{t-1} > (q+3)/4$

We have just shown that the outcome (ii) exists. We consider now other outcomes. Denote by $p'_1$ and $p'_0$ prices when $x_{t-1} < (q+1)/4$, and by $p''_1$ and $p'_0$ prices when $x_{t-1} > (q+3)/4$.

Ad. i) Here, the entrant has always a small market share, and his profits are at most[11]

$$\Pi_1 = \frac{1}{1-d} p_1^t \frac{q+1-\sqrt{(q+1)^2 - 8(q - p_1^t + p'_0)}}{4}.$$

Assuming that the market share of the entrant must stay small, this profit is maximized at[12]

$$p_1^t = p'_0 + q - (q+1)^2/8 + e \text{, or } p'_0 = p_1^t + (q+1)^2/8 - q - e,$$

where ε is a small positive number. Consider now the incumbent. His profits are

$$\Pi_0 = \frac{1}{1-d} p_0^t \frac{3-q+\sqrt{(q+1)^2 - 8(q - p'_1 + p_0^t)}}{4}.$$

[10] See footnote 6.
[11] See footnote 7
[12] See footnote 8

Taking the first order condition we see that this profit is decreasing at $p_0^t = p'_1 + (q+1)^2/8 - q - e$. Hence, at optimum $p_0^t < p'_1 + (q+1)^2/8 - q - e$.

Therefore, the only equilibrium prices can be

$$p_0^t = 0 \text{ and } p_1^t = q - (q+1)^2/8 + e.$$

At these prices, the entrant's total profits will be

$$\Pi_1 = \frac{1}{1-d}\left(q - \frac{(q+1)^2}{8}\right)\left(\frac{q+1}{4}\right),$$

which is lower than what he can earn by taking over and keeping the market,

$$\Pi_1 = q - \frac{(q+1)^2}{8} + \frac{d \min\{p''_0 + q, 1\}}{1-d}.$$

In the same way we show that outcome (iv) is not possible.

Ad. iii). Here, the market share follows a cycle: $x_t = 0, x_{t+1} = 1, x_{t+2} = 0, ...$etc.[13] Total discounted profits are equal to

$$\Pi_1 = \frac{d}{1-d^2}\min\{p'_0 + q - (q+1)^2/8, 1\}.$$

If, on the other hand, the entrant keeps $x_t = 1$, his discounted profits will be

$$\Pi_1 = \frac{1}{1-d}\min\{p''_0 + q, 1\}.$$

We know that $\min\{p''_0 + q, 1\} \geq 0.5$, $\min\{p'_0 + q - (q+1)^2/8, 1\} \leq 1$ and $d/(1+d) \leq 0.5$.

Then, it follows that $\Pi_1 = \frac{d}{1-d^2}\min\{p'_0 + q - (q+1)^2/8, 1\} \leq \frac{1}{1-d}\min\{p''_0 + q, 1\}$.

This shows that no other outcome than $x_t = 1$ for every $x_{t-1}$ can arise in the equilibrium. This completes the proof. ///

*Proof of Proposition 4.* From Lemma 1 we know that the entrant's market share will never be $(q+1)/4 < x_t < (q+3)/4$, hence we do not consider these cases. For the remaing initial market shares we first show that the proposed strategies form an equilibrium when $d = 0$. Next, we show that they also form an equilibrium when $d > 0$, and we use the restrictions on $q_1$ and $q_0$ to show that in this equilibrium the market is covered. Subsequently, we show that no other outcome with a covered market can arise in a Markov equilibrium, and since it is always optimal to cover the market, no equilibria with the market not covered exist.

Suppose $d = 0$. We consider different initial market shares in turn.

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[13]If $x_{t-1} > (q+3)/4$, the only possible $x_t < (q+1)/4$ is $x_t = 0$

i) $d = 0$ and $x_{t-1} < (q+1)/4$. The proof is similar to the case (i) of Propositions 2 and 3.

ii) $d = 0$ and $x_{t-1} > (q+3)/4$. We initially assume that the market is covered and look at the entrant and the incumbent in turn.

*a)* The profit function of the entrant is

$$\Pi_1^t = \begin{cases} p_1^t & , \quad \text{if} \quad p_1^t \le p_0^t + q \\ p_1^t \dfrac{q+3+\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}}{4}, & \text{if} \quad p_0^t + q < p_1^t \le p_0^t + \dfrac{(q+3)^2}{8} - 1, \\ 0 & , \quad \text{otherwise} \end{cases}$$

From this we derive the optimal response of the entrant. First, note that both $p_1^t > p_0^t + (q+3)^2/8 - 1$ and $p_1^t < p_0^t + q$ cannot be optimal. When $p_0^t + q \le p_1^t \le p_0^t + (q+3)^2/8 - 1$, the profit function is concave and decreasing at the upper boundary of the domain. Hence, two optima are possible: [14]

- if $p_0^t < 1 - 2q$, there is an interior solution, given by the first order condition,

$$\frac{d\Pi_1^t}{dp_1^t} = \frac{q+3+\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}}{4} - \frac{p_1^t}{\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}} = 0.$$

Solving for the optimal price, we get

$$p_1^t = \frac{12p_0^t + (q+3)^2 - 12 + (q+3)\sqrt{(q+3)^2 - 6 + 6p_0^t}}{18}$$

- if $p_0^t \ge 1 - 2q$, there is a corner solution at the lower boundary: $p_1^t = p_0^t + q$.

*b)* The profit function of the incumbent is

$$\Pi_0^t = \begin{cases} p_0^t & , \quad \text{if} \quad p_0^t < p_1^t + 1 - \dfrac{(q+3)^2}{8} \\ p_0^t \dfrac{1-q-\sqrt{(q+3)^2 - 8(1+p_1^t - p_0^t)}}{4}, & \text{if} \quad p_1^t + 1 - \dfrac{(q+3)^2}{8} < p_0^t < p_1^t - q \\ 0 & , \quad \text{otherwise} \end{cases}$$

Note that it is never optimal to set $p_0^t \ge p_1^t - q$, or $p_0^t$ lower than

[14] We can see now the main difference between $q \ge 0.5$ and $q < 0.5$. When $q \ge 0.5$, $p_0^t \ge 1 - 2q$ for any $p_0^t \ge 0$, and it is always optimal to set $p_1^t = p_0^t + q$ and keep the whole market. When $q < 0.5$, $p_0^t < 1 - 2q$ and an interior solution with less than the whole market for the entrant is possible.

$p_1^t + 1 - (q+3)^2/8 - e$ for any $e > 0$. The profit function is discontinuous at $p_0^t = p_1^t + 1 - (q+3)^2/8$.[15] For $p_1^t + 1 - (q+3)^2 \le p_0^t < p_1^t - q$, the profit function is concave if $p_0^t > \frac{4}{3}\left(p_1^t + 1 - (q+3)^2/8\right)$, and convex otherwise. Moreover, it is decreasing at the upper boundary of its domain, $p_1^t - q$. Therefore, two optimal values can arise:

- an interior solution, given by the first order condition

$$\frac{d\Pi_0^t}{dp_0^t} = \frac{1 - q - \sqrt{(q+3)^2 - 8(1 + p_1^t - p_0^t)}}{4} - \frac{p_0^t}{\sqrt{(q+3)^2 - 8(1 + p_1^t - p_0^t)}} = 0,$$

provided that the function is in its concave range. Solving for the optimal price, we get

$$p_0^t = \frac{12p_1^t - (1-q)^2 - 12q + (1-q)\sqrt{(1-q)^2 + 6q - 6p_1^t}}{18};$$

- a corner solution at the lower boundary, $p_0^t = p_1^t + 1 - (q+3)^2/8 - e$, where $e$ is a small positive number.

It is easy to check that if the optimal price of at least one of the sellers is a corner solution, there is no equilibrium. Therefore, if there exist equilibrium prices, they must be both interior solutions to the optimization problem. It follows from the above analysis, that for the interior solutions to be optimal, the equilibrium prices must satisfy:

i) $p_0^t < 1 - 2q$, so that the interior solution of the entrant is optimal;

ii) At the equilibrium price of the entrant, the incumbent must earn more if he plays the interior solution than in the corner solution. That is,

$$\frac{1}{216}\left(q^2 + 10q + 1 - 12p_1^t - (1-q)\sqrt{(1-q)^2 + 6q - 6p_1^t}\right)\cdot$$
$$\left(3q - 3 + \sqrt{5q^2 + 14q + 5 - 24p_1^t - 4(q-1)\sqrt{(1-q)^2 + 6q - 6p_1^t}}\right) \ge p_1^t + 1 - \frac{(q+3)^2}{8}.$$

The left side of this inequality has been obtained by substituting the interior optimal price of the incumbent into his profit function. Solving this inequality, we get the following condition for an equilibrium:

$$p_1^t \le \frac{-5q - 18 + (7+2q)\sqrt{7+2q}}{4}$$

[15] $\Pi_0^t$ approaches $\left(p_1^t + 1 - \frac{(q+3)^2}{8}\right)\left(\frac{1-q}{2}\right) < p_1^t + 1 - \frac{(q+3)^2}{8}$ as $p_0^t$ approaches $p_1^t + 1 - \frac{(q+3)^2}{8}$ from above.

Substituting the incumbent's optimal response function into the entrant's optimal response function, solving for $p_0^t$ and $p_1^t$ and eliminating solutions with negative prices we obtain the following equilibrium:

$$p_0^t = \frac{-3q^2 - 46q - 23 + (9-q)\sqrt{9q^2 + 38q + 9}}{100},$$

$$p_1^t = \frac{3q^2 + 6q - 17 + (11+q)\sqrt{9q^2 + 38q + 9}}{100}.$$

We have to check whether the conditions which make sure that these are interior solutions are satisfied. It is easy to check that in the equilibrium the incumbent's price satisfies $p_0^t < 1 - 2q$. The entrant's price must satisfy

$$p_1^t = \frac{3q^2 + 6q - 17 + (11+q)\sqrt{9q^2 + 38q + 9}}{100} < \frac{-5q - 18 + (7+2q)\sqrt{7+2q}}{4}.$$

Solving for $q$, we obtain the following condition for the existence of an equilibrium:

$$q > \hat{q} = -\frac{7}{2} + \frac{1}{2}\left[\frac{1}{3}\left(41 + 2\sqrt{238}\right)^{1/3} + \frac{3}{\left(41 + 2\sqrt{238}\right)^{1/3}} + \frac{2}{3}\right]^2 \approx 0.3477$$

If $q$ satisfies this condition, there exist equilibrium prices such that for both sellers the interior solution to the profit maximization problem is optimal. Hence, if $q > \hat{q}$ and $d = 0$, the proposed strategies form an equilibrium.

We show now that the strategies described in the proposition form an equilibrium if $d > 0$. Suppose that the entrant plays the equilibrium strategy and consider the incumbent. If he plays the equilibrium strategy, and if $x_{t-1} < (q+1)/4$, he earns

$$\Pi_0 = \frac{d}{1-d}\hat{\Pi}_0,$$

where $\hat{\Pi}_0$ denotes the profit in period $t$ if $x_{t-1} > (q+3)/4$ and both sellers play the equilibrium strategy. Note that no other strategy can give him higher profits: when $x_{t-1} < (q+1)/4$ and the entrant plays the equilibrium strategy, the incumbent's maximum profits are 0. When $x_{t-1} > (q+3)/4$ and the entrant plays his equilibrium strategy, the incumbent's maximum profits are $\hat{\Pi}_0$. Hence, no strategy can give higher profits than equilibrium strategy.

Suppose now that the incumbent plays the equilibrium strategy and consider the entrant. When he plays his equilibrium strategy, his profits are:

$$\Pi_1 = q - \frac{(q+1)^2}{8} + \frac{d\hat{\Pi}_1}{1-d},$$

where $\hat{\Pi}_1$ denotes the profit in period $t$ if $x_{t-1} > (q+3)/4$ and both sellers play the equilibrium strategy. Note that if the incumbent plays the equilibrium strategy, the entrant can never earn higher per period profits than $\hat{\Pi}_1$. It is obviously true if $x_{t-1} > (q+3)/4$. When $x_{t-1} < (q+1)/4$, the maximum profit is $q-(q+1)^2/8$. Observe that if $x_{t-1} > (q+3)/4$, the entrant can always earn $q$ by setting $p_1^t = q$, which is not the optimal price. Hence, $\hat{\Pi}_1 > q > q-(q+1)^2/8$. Hence, a strategy different than the equilibrium strategy will give either lower current profits, or lower future profits, or both. This shows that under the assumption of the market being covered, the proposed strategies form an equilibrium for any $d$.

The next step shows that in the described equilibrium the market is covered. Note that when $p_1^t + p_0^t \le 1$, the market in period $t$ is covered. Everyone consumes one of the goods, if for all $q$

$$(1-q)q_1 + qx_t - p_1^t \ge 0 \text{ or } (1-q)q_0 + q(1-x_t) - p_0^t \ge 0.$$

This is surely true, if (but not only if) the sum of these two expressions is nonnegative,

$$(1-q)(q_1+q_0) + q - p_1^t - p_0^t \ge 0,$$

which, since $q_1 > q_0 > 0.5$, is certainly the case if $p_1^t + p_0^t \le 1$. It is easy to check that in the proposed equilibrium $p_1^t + p_0^t \le 1$ if $x_t < 0.5$. If $x_t > (q+3)/4$, the sum of equilibrium prices is $p_1^t + p_0^t = \dfrac{\sqrt{9q^2+38q+9}-q-1}{100}$, which is obviously smaller than 1.

Next, we show that no other outcome with a covered market can arise in an equilibrium. In any period two types of market shares can arise: $x_t < (q+1)/4$ or $x_t > (q+3)/4$. Hence, several equilibrium outcomes are possible:

i) $x_t < (q+1)/4$ for every $x_{t-1}$;

ii) $x_t > (q+3)/4$ for every $x_{t-1}$;

iii) $x_t > (q+3)/4$ if $x_{t-1} < (q+1)/4$ and $x_t < (q+1)/4$ if $x_{t-1} > (q+3)/4$;

iv) $x_t < (q+1)/4$ if $x_{t-1} < (q+1)/4$ and $x_t > (q+3)/4$ if $x_{t-1} > (q+3)/4$.

We have just shown that an equilibrium outcome described in (ii) exists. We consider now the other cases.

Ad. i) The proof is similar to the analogous case in the proof of Proposition 3, where we showed that in this case the optimal $p_1^t = p_0^t + q - (q+1)^2/8 + e$, while optimal $p_0^t < p_0^t + (q+1)^2/8 - q - e$. Hence, the only equilibrium prices are $p_1^t = q - (q+1)^2/8 + e$ and $p_0^t = 0$. At these prices, the entrant's profits are

$$\Pi_1 = \frac{1}{1-d}\left(q - \frac{(q+1)^2}{8}\right)\left(\frac{q+1}{4}\right).$$

which is lower than the profits that he can earn if he takes over the market and keeps a large market share,

$$\Pi_1 = q - (q+1)^2/8 + \frac{d}{1-d}\Pi_1^t,$$

where $\Pi_1^t > q$. [16]

The same reasoning rules out an equilibrium outcome of type (iv).

Ad.(iii) In this case, the market share follows a cycle of the type: $x_t = 0, x_{t+1} = 1, x_{t+2} = 0, \ldots$etc. [17] Denote by $p'_0$ and $p'_1$ equilibrium prices if $x_{t-1} < (q+1)/4$, and by $p''_0$ and $p''_1$ equilibrium prices if $x_{t-1} > (q+1)/4$. The discounted profits and prices of both sellers are:

$$\Pi_1 = \frac{p'_1}{1-d^2} \quad \text{and} \quad p'_1 = p'_0 + q - \frac{(q+1)^2}{8};$$

$$\Pi_0 = \frac{dp''_0}{1-d^2} \quad \text{and} \quad p''_0 = p''_1 + 1 - \frac{(q+3)^2}{8}.$$

This could be an equilibrium outcome if none of the sellers could increase his profits by playing a strategy leading to a different outcome. Consider the entrant. His profits in this equilibrium must be larger than what he can earn, given the price of the incumbent, if $x_t > (q+3)/4$ forever, i.e.,

$$\frac{p'_1}{1-d^2} > p'_1 + \frac{d}{1-d}\max_{p_1} p_1 \frac{q+3+\sqrt{(q+3)^2 - 8(1+p_1-p''_0)}}{4}.$$

Similarly, the incumbent must earn higher profits than when $x_t < (q+1)/4$, i.e.,

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[16] If $x_t > (q+3)/4$, the entrant can always set the price $q$ and keep the whole market, which is not optimal.

[17] See footnote 13

$$\frac{p''_0}{1-d^2} > p''_0 + \frac{d}{1-d} \max_{p_0} p_0 \frac{3-q+\sqrt{(q+1)^2 - 8(1+p'_1 - p_0)}}{4}.$$

Since

$$\max_{p_1} p_1 \frac{q+3+\sqrt{(q+3)^2 - 8(1+p_1 - p''_0}}{4} \geq p''_0 + q, \text{ and}$$

$$\max_{p_0} p_0 \frac{3-q+\sqrt{(q+1)^2 - 8(q - p'_1 + p_0)}}{4} \geq p'_1 - q,$$ [18]

this requires

$$\frac{dp'_1}{1+d} > p''_0 + q \quad \text{and} \quad \frac{dp''_0}{1+d} > p'_1 - q,$$

which leads to a contradiction for every $d$. Hence, market share cycles of this type cannot be an equilibrium outcome.

Therefore, we have shown that the only possible equilibrium outcome with a covered market is one with $x_t > (q+3)/4$ for every $x_{t-1}$.

Finally, we show that it is always optimal for at least one of the sellers to cover the market, and hence no equilibrium with a non-covered market exists. We consider several initial market shares:

i) $x_{t-1} < (q+1)/4$.[19] We show that if $0.5 \leq q_1 \leq 3$ it is always optimal for the entrant to cover the market. We consider several cases:

*a)* $p_0^t + q - (q+1)^2/8 \geq 1$. Suppose $0.5 \leq q_1 \leq 3$. Then, if $p_1^t = 1$, the entrant will have the whole market and earn $\Pi_1^t = 1$, while if $p_1^t > 1$, $\Pi_1^t < 1$ and the market share will be less than 1. Hence, $p_1^t = 1$ is better both for present and future profits.

*b)* $(q+1)^2/6 - q < p_0^t < 1 - q + (q+1)^2/8$ .

Then, $\Pi_1^t = p_1^t = p'_0 + q - (q+1)^2/8$ is higher than the maximum profit that can be achieved when $p_1^t > p'_0 + q - (q+1)^2/8$, and it also results in larger market share[20]. Hence, $p_1^t = p'_0 + q - (q+1)^2/8$, at which price the market is covered, is optimal for current and future profits.

ii) $x_{t-1} > (q+3)/4$. We show that if $0.5 \leq q_0 \leq 3$, it is always optimal for the incumbent to cover the market. We consider several cases.

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18 $p_1 \frac{q+3+\sqrt{(q+3)^2 - 8(1+p_1 - p''_0)}}{4} = p''_0 + q$ if $p_1 = p''_0 + q$ and $p_0 \frac{3-q+\sqrt{(q+1)^2 - 8(q - p'_1 + p_0)}}{4} = p'_1 - q$ if $p_0 = p'_1 - q$

19 The proof for $(q+1)/4 < x_{t-1} < 0.5$ is analogous.

*a)* $p_1^t + 1 - (q+3)^2/8 \geq 1$. Then, if $0.5 \leq q_0 \leq 3$, for the same reasons as for the incumbent above it is optimal for the incumbent to set $p_0^t = 1$ and cover the market.

*b)* $(q+3)^2/8 < p_1^t < (q^2+4q+1)/6$. Here, $p_0^t = p_1^t + 1 - (q+3)^2/8$ gives the highest current profits and the highest market share, [21] hence it is optimal.

*c)* $p_1^t < (q^2+4q+1)/6$. Since $p_0^t < p_1^t - q$ (else the incumbent has zero profits now and 0 market share in at the beginning of the next period), $p_0^t + p''_1 < (1-q)^2/6 < 1$. Hence, the market is covered. This shows that it is always optimal for at least one of the sellers to cover the market, which completes the proof. ///

*Proof of Proposition 5.* We show that no pure strategy Markov equilibrium with a covered market exists. Moreover, it is always optimal for at least one of the sellers to cover the market. Hence, no pure-strategy equilibrium with a non-covered market exists.

We consider all potential equilibrium outcomes with the market covered and show that none of them can arise in an equilibrium. We begin with the case $\boldsymbol{d} = 0$, and then extend the argument to higher $\boldsymbol{d}$.

Assume that $\boldsymbol{d} = 0$ and consider two types of initial market shares: $x_{t-1} > (q+3)/4$ and $x_{t-1} < (q+1)/4$.[22]

i) $\boldsymbol{d} = 0$ and $x_{t-1} > (q+3)/4$. We have shown in the proof of Proposition 4 that a necessary condition for the existence of equilibrium is $q > \hat{q}$. Otherwise, there does not equilibrium prices at which the incumbent would prefer having a small market share to taking over the market. Hence, if $q < \hat{q}$, there is no pure strategy equilibrium.

ii) $\boldsymbol{d} = 0$ and $x_{t-1} < (q+1)/4$. We have shown earlier[23] that when $q > 3 - 2\sqrt{2}$, the entrant always finds it optimal to take over the market. Therefore, this must be the equilibrium outcome. Then, $x_t > (q+3)/4$, and in the next period there is no pure strategy equilibrium. Hence, when $q > 3 - 2\sqrt{2}$, there does not exist a pure strategy Markov equilibrium. Therefore, an equilibrium could only exist if $q < 3 - 2\sqrt{2}$, and the outcome would have to be $x_t < (q+1)/4$ in every period. We show now that such an equilibrium does not exist.

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[20] see footnote 8.

[21] When $p''_1 > (q^2+4q+1)/6$, the profit function is decreasing if $p_0^t > p''_1 + 1 - (q+3)^2/8$ .

[22] Since $(q+1)/4 \leq x_{t-1} \leq (q+3)/4$ can never occur, we do not consider it. .

[23] see footnote 8.

In an equilibrium, both sellers must optimize given the other's strategy. The profit functions are

$$\Pi_1^t = p_1^t \frac{q+1-\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{4};$$

$$\Pi_0^t = p_0^t \frac{3-q+\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{4}.$$

We obtain now optimal response functions of the sellers.

a) *incumbent*. The profit function is concave and increasing at the lower boundary of its domain. Hence, two optima are possible:

- an interior solution, given by the first order condition

$$\frac{d\Pi_0^t}{dp_0^t} = \frac{3-q+\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{4} - \frac{p_0^t}{\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}} = 0.$$

Solving for $p_0^t$, we obtain

$$p_0^t = \frac{q^2 - 6q - 3 + 12p_1^t + (3-q)\sqrt{(3-q)^2 - 6 + 6p_1^t}}{18}.$$

- a corner solution, $p_0^t = p_1^t - q$.

b) *entrant*. The profit function of the entrant is concave when $p_1^t > \frac{4}{3}\left(p_0^t + q - \frac{(q+1)^2}{8}\right)$ and convex otherwise. Besides, it is decreasing at the upper boundary of its domain, $p_0^t + q$. Hence, two optimum values could arise:

- an interior solution, given by the first order condition

$$\frac{d\Pi_1^t}{dp_1^t} = \frac{q+1-\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}}{4} - \frac{p_1^t}{\sqrt{(q+1)^2 - 8(q - p_1^t + p_0^t)}} = 0.$$

Solving for $p_1^t$, we get

$$p_1^t = \frac{-q^2 + 10q - 1 + 12p_0^t + (q+1)\sqrt{(q+1)^2 - 6q - 6p_0^t}}{18}.$$

- A corner solution at the lower boundary of the domain, $p_1^t = p_0^t + q - \frac{(q+1)^2}{8}$.

Since in an equilibrium $x_t < (q+1)/4$, the optimal responses of both sellers must be interior. Denote the interior optimal prices by $p_1^*(p_0)$ and $p_0^*(p_1)$.[24]

[24] Note that for the entrant a global optimum may be different (corner solution).

Figure A. 1 shows the interior optimal prices as functions of the competitor's price. The figure is meant to help understanding how the proof works.

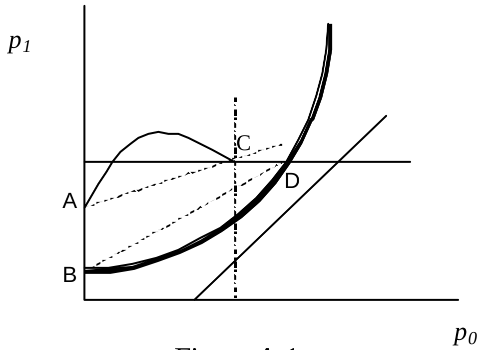


Figure A.1

The bold curve represents the incumbent's reaction function, $p_0^*(p_1)$. The single line represents solutions to the entrant's optimization problem: the left, curved one is the interior solution, $p_1^*(p_0)$, while the right, straight line is the corner solution, $p_1^t = p_0^t + q$. Note that for some $p_0$ both solutions exist. In that interval only the solution that gives higher profits is relevant.

We have shown that an equilibrium with a corner solution does not exist. Hence, to prove that there is no equilibrium, it is enough to show that $p_0^*(p_1)$ and $p_1^*(p_0)$ do not cross. The proof consists of several steps. We show that

*a)* Point A lies above point B;

*b)* Point D lies to the right of point C;

*c)* $p_1^*(p_0)$ is concave in $p_0$, and therefore all values of $p_1^*(p_0)$ lie above the straight line connecting A and C;

*d)* $p_0^*(p_1)$ is concave in $p_1$. Hence, its inverse, $p_0^{*-1}(p_0)$, is convex in $p_0$. Therefore, all values of the inverse reaction function lie below the straight line connecting B and D.

*e)* (a) to (d) imply that the curve representing the interior optimum of the entrant lies above the reaction curve of the incumbent, and hence the two curves do not cross. This shows that a pure strategy equilibrium does not exist.

ad. *a)* We show that the $p_1$ coordinate of A is always larger than $p_1$ coordinate of B. That is, $p_1^*(0) > p_0^{*-1}(0)$

The incumbent's optimal response is equal to 0, if

$$p_0^t = \frac{q^2 - 6q - 3 + 12p_1^t + (3-q)\sqrt{(3-q)^2 - 6 + 6p_1^t}}{18} = 0\,,$$

which is true if $p_1 = q-(q+1)^2/8$ (and for all lower $p_1$). Hence, $p_0^{*-1}(0) = q-(q+1)^2/8$. [25]

The entrant's optimal response to 0 is

$$p_1^*(0) = \frac{-q^2+10q-1+(q+1)\sqrt{(q+1)^2-6q}}{18}.$$

Comparing these two values of $p_1$ we see that $p_1^*(0) > p_0^{*-1}(0)$ for every relevant $q$.

*b)* Here we show that the highest $p_0$ for which the entrant's interior optimum exists, $\overline{p}_0$, is lower than $p_0^*(p_1^*(\overline{p}_0))$. We know that when $p_0 > \frac{(q+1)^2}{6}-q$, the entrant's profit function is always decreasing, and hence there is no interior solution. On the other hand, when $p_0 < \frac{(q+1)^2}{6}-q$, the profit function is increasing at the lower boundary so that an interior solution exists. Therefore, $\overline{p}_0 = \frac{(q+1)^2}{6}-q$ and

$$p_0^*(p_1^*(\overline{p}_0)) = p_0\left(\frac{(q+1)^2}{18}\right) = \frac{5q^2-14q-1+(3-q)\sqrt{12q^2-48q+30}}{54}.$$

Comparing these two values, we see that $p_0^*(p_1^*(\overline{p}_0))$ is larger than $\overline{p}_0$ for all relevant $q$.

*c)* $p_0(p_1)$ is concave.

$$\frac{d^2 p_0}{dp_1^2} = -\frac{1}{2}\frac{3-q}{\sqrt{q^2-6q+3+6p_1}} < 0 \text{ for all relevant } q$$

*d)* $p_1(p_0)$ is concave

$$\frac{d^2 p_1}{dp_0^2} = -\frac{1}{2}\frac{q+1}{\sqrt{q^2-4q+1-6p_0}} < 0 \text{ for all relevant } q.$$

This shows that when $d = 0$, there does not exist a pure strategy Markov equilibrium. We now extend the argument to the case when $d > 0$. Since any period two types of market shares can arise, $x_t < (q+1)/4$ or $x_t > (q+3)/4$, several market outcomes are possible:

i) $x_t < (q+1)/4$ for every $x_{t-1}$;

ii) $x_t > (q+3)/4$ for every $x_{t-1}$;

iii) $x_t > (q+3)/4$ if $x_{t-1} < (q+1)/4$ and $x_t < (q+1)/4$ if $x_{t-1} > (q+3)/4$;

[25] If $q-(q+1)^2/8<0$, $p_0^*(p_1)>0$ for any $p_1$. Hence, the beginning of the incumbent's reaction curve lies obviously below the entrant' reaction curve.

iv) $x_t < (q+1)/4$ if $x_{t-1} < (q+1)/4$ and $x_t > (q+3)/4$ if $x_{t-1} > (q+3)/4$.

Ad.i) In this case, the total profits of sellers are: $\Pi_1 = \frac{1}{1-d}\Pi_1^t$ and $\Pi_0 = \frac{1}{1-d}\Pi_0^t$, where $\Pi_1^t$ and $\Pi_0^t$ are per-period profits if $x_{t-1} < (q+1)/4$ and $x_t < (q+1)/4$. Their maximization leads to the same optimal responses as maximization of $\Pi_1^t$ and $\Pi_0^t$. Since in that case no pure strategy equilibrium exists, it also does not exist when $d > 0$. Ad. ii) The argument is similar to that in i). Ad iii). Follows from i) and ii). Ad. iv) The same argument as in the proof of proposition 4 shows that a pure strategy Markov equilibrium with market share cycle does not exist.

We have shown that none of the possible outcomes can arise in the equilibrium. Hence, if $0 \le q < \hat{q}$ there does not exist a pure strategy Markov equilibrium with a covered market. A similar argument as in the Proposition 4 shows that it is always optimal to cover the market. Hence, there does not exist an equilibrium with not covered market. This completes the proof. ///