

A PROBABILISTIC ANALYSIS OF THE MULTIKNAPSACK VALUE FUNCTION

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The optimal solution value of the multiknapsack problem as a function of the knapsack capacities is studied under the assumption that the profit and weight coefficients are generated by an appropriate random mechanism. A strong asymptotic characterization is obtained, that yields a closed form expression for certain special cases.

Key words: Multiknapsack value function, Lagrangean relaxation, asymptotic characterization, uniform strong law of large numbers.

1. Introduction

The *multiknapsack* (MK) problem is to select a collection of *items* that yields maximal *profit* subject to a number of *capacity constraints*. More precisely, let us suppose that item j ($j = 1, \dots, n$) requires a_{ij} units of space in the i th knapsack ($i = 1, \dots, m$) and yields c_j units of profit upon inclusion, and that the i th knapsack has capacity b_i ($i = 1, \dots, m$). Then the multiknapsack problem can be written as:

$$\begin{aligned} \text{(MK)} \quad & \max \sum_{j=1}^n c_j x_j \\ & \text{s.t.} \quad \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, 2, \dots, m), \\ & \quad \quad x_j \in \{0, 1\} \quad (j = 1, 2, \dots, n). \end{aligned}$$

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MK has been used to model problems in the areas of scheduling and capital budgeting [9]. The problem is known to be NP-hard [6]; it is a generalization of the standard knapsack problem ($m = 1$). MK can be solved by a polynomial approximation scheme [4], but a fully polynomial one cannot exist unless $P = NP$ [5].

In this paper, we are interested in the behavior of the solution value of MK with respect to changing knapsack capacities. We shall show that if the coefficients c_j and a_{ij} ($j = 1, \dots, n$, $i = 1, \dots, m$) are generated by an appropriate random mechanism, then the sequence of optimal values, properly normalized, converges *with probability one* (w.p. 1) to a function of the b_i 's, as n goes to infinity and m remains fixed. The function will be computed in closed form for some special cases.

A number of probabilistic analyses of algorithms for the single knapsack problem ($m = 1$) has been carried out in the past [1, 2, 7, 10], but the solution value has not yet been asymptotically characterized in the above fashion. A similar comment applies to the probabilistic analysis in [4] for MK under a stochastic model less general than the one considered here. Random variables will be indicated by boldface characters.

2. Upper and lower bounds

Let us assume that the profit coefficients c_1, c_2, \dots , are nonnegative independent identically distributed (i.i.d.) random variables with common distribution function F and finite expectation Ec_1 . Let $a_j = (a_{1j}, a_{2j}, \dots, a_{mj})$ for $j = 1, 2, \dots$. We assume that $\mathbf{a}_1, \mathbf{a}_2, \dots$, are nonnegative i.i.d. random vectors. Moreover, the profit coefficients and the requirement coefficients with different indices are assumed to be independent of each other.

We shall use the *independent model* approach for the sampling of a series of problem instances of growing size with the number m of restrictions held fixed. Thus, each successive problem instance is drawn independently from the other problem instances. This is in contrast with the *incremental model*, in which a problem instance of size n is built up from the coefficients of the problem instance of size $n - 1$ by adding an extra set of coefficients (c_n, \mathbf{a}_n) that is drawn independently from the previous ones. We refer to [12] for an overview of the relation between these models. In particular, it is shown that convergence results in the independent model imply convergence results in the incremental model, thus making the independent model the more general of the two.

It is reasonable to assume that the capacities b_i grow proportionally with the number of items. Specifically, let $b_i = n\beta_i$ ($i = 1, 2, \dots, m$) for $\beta \in V := \{\beta \mid 0 < \beta < E\mathbf{a}_1\}$. As remarked in [10], the i th constraint would tend to be redundant if $\beta_i > E\mathbf{a}_{i1}$, in the sense that it would asymptotically be satisfied with probability 1 even if all items were included.

We define z_n^1 to be the optimal solution value of MK and z_n^{LP} to be the value of the LP-relaxation MKLP. Let $c_{\max} = \max_{j=1, \dots, n} \{c_j\}$.

Lemma 2.1. *For every realization $(c_1, a_1), (c_2, a_2), \dots, (c_n, a_n)$ of $(c_1, a_1), (c_2, a_2), \dots, (c_n, a_n)$, we have*

$$z_n^{\text{LP}} \geq z_n^{\text{I}} \geq z_n^{\text{LP}} - mc_{\max}. \tag{2.1}$$

Proof. The first inequality is trivial. An optimal basic solution of MKLP has at most m basic variables that are fractional. Rounding down these values yields a feasible integer solution whose value is given by z_n^{LP} decreased by at most mc_{\max} . \square

Division of (2.1) by n yields

$$z_n^{\text{LP}}/n \geq z_n^{\text{I}}/n \geq z_n^{\text{LP}}/n - mc_{\max}/n.$$

Since, $Ec_1 < \infty$, we have that $\lim_{n \rightarrow \infty} c_{\max}/n = 0$ w.p. 1. From these two observations asymptotic equivalence of z_n^{I}/n and z_n^{LP}/n w.p. 1 follows easily.

Lemma 2.2.

$$\lim_{n \rightarrow \infty} \left| \frac{z_n^{\text{I}}}{n} - \frac{z_n^{\text{LP}}}{n} \right| = 0 \quad \text{w.p. 1.} \quad \square$$

3. Asymptotic characterization of the solution value

In order to derive a function of b_i ($i = 1, 2, \dots, m$) to which z_n^{LP} and hence z_n^{I} is asymptotic w.p. 1, we consider the *Lagrangean relaxation* of MKLP, defined by

$$w_n(\lambda) := \max \left\{ \sum_{i=1}^m \lambda_i b_i + \sum_{j=1}^n \left(c_j - \sum_{i=1}^m \lambda_i a_{ij} \right) x_j \mid 0 \leq x_j \leq 1 \ (j = 1, 2, \dots, n) \right\},$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$.

For every realization of the stochastic parameters, the function $w_n(\lambda)$ is convex over the region defined by $\lambda \geq 0$. Moreover,

$$\min_{\lambda \geq 0} w_n(\lambda) = z_n^{\text{LP}}. \tag{3.1}$$

If we define the random variables

$$x_j^{\text{I}}(\lambda) := \begin{cases} 1 & \text{if } c_j - \sum_{i=1}^m \lambda_i a_{ij} > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (j = 1, 2, \dots, n),$$

then

$$w_n(\lambda) = \sum_{i=1}^m \lambda_i b_i + \sum_{j=1}^n \left(c_j - \sum_{i=1}^m \lambda_i a_{ij} \right) x_j^{\text{I}}(\lambda).$$

We also define $L_n(\lambda) := w_n(\lambda)/n$, and for $c \in \mathbb{R}$, $a \in \mathbb{R}^m$ the function $f_\lambda : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ for any $\lambda \in \mathbb{R}^m$, as

$$f_\lambda(c, a) = \lambda^T \beta + (c - \lambda^T a)x^L(\lambda),$$

with

$$x^L(\lambda) = \begin{cases} 1 & \text{if } c > \lambda^T a, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L_n(\lambda)$ can be viewed as the *mean value* of the random variable $f_\lambda(c, a)$ over n independent observations $(c_1, a_1), (c_2, a_2), \dots, (c_n, a_n)$. Because of the independence of $(c_1, a_1), (c_2, a_2), \dots, (c_n, a_n)$, the random variables $x_1^L(\lambda), x_2^L(\lambda), \dots, x_n^L(\lambda)$ are also independent. Hence, the strong law of large numbers [8, 17.3] implies that, for every $\lambda \geq 0$, $L_n(\lambda)$ converges with probability 1 to the *expectation*

$$Ef_\lambda(c, a) = \lambda^T \beta + Ec_1 x_1^L(\lambda) - \lambda^T E a_1 x_1^L(\lambda). \tag{3.2}$$

Let us write $L(\lambda) := Ef_\lambda(c, a)$ and let λ^* be a minimizer of $L(\lambda)$.

Let λ_n^* be a vector of multipliers minimizing $w_n(\lambda)$. We shall now prove that $L_n(\lambda_n^*) \rightarrow L(\lambda^*)$ with probability 1 by first strengthening our previous result from pointwise to uniform convergence with probability 1. Since the functions $L_n(\lambda)$ are convex for every realization of the stochastic parameters we may apply Theorem 10.8 from [11, p. 90], which states that pointwise convergence of a sequence of convex functions on any compact subset of their domain implies uniform convergence on this subset to a function that is also convex. Hence, all we have to show is that a minimizing sequence $\{\lambda_n^*\}$ ($n = 1, 2, \dots$) and λ^* are contained in a compact subset of \mathbb{R}_+^m .

Lemma 3.1. *There exists a number n_1 such that for all $n \geq n_1$, $L_n(\lambda)$ attains its minimum within the set*

$$S := \{\lambda \mid \lambda \geq 0, \lambda^T \beta \leq Ec_1 + 1\}$$

with probability 1. $L(\lambda)$ also attains its minimum value within S .

Proof. As a result of the strong law of large numbers we have that

$$\Pr \left\{ \exists n_1: \forall n \geq n_1, Ec_1 + 1 \geq \frac{1}{n} \sum_{j=1}^n c_j = L_n(0) \right\} = 1.$$

Since $L_n(\lambda) \geq \lambda^T \beta$, we have for any $\lambda \notin S$, $\lambda \geq 0$, that $L_n(\lambda) > Ec_1 + 1$. This together with the above probability statement yields that for any $\lambda \notin S$,

$$\Pr \{ \exists n_1: \forall n \geq n_1, L_n(\lambda) \geq Ec_1 + 1 \geq L_n(0) \} = 1.$$

Thus, for all $n \geq n_1$, $\lambda_n^* \in S$ with probability 1. For $L(\lambda)$ we have that $L(\lambda) \geq \lambda^T \beta > Ec_1 = L(0)$ for λ such that $\lambda^T \beta > Ec_1 + 1$, so that $\lambda^* \in S$ as well. \square

Since we assumed that Ec_1 exists, S is indeed a compact set. We therefore have the following lemma.

Lemma 3.2 [11, Theorem 10.8]. $L_n(\lambda) \rightarrow L(\lambda)$ w.p. 1 uniformly on S . \square

We are now in a position to prove the required result.

Theorem 3.1. $L_n(\lambda_n^*) \rightarrow L(\lambda^*)$ w.p. 1.

Proof. Uniform convergence w.p. 1 on S can be written as

$$\Pr\left\{\forall \varepsilon > 0 \exists n_0: \forall n \geq n_0, \sup_{\lambda \in S} |L_n(\lambda) - L(\lambda)| < \varepsilon\right\} = 1. \tag{3.3}$$

If $L_n(\lambda_n^*) > L(\lambda^*)$, then $|L_n(\lambda_n^*) - L(\lambda^*)| \leq L_n(\lambda^*) - L(\lambda^*)$. A similar argument for the case that $L_n(\lambda_n^*) < L(\lambda^*)$ leads to the conclusion that

$$|L_n(\lambda_n^*) - L(\lambda^*)| \leq \sup_{\lambda \in S} |L_n(\lambda) - L(\lambda)|. \tag{3.4}$$

Combination of (3.3) and (3.4) yields

$$\Pr\{\forall \varepsilon > 0 \exists n_0: \forall n \geq n_0, |L_n(\lambda_n^*) - L(\lambda^*)| < \varepsilon\} = 1. \quad \square$$

Together (3.1), Theorem 3.1 and Lemma 2.2 imply the following result.

Theorem 3.2. $n^{-1}z_n^j \rightarrow L(\lambda^*)$ w.p. 1. \square

This theorem gives the required asymptotic characterization of the solution value of MK. $L(\lambda^*)$ is a function of the righthand sides b_i ($i = 1, 2, \dots, m$) and is defined implicitly by minimization of $L(\lambda)$ over S . Its explicit computation will be considered in Section 5. We note that the only assumptions that we made hitherto were the i.i.d. property of the stochastic parameters and the finiteness of the first moment of the profit coefficients. Under these assumptions also the following stronger version of Theorem 3.1 can be proved, indicating regular asymptotic behavior of the sequence of optimal Lagrangean multipliers λ_n^* .

Theorem 3.3. *With probability 1, the accumulation points of the series $\lambda_1^*, \lambda_2^*, \dots$, are contained in the set of values of λ that minimize $L(\lambda)$.*

Proof. We consider the following inequality:

$$|L(\lambda_n^*) - L(\lambda^*)| \leq |L(\lambda_n^*) - L_n(\lambda_n^*)| + |L_n(\lambda_n^*) - L(\lambda^*)|.$$

Lemma 3.2 and Theorem 3.1 applied to the right-hand side imply immediately that $L(\lambda_n^*) \rightarrow L(\lambda^*)$ w.p. 1. The proof can now be completed by application of Corollary 27.2.1 in [11, p. 266]. \square

In case λ^* is the unique minimizer of $L(\lambda)$, we have that $\lambda_n^* \rightarrow \lambda^*$ w.p. 1 (see [11, Corollary 27.2.2, p. 266]).

4. Smoothness properties

In this section we investigate some properties of $L(\lambda)$ related to properties of the distributions of the stochastic parameters.

Recall the formulation (3.2) of $L(\lambda)$ and rewrite it as

$$L(\lambda) = \lambda^T \beta + E((c_1 - \lambda^T \mathbf{a}_1) x_1^L(\lambda)).$$

Let us assume that c and \mathbf{a} are independent. Then integration of $L(\lambda)$ by parts yields

$$L(\lambda) = \lambda^T \beta + E \int_{\lambda^T \mathbf{a}_1}^{\infty} (c - \lambda^T \mathbf{a}_1) dF(c) = \lambda^T \beta + E \int_{\lambda^T \mathbf{a}_1}^{\infty} (1 - F(c)) dc.$$

We notice that, under our assumption that $E c_1 < \infty$, $L(\lambda)$ is finite for each $\lambda \geq 0$. Additional assumptions with respect to the distribution of c and \mathbf{a} allow us to establish other smoothness properties of $L(\lambda)$ as shown below. Throughout this section we will assume that c and \mathbf{a} are independent and that $E \mathbf{a}_1$ is finite.

Lemma 4.1. *If the distribution function F of c is continuous, then $L(\lambda)$ is differentiable and its gradient is given by*

$$\nabla L(\lambda) = \beta - E \mathbf{a}_1 x_1^L(\lambda).$$

Proof. Differentiability of $L(\lambda)$ is ensured through Theorem 23.1 in [11, p. 213]. To compute $\partial L / \partial \lambda_k$ ($k = 1, \dots, m$) suppose for the moment that differentiation and integration may be interchanged. Then

$$\frac{\partial L(\lambda)}{\partial \lambda_k} = \beta_k - E \left((1 - F(\lambda^T \mathbf{a}_1)) \frac{\partial \lambda^T \mathbf{a}_1}{\partial \lambda_k} \right) = \beta_k - E(\mathbf{a}_{k1} (1 - F(\lambda^T \mathbf{a}_1))),$$

which, from the independence of c and \mathbf{a} , is equal to

$$\frac{\partial L(\lambda)}{\partial \lambda_k} = \beta_k - E \mathbf{a}_{k1} x_1^L(\lambda). \quad (4.1)$$

Under the assumption that $E \mathbf{a}_1$ is finite the interchange of expectation and derivative can be justified by the Dominated Convergence Theorem as in Application 3° of [8, p. 126]. \square

Since the constraints $\lambda \geq 0$ satisfy the first-order constraint qualifications in λ^* [3], Kuhn-Tucker conditions hold at λ^* .

Lemma 4.2. λ^* satisfies the following conditions for $i = 1, 2, \dots, m$:

- (i) $\lambda_i^* (\beta_i - E \mathbf{a}_{i1} x_1^L(\lambda^*)) = 0$,
- (ii) $E \mathbf{a}_{i1} x_1^L(\lambda^*) \leq \beta_i$. \square

Let f be the density function of \mathbf{c} .

Lemma 4.3. *If F is continuous, f is bounded, and $E(\mathbf{a}_1 \mathbf{a}_1^T) < \infty$, then $L(\lambda)$ is twice differentiable.*

Proof. Differentiation of (4.1) with respect to λ_l yields

$$\frac{\partial L(\lambda)}{\partial \lambda_k \partial \lambda_l} = -\frac{\partial}{\partial \lambda_l} E(\mathbf{a}_{k1}(1 - F(\lambda^T \mathbf{a}_1))) = E(\mathbf{a}_{k1} \mathbf{a}_{l1} f(\lambda^T \mathbf{a}_1)), \quad (4.2)$$

since expectations and derivatives may be interchanged under the assumptions of the lemma, which cause the last term of (4.2) to be bounded. Again the interchange is justified by the Dominated Convergence Theorem [8, p. 126]. \square

Lemma 4.4. *Assume that \mathbf{c} has bounded density function f and that F is continuous, has bounded non-singular support $[\mu, \nu]$, and is strictly increasing over $[\mu, \nu]$. Furthermore assume that the random vector \mathbf{a} has a density function with positive density over a convex and open set, and that $E(\mathbf{a}_1 \mathbf{a}_1^T) < \infty$. Then $L(\lambda)$ has a unique minimum.*

Proof. From Lemma 3.1 we know that $L(\lambda)$ has a minimizer, which we denote by λ^* . Suppose for the moment that there exists a vector \mathbf{a}_0 with positive density such that $\mu < \lambda^{*T} \mathbf{a}_0 < \nu$. Then there is a neighbourhood N_{λ^*} of λ^* such that for any $\lambda \in N_{\lambda^*}$ we have $\mu < \lambda^T \mathbf{a}_0 < \nu$. Since the assumptions of Lemma 4.3 are satisfied, equation (4.2) holds. Moreover, since F is strictly increasing over $[\mu, \nu]$, f is positive almost everywhere over $[\mu, \nu]$ so that for all $\lambda \in N_{\lambda^*}$ the second order derivative matrix of $L(\lambda)$ is positive definite. Therefore λ^* is a unique local minimum and by the convexity of $L(\lambda)$ also the unique global minimum.

We show that a vector \mathbf{a}_0 exists for which $\mu < \lambda^{*T} \mathbf{a}_0 < \nu$. First suppose that no such vector exists, then either $\lambda^{*T} \mathbf{a} > \nu$ for all \mathbf{a} with positive density or $\lambda^{*T} \mathbf{a} < \mu$ for all \mathbf{a} with positive density (since this set is open and convex). In the former case $L(\lambda) = \lambda^T \beta$ for all λ in some neighbourhood of λ^* . Since $\lambda^{*T} \mathbf{a} > \nu$ implies that $\lambda^* \neq 0$ it is easy to see that by decreasing positive components of λ^* we obtain a function value of $L(\lambda)$ smaller than $L(\lambda^*)$. In the other case, $\lambda^{*T} \mathbf{a} < \mu$, we have that $L(\lambda) = \lambda^T \beta + E \mathbf{c}_1 - \lambda^T E \mathbf{a}_1$ for all λ in some neighbourhood of λ^* . Since we assumed that $\beta < E \mathbf{a}_1$, $L(\lambda)$ can be decreased below $L(\lambda^*)$ by increasing λ^* , again providing a contradiction. Thus, there exists a vector \mathbf{a}'_0 with positive density such that $\mu \leq \lambda^{*T} \mathbf{a}'_0 \leq \nu$. By the assumption that the set on which the density of \mathbf{a} is positive is open and convex there also exists a vector \mathbf{a}_0 with positive density such that $\mu < \lambda^{*T} \mathbf{a}_0 < \nu$, which completes the proof. \square

5. Special cases: $m = 1$, $m = 2$

In this section, we assume (as in [4, 7, 10]), that the profit coefficients \mathbf{c}_j as well as the requirements \mathbf{a}_{ij} are independent uniformly distributed over $[0, 1]$, $j = 1, \dots, n$, $i = 1, \dots, m$, and carry out some actual computations on the solution value of MK.

In the case that $m = 1$, straightforward calculations lead to the following formulae:

$$Eax^L(\lambda) = \begin{cases} \frac{1}{2} - \frac{1}{3}\lambda & \text{if } \lambda \leq 1, \\ 1/(6\lambda^2) & \text{if } \lambda > 1, \end{cases} \quad Ecx^L(\lambda) = \begin{cases} \frac{1}{2} - \frac{1}{6}\lambda^2 & \text{if } \lambda \leq 1, \\ 1/(3\lambda) & \text{if } \lambda > 1, \end{cases}$$

$$L(\lambda) = \begin{cases} \lambda(\beta - \frac{1}{2} + \frac{1}{3}\lambda) + \frac{1}{2} - \frac{1}{6}\lambda^2 & \text{if } \lambda \leq 1, \\ \lambda(\beta - 1/(6\lambda^2)) + 1/(3\lambda) & \text{if } \lambda > 1, \end{cases}$$

$$\lambda^* = \begin{cases} (6\beta)^{-1/2} & \text{if } 0 < \beta < \frac{1}{6}, \\ \frac{3}{2} - 3\beta & \text{if } \frac{1}{6} \leq \beta < \frac{1}{2}, \end{cases}$$

$$L(\lambda^*) = \begin{cases} (\frac{2}{3}\beta)^{1/2} & \text{if } 0 < \beta < \frac{1}{6}, \\ -\frac{3}{2}\beta^2 + \frac{3}{2}\beta + \frac{1}{8} & \text{if } \frac{1}{6} \leq \beta < \frac{1}{2}. \end{cases}$$

The graph of $L(\lambda^*)$ as a function of β is shown in Figure 1. Notice that in accordance with Lemmas 4.1 and 4.2, $Eax_1^L(\lambda^*) - \beta = 0$.

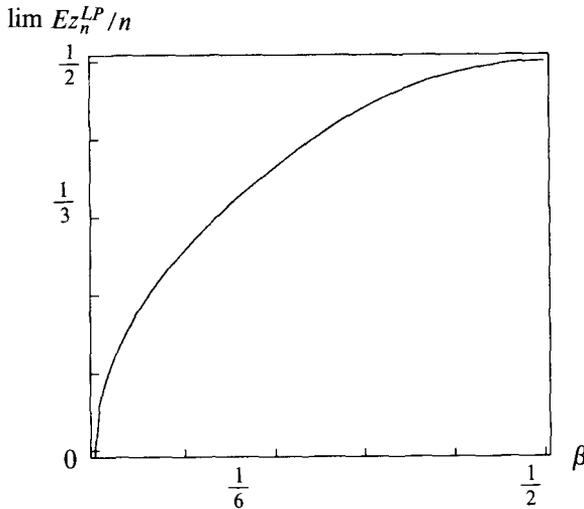


Fig. 1.

In the case that $m = 2$, $Ecx_1^L(\lambda)$, $Ea_1x_1^L(\lambda)$ and $Ea_2x_1^L(\lambda)$ take different forms over different regions. If $\beta_1 \cong \beta_2$, these are defined as follows:

$$A := \{(\beta_1, \beta_2) \mid \beta_1 \cong \beta_2, \beta_2 \cong 24\beta_1^2\},$$

$$B := \{(\beta_1, \beta_2) \mid \beta_2 \leq \frac{8}{3}\beta_1^2, \beta_2 \leq \frac{1}{6}, \beta_1 < \frac{1}{2}\},$$

$$C := \{(\beta_1, \beta_2) \mid \beta_2 \leq \frac{4}{3}\beta_1 - \frac{1}{6}, \beta_2 > \frac{1}{6}, \beta_1 < \frac{1}{2}\},$$

$$D := \{(\beta_1, \beta_2) \mid \beta_1 \cong \beta_2, \beta_2 > \frac{4}{3}\beta_1 - \frac{1}{6}, \beta_1 + \beta_2 \cong \frac{5}{12}\},$$

$$E := \{(\beta_1, \beta_2) \mid \beta_1 \cong \beta_2, \beta_2 > \frac{8}{3}\beta_1^2, \beta_1 + \beta_2 < \frac{5}{12}\}.$$

The regions are depicted in Figure 2.

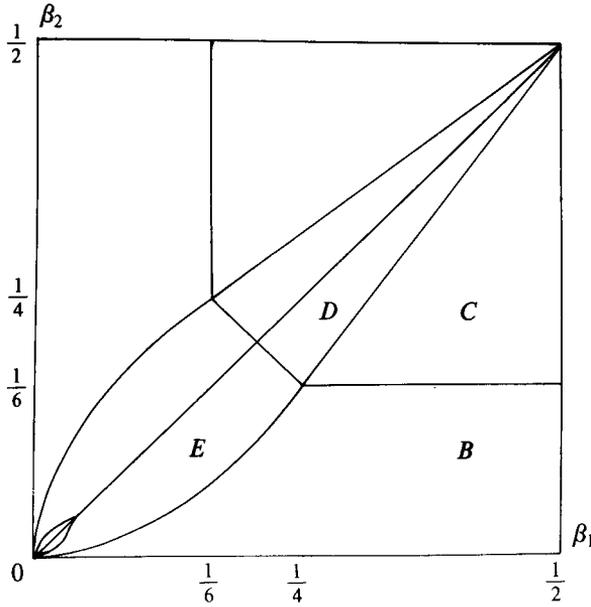


Fig. 2.

The values of λ^* and $L(\lambda^*)$ in the corresponding five regions where $\beta_1 < \beta_2$ can be obtained by exchanging β_1 with β_2 and λ_1 with λ_2 in the formulae given below.

Region A: $\lambda_1^* = (\beta_2 / (24\beta_1^2))^{1/3}$, $\lambda_2^* = (\beta_1 / (24\beta_2^2))^{1/3}$,
 $L(\lambda^*) = \frac{1}{2}(9\beta_1\beta_2)^{1/3}$;

Region B: $\lambda_1^* = 0$, $\lambda_2^* = (1 / (6\beta_2))^{1/2}$, $L(\lambda^*) = (\frac{2}{3}\beta_2)^{1/2}$;

Region C: $\lambda_1^* = 0$, $\lambda_2^* = \frac{3}{2} - 3\beta_2$, $L(\lambda^*) = \frac{3}{2}\beta_2^2 + \frac{3}{2}\beta_2 + \frac{1}{8}$;

Region D: $\lambda_1^* = \frac{1}{7}(36\beta_2 - 48\beta_1 + 6)$, $\lambda_2^* = \frac{1}{7}(36\beta_1 - 48\beta_2 + 6)$,
 $L(\lambda^*) = \frac{1}{7}(-24\beta_2^2 + 36\beta_1\beta_2 + 6\beta_2 + \frac{1}{2})$.

Region E: A closed form equation for the values λ^* , $L(\lambda^*)$ with β lying in E is complicated, though not impossible, since it involves the solution of an equation of degree four. Numerical evaluation is easier through use of an appropriate non-linear programming routine.

A picture of the surface $L(\lambda^*)$, defined over the (β_1, β_2) plane and evaluated either analytically or numerically, is presented in Figure 3.

Calculation of $\beta_i - Ea_{i1}x_1^i(\lambda^*)$ ($i = 1, 2$) for the regions A, B, C and D yields the following result. In A and D both $\beta_1 - Ea_{11}x_1^1(\lambda^*) = 0$ and $\beta_2 - Ea_{21}x_1^1(\lambda^*) = 0$, while in B and C only $\beta_2 - Ea_{21}x_1^1(\lambda^*) = 0$ and $\beta_1 - Ea_{11}x_1^1(\lambda^*) > 0$. This says that in A and D the expected slack in both constraints tend to 0 in an optimal solution, while in B and C this is only true for the tighter constraint. This corresponds to intuition: when β_2 is sufficiently small with respect to β_1 , as in B and C, the first constraint

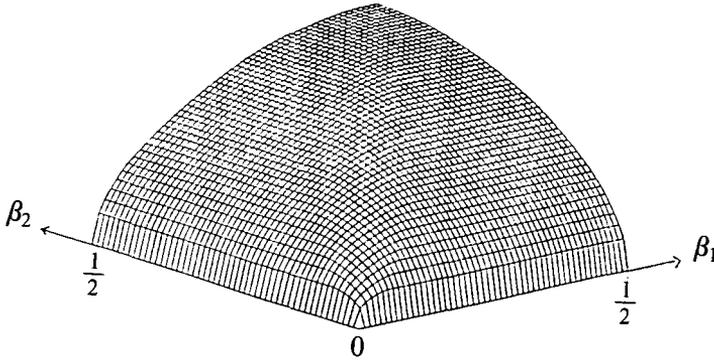


Fig. 3.

can be disregarded and MK is reduced to a single knapsack problem. To support this conclusion, observe that the values of λ_2^* and $L(\lambda^*)$ obtained for the regions B and C are identical to the corresponding ones derived for the case $m = 1$.

Similar calculations can be carried out for $m \geq 3$, even though in these cases only numerical approximation of λ^* and $L(\lambda^*)$ is possible for many values of β . The computation of $L(\lambda)$ and $\nabla L(\lambda)$ amounts to integrating the density function over regions defined by linear inequalities. In many situations, closed form expressions for these integrals can be derived in principle.

The implication of the results presented for the analysis of certain generalized greedy solution methods for MK will form the subject of a future paper.

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