Linear Parametric Sensitivity Analysis of the Constraint Coefficient Matrix in Linear Programs

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ABSTRACT AND KEYWORDS	
Abstract	Sensitivity analysis is used to quantify the impact of changes in the initial data of linear programs on the optimal value. In particular, parametric sensitivity analysis involves a perturbation analysis in which the effects of small changes of some or all of the initial data on an optimal solution are investigated, and the optimal solution is studied on a so-called critical range of the initial data, in which certain properties such as the optimal basis in linear programming are not changed. Linear one-parameter perturbations of the objective function or of the so-called "right-hand side" of linear programs and their effect on the optimal value is very well known and can be found in most college textbooks on Management Science or Operations Research. In contrast, no explicit formulas have been established that describe the behavior of the optimal value under linear one-parameter perturbations of the constraint coefficient matrix. In this paper, such explicit formulas are derived in terms of local expressions of the optimal value function and intervals on which these expressions are valid. We illustrate this result using two simple examples.
Free Keywords	Linear parametric programming, Linear programming, Sensitivity analysis, Rational matrix function
Availability	
Classifications	

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September 22, 2005

Abstract

Sensitivity analysis is used to quantify the impact of changes in the initial data of linear programs on the optimal value. In particular, parametric sensitivity analysis involves a perturbation analysis in which the effects of small changes of some or all of the initial data on an optimal solution are investigated, and the optimal solution is studied on a so-called critical range of the initial data, in which certain properties such as the optimal basis in linear programming are not changed. Linear one-parameter perturbations of the objective function or of the so-called "right-hand side" of linear programs and their effect on the optimal value is very well known and can be found in most college textbooks on Management Science or Operations Research. In contrast, no explicit formulas have been established that describe the behavior of the optimal value under linear one-parameter perturbations of the constraint coefficient matrix. In this paper, such explicit formulas are derived in terms of local expressions of the optimal value function and intervals on which these expressions are valid. We illustrate this result using two simple examples.

1 Introduction

Sensitivity analysis is used to quantify the impact of changes in the initial data of linear programs on optimal solutions, whenever they exist. In particular, parametric sensitivity analysis involves a perturbation analysis in which the effects of small changes of some or all of the initial data on an optimal solution are investigated, and the optimal solution is studied on a so-called critical range of the initial data, in which certain properties such as the optimal basis in linear

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programming are not changed [2]. In general, one may consider parameterized linear programs

$$Z(\lambda) = \max\{c(\lambda)^T x : \; A(\lambda)x = b(\lambda), \; x \geq 0\}, \tag{1}$$

in which λ runs through a subset Λ of a metric space, and where $A : \Lambda \to \mathbb{R}^{m,n}$, $b : \Lambda \to \mathbb{R}^m$, and $c : \Lambda \to \mathbb{R}^n$ are functions. We may assume without loss of generality that $A(\lambda)$ has full rank m for all $\lambda \in \Lambda$ and that $m \leq n$. Indeed, any linear program of the form

$$Z(\lambda) = \max\{c_1(\lambda)^T x_1 : \; A_1(\lambda)x_1 \leq b_1(\lambda), \; x_1 \geq 0\},$$

can be rewritten as in (1) with $\operatorname{rank} A(\lambda) = m$, by putting

$$A(\lambda) = \begin{pmatrix} A_1(\lambda) & I_m \end{pmatrix}, \quad b(\lambda) = b_1(\lambda), \quad c(\lambda) = \begin{pmatrix} c_1(\lambda) \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ s \end{pmatrix}.$$

The parameterized linear programs

$$Z(\lambda) = \max\{c^T x : \; Ax = b(\lambda), \; x \geq 0\},$$

and

$$Z(\lambda) = \max\{c(\lambda)^T x : \; Ax = b, \; x \geq 0\},$$

where $b(\lambda) = b + \lambda d$ and $c(\lambda) = c + \lambda e$ are linear perturbations, are well understood; see for example [4]. In this study, we consider the case

$$Z(\lambda) = \max\{c^T x : \; A(\lambda)x = b, \; x \geq 0\}, \tag{2}$$

where $A(\lambda) = A + \lambda F$ is a parameterized set of $m \times n$ matrices with a one-dimensional parameter $\lambda \in \Lambda \subseteq \mathbb{R}$. The initial data $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are fixed. We study local behavior of the function $Z : \Lambda \to \mathbb{R}$. It has been observed that Z is locally rational; for a brief survey, see [2]. In this paper, this observation is extended to an explicit description of the rational functions involved using realization theory; see [1]. The set-up of this paper is as follows. In the remainder of this section, notation is fixed. In Section 2, we provide an introduction to realization theory for scalar rational functions. In Section 3, we apply realization theory to parametric sensitivity analysis in order to derive the main result of this paper. We illustrate the main result using two simple examples in Section 4.

The symbol I_m denotes the $m \times m$ identity matrix. In case the size of the identity matrix is not relevant or obvious, we use shorthand notation I. If B is a square matrix, then $\rho(B) \subset \mathbb{C}$ indicates the resolvent set of B consisting of those complex numbers μ for which $\mu I - B$ is invertible, and the spectrum or the set of eigenvalues $\sigma(B)$ of B consists of those complex numbers μ for which the set of equations $Bx = \mu x$ has a nonzero solution x.

2 Realization Theory

In this section, we discuss realization theory for scalar rational functions. In Chapter 3 in [1], a more extensive discussion can be found on realization theory for functions which are matrix or operator valued. A fundamental observation in realization theory is that when $b, c \in \mathbb{R}^m$ and A an $m \times m$ matrix, then the function

$$f(\lambda) = 1 + \lambda c^T (I_m + \lambda A)^{-1} b$$

is a rational function that can be described completely in terms of eigenvalues of two matrices. In order to prove this, we use a property of the determinant, namely $\det(I + BC) = \det(I + CB)$ whenever both BC and CB are square matrices, not necessarily of the same size. We arrive at

$$f(\lambda) = \det\ f(\lambda) = \det(1 + \lambda c^T (I_m + \lambda A)^{-1} b) = \det(I_m + bc^T (I_m + \lambda A)^{-1}) =$$

$$\frac{\det(I_m + \lambda(A + bc^T))}{\det(I_m + \lambda A)} = \frac{\det(I_m + \lambda A^\times)}{\det(I_m + \lambda A)},$$

where we have written $A^\times = A + bc^T$. More explicitly, when $\alpha_1, \ldots, \alpha_m$ are the eigenvalues of A and $\alpha_1^\times, \ldots, \alpha_m^\times$ are the eigenvalues of $A^\times$, counted according to their multiplicities, then

$$f(\lambda) = \prod_{j=1}^{m} \frac{1 + \lambda \alpha_j^\times}{1 + \lambda \alpha_j}. \tag{3}$$

Observe that when A and $A^\times$ have no common eigenvalues, then the number m of factors in the enumerator and denominator on the right hand side of (3) is minimal. Conversely, when $f(\lambda)$ is given by (3), then we may construct a realization

$$f(\lambda) = 1 + \lambda c^T (I_m + \lambda A)^{-1} b,$$

where $b, c \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m,m}$ are given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{m-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} a_0 - a_0^\times \\ a_1 - a_1^\times \\ \vdots \\ \vdots \\ a_{m-1} - a_{m-1}^\times \end{pmatrix}.$$

The parameters a_k and $a_k^\times$ for $k = 0, \ldots, m-1$ are derived from

$$\prod_{j=1}^{m} (1 + \lambda \alpha_j) = 1 + \sum_{k=1}^{m} a_{m-k} (-\lambda)^k, \quad \prod_{j=1}^{m} (1 + \lambda \alpha_j^\times) = 1 + \sum_{k=1}^{m} a_{m-k}^\times (-\lambda)^k.$$

For a given rational function, the realization parameters A, b, c are not unique. First of all, the size m of the matrix A can be minimized by cancellations whenever A and $A^{\times}$ have common eigenvalues. Further, given realizations

$$f(\lambda) = 1 + \lambda c_1^T (I_m + \lambda A_1)^{-1} b_1 = 1 + \lambda c_2^T (I_m + \lambda A_2)^{-1} b_2$$

of minimal size m, there exists an invertible $m \times m$ matrix S, such that $c_2^T = c_1^T S$, $A_2 = S^{-1} A_1 S$, and $b_2 = S^{-1} b_1$; see Chapter 3 in [1]. In other words, minimal realizations, i.e. realizations of minimal size, are mutually similar.

3 Parametric Sensitivity Analysis

In order to introduce the concept of basic feasible solutions of the linear program (2), we provide some notation. Given $\pi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ injective, we define

$$E_\pi = \begin{pmatrix} e_{\pi(1)} & \cdots & e_{\pi(m)} \end{pmatrix} : \mathbb{R}^m \to \mathbb{R}^n,$$

$A_\pi(\lambda) = A(\lambda) E_\pi : \mathbb{R}^m \to \mathbb{R}^m$, and $c_\pi = E_\pi^T c \in \mathbb{R}^m$. We assume that π is strictly increasing. We define $\overline{\pi} : \{1, \ldots, n-m\} \to \{1, \ldots, n\}$ as the strictly increasing mapping such that $\operatorname{ran}(\pi) \cup \operatorname{ran}(\overline{\pi}) = \{1, \ldots, n\}$. If $A_\pi(\lambda_0)$ is invertible for a fixed $\lambda_0 \in \Lambda$ and if $A_\pi(\lambda_0)^{-1} b \geq 0$, then $x_\pi(\lambda_0) = E_\pi A_\pi(\lambda_0)^{-1} b$ is a basic feasible solution to the linear program

$$Z(\lambda_0) = \max\{c^T x : \; A(\lambda_0) x = b, \; x \geq 0\}, \tag{4}$$

and all basic feasible solutions to this program arise in this manner; see for example [4]. Moreover, $x_\pi(\lambda_0)$ provides an optimal solution to the program if and only if the reduced costs satisfy

$$c_{\overline{\pi}}^T - c_\pi^T A_\pi(\lambda_0)^{-1} A_{\overline{\pi}}(\lambda_0) \geq 0.$$

We now study the local behavior of the function Z in a neighborhood of λ_0. Let π be given such that $x_\pi(\lambda_0)$ is an optimal solution to (4), i.e. $Z(\lambda_0) = c^T x_\pi(\lambda_0)$, which is equivalent to

(1) $A_\pi(\lambda_0) = A_\pi + \lambda_0 F_\pi$ is invertible,

(2) $x_\pi(\lambda_0) = E_\pi A_\pi(\lambda_0)^{-1} b \geq 0$,

(3) $c_{\overline{\pi}}^T - c_\pi^T A_\pi(\lambda_0)^{-1} A_{\overline{\pi}}(\lambda_0) \geq 0$.

We aim to identify the neigborhood $\Lambda \supseteq \Lambda_\pi \ni \lambda_0$, on which an optimal solution to (2) is given by $Z(\lambda) = c^T x_\pi(\lambda)$ for all $\lambda \in \Lambda_\pi$. Equivalently, we put

(1) $A_\pi(\lambda) = A_\pi + \lambda F_\pi$ is invertible,

(2) $x_\pi(\lambda) = E_\pi A_\pi(\lambda)^{-1} b \geq 0$,

(3) $c_{\overline{\pi}}^T - c_\pi^T A_\pi(\lambda)^{-1} A_{\overline{\pi}}(\lambda) \geq 0$,

for all $\lambda \in \Lambda_\pi$. We remark that in principle, the neighborhood Λ_π may turn out to be a single point, i.e., $\Lambda_\pi = \{\lambda_0\}$. The method presented here does not relieve this issue, which may result in computational inefficiencies in the degenerate case; see [4].

We shall translate the conditions (1-3) in terms of properties of eigenvalues of specific matrices using realization theory as discussed in Section 2.

Condition (1) We first discuss the condition

$$\det A_\pi(\lambda) = \det(A_\pi + \lambda F_\pi) \neq 0.$$

Note that

$$\det(A_\pi + \lambda F_\pi) = \det(A_\pi(\lambda_0) + (\lambda - \lambda_0) F_\pi) =$$
$$(\lambda - \lambda_0) \cdot \det(A_\pi(\lambda_0)) \cdot \det\left(A_\pi(\lambda_0)^{-1} F_\pi - \frac{1}{\lambda_0 - \lambda} I_m\right),$$

which implies that $A_\pi(\lambda)$ is invertible if and only if

$$\frac{1}{\lambda_0 - \lambda} \in \rho\left(A_\pi(\lambda_0)^{-1} F_\pi\right),$$

which comes down to the fact that

$$1 + \alpha_j(\lambda - \lambda_0) \neq 0, \quad j = 1, \ldots, m, \tag{5}$$

where $\alpha_1, \ldots, \alpha_m$ are the eigenvalues of $A_\pi(\lambda_0)^{-1} F_\pi$.

Condition (2) In order to translate the second condition, observe that for $1 \leq q \leq m$, the inequality $e_q^T A_\pi(\lambda)^{-1} b \geq 0$ holds true if and only if

$$1 + (\lambda - \lambda_0) e_q^T A_\pi(\lambda)^{-1} b \begin{cases} \geq 1, & \lambda - \lambda_0 \geq 0 \\ \leq 1, & \lambda - \lambda_0 \leq 0 \end{cases}.$$

Further,

$$1 + (\lambda - \lambda_0) e_q^T A_\pi(\lambda)^{-1} b = 1 + (\lambda - \lambda_0) e_q^T (I_m + (\lambda - \lambda_0) A_\pi(\lambda_0)^{-1} F_\pi)^{-1} A_\pi(\lambda_0)^{-1} b =$$
$$\prod_{j=1}^m \frac{1 + (\lambda - \lambda_0)\beta_{q,j}^\times}{1 + (\lambda - \lambda_0)\alpha_j},$$

where $\beta_{q,1}^\times, \ldots, \beta_{q,m}^\times$ are the eigenvalues of $A_\pi(\lambda_0)^{-1}(F_\pi + b e_q^T)$. This implies

$$\prod_{j=1}^m \frac{1 + (\lambda - \lambda_0)\beta_{q,j}^\times}{1 + (\lambda - \lambda_0)\alpha_j} \begin{cases} \geq 1, & \lambda - \lambda_0 \geq 0 \\ \leq 1, & \lambda - \lambda_0 \leq 0 \end{cases}. \tag{6}$$

Condition (3)
For $p \in \mathrm{ran}(\overline{\pi})$, we find that

$$c_p - c_\pi^T A_\pi(\lambda)^{-1}(a_p + \lambda f_p) \geq 0$$

can be rewritten as

$$1+(\lambda-\lambda_0)c_\pi^T A_\pi(\lambda)^{-1} f_p + \frac{1}{\lambda-\lambda_0}\left\{1+(\lambda-\lambda_0)c_\pi^T A_\pi(\lambda)^{-1}(a_p+\lambda_0 f_p)\right\} \leq c_p + 1 + \frac{1}{\lambda-\lambda_0},$$

or

$$\prod_{j=1}^{m} \frac{1+(\lambda-\lambda_0)\gamma_{p,j}^\times}{1+(\lambda-\lambda_0)\alpha_j} + \frac{1}{\lambda-\lambda_0}\prod_{j=1}^{m} \frac{1+(\lambda-\lambda_0)\delta_{p,j}^\times}{1+(\lambda-\lambda_0)\alpha_j} \leq c_p + 1 + \frac{1}{\lambda-\lambda_0}, \qquad (7)$$

where $\gamma_{p,1}^\times, \dots, \gamma_{p,m}^\times$ are the eigenvalues of $A_\pi(\lambda_0)^{-1}(F_\pi + f_p c_\pi^T)$, and $\delta_{p,1}^\times, \dots, \delta_{p,m}^\times$ are the eigenvalues of $A_\pi(\lambda_0)^{-1}(F_\pi + (a_p + \lambda_0 f_p)c_\pi^T)$.

In case (5), (6), and (7) hold true, we find that

$$1+(\lambda-\lambda_0)Z(\lambda) = 1+(\lambda-\lambda_0)c_\pi^T(I_m + (\lambda-\lambda_0)A_\pi(\lambda_0)^{-1}F_\pi)^{-1}A_\pi(\lambda_0)^{-1}b = \prod_{j=1}^{m} \frac{1+(\lambda-\lambda_0)\alpha_j^\times}{1+(\lambda-\lambda_0)\alpha_j},$$

where $\alpha_1, \dots, \alpha_m$ are the eigenvalues of $A_\pi(\lambda_0)^{-1}F_\pi$, and $\alpha_1^\times, \dots, \alpha_m^\times$ are the eigenvalues of $A_\pi(\lambda_0)^{-1}(F_\pi + bc_\pi^T)$. This implies

$$Z(\lambda) = \frac{1}{\lambda-\lambda_0}\left(\prod_{j=1}^{m} \frac{1+(\lambda-\lambda_0)\alpha_j^\times}{1+(\lambda-\lambda_0)\alpha_j} - 1\right). \qquad (8)$$

We introduce a short-hand notation where we define real polynomials $P_\zeta(\lambda) = \prod_{j=1}^{m}(1+\zeta_j\lambda)$. In terms of these polynomials, we get

$$Z(\lambda) = \frac{1}{\lambda-\lambda_0}\left(\frac{P_{\alpha^\times}(\lambda-\lambda_0)}{P_\alpha(\lambda-\lambda_0)} - 1\right)$$

for $\lambda \in \Lambda_\pi$, $\lambda \neq \lambda_0$, for which

(1) $P_\alpha(\lambda-\lambda_0) > 0$,

(2) $P_{\beta_q}^\times(\lambda-\lambda_0)\begin{cases} \geq P_\alpha(\lambda-\lambda_0), & \lambda \geq \lambda_0 \\ \leq P_\alpha(\lambda-\lambda_0), & \lambda \leq \lambda_0 \end{cases}, q = 1, \dots, m.$

(3) $P_{\gamma_p^\times}(\lambda-\lambda_0) + \frac{1}{\lambda-\lambda_0}P_{\delta_p}(\lambda-\lambda_0) \leq \left(c_p + 1 + \frac{1}{\lambda-\lambda_0}\right)P_\alpha(\lambda-\lambda_0)$ for $p \in \mathrm{ran}(\overline{\pi})$.

4 Illustrative Examples

In this section, we illustrate the results from the paper using two simple examples.

Example 1 The family of linear programs which maximizes $x_1 + 2x_2$ under the constraints $x_2 + s_1 = 2$, $x_1 + \lambda x_2 + s_2 = 2$, and $x_1, x_2, s_1, s_2 \geq 0$, is parameterized with the parameter $\lambda \geq 0$. We rewrite the linear program as in (2) with data

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad c^T = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}.$$

We start with $\lambda_0 = 0$. The maximum value 6 is attained at the vertex $(x_1, x_2, s_1, s_2) = (2, 2, 0, 0)$ which corresponds to $\pi(1) = 1$ and $\pi(2) = 2$, since x_1 and x_2 are nonzero. We get

$$A_\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad F_\pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, c_\pi = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The eigenvalues of

$$A_\pi(\lambda_0)^{-1} F_\pi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

read $\alpha_1 = \alpha_2 = 0$, so that Condition (1) is satisfied automatically. To verify Condition (2), we compute the eigenvalues of

$$A_\pi(\lambda_0)^{-1}(F_\pi + b e_1^T) = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix},$$

which read $\beta_{1,1} = 1 - \sqrt{3}$ and $\beta_{1,2} = 1 + \sqrt{3}$, and of

$$A_\pi(\lambda_0)^{-1}(F_\pi + b e_2^T) = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix},$$

which read $\beta_{2,1} = 0$ and $\beta_{2,2} = 2$.

Condition (2) is equivalent to $\lambda \leq 1$. Verification of Condition (3) requires the computation of the eigenvalues of some more matrices. These matrices with their eigenvalues are

$$A_\pi(\lambda_0)^{-1}(F_\pi + f_3 c_\pi^T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma_{1,1} = 0, \ \gamma_{1,2} = 0,$$

$$A_\pi(\lambda_0)^{-1}(F_\pi + f_4 c_\pi^T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma_{2,1} = 0, \ \gamma_{2,2} = 0,$$

$$A_\pi(\lambda_0)^{-1}(F_\pi + (a_3 + \lambda_0 f_3) c_\pi^T) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \delta_{1,1} = 1 - \sqrt{2}, \ \delta_{1,2} = 1 + \sqrt{2},$$

$$A_\pi(\lambda_0)^{-1}(F_\pi + (a_4 + \lambda_0 f_4)c_\pi^T) = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, \quad \delta_{2,1} = 1,\ \delta_{2,2} = 0.$$

Condition (3) comes down to $\lambda \leq 2$. We have established that $\lambda_\pi = [0, 1]$. To derive the optimum value as a function of $\lambda \in [0, 1]$, we compute the eigenvalues of the matrix

$$A_\pi(\lambda_0)^{-1}(F_\pi + bc_\pi^T) = \begin{pmatrix} 2 & 5 \\ 2 & 4 \end{pmatrix}$$

being $\alpha_1^\times = 3 + \sqrt{11}$ and $\alpha_2^\times = 3 - \sqrt{11}$. As a result, we get

$$Z(\lambda) = (1 + (\lambda - \lambda_0)\alpha_1^\times)(1 + (\lambda - \lambda_0)\alpha_2^\times) = 6 - 2\lambda.$$

We continue with $\lambda_0 = 1$, for which the maximum value is equal to 4, attained at the vertex $(0, 2, 0, 0)$ which is a degenerate basic feasible solution. We choose $\pi(1) = 2$ and $\pi(2) = 3$, providing a nontrivial interval on which the local behavior of the optimal value can be defined. The eigenvalues and corresponding restrictions on λ are summarized in Table 1.

$\alpha_1 = 1,\ \alpha_2 = 0$	$\beta_{1,1} = 3,\ \beta_{1,2} = 0,$ $\beta_{2,1} = 1/2(1 + i\sqrt{7})$ $\beta_{2,2} = 1/2(1 - i\sqrt{7})$	$\gamma_{1,1} = 1,\ \gamma_{1,2} = 0,\ \delta_{1,1} = 3,\ \delta_{1,2} = 0,$ $\gamma_{2,1} = 1,\ \gamma_{2,2} = 0,\ \delta_{2,1} = 3,\ \delta_{2,2} = 0$
	$\lambda \geq 1$	$\lambda \leq 2$

Table 1: eigenvalues and corresponding restrictions on λ with $\lambda_0 = 1$

By computing $\alpha_1^\times = 5$ and $\alpha_2^\times = 0$, we establish that $Z(\lambda) = 4/\lambda$ for $1 \leq \lambda \leq 2$.

As we continue with $\lambda_0 = 2$ with maximum value equal to 2 and vertex $(2, 0, 2, 0)$, hence $\pi(1) = 1$ and $\pi(2) = 3$, we arrive at eigenvalues and corresponding restrictions on λ as summarized in Table 2.

$\alpha_1 = 0,\ \alpha_2 = 0$	$\beta_{1,1} = 2,\ \beta_{1,2} = 0,$ $\beta_{2,1} = 2,\ \beta_{2,2} = 0$	$\gamma_{1,1} = 1,\ \gamma_{1,2} = 0,\ \delta_{1,1} = 32,\ \delta_{1,2} = 0,$ $\gamma_{2,1} = 0,\ \gamma_{2,2} = 0,\ \delta_{2,1} = 1,\ \delta_{2,2} = 0$
		$\lambda \geq 2$

Table 2: eigenvalues and corresponding restrictions on λ with $\lambda_0 = 2$

By computing $\alpha_1^\times = 2$ and $\alpha_2^\times = 0$, we establish that $Z(\lambda) = 2$ for $\lambda \geq 2$. We observe that $Z(\lambda)$ is a piecewise rational function; see Figure 2.

Example 2 The family of linear programs which maximizes $x_1 + x_2$ under the constraints $-x_1 + x_2 + s_1 = 1,\ x_1 - \lambda x_2 + s_2 = 1,$ and $x_1, x_2, s_1, s_2 \geq 0,$ is

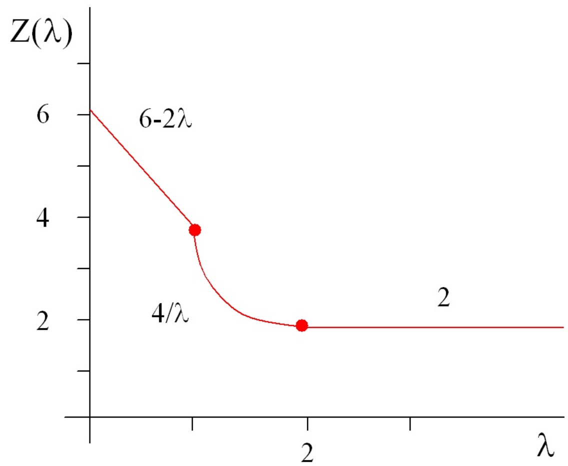

Figure 1: optimal value function $Z(\lambda)$

parameterized with the parameter $\lambda \in \mathbb{R}$. We rewrite the linear program as in (2) with data

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c^T = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}.$$

We start with $\lambda_0 = 0$ for which the maximum value is equal to 3, attained at the vertex $(1, 2, 0, 0)$, hence $\pi(1) = 1$ and $\pi(2) = 2$, providing a nontrivial interval on which the local behavior of the optimal value can be defined. The eigenvalues and corresponding restrictions on λ are summarized in Table 3.

$\alpha_1 = 0$, $\alpha_2 = -1$	$\beta_{1,1} = i$, $\beta_{1,2} = -i$, $\beta_{2,1} = 0$, $\beta_{2,2} = 1$	$\gamma_{1,1} = 0$, $\gamma_{1,2} = -1$, $\delta_{1,1} = i$, $\delta_{1,2} = -i$, $\gamma_{2,1} = 0$, $\gamma_{2,2} = -1$, $\delta_{2,1} = 1$, $\delta_{2,2} = 0$
$\lambda \neq 1$	$-1 \leq \lambda < 1$	$-1 \leq \lambda < 1$

Table 3: eigenvalues and corresponding restrictions on λ with $\lambda_0 = 0$

By computing $\alpha_1^\times = 1$ and $\alpha_2^\times = 1$, we establish that $Z(\lambda) = \frac{3+\lambda}{1-\lambda}$ for $-1 \leq \lambda < 1$.

We continue with $\lambda_0 = -1$ for which the maximum value is equal to 1, attained at the vertex $(1, 0, 2, 0)$, hence $\pi(1) = 1$ and $\pi(2) = 3$, providing a nontrivial interval on which the local behavior of the optimal value can be defined. The eigenvalues and corresponding restrictions on λ are summarized in Table 4.

$\alpha_1 = 0,\ \alpha_2 = 0$	$\beta_{1,1} = 1,\ \beta_{1,2} = 0,$ $\beta_{2,1} = 0,\ \beta_{2,2} = 2$	$\gamma_{1,1} = -1,\ \gamma_{1,2} = 0,\ \delta_{1,1} = 1,\ \delta_{1,2} = 0,$ $\gamma_{2,1} = 0,\ \gamma_{2,2} = 0,\ \delta_{2,1} = 1,\ \delta_{2,2} = 0$
		$\lambda \leq -1$

Table 4: eigenvalues and corresponding restrictions on λ with $\lambda_0 = -1$

By computing $\alpha_1^\times = 1$ and $\alpha_2^\times = 0$, we establish that $Z(\lambda) = 1$ for $\lambda \leq -1$. For $\lambda_0 > 1$, no feasible optimal solution will be found

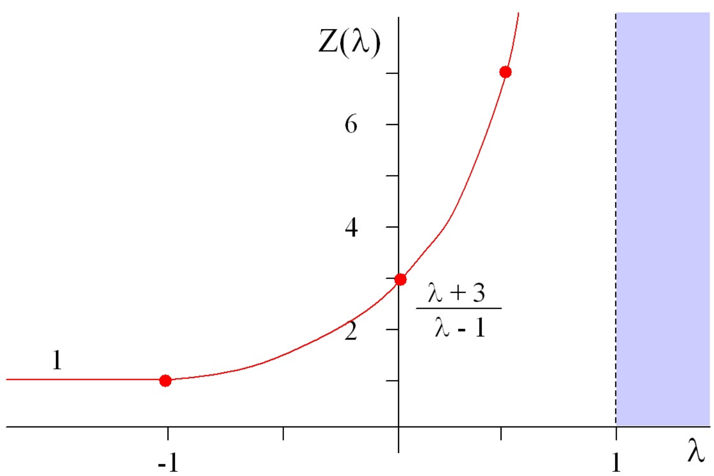

Figure 2: optimal value function $Z(\lambda)$

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