Stable probability distributions and their domains of attraction: a direct approach

J.L. Geluk and L. de Haan

Econometric Institute
Erasmus University Rotterdam
P.O. Box 1738
NL-3000 DR Rotterdam
The Netherlands

August 15, 1997

Abstract

The theory of stable probability distributions and their domains of attraction is derived in a direct way (avoiding the usual route via infinitely divisible distributions) using Fourier transforms. Regularly varying functions play an important role in the exposition.

Keywords: Domain of attraction, (generalised) regular variation, Characteristic function

1 Introduction and main results

Let $X, X_1, X_2, \ldots$ be independent random variables all of them from the same probability distribution with distribution function F. Consider the sequence $S_n := X_1 + X_2 + \ldots + X_n,\ n = 1, 2, \ldots$ and suppose that for some sequences of norming constants $a_n > 0$ and b_n $(n = 1, 2, \ldots)$ the sequence $S_n/a_n - b_n$ has a non-degenerate limit distribution.

In this note we shall find the general form of all the possible limit distributions and for each of these limit distributions we shall give necessary and sufficient conditions on the distribution function F in order that $S_n/a_n - b_n$ converges to that particular distribution function.

The limit distributions are called stable distributions and the set of distribution functions F such that $S_n/a_n - b_n$ converges to a particular stable distribution is called its domain of attraction. Thus we shall identify all stable distributions and their domains of attraction.

The indicated results have been developed more than sixty years ago. One of the earliest systematic treatments is in Paul Lévy's famous book 'Théorie de l'addition des variables aléatoires" [13]. A well known complete description of the theory is the book by Gnedenko and Kolmogorov [8]. Various standard texts in probability theory offer an exposition of the subject, for example Breiman [2], Feller [6], Dudley [4]. In these texts the theory of stable distributions is treated as part of the (more general and more involved) theory of infinitely divisible distributions. Although infinitely divisible distributions form an interesting and useful subject of probability theory, the stable distributions have attracted far more attention, both in theoretical research (see e.g. the books by Zolotarev [18] and Samorodnitsky and Taqqu [16]) as well as in applied research (see e.g. Fama [5], Kunst [12], Mandelbrot [14], Samuelson [17]).

In contrast to the mentioned references, in this note the theory is developed ab initio, independent of results from the theory of infinitely divisible distributions which is too complicated to be included in a standard course of probability theory. We have tried to present the theory of stable distributions in a sufficiently streamlined form for presentation in such a course.

We now set out to develop some preliminary results that allow us to formulate the two main theorems. We start from the limit relation:

$$\frac{S_n}{a_n} - b_n \xrightarrow{d} Y,$$

or equivalently

$$\lim_{n\to\infty} P(\frac{S_n}{a_n} - b_n \le x) = G(x) \tag{1}$$

for all continuity points x of G, the distribution function of the non-degenerate random variable Y. The first question is if it is possible to have different limit distributions for different choices of a_n and b_n. Khinchine's convergence to types theorem (Feller [6], Ch. VIII.2, lemma 1) says that a different choice of the norming constants can only result in a limit distribution function of the form $G(Ax+B)$ with $A>0$ and B real. The set of all such transforms of G will be called the type of G. From now on when we talk about a limit distribution we shall mean the entire type so that no confusion is possible.

First of all we are going to reformulate the limit relation (1) in terms of the characteristic functions (or Fourier transforms). Define for $s \in \mathbb{R}$ the characteristic functions

$$\phi(s) := Ee^{isX} = \int_{-\infty}^{\infty} e^{isx} dF(x)$$

and

$$\psi(s) := Ee^{isY} = \int_{-\infty}^{\infty} e^{isx} dG(x),$$

or, what is more convenient in the present setup

$$\lambda(t) := \phi(1/t)$$

and

$$g(t) := \psi(1/t)$$

for $t \in [-\infty, \infty] \setminus \{0\}$. By Lévy's continuity theorem for characteristic functions (Feller [6] Ch. XV.3) relation (1) is equivalent to

$$\lim_{n\to\infty} e^{-ib_n/t}\lambda^n(a_n t) = g(t), \; t \in [-\infty, \infty] \setminus \{0\} \tag{2}$$

uniformly on neighborhoods of $\pm\infty$. Note that for $t = \pm\infty$ both sides equal 1.

We start with a definition and a preliminary result.

Definition A positive measurable function f is regularly varying if there exists a constant $\gamma \in \mathbb{R}$, the index (or order) such that

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^{\gamma} \text{ for all } x > 0. \tag{3}$$

In this case we will use the notation $f \in RV_{\gamma}$. A function in RV_0 is called slowly varying. For positive measurable f the limit in (3) is either identically 0 or of the form given above.

Proposition If (1) holds, then $|g(t)|^2 = e^{-c|t|^{-\alpha}}$ for some $\alpha \in (0, 2]$ and $c > 0$. Moreover

$$\lim_{t\to\infty} \frac{-\log|\lambda(tx)|}{-\log|\lambda(t)|} = x^{-\alpha} \text{ for } x > 0, \tag{4}$$

i.e. $-\log|\lambda|$ is regularly varying with index $-\alpha$.

Proof. From (2) we have

$$\lim_{n\to\infty} |\lambda(a_n t)|^n = |g(t)|$$

locally uniformly near $\pm\infty$. It follows that we have

$$\lim_{n\to\infty} -n \log|\lambda(a_n t)| = -\log|g(t)|, \tag{5}$$

for each $t \in \mathbb{R}, t \neq 0$ for which $g(t) \neq 0$. For such t it follows that $\log|\lambda(a_n t)| \to 0$, hence $a_n \to +\infty$ (note that $a_n > 0$ by assumption). Moreover replacing n with $n+1$ gives

$$\lim_{n\to\infty} -(n+1)\log|\lambda(\frac{a_{n+1}}{a_n} a_n t)| = -\log|g(t)|,$$

which in combination with (5) implies $a_{n+1}/a_n \to 1$ as $n \to \infty$ since convergence in (5) is uniform on neighborhoods of infinity. Application of Lemma 9 below then shows that the function $-\log|\lambda|$ is regularly varying and its order, say $-\alpha$ has to be non-positive since $\lim_{t\to\infty} -\log|\lambda(t)| = 0$ by (5). Dividing (5) by its counterpart for $t = 1$ we find

$$\lim_{n\to\infty} \frac{-\log|\lambda(a_n t)|}{-\log|\lambda(a_n)|} = \frac{\log|g(t)|}{\log|g(1)|},$$

hence $\log|g(t)|/\log|g(1)| = t^{-\alpha}$ for $t > 0$. Since $|g(t)|^2 = g(t)g(-t)$ is an even function we have $\log|g(t)|/\log|g(1)| = |t|^{-\alpha}$ for $t \neq 0$. Note that $|g(t)|^2$ is a characteristic function as a product of two characteristic functions.

The restriction $\alpha > 0$ stems from the fact that Y is non-degenerate. Next we show that necessarily $\alpha \leq 2$: the assumption $\alpha > 2$ would lead to a non-constant characteristic function with a vanishing second order derivative at 0, which is a contradiction.

Definition Any probability distribution function G with characteristic function g satisfying

$$|g(t)|^2 = e^{-c|t|^{-\alpha}} \text{ for some } \alpha \in (0, 2] \tag{6}$$

is called a stable distribution with index α or α-stable distribution.

Definition The class of distribution functions F for which (1) holds with a limit distribution G satisfying (6) is called the α-stable domain of attraction.

Notation: $F \in D_\alpha$.

The classes of distributions introduced above are useful for the rest of this note. However the α-stable distributions do not form one type. We shall see that we need another (skewness) parameter to describe the fulll class of all stable distributions. Note that the characteristic functions $|g(t)|^2$ from (6) represent probability distributions that are symmetric about zero.

We are now in a position to formulate the main results. Define

$$U(t) := Re\lambda(t)$$

and

$$V(t) := Im\lambda(t)$$

and for $0 < \alpha < 2$

$$s_\alpha = \int_0^\infty x^{-\alpha} \sin x dx$$

and

$$c_\alpha = \int_1^\infty x^{-\alpha} \cos x dx + \int_0^1 x^{-\alpha}(\cos x - 1) dx.$$

The constants s_α and c_α can also be written in terms of the gamma function. We have for $0 < \alpha < 2, \alpha \neq 1$

$$s_\alpha = \,, (1-\alpha) \cos \frac{\alpha\pi}{2}$$

and

$$c_\alpha = \,, (1-\alpha) \sin \frac{\alpha\pi}{2} - \frac{1}{1-\alpha}.$$

In case $\alpha = 1$ one should replace the formulae with the corresponding limit as $\alpha \to 1$: $s_1 = \pi/2$ and $c_1 = \,, '(1)$. ($-, '(1)$ is Euler's constant.)

Further we adopt the convention that the function $(t^a - 1)/a$ is defined for all $t > 0, a \in \mathbb{R}$ and reads as $\log t$ for $a = 0$ (by continuity). Also the function $\psi_{\alpha,p}(t)$ in formula (7) below is defined to be 1 at $t = 0$ (by continuity).

Theorem 1 *Suppose $0 < \alpha < 2$. Every α-stable distribution (or rather distribution type) has a characteristic function of the following form:*

$$\psi_{\alpha,p}(s) = \exp -\{|s|^\alpha + is\frac{2p-1}{s_\alpha}(1 + (1-\alpha)c_\alpha)\frac{|s|^{\alpha-1} - 1}{\alpha - 1}\} \qquad (7)$$

with $0 \leq p \leq 1$.

The following statements are equivalent.

(i) $F \in D_\alpha$

(ii) $1 - F(t) + F(-t) \in RV_{-\alpha}$ and there exists a constant $p \in [0,1]$ such that the tail balance condition

$$\lim_{t\to\infty} \frac{1 - F(t)}{1 - F(t) + F(-t)} = p$$

holds.

(iii) $1 - U(t) \in RV_{-\alpha}$ and there exists a constant $p \in [0,1]$ such that

$$\lim_{t\to\infty} \frac{txV(tx) - tV(t)}{t(1-U(t))} = \frac{(2p-1)[1 + (1-\alpha)c_\alpha]}{s_\alpha} \, \frac{|x|^{1-\alpha} - 1}{1-\alpha}, \quad x \in \mathbb{R} \setminus \{0\}. \quad (8)$$

Further, if any of (i),(ii) or (iii) holds, then

$$\lim_{t\to\infty} \frac{1 - U(t)}{1 - F(t) + F(-t)} = s_\alpha \qquad (9)$$

and

$$\lim_{t\to\infty} \frac{V(t) - \frac{1}{t}\int_0^t (1-F(s)-F(-s))ds}{1-F(t)+F(-t)} = (2p-1)c_\alpha. \tag{10}$$

Remark 1 The parameter α is the same in the three equivalent statements of Theorem 1. The theorem is also true if one keeps α and p fixed in the three statements. Statement (i) then reads: (1) holds with G such that its characteristic function ψ is as in (7).

Remark 2 Unlike in other texts here and in the proof we do not treat the case $\alpha = 1$ separately. However for $\alpha \neq 1$ the statements of the theorem can be simplified: line (7) reads: (remember we need only one member of the type)

$$\psi_{\alpha,p}(s) = \exp -\{|s|^\alpha + is\frac{2p-1}{s_\alpha(\alpha-1)}(1+(1-\alpha)c_\alpha)|s|^{\alpha-1}.\}$$

From Lemma 1 below it follows that in case $0 < \alpha < 1$ (iii) is equivalent to $1-U(t) \in RV_{-\alpha}$ and $V(t) \sim c^*_\alpha(1-U(t))/(1-\alpha)$ as $t\to\infty$, where

$$c^*_\alpha = (2p-1)[1+(1-\alpha)c_\alpha]/s_\alpha. \tag{11}$$

If $1 < \alpha < 2$, then (iii) is equivalent to $1-U(t) \in RV_{-\alpha}$, $tV(t) \to \mu$ for some constant μ and $\mu - tV(t) \sim c^*_\alpha(1-U(t))/(\alpha-1)$ as $t\to\infty$. In view of (10) we must have $\mu = EX$, which is finite in this case.

Remark 3 Suppose any of (i),(ii) or (iii) holds. We now indicate how to choose the normalizing constants $a_n > 0$ and b_n in terms of either the distribution function F or the characteristic function ϕ (i.e. in terms of the functions U and V).

The relation (1) holds with G such that the function ψ is exactly as in (7) (i.e. this distribution and not another one of the same type) if we choose a_n, b_n such that

$$\lim_{n\to\infty} ns_\alpha(1-F(a_n)+F(-a_n)) = 1$$

and

$$b_n = \frac{n}{a_n}\int_0^{a_n}(1-F(s)-F(-s))ds + \frac{2p-1}{s_\alpha}c_\alpha.$$

See (9) and part (iii)$\to$(i) of the proof. Note that the above choice of the sequence a_n is always possible since $1-F(x)+F(-x)$ is regularly varying. It follows from relations (9) and (10) that the same limit distribution is obtained with the alternative choices of a_n and b_n

$$\lim_{n\to\infty} n(1-U(a_n)) = 1$$

and

$$b_n = nV(a_n).$$

Remark 4 The behavior of U and V at $-\infty$ follows from (9) and (10) since U is an even and V is an odd function.

The case $\alpha = 2$ is covered by the following result.

Theorem 2 *Every 2-stable distribution (or rather distribution type) has a characteristic function of the following form:*

$$\psi_2(s) = \exp(-s^2), \tag{12}$$

corresponding to the normal distribution.

The following statements are equivalent

(i) $F \in D_2$.
(ii) The function $H_1(t) := \int_0^t u(1 - F(u) + F(-u))du$ *is slowly varying.*
(iii) $1 - U(t) \in RV_{-2}$ *and for* $x > 0$

$$\frac{\mu - tV(t)}{t(1 - U(t))} \to 0, \quad t \to \infty. \tag{13}$$

If (i) holds, then as $t \to \infty$

$$1 - U(t) \sim \frac{H_1(t)}{t^2} \tag{14}$$

and

$$V(t) - \frac{\mu}{t} = o(\frac{H_1(t)}{t^2}), \tag{15}$$

where $\mu = EX$.

Remark The behavior of U and V at $-\infty$ follows from (14) and (15) since U is an even and V is an odd function.

Using the results of the Theorems 1 and 2 one verifies easily that the stable distribution functions are precisely those distribution functions G such that if $Y, Y_1, Y_2, \ldots$ are i.i.d. G, there exist constants $A_n > 0$ and B_n such that for $n \geq 1$ $(Y_1 + Y_2 + \ldots + Y_n)/A_n - B_n$ has the same distribution as Y.

2 Auxiliary results

Before we prove the theorems we collect some basic facts about regularly varying functions in a sequence of lemmas. Lemmas 1 up to 7 are standard results that are useful in other contexts as well. Lemma 8 (preparing for the use of Lebesgue's theorem on dominated convergence) and 9 (on replacing a sequence by a continuous variable in the limit relation) are specific for the present setup.

Lemma 1 *(see [7], theorem 1.9, 1.10)*

Suppose f is a measurable function and there is a positive function a such that for all $x > 0$

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma} \tag{16}$$

where γ is a real parameter. (The right hand side is interpreted as $\log x$ for $\gamma = 0$.)

If (16) holds with $\gamma > 0$, then $a(t) \sim \gamma f(t)$ as $t \to \infty$, both functions tend to infinity and hence $f \in RV_\gamma$.

If (16) holds with $\gamma < 0$, then $\lim_{t\to\infty} f(t) =: f(\infty)$ exists and $a(t) \sim -\gamma(f(\infty) - f(t)) \to 0$ $(t \to \infty)$. Hence $f(\infty) - f(t)$ is regularly varying of order γ.

If (16) holds with $\gamma = 0$, then $a(t) = o(f(t))(t \to \infty)$ and a is regularly varying of order 0, i.e. slowly varying. Also $\lim_{t\to\infty} f(t) =: f(\infty)$ exists (finite or $+\infty$). If $f(\infty) = \infty$, then $f \in RV_0$. If $f(\infty) < \infty$, then $f(\infty) - f(t)$ is slowly varying and $a(t) = o(f(\infty) - f(t))$ as $t \to \infty$.

Remark 1 For f measurable the limit in (16), if not identically zero, is necessarily of the form given.

Remark 2 If the limit in (16) exists and is identically 0 for $x > 0$ with $a \in RV_\gamma$, then

if $\gamma > 0, f(t) = o(a(t))$ as $t \to \infty$,

if $\gamma < 0, f(\infty)$ exists and $f(\infty) - f(t) = o(a(t))$ as $t \to \infty$.

Lemma 2 *(see [7], theorem 1.20) Suppose that the function f is integrable over finite intervals and that (16) holds with $\gamma = 0$.*

(i) Let $k : \mathbb{R}^+ \to \mathbb{R}$ be a function which is bounded on $[0, A]$ for some $A > 0$. Then as $t \to \infty$

$$\int_0^A \frac{f(ts) - f(t)}{a(t)} k(s)ds \to \int_0^A \log s\ k(s)ds.$$

(ii) Let $k : \mathbb{R}^+ \to \mathbb{R}$ be a function such that $\int_A^\infty s^\varepsilon k(s)ds < \infty$ for some $A, \varepsilon > 0$. Then

$$\int_A^\infty \frac{f(ts) - f(t)}{a(t)} k(s)ds \to \int_A^\infty \log s\ k(s)ds.$$

Lemma 3 *(cf. Bingham et al. [1], Ch. 4) Suppose that the function g is integrable over finite intervals and that (3) holds with f positive. Assume $g(t)/f(t) \to c \geq 0$ as $t \to \infty$.*

(i) *Suppose* $\gamma > -1$ *in (3). Let* $k : \mathbb{R}^+ \to \mathbb{R}$ *be a function which is bounded on* $[0, A]$ *for some* $A > 0$*. Then as* $t \to \infty$

$$\int_0^A \frac{g(tx)}{f(t)} k(x)dx \to c \int_0^A x^\gamma\ k(x)dx.$$

(ii) *Let* $k : \mathbb{R}^+ \to \mathbb{R}$ *be a function such that* $\int_A^\infty x^{\gamma+\varepsilon}|k(x)|dx < \infty$ *for some* $A, \varepsilon > 0$*. Then*

$$\int_A^\infty \frac{g(tx)}{f(t)} k(x)dx \to c \int_A^\infty x^\gamma\ k(x)dx.$$

Remark

If the limits in (16) and (3) are identically zero, then the limits in Lemma 2 and Lemma 3 are also identically zero.

Lemma 4 *(see e.g. Ibragimov and Linnik [10], proof of Lemma 2.6.1) Suppose* g *is a non-increasing function and* $g(t)/f(t) \to c \in [0, \infty)$ *as* $t \to \infty$ *for some function* $f \in RV_{-\alpha}$ $(0 < \alpha < 2)$*. For any* $\varepsilon > 0$ *there exist constants* A_0 *and* t_0 *such that for all* $t \geq t_0$ *and* $A > A_0$

$$|\int_A^\infty \frac{g(tx)}{f(t)} \sin x dx| < \varepsilon$$

and

$$|\int_A^\infty \frac{g(tx)}{f(t)} \cos x dx| < \varepsilon.$$

Proof By the second mean value theorem for all $B > A$

$$\int_A^B \frac{g(tx)}{f(t)} \sin x dx = \frac{g(tA)}{f(t)} \int_A^\xi \sin x dx$$

for some $\xi \in [A, B]$, hence

$$|\int_A^B \frac{g(tx)}{f(t)} \sin x dx| \leq 2 \frac{g(tA)}{f(tA)} \frac{f(tA)}{f(t)} \to 2cA^{-\alpha}$$

as $t \to \infty$. The proof of the first statement is complete since the right hand side tends to zero as $A \to \infty$. The proof of the second statement is similar.

Next we give a version of the monotone density theorem (see e.g. Bingham et al. [1], Ch. I.7.3).

Lemma 5 *If* $f(t) := \int_0^t \psi(s)ds$ *is regularly varying with index* $\alpha > 0$ *and* ψ *is monotone, then* $\psi \in RV_{\alpha-1}$*.*

In the sequel we need a modification of the above Lemma.

Lemma 6 *Suppose f is non-decreasing. If there exists $\beta \geq 0$ and a positive function a such that the function $\overline{f}$ defined by $\overline{f}(t) := \frac{1}{t}\int_0^t f(s)ds$ satisfies*

$$\frac{\overline{f}(tx) - \overline{f}(t)}{a(t)} \to \frac{x^\beta - 1}{\beta} \text{ for } x > 0, t \to \infty, \tag{17}$$

then

$$\frac{f(tx) - f(t)}{a(t)} \to \frac{x^\beta - 1}{\beta} \text{ for } x > 0, t \to \infty.$$

Proof. Define the function ψ by

$$\psi(t) := tf(t) - \int_0^t f(s)ds \ (t > 0). \tag{18}$$

It is easy to see that this definition implies

$$\bar{f}(t) = \int_0^t \psi(s)\frac{ds}{s^2}.$$

Hence we have for $x > 0$ and $t \to \infty$

$$\int_1^x \frac{\psi(ts)}{ta(t)}\frac{ds}{s^2} = \frac{\overline{f}(tx) - \overline{f}(t)}{a(t)} \to \frac{x^\beta - 1}{\beta}.$$

Since ψ is non-decreasing, for $x > 1$ the left hand side is at least $\frac{\psi(t)}{ta(t)}(1 - x^{-1})$, hence

$$\limsup_{t\to\infty} \frac{\psi(t)}{ta(t)} \leq \frac{x^\beta - 1}{\beta(1 - x^{-1})}.$$

This shows that $\limsup_{t\to\infty} \psi(t)/ta(t) \leq 1$ by letting $x \downarrow 1$. Starting with $0 < x < 1$ and applying a similar inequality we get $\liminf_{t\to\infty} \psi(t)/ta(t) \geq 1$. It follows that $\psi(t) \sim ta(t)(t \to \infty)$ which combined with (18) gives

$$\lim_{t\to\infty} \frac{f(t) - \bar{f}(t)}{a(t)} = 1.$$

Hence as $t \to \infty$

$$\frac{f(tx) - f(t)}{a(t)} = \frac{\overline{f}(tx) - \overline{f}(t)}{a(t)} + o(1) \to \frac{x^\beta - 1}{\beta}.$$

Remark

If the limit in (17) is identically zero, the corresponding limit for f is also identically zero.

The next Lemma is a special case of Feller [6], Chapter VIII.9, theorem 2.

Lemma 7 *Suppose F_0 is a distribution function on $[0,\infty)$. The function U_2 is defined by $U_2(t) := \int_0^t s^2 dF_0(s)$. Then $U_2 \in RV_0$ if and only if*

$$\frac{t^2(1-F_0(t))}{U_2(t)} \to 0 \ (t \to \infty).$$

Remark

An integration by parts shows that the above statements are also equivalent to

$$\frac{t^2(1-F_0(t))}{\int_0^t s(1-F_0(s))ds} \to 0 \ (t \to \infty).$$

The following result is a modification of a result in Pitman [15].

Lemma 8 *Assume the conditions of theorem 1 (iii) (or theorem 2(iii)) are satisfied. For every $y > 0$ there is a constant c such that for every $T > 0$ and $0 \le x \le y$*

$$|\int_0^T \frac{V(1/t)}{t} \cos txdt| \le c,$$

$$|\int_0^T \frac{V(1/t)}{t^2} \sin txdt| \le c$$

and

$$|\int_0^T \frac{1-U(1/t)}{t} x \sin txdt| \le c.$$

Proof Since the other statements can be proved similarly, we only prove the first statement. Note that if (8) holds with $0 < \alpha \le 1$, there exists t_0 such that $|V(1/t)| \le t^{\alpha/2}$ for $0 < t < t_0 < 1$. Define

$$|\int_0^T \frac{V(1/t)}{t} \cos txdt| =: \mathrm{L}_1 + \mathrm{L}_2, \tag{19}$$

where L_1 and L_2 are the integrals over $(0, t_0)$ and (t_0, T) respectively. It follows that L_1 is bounded if $0 < \alpha \le 1$. For $1 < \alpha < 2$ it follows from (8) that $\lim_{t\to\infty} tV(t) =: \mu$ exists, in case $\alpha = 2$ this follows from (14). Hence L_1 is bounded. Next we estimate L_2. Integration by parts gives

$$\begin{aligned} V(1/t) &= \int_{-\infty}^{\infty} \sin txdF(x) = \int_{-\infty}^{0} \sin txdF(x) + \int_0^{\infty} \sin txd(F(x)-1) = \\ &= t\int_0^{\infty} K(y) \cos tydy, \end{aligned}$$

where $K(y) := 1 - F(y) - F(-y)$.

Hence

$$\mathrm{L}_2 = \int_{t_0}^{T} \int_0^{\infty} K(y) \cos ty \cos txdydt. \tag{20}$$

Using the second mean value theorem for each $M > 0$ there exists $\xi \in [0, M]$ such that

$$
\begin{aligned}
&| \int_0^M (1 - F(y)) \cos ty \cos tx dy| \\
&\quad = (1 - F(0)) | \cos tx \int_0^\xi \cos ty dy| \leq 2/t \leq 2/t_0 \text{ for } t_0 \leq t \leq T.
\end{aligned}
$$

Note that a similar argument holds for the integral containing $F(-y)$. Hence we may reverse the order of integration in (20) to find

$$
\begin{aligned}
\mathrm{L}_2 &= \int_0^\infty \int_{t_0}^T K(y) \cos ty \cos tx dt dy \\
&= \frac{1}{2} \int_0^\infty K(y) \left(\frac{\cos T(x+y)}{x+y} + \frac{\cos T(x-y)}{x-y} - \frac{\cos t_0(x+y)}{x+y} - \frac{\cos t_0(x-y)}{x-y} \right) dy.
\end{aligned}
$$

The latter integral is bounded since

$$
\int_{-\infty}^\infty \frac{\cos T(x+y) - \cos t_0(x+y)}{x+y} dy = \int_{-\infty}^\infty \frac{\cos Ty - \cos t_0 y}{y} dy
$$

exists as a finite (semiconvergent) integral for all real x.

Lemma 9 *(extension of Kendall [11], cf. Bingham et al. [1], Ch. 1.9) Suppose*

$$
\limsup_{n\to\infty} x_n = \infty, \ \limsup_{n\to\infty} x_{n+1}/x_n = 1
$$

and f is a continuous function .

1. *Suppose $0 < b < c < \infty$ and for some sequence a_n*

   $$
   a_n f(\lambda x_n) \to \psi(\lambda) \in (0, \infty) \text{ for all } \lambda \in (b, c) \text{ as } n \to \infty,
   $$

   then f varies regularly.

2. *Suppose $0 < b < c < \infty$, the function a is regularly varying and*

   $$
   \lim_{n\to\infty} \frac{f(\lambda x_n) - f(x_n)}{a(x_n)} \to \psi(\lambda) \text{ for all } \lambda \in (b, c),
   $$

   then there exist constants $c, \gamma \in \mathbb{R}$ such that

   $$
   \frac{f(tx) - f(t)}{a(t)} \to c \frac{x^\gamma - 1}{\gamma}, \quad t \to \infty, x > 0.
   $$

Proof The continuity of f is the key assumption.

1. With $V = (b, c)$ there exists a non-empty interval K such that $V \cap u^{-1}V \neq 0$ for all $u \in K$. If $t, ut \in V$ we have

$$\frac{f(x_n ut)}{f(x_n t)} \to \frac{\psi(ut)}{\psi(t)} \text{ as } n \to \infty.$$

Hence if we write $f^*(t) = f(ue^t)/f(e^t)$ for $u > 0$ fixed and $x_n^* = \log x_n$, then $f^*(t + x_n^*)$ converges as $n \to \infty$ for all t in a non-empty open interval J. Choose $\varepsilon > 0$ and define for $k \in Z, m \in N$ the closed sets

$$C_{k,m} := \cap_{n \geq m} \{t \in \mathrm{IR}; \mathrm{f}^*(\mathrm{t} + \mathrm{x}_\mathrm{n}^*) \in [\mathrm{k}\varepsilon - \varepsilon, \mathrm{k}\varepsilon + \varepsilon]\}.$$

By Baire's category theorem (see Hewitt and Stromberg [9]), since J is nonempty and open one of the sets $C_{k,m}$ contains an open interval I. This means that

$$k\varepsilon - \varepsilon \leq f^*(t + x_n^*) \leq k\varepsilon + \varepsilon \text{ for } n \geq m, t \in I.$$

Since by assumption $x_n^* \to \infty, x_{n+1}^* - x_n^* \to 0$, it follows that $\cup_{n \geq m} x_n^* + I$ contains an interval of the form $[t_0, \infty]$, hence

$$k\varepsilon - \varepsilon \leq f^*(t) \leq k\varepsilon + \varepsilon \text{ for all } t \geq t_0.$$

Hence $\lim_{t \to \infty} f^*(t)$ exists and is finite and positive for all $u \in K$, i.e.

$$\lim_{t \to \infty} \frac{f(ue^t)}{f(e^t)}$$

exists and is finite for all $u \in K$. It follows that the function f is regularly varying.

2. In a similar way as above, using the fact that a is regularly varying we obtain for $u > 0$ fixed and all t in a non-empty open interval

$$\lim_{n \to \infty} \frac{f(x_n tu) - f(x_n u)}{a(x_n u)} = \psi^*(t).$$

Define for $u > 0$ fixed the function

$$f^*(t + x_n^*) := \frac{f(x_n ue^t) - f(x_n e^t)}{a(x_n e^t)}$$

(with $x_n^* = \log x_n$ as before). Then in a similar way as above we find $\lim_{t \to \infty} f^*(t)$ exists and is finite.

3 Proof of the main theorems

Proof of theorem 1 We first prove the equivalence of (i), (ii) and (iii). In the part (iii) $\to$ (i) of the proof we obtain the characterization (7) (see (46)).

Proof of theorem 1 (i)$\to$ (iii) It follows from (2) that for all real $t \neq 0$

$$\lim_{n\to\infty} n \log \lambda(a_n t) - ib_n/t = \log g(t), \tag{21}$$

hence

$$\lim_{n\to\infty} nR(a_n t) = -Re(\log g(t)), \tag{22}$$

and

$$\lim_{n\to\infty} nI(a_n t) - b_n/t = Im(\log g(t)), \tag{23}$$

where $R(t) = -Re \log \lambda(t)$ and $I(t) = Im \log \lambda(t)$ (Re and Im denote the real and the imaginary part respectively). Note that there exists a unique version of $\log \lambda$ ($\log g$) satisfying $\log \lambda(t) \to 1$ ($\log g(t) \to 1$) as $t \to \infty$ (see e.g. Feller [6], Ch. XV.)

Application of Lemma 9 (note that $a_n \to \infty, a_{n+1}/a_n \to 1$ $(n \to \infty)$ as in the proof of the proposition) shows that the function R is regularly varying and $-Re(\log g(t)) = |t|^{-\alpha}$ for $t \neq 0$.

Next we focus on (23). By setting $t = 1$ we get

$$\lim_{n\to\infty} nI(a_n) - b_n = Im \log g(1),$$

hence

$$\lim_{n\to\infty} n[I(a_n t) - t^{-1} I(a_n)] = Im \log g(t) - t^{-1} Im \log g(1).$$

Combining this with (22) for $t = 1$, we get for all real $t \neq 0$

$$\lim_{n\to\infty} \frac{a_n t I(a_n t) - a_n I(a_n)}{a_n R(a_n)} = \frac{t Im \log g(t) - Im \log g(1)}{-Re \log g(1)} =: \tau(t). \tag{24}$$

In a similar way, using Lemma 9 this implies

$$\frac{txI(tx) - xI(x)}{xR(x)} \to c\frac{t^\gamma - 1}{\gamma}, \quad x \to \infty, t > 0, \tag{25}$$

where $c \in \mathbb{R}$ is a constant. Since $R \in RV_{-\alpha}$ it follows from Lemma 1 that $c = 0$ or, if $c \neq 0$, then $\gamma = 1 - \alpha$. Using the fact that I is an odd function we now have

$$\frac{txI(tx) - xI(x)}{xR(x)} \to c\frac{|t|^\gamma - 1}{\gamma}, \quad x \to \infty, t \in \mathbb{R} \setminus \{0\}. \tag{26}$$

We have now (iii) with $1 - U$ replaced with R and V replaced with I.

Since for complex z, $|z| < 1/2$

$$|e^z - 1 - z| \le |z|^2$$

we have

$$|\lambda(a_n) - 1 - \log\lambda(a_n)| \le |\log\lambda(a_n)|^2$$

for n sufficiently large. From (22) and (23) we obtain

$$n|\log\lambda(a_n)|^2 = n(R(a_n) + I(a_n))^2 \le 2n(R(a_n)^2 + I(a_n)^2) \to 0(n \to \infty). \quad (27)$$

It follows that

$$\lim_{n\to\infty} n|-\log\lambda(a_n) - 1 + \lambda(a_n)| = 0, \quad (28)$$

hence we may replace $-\log\lambda$ in (21) with $1-\lambda$. Now that we know that (21) holds for $1-\lambda$ instead of $-\log\lambda$, we can repeat the above argument with $-Re\log\lambda$ replaced with $1-U$ and $Im\log\lambda$ replaced with V to obtain (iii).

Proof of theorem 1 (ii)$\to$ (iii) Define the functions H and K by

$$H(t) := 1 - F(t) + F(-t)$$

and

$$K(t) := 1 - F(t) - F(-t).$$

First we prove that

$$\lim_{t\to\infty} \frac{\lambda(t) - 1 - \frac{i}{t}\int_0^t K(s)ds}{H(t)} = -s_\alpha + i(2p-1)c_\alpha. \quad (29)$$

Now for any $A > 0$

$$\frac{\lambda(t) - 1 - \frac{i}{t}\int_0^t K(s)ds}{H(t)}$$
$$= -\int_0^A \sin x \frac{H(tx)}{H(t)}dx - i\int_0^A (1-\cos x)\frac{K(tx)}{H(t)}dx + i\int_1^A \frac{K(tx)}{H(t)}dx$$
$$-\int_A^\infty \sin x \frac{H(tx)}{H(t)}dx + i\int_A^\infty \cos x \frac{K(tx)}{H(t)}dx.$$

Take $\varepsilon > 0$. By Lemma 4 the last two integrals are less than ε for $t > t_0$ and $A > A_0$. For fixed $A > 0$ the first three integrals converge by Lemma 3 to

$$-\int_0^A \sin x \frac{dx}{x^\alpha} - i(2p-1)\int_0^A (1-\cos x)\frac{dx}{x^\alpha} + i(2p-1)\int_1^A \frac{dx}{x^\alpha}.$$

Now (29) follows if we take $A \to \infty$. By separating the real and imaginary parts in (29) we get the limiting behavior as $t \to +\infty$ in (9) and (10) . The limiting behavior as $t \to -\infty$ follows since U is an even and V an odd function. Obviously (9) implies that $1-U \in RV_{-\alpha}$ (since $H \in RV_{-\alpha}$). Note that $s_\alpha \neq 0$ for $0 < \alpha < 2$. Now (10) implies that for $x > 0$

$$\lim_{t\to\infty} \frac{txV(tx) - \int_0^{tx} K(s)ds}{tH(t)} = (2p-1)c_\alpha x^{1-\alpha}$$

(use $H \in RV_{-\alpha}$). Combination with (10) gives

$$\lim_{t\to\infty} \frac{txV(tx) - tV(t)}{tH(t)} - \int_1^x \frac{K(ts)}{H(t)} ds = (2p-1)c_\alpha (x^{1-\alpha} - 1). \tag{30}$$

Note that the integral on the left hand side converges to $(2p-1)\frac{x^{1-\alpha}-1}{1-\alpha}$ as $t \to \infty$ by Lemma 3. Now (8) follows since $1-U$ satisfies (9).

Proof of theorem 1 (iii)→ (ii) In this part of the proof c denotes a constant which may take different values at each occurrence. In order to prove the results in this part we make use of Lévy's inversion relation

$$F(x+h) - F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\zeta)\frac{1-e^{-i\zeta h}}{i\zeta} e^{-i\zeta x} d\zeta, \tag{31}$$

valid for all $x, x+h$ for which F is continuous. See e.g. Chow and Teicher [3]. Note that the above integral is to be understood as the limit as $A \to \infty$ of the integral over $(-A, +A)$. A similar remark holds for the other inversion integrals below. Using the relation (31), $\phi(t) = \lambda(1/t) = U(1/t) + iV(1/t)$ and the fact that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ we obtain the following inversion formula for H

$$H(x) = \frac{2}{\pi}\int_0^\infty \frac{1 - U(1/t)}{t} \sin tx dt, \ x > 0 \tag{32}$$

First we prove that H is regularly varying with order $-\alpha$. For $t \geq 0$ define

$$H_2(t) = \int_0^t H_1(x)dx,$$

where $H_1(t) = \int_0^t xH(x)dx = \frac{1}{2}\int_0^{t^2} H(\sqrt{u})du$ as in Theorem 2(ii). Using (32) it follows that

$$H_1(t) = \frac{2}{\pi}\int_0^t \int_0^\infty \frac{1-U(1/s)}{s} x \sin sx ds dx.$$

From Lemma 8 it follows that we may reverse the order of integration, and so

$$H_1(t) = \frac{2}{\pi}\int_0^\infty \{1 - U(1/s)\} \frac{\sin ts - ts\cos ts}{s^3} ds.$$

Since this integral is absolutely convergent, by Fubini's theorem

$$\begin{aligned} H_2(t) &= \frac{2}{\pi}\int_0^\infty \int_0^t \{1-U(1/s)\}\frac{\sin xs - xs\cos xs}{s^3}dxds \\ &= \frac{2}{\pi}\int_0^\infty \{1-U(1/s)\}\frac{2(1-\cos ts)-ts\sin ts}{s^4}ds. \end{aligned} \tag{33}$$

Hence

$$\frac{H_2(t)}{t^3(1-U(t))} = \frac{2}{\pi}\int_0^\infty \frac{1-U(t/s)}{1-U(t)}\frac{2(1-\cos s)-s\sin s}{s^4}ds.$$

Since $1-U$ is regularly varying with index $-\alpha$, in view of Lemma 3 (substitute $s=x^{-1}$) the right hand side converges to $\frac{2}{\pi}\int_0^\infty \frac{2(1-\cos s)-s\sin s}{s^{4-\alpha}}ds$ as $t\to\infty$. As a consequence $H_2 \in RV_{3-\alpha}$. By the monotone density theorem (Lemma 5) it follows that $H_1 \in RV_{2-\alpha}$, then $H \in RV_{-\alpha}$. In order to prove the tail balance condition we need an inversion relation for K. Similar to the inversion relation for H we obtain

$$K(x)-K(y) = \frac{2}{\pi}\int_0^\infty \frac{V(1/t)}{t}(\cos tx - \cos ty)dt,\ x,y>0, \tag{34}$$

hence by Lemma 8 the function $K(x) - \frac{2}{\pi}\int_0^\infty \frac{V(1/t)}{t}\cos tx dt$ is constant for $x>0$. The constant is necessarily 0. This follows by taking the limit as $x\to\infty$ and applying the Riemann-Lebesgue lemma in (19). See e.g. Feller [6], Ch. XV.4. For $y>0$ we have

$$\begin{aligned} K_1(y) :&= \int_0^y K(x)dx = \frac{2}{\pi}\int_0^y\int_0^\infty \frac{V(1/t)}{t}\cos txdtdx \\ &= \frac{2}{\pi}\int_0^\infty\int_0^y \frac{V(1/t)}{t}\cos txdxdt = \frac{2}{\pi}\int_0^\infty \frac{V(1/t)}{t}\frac{\sin ty}{t}dt. \end{aligned} \tag{35}$$

Interchanging the order of integration is justified by Lemma 8. Now we integrate once more, use (35) and Lemma 8 to find for $t>0$

$$\overline{K}_1(t) := \frac{1}{t}\int_0^t K_1(y)dy = \frac{2}{\pi t}\int_0^\infty \frac{V(1/s)(1-\cos st)}{s^3}ds.$$

It follows that for $b,t>0$

$$\frac{\overline{K}_1(bt)-\overline{K}_1(t)}{a(t)} = \frac{2}{\pi}\int_0^\infty \frac{btsV(bts)-tsV(ts)}{a(ts)}\frac{a(ts)}{a(t)}\{1-\cos(s^{-1})\}ds, \tag{36}$$

where $a(t) := t(1-U(t))$.

Taking the limit as $t\to\infty$, using (8) and the Lemmas 1,2 and 3 we find for $x>0$

$$\lim_{t\to\infty}\frac{\bar{K}_1(tx)-\bar{K}_1(t)}{a(t)} = c\frac{x^{1-\alpha}-1}{1-\alpha}. \tag{37}$$

Application of Lemma 6 then shows that

$$\lim_{t\to\infty} \frac{K_1(tx)-K_1(t)}{a(t)} = c\frac{x^{1-\alpha}-1}{1-\alpha},\ x>0. \tag{38}$$

It follows from (9), since $H \in RV_{-\alpha}$, that as $t\to\infty$ for $x>0$

$$\frac{\int_0^{tx} H(s)ds - \int_0^t H(s)ds}{a(t)} = \frac{tH(t)}{t(1-U(t))}\int_1^x \frac{H(ts)}{H(t)}ds \to c\frac{x^{1-\alpha}-1}{1-\alpha}. \tag{39}$$

Adding both sides of (37) and (39), it follows that for $x>0$, as $t\to\infty$

$$t\int_1^x \frac{1-F(ts)}{a(t)}ds \to c\frac{x^{1-\alpha}-1}{1-\alpha}.$$

In view of (9) this implies

$$\int_1^x \frac{1-F(ts)}{1-F(t)+F(-t)}ds \to c\frac{x^{1-\alpha}-1}{1-\alpha},\ t\to\infty, x>0. \tag{40}$$

For $x>1$ the left hand side is at most $(x-1)(1-F(t))/(1-F(t)+F(-t))$. Hence

$$\liminf_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} \geq c\frac{x^{1-\alpha}-1}{(1-\alpha)(x-1)}.$$

Letting $x\downarrow 1$ then gives

$$\liminf_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} \geq c. \tag{41}$$

Starting with $0<x<1$ in (40) and applying similar inequalities we obtain $\limsup_{t\to\infty}(1-F(t))/(1-F(t)+F(-t)) \leq c$ where c equals the constant in (41).

Proof of theorem 1 (iii)→ (i) Define the sequence $a_n, n=1,2,\ldots$ such that

$$\lim_{n\to\infty} s_\alpha n(1-F(a_n)+F(-a_n)) = 1. \tag{42}$$

Note that this is possible since $1-F(t)+F(-t)$ is regularly varying. Moreover $a_n\to\infty$ as $n\to\infty$. By (9) we have, since $1-U\in RV_{-\alpha}$,

$$\lim_{n\to\infty} n(1-U(a_nt)) = |t|^{-\alpha} \text{ for all } t\in \mathbb{R}, t\neq 0. \tag{43}$$

Define the sequence $b_n, n=1,2,\ldots$ by

$$b_n = \frac{n}{a_n}\int_0^{a_n} K(s)ds + \frac{2p-1}{s_\alpha}c_\alpha.$$

Then as $n \to \infty$ for all $t \in \mathbb{R}, t \neq 0$ by (9) and (43)

$$
\begin{aligned}
nV(a_n t) \quad - \quad \frac{b_n}{t} &= \frac{n}{a_n t}[a_n t V(a_n t) - a_n V(a_n)] + \frac{n}{t}[V(a_n) - \frac{b_n}{n}] - \frac{2p-1}{ts_\alpha} c_\alpha. \\
&\sim \frac{1}{t}\frac{a_n t V(a_n t) - a_n V(a_n)}{a_n(1-U(a_n))} + \frac{1}{ts_\alpha}\frac{V(a_n) - \frac{1}{a_n}\int_0^{a_n} K(s)ds}{H(a_n)} - \frac{2p-1}{ts_\alpha}c_\alpha.
\end{aligned}
$$

Substituting relations (8) and (10) on the right hand side we find

$$\lim_{n\to\infty}\{nV(a_n t) - \frac{b_n}{t}\} = \frac{2p-1}{ts_\alpha}\{[1+(1-\alpha)c_\alpha]\frac{|t|^{1-\alpha}-1}{1-\alpha}\}. \tag{44}$$

Combining (43) and (44) we get

$$\lim_{n\to\infty} n(1-\lambda(a_n t)) + ib_n/t = |t|^{-\alpha} - \frac{i(2p-1)}{ts_\alpha}\{[1+(1-\alpha)c_\alpha]\frac{|t|^{1-\alpha}-1}{1-\alpha}\}. \tag{45}$$

We want to prove

$$\lim_{n\to\infty} \lambda^n(a_n t)e^{-ib_n/t} = \exp -\{|t|^{-\alpha} - \frac{i(2p-1)}{ts_\alpha}\{[1+(1-\alpha)c_\alpha]\frac{|t|^{1-\alpha}-1}{1-\alpha}\}. \tag{46}$$

Now for $|z| < 1/2$ we have $|e^z - 1 - z| \leq |z|^2$. In particular for fixed $t \in \mathbb{R}, t \neq 0$ there exists n_0 such that for $n > n_0$

$$|e^{-1+\lambda(a_n t)} - \lambda(a_n t)| \leq |1-\lambda(a_n t)|^2$$

and hence

$$e^{-n(1-\lambda(a_n t))}e^{-ib_n/t} = \lambda^n(a_n t)e^{-ib_n/t}\left\{1 + O\left(\frac{|1-\lambda(a_n t)|^2}{\lambda(a_n t)}\right)\right\}^n.$$

So it is sufficient to prove that $n|1-\lambda(a_n t)|^2 \to 0$ as $n \to \infty$. This follows from (43) and (44).

Proof of theorem 2 (i) → (iii). Following the reasoning of the proof of Theorem 1, part (i) → (iii) we find that $1-U \in RV_{-2}$. Since (25) now holds with $\gamma = -1$, application of Lemma 1 (or its extended form from Remark 2 following the Lemma) shows that $\lim_{t\to\infty} tI(t) =: c_0$ exists, hence (26) holds with c possibly 0 and the right hand side equals $\tau(t) = tIm\log g(t) - Im\log g(1) = -c(|t|^{-1}-1)$. Since $-Re\log g(t) = t^{-2},\ t \neq 0$, we have

$$g(t) = \exp\{-t^{-2} + it^{-1}(c_3 + c_4|t|^{-1})\},$$

where c_3, c_4 are constants. Since any bounded continuous function ω with $\omega(0) = 1$ is a characteristic function only if for all x and $\epsilon > 0$

$$\int_{-\infty}^{\infty} e^{-i\zeta x}\omega(\zeta)e^{-\epsilon\zeta^2}d\zeta \geq 0,$$

(see Feller [6], Ch. XIX.2) we must have $c_3 = c_4 = 0$, hence $\psi(t) = g(t^{-1}) = e^{-t^2}$ and (25) holds with $c = 0$. Remark 2 following Lemma 1 now shows that $\lim_{t\to\infty} tV(t) =: \mu$ exists and (13) holds.

Proof of theorem 2 (ii) → (iii). By Lemma 7 (take $F_0(t) = 1 - H(t),\ t \geq 0$) $t^2 H(t)/H_1(t) \to 0$ as $t \to \infty$. Note that

$$\frac{\lambda(t) - 1 - \frac{i\mu}{t} + \frac{H_1(t)}{t^2}}{\frac{H_1(t)}{t^2}} =$$

$$-\int_0^1 \frac{(tx)^2 H(tx)}{H_1(t)} \frac{\sin x - x}{x^2} dx - \int_1^\infty \frac{(tx)^2 H(tx)}{H_1(t)} \frac{\sin x}{x^2} dx$$
$$+i \int_0^1 \frac{(tx)^2 K(tx)}{H_1(t)} \frac{\cos x - 1}{x^2} dx + i \int_1^\infty \frac{(tx)^2 K(tx)}{H_1(t)} \frac{\cos x - 1}{x^2} dx.$$

Application of Lemma 3 shows that the integrals on the right hand side all tend to zero as $t \to \infty$. Now (14) and (15) follow by taking the real and imaginary part and (iii) follows from (ii), (14) and (15).

Proof of theorem 2 (iii) → (ii). Compared to the corresponding part in the proof of Theorem 1 we have to integrate once more in order to get an absolutely convergent integral. For the function H_3 defined by $H_3(t) = \int_0^t H_2(s)ds$ an expression similar to (33) can be given. A similar calculation shows that H_1 is slowly varying.

Proof of theorem 2 (iii) → (i). With the sequences $a_n, b_n, n = 1, 2, \ldots$ defined by

$$\frac{nH_1(a_n)}{a_n^2} \to 1$$

as $n \to \infty$ and $b_n = n\mu/a_n$, the proof is similar to the proof of the corresponding part of Theorem 1. We omit the details.

References

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular variation,* Encycl. Math. Appl, 27 Cambridge Univ. Press (1987).

[2] L. Breiman (1968), *Probability*, Addison-Wesley, Reading MA.

[3] Y.S. Chow, H.Teicher (1978), *Probability theory, independence, interchangeability, martingales*, Springer, Berlin.

[4] R.M. Dudley (1989), *Real analysis and probability*, Wadsworth and Brooks/Cole.

[5] E. Fama (1965), The behavior of stock prices, Journal of Business, 38, 34-105.

[6] W. Feller (1971). *An introduction to probability theory and its applications 2,* 2nd ed. Wiley, New York.

[7] J.L. Geluk, L. de Haan, *Regular variation, extensions and Tauberian theorems*, CWI tract 40 Amsterdam (1987).

[8] B.V. Gnedenko, A.N. Kolmogorov (1954), *Limit distributions for sums of independent random variables*, Addison-Wesley, Reading MA.

[9] E. Hewitt, K. Stromberg (1969), *Real and abstract analysis*, Springer Verlag, Berlin.

[10] I.A. Ibragimov, Yu. V. Linnik (1971), *Independent and stationary sequences of random variables*, Wolters-Noordhoff, Groningen.

[11] D.G. Kendall, (1968), Delphic semigroups, infinitely divisible regenerative phenomena and the arithmetic of p-funtions, Z. f. Wahrsch., 9, 163-195.

[12] R.M. Kunst (1993), Apparently stable increments in finance data: Could ARCH effects be the cause?, J. Stat. Comp. Sim., 45, 121-127.

[13] P. Lévy (1954), *Théorie de l'addition des variables aléatoires*, 2nd edn, Gauthier Villars, Paris.

[14] B. Mandelbrot (1963), The variation of certain speculative prices, Journal of Business, 36, 394-419.

[15] E.J.G. Pitman (1968), On the behaviour of the characteristic function of a probability distribution in the neighbourhood of the origin, J. Austr. Math. Soc. (ser. A), 29, 337-347.

[16] G. Samorodnitsky, M.S. Taqqu (1994), *Stable non-Gaussian random processes*, Chapman and Hall, London.

[17] P. Samuelson (1967), Efficient portfolio selection for Pareto-Lévy investments, Journal of Financial and Quantitative Analysis, 2, 107-117.

[18] V.M. Zolotarev (1986), *One-dimensional stable distributions*, Translations of mathematical monographs, Vol. 65, American Mathematical Society.