Pricing Double Barrier Options: An Analytical Approach

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Abstract

Double barrier options have become popular instruments in derivative markets. Several papers have already analysed double knock-out call and put options using different methods. In a recent paper, Geman and Yor (1996) derive expressions for the Laplace transform of the double barrrier option price. However, they have to resort to numerical inversion of the Laplace transform to obtain option prices. In this paper, we are able to solve, using contour integration, the inverse of the Laplace transforms analytically thereby eliminating the need for numerical inversion routines. To our knowledge, this is one of the first applications of contour integration to option pricing problems. To illustrate the power of this method, we derive analytical valuation formulas for a much wider variety of double barrier options than has been treated in the literature so far. Many of these variants are nowadays being traded in the markets. Especially, options which pay a fixed amount of money (a "rebate") as soon as one of the barriers is hit and double barrier knock-in options.

Key Words: double barrier options, option pricing, partial differential equations, Laplace transform, Cauchy's Residue Theorem.

1. Introduction

Barrier options have become very popular instruments in derivative markets. It is relatively straightforward to price and hedge "single barrier" options. Valuation formulas have been available in the literature for quite a while, see Merton (1973) or Goldman et al. (1979). The valuation and hedging formulas have been incorporated in standard market software for options traders and clients. In fact, most derivatives firms view "single barrier" options nowadays more like vanilla than exotic options.

One of the reasons why barrier options have become so popular, is the fact that they are cheaper than standard options, but offer a similar kind of protection. A natural extension to "single barrier" options is to consider double barrier options. These are options which have a barrier above and below the price of the underlying, and the option gets knocked in or out as soon as one of the two barriers is hit.

Several papers have already analysed double knock-out call and put options using different methods. Kunitomo and Ikeda (1992) derive the probability density for staying between two (exponentially) curved boundaries. They express the density as an infinite sum of normal density functions, and the prices for double knock-out call and puts are derived by integrating with respect to this density. Furthermore they show that each of the terms in the infinite sums fall to zero very rapidly, hence only a small number of terms needs to be evaluated to obtain an accurate value.

In a recent paper, Geman and Yor (1996) derive expressions for the Laplace transform of the double barrrier option price. They invert the Laplace transform numerically to obtain option prices.

These papers deal, however, with only one type of double barrier option: double barrier knock-out calls and puts. In the markets a much wider variety of double barrier options is being traded. Especially, options which pay a fixed amount of money (a "rebate") as soon as one of the barriers is hit and double barrier knock-in options.

In this paper we derive analytical formulas for pricing a wide variety of double barrier options. We find formulas for options which give a constant payoff either "at hit" or at maturity, we derive pricing formulas for double barrier options where the final payoff can be expressed as any power of the underlying value and we find valuation formulas for knock-in options.

We derive our results by considering, just like Geman and Yor (1996), Laplace transforms. Unlike Geman and Yor (1996), we do not consider the Laplace tranform of the option price, but the Laplace transform of the density functions of hitting the upper or lower barrier. We are able to find analytical expressions for the density functions, using contour integration, thereby eliminating the need for numerical inversion routines. Option prices are then calculated by integrating the option payoff with respect to the density functions. To our knowledge this is one of the first applications of contour integration to the area of option pricing theory.

The paper is organised as follows. In Section 2 we review well known results on how the probability density function for staying between two barriers can be expressed in several ways. In Section 3 we derive our analytical expressions for the probability density of the first passage time for the upper and lower barriers. In Section 4 we derive some pricing formulas for different kinds of double barrier options. In Section 5 we compare our results to Kunitomo and Ikeda (1992) for double barrier knock-out options whith no rebate, and to the Crank-Nicholson finite difference method for double barrier options with do pay a rebate at hit. Finally, we conclude in Section 6.

2. Transition Density Function

If we make the assumption (which is standard) that the underlying asset of the option can be modeled as a geometric Brownian motion, we can model the log of the asset price (under the equivalent martingale measure) by the following stochastic differential equation

$$dz = \mu\, dt + \sigma\, dW, \tag{1}$$

where μ and σ are constants.

The case we want to consider is more complicated. We want to value a double barrier options. This can be modeled by assuming that the process z is killed as soon as it hits one of the two barriers. Suppose we have two barriers, the lower barrier is at 0, the upper barrier at the level l. This specification is general, since we can always shift the process z by a constant such that the lower barrier is placed at 0.

The two barriers are so called *absorbing barriers*, since the process z is killed as soon as it hits one of the barriers.

Let us consider the transition density function $p(t, x; s, y)$. It describes the probability density that the process z starts at time t at $z(t) = x$ and survives until time s and ends up at $z(s) = y$. Of course we have, $t \leq s$ and $0 \leq x, y \leq l$.

This transition density function satisfies the forward and backward equations, see Arnold (1992). The backward equation is given by (with subscripts denoting derivatives)

$$p_t + \mu p_x + \tfrac{1}{2}\sigma^2 p_{xx} = 0, \tag{2}$$

subject to the boundary conditions $p(t, 0; ., .) = p(t, l; ., .) = 0$, and $p(s, x; s, y) = \delta(y - x)$, where δ is the Dirac delta function.

The last condition is standard, and states that the density function must collapse into a delta-function at time $t = s$, since there is no uncertainty left in the process. The first two conditions specify the absorbing barriers at 0 and l. If the process hits one of the barriers, it gets killed and there is no probability of making it back to y at time s.

The forward equation is given by

$$-p_s - \mu p_y + \tfrac{1}{2}\sigma^2 p_{yy} = 0, \tag{3}$$

subject to the boundary conditions $p(., .; s, 0) = p(., .; s, l) = 0$, and $p(t, x; t, y) = \delta(x - y)$.

The solution to the backward or forward equation can be represented in several ways. Kunitomo and Ikeda (1992) use the representation which is obtained by the "method of images", and express the probability density in terms of a doubly infinite sum of normal density functions. It is also well known, see for example Cox and Miller (1965), Chapter 5.7, that another representation of the solution can be obtained by the method of "separation of variables". The solution is then represented in terms of a Fourier series:

$$\begin{aligned} p(t,x;s,y) &= e^{\frac{\mu}{\sigma^2}(y-x)} \frac{2}{l} \sum_{k=1}^{\infty} e^{-\lambda_k (s-t)} \sin(k\pi \frac{x}{l}) \sin(k\pi \frac{y}{l}) \\ \lambda_k &= \frac{1}{2} \left(\frac{\mu^2}{\sigma^2} + \frac{k^2 \pi^2 \sigma^2}{l^2} \right). \end{aligned} \tag{4}$$

Substitution in the backward equation (2) or the forward equation (3) will confirm that this is indeed a valid solution, that satisfies the boundary conditions. Furthermore, the series representation is absolutely convergent, hence we are allowed to perform differentiation and integration on a term-by-term basis.

The choice for represention (4) of the solution, has the additional advantage that analytical expressions (on a term-by-term basis) can be found for calculating options prices. Hence, there is no need to work with approximations, as in the case of the cumulative normal distribution function.

The solution (4) looks very complicated to evaluate. However, the term λ_k grows quadratically in k, hence $\exp\{-\lambda_k(s-t)\}$ vanishes to zero very rapidly for increasing k. So, only very few terms have to be summed to obtain an accurate answer.

3. Barrier Densities

We have now characterised the density function of surviving until time s. This densitiy is used for pricing double knock-out options which get nullified as soon as one of the barriers gets hit.

We are also interested in the density functions of hitting the upper and the lower barrier. These densities are used for pricing options which have a non-zero payoff as soon as one of the barriers is hit.

Let $g^+(t,x;s)$ denote the probability density function of first hitting the upper barrier at time s before the lower barrier is hit, given that the process started at (t,x). Let $g^-(t,x;s)$ denote the probability density of first hitting the lower barrier, before the upper barrier is hit.

Given the fact that the process z can either hit the upper barrier, or the lower barrier, or survive, we can derive the following identity for all $T > t$

$$\int_t^T g^+(t,x;s)\,ds + \int_t^T g^-(t,x;s)\,ds + \int_0^l p(t,x;T,y)\,dy \equiv 1. \tag{5}$$

Taking the derivative with respect to T yields

$$g^+(t,x;T) + g^-(t,x;T) = -\frac{\partial}{\partial T}\int_0^l p(t,x;T,y)\,dy. \tag{6}$$

Using this expression, we can express the sum of the two densities using (4) as follows

$$\begin{aligned} g^+(t,x;s) + g^-(t,x;s) = \quad & e^{\frac{\mu}{\sigma^2}(l-x)}\frac{\sigma^2}{l^2}\sum_{k=1}^{\infty} e^{-\lambda_k(s-t)}k\pi\sin(k\pi\frac{l-x}{l}) \\ & + e^{-\frac{\mu}{\sigma^2}x}\frac{\sigma^2}{l^2}\sum_{k=1}^{\infty} e^{-\lambda_k(s-t)}k\pi\sin(k\pi\frac{x}{l}). \end{aligned} \tag{7}$$

Although we have tentatively grouped the terms, we cannot determine from this expression what the individual barrier densities are. To derive expressions for the individual densities we have to use a different approach.

3.1. Derivation of g^+

The density $g^+(t,x;s)$ must satisfy the backward equation:

$$g_t^+ + \mu g_x^+ + \tfrac{1}{2}\sigma^2 g_{xx}^+ = 0. \tag{8}$$

Due to the fact that μ and σ are constants, we know that the function g^+ depends only on $s-t$. If we set $\tau = s-t$, with $\tau \geq 0$, we can write $g^+(t,x;s) = g^+(\tau,x)$, which solves

$$-g^+_\tau + \mu g^+_x + \tfrac{1}{2}\sigma^2 g^+_{xx} = 0, \tag{9}$$

subject to the boundary conditions $g^+(\tau,l) = \delta(\tau)$, $g^+(0,x) = \delta(l-x)$ and $g^+(\tau,0) = 0$.

To obtain a solution for (9) we consider the Laplace transform[1] $\gamma^+(x)$

$$\gamma^+(x;v) = \int_0^\infty e^{-v\tau} g^+(\tau,x)\, d\tau,$$

for any $v \geq 0$. Substituting γ^+ into (9) and the boundary conditions yields an ordinary differential equation

$$-v\gamma^+ + \mu\gamma^+_x + \tfrac{1}{2}\sigma^2\gamma^+_{xx} = 0, \tag{10}$$

subject to the boundary conditions $\gamma^+(0) = 0$ and $\gamma^+(l) = 1$.

By considering the Laplace transform, we have managed to reduce the partial differential equation (9) to the second order ordinary differential equation (10). Differential equations of this kind are easy to solve. It is well known (or substitution will confirm) that the solution can be expressed as

$$\gamma^+(x) = e^{-\frac{\mu}{\sigma^2}x}\big(A\sinh(\theta x) + B\cosh(\theta x)\big), \tag{11}$$

with $\theta = \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\sigma^2 v}$. The constants A and B have to be determined from the boundary conditions. Solving for the boundary conditions yields: $B = 0$ and $A = \exp\{\frac{\mu}{\sigma^2}l\}/\sinh(\theta l)$. Hence, the solution to (10) and the boundary conditions is given by

$$\begin{aligned} \theta(v) &= \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\sigma^2 v} \\ \gamma^+(x;v) &= e^{\frac{\mu}{\sigma^2}(l-x)}\frac{\sinh\big(\theta(v)x\big)}{\sinh\big(\theta(v)l\big)}. \end{aligned} \tag{12}$$

We have written $\theta(v)$ to emphasize the dependence of θ on v.

To obtain the density for the upper barrier g^+, we now have to invert the Laplace transform γ^+. This can be done using Bromwich's Integral (see, Duffy (1994), Chapter 2.1.)

$$g^+(\tau,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\tau z}\gamma^+(x;z)\, dz, \tag{13}$$

[1] For an introduction to methods for solving partial differential equations, see Williams (1980) or Duffy (1994).

where c lies to the right of any of the singularities of the function γ^+. Note, that from this moment on, we view $\gamma^+(x;z)$ as a function in the complex variable z, with x as a parameter.

The integral (13) can be evaluated as follows. We can transform the (line)integral into a contour integral by adding a circular arc in the second and third quadrant. (This arc goes counter-clockwise from the positive imaginary axis to the negative imaginary axis.) The contribution of this arc vanishes when it's radius goes to infinity.

The value of the contour integral we have constructed can now be determined by Cauchy's Residue Theorem (see, Duffy (1994), Chapter 1.4):

Cauchy's Residue Theorem. *If $f(z)$ is analytic inside a closed contour C (taken in the positive sense) except at points z_k where f has singularities, then*

$$\oint_C f(z)\,dz = 2\pi i \sum_k \textit{Residue of } f(z) \textit{ at } z_k.$$

As is well known from complex function theory, the residue of a singularity z_k equals the coefficient a_{-1} from the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_k)^n \tag{14}$$

around a singularity z_k. The positive part of the summation is the familiar Taylor expansion, the negative part involves negative powers of $z - z_k$ and gives the behaviour at the singularity.

The largest negative power in the Laurent expansion, gives the order of the singularity. For a first order singularity (the only case we will encounter here), the residue can be computed as

$$\mathrm{Res}(z_k) = \lim_{z \to z_k} (z - z_k) f(z). \tag{15}$$

This results follows directly after substituting into the Laurent expansion, and applying De l'Hospital's Rule.

Since the arc we have added to transform the line-integral (13) into a contour integral makes no contribution to the integral, the line-integral must be equal, by the Residue Theorem, to the sum of the residues of the singularities enclosed in the contour.

Let us therefore find the singularities of the function $e^{\tau z}\gamma^+(x;z)$. Singularities can only be caused, if the term in the denominator of γ^+ goes to zero. Using the identity $\sinh(z) = -i\sin(iz)$, we find that $\sinh\big(\theta(z)l\big)$ is zero if $i\theta(z)l = k\pi$, for k integer. Solving for z yields

$$z_k = -\frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2}\right). \tag{16}$$

So, for $k = 0, 1, 2, \ldots$ we have identified all the singularities z_k of the function $e^{\tau z}\gamma^+(x;z)$. The residue for each singularity z_k can be obtained via

$$\begin{aligned}
\mathrm{Res}(z_k) &= \lim_{z\to z_k} e^{\tau z} e^{\frac{\mu}{\sigma^2}(l-x)} \sinh(\theta x)\frac{z - z_k}{\sinh(\theta l)} \\
&= \lim_{z\to z_k} e^{\tau z} e^{\frac{\mu}{\sigma^2}(l-x)} \sinh(\theta x)\frac{1}{\cosh(\theta l)\frac{\partial\theta}{\partial z}l} \\
&= e^{\tau z_k} e^{\frac{\mu}{\sigma^2}(l-x)} \sinh(k\pi i\frac{x}{l})(-1)^k\frac{\sigma^2}{l^2}k\pi i \\
&= e^{\tau z_k} e^{\frac{\mu}{\sigma^2}(l-x)} \frac{\sigma^2}{l^2}k\pi \sin(k\pi\frac{l-x}{l}).
\end{aligned} \tag{17}$$

Thus, summing up all of the residues gives the following expression for the density function of hitting the upper barrier $g^+(t,x;s)$ (remember that we set $\tau = s - t$):

$$g^+(t,x;s) = e^{\frac{\mu}{\sigma^2}(l-x)}\frac{\sigma^2}{l^2}\sum_{k=1}^{\infty} e^{z_k(s-t)}k\pi\sin(k\pi\frac{l-x}{l}). \tag{18}$$

3.2. Derivation of g^-

An expression for the density g^- can be derived in a similar fashion. The Laplace transform γ^- satisfies also the ordinary diferential equation (10), however with respect to the boundary conditions $\gamma^-(0) = 1$ and $\gamma^-(l) = 0$. Solving the differential equation with respect to these boundary conditions yields

$$\gamma^-(x;v) = e^{-\frac{\mu}{\sigma^2}x}\frac{\sinh\big(\theta(v)(l-x)\big)}{\sinh\big(\theta(v)l\big)}. \tag{19}$$

We see that $\gamma^-(x) = \exp\{-2\frac{\mu}{\sigma^2}x\}\gamma^+(l-x)$. Hence, by substitution into (18) we obtain immediately

$$g^-(t,x;s) = e^{-\frac{\mu}{\sigma^2}x}\frac{\sigma^2}{l^2}\sum_{k=1}^{\infty} e^{z_k(s-t)}k\pi\sin(k\pi\frac{x}{l}). \tag{20}$$

It is easy to verify that the sum of g^+ and g^- is indeed equal to the expression (7) derived before.

4. Valuation Formulas

With the analytical expressions we have derived for the transition density p and the barrier densities g^+ and g^- we can calculate prices for various types of double barrier options.

In the cases we will analyse, we take the underlying to be an F/X-rate.[2] Let $S(t)$ be the spot exchange rate today. Let r_d be the domestic interest rate, r_f the foreign interest rate and σ the volatility of the exchange rate. Let U be the upper barrier and L be the lower barrier with $L < S(t) < U$.

If we divide by L and take logarithms we obtain for $s > t$ that $z(s) = \log(S(s)/L)$ where z is the process defined in (1) with $x = z(t) = \log(S(t)/L)$ and $l = \log(U/L)$. The drift-term μ of the process z (under the equivalent martingale measure) is equal to $\mu = r_d - r_f - \frac{1}{2}\sigma^2$. For all options we denote the maturity date by T.

4.1. Constant payoff at maturity

The simplest kind of double barrier is an option which pays a constant amount at the maturity of the option. Suppose we receive an amount K_U if the upper barrier is hit first, an amount K_L if the lower barrier is hit first and an amount K is neither barrier is hit during the life. All amounts are payed at maturity T. The value $V_{\mathrm{CPM}}(t)$ of this option is equal to

$$V_{\mathrm{CPM}}(t) = e^{-r_d(T-t)}\Big(K_U P^+(T) + K_L P^-(T) + K\big(1 - P^+(T) - P^-(T)\big)\Big), \tag{21}$$

where $P^+(T)$ and $P^-(T)$ denote the probability of hitting first the upper and the lower barrier respectively before time T. The probability of surviving until time T is given by (5) as $1 - P^+ - P^-$. To find P^+ and P^- we have to integrate over the barrier densities. To find an expression for these integrals we rewrite them as

$$\begin{aligned} P^{\pm}(T) = \int_t^T g^{\pm}(t,x;s)\,ds &= \int_t^\infty g^{\pm}(t,x;s)\,ds - \int_T^\infty g^{\pm}(t,x;s)\,ds \\ &= \gamma^{\pm}(x;0) - \int_T^\infty g^{\pm}(t,x;s)\,ds. \end{aligned} \tag{22}$$

[2] In stead of an F/X-rate, the formulas can also be applied to equity or commodities with a continuous dividend-yield δ, by setting $r_f = \delta$.

Integrating on a term-by-term basis, we find for

$$\begin{aligned} P^+(T) &= e^{\frac{\mu}{\sigma^2}(l-x)} \left(\frac{\sinh(\frac{\mu}{\sigma^2}x)}{\sinh(\frac{\mu}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda_k(T-t)}}{\lambda_k} k\pi \sin(k\pi \frac{l-x}{l}) \right), \\ P^-(T) &= e^{-\frac{\mu}{\sigma^2}x} \left(\frac{\sinh(\frac{\mu}{\sigma^2}(l-x))}{\sinh(\frac{\mu}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda_k(T-t)}}{\lambda_k} k\pi \sin(k\pi \frac{x}{l}) \right). \end{aligned} \tag{23}$$

4.2. Rebate at hit

A more realistic payoff scheme, which is used often in knock-out options, is to offer a rebate as soon as the option hits one of the barriers. Suppose we receive an amount K_U at the moment the upper barrier is hit first. The value $V_{\mathrm{RAHU}}(t)$ is given by

$$V_{\mathrm{RAHU}}(t) = K_U \int_t^T e^{-r_d(s-t)} g^+(t,x;s)\, ds. \tag{24}$$

Solving this integral involves finding a primitive for terms of the form $e^{-r_d(s-t)}e^{-\lambda_k(s-t)}$. We obtain a value for the integral in a simpler way, if we bring r_d inside the λ_k as follows

$$r_d + \lambda_k = \frac{1}{2}\left(\frac{2\sigma^2 r_d + \mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2} \right) = \frac{1}{2}\left(\frac{\mu'^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2} \right) = \lambda'_k \tag{25}$$

with

$$\mu' = \sqrt{\mu^2 + 2\sigma^2 r_d} \tag{26}$$

If we denote g'^+ as the barrier density with drift μ', then we obtain

$$\begin{aligned} V_{\mathrm{RAHU}}(t) &= K_U e^{\frac{\mu-\mu'}{\sigma^2}(l-x)} \int_t^T g'^+(t,x;s)\, ds \\ &= K_U e^{\frac{\mu}{\sigma^2}(l-x)} \left(\frac{\sinh(\frac{\mu'}{\sigma^2}x)}{\sinh(\frac{\mu'}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda'_k(T-t)}}{\lambda'_k} k\pi \sin(k\pi \frac{l-x}{l}) \right). \end{aligned} \tag{27}$$

Similarly, we find that the value of an amount K_L received as soon as the lower barrier is hit first, can be expressed as

$$\begin{aligned} V_{\mathrm{RAHL}}(t) &= K_L e^{-\frac{\mu-\mu'}{\sigma^2}x} \int_t^T g'^-(t,x;s)\, ds \\ &= K_L e^{-\frac{\mu}{\sigma^2}x} \left(\frac{\sinh(\frac{\mu'}{\sigma^2}(l-x))}{\sinh(\frac{\mu'}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda'_k(T-t)}}{\lambda'_k} k\pi \sin(k\pi \frac{x}{l}) \right). \end{aligned} \tag{28}$$

4.3. Double knock-out

Another payoff we want to consider, are double knock-out options.[3] Suppose we have a double knock-out call, with a payoff $\max\{S(T)-K,0\}$, if the price of S hit neither barrier during the life $[t,T]$ of the option. The value at t is given by

$$V_{\text{DKOC}}(t) = e^{-r_d(T-t)} \int_0^l \max\{Le^y - K, 0\} p(t,x;T,y)\, dy. \tag{29}$$

The option is in-the-money for $Le^y > K \iff y > \log(K/L) = d$. If we assume $0 \le d \le l$ (the other cases are trivial) then we get

$$\begin{aligned} V_{\text{DKOC}}(t) &= e^{-r_d(T-t)} \int_d^l (Le^y - K) p(t,x;T,y)\, dy \\ &= e^{-r_d(T-t)} \left(L \int_d^l e^y p(t,x;T,y)\, dy - K \int_d^l p(t,x;T,y)\, dy \right). \end{aligned} \tag{30}$$

Both integrals involve finding the primitive for terms of the form $e^{ay}\sin(by)$. The primitive for these terms is given by

$$\int e^{ay}\sin(by)\, dy = e^{ay} \frac{a\sin(by) - b\cos(by)}{a^2+b^2}.$$

Hence, if we set $Q(\alpha, y) = \int e^{\alpha y} p(t,x;T,y)\, dy$, we obtain for Q

$$Q(\alpha,y) = \frac{2}{l} e^{\frac{\mu}{\sigma^2}(y-x)} e^{\alpha y} \sum_{k=1}^{\infty} e^{-\lambda_k (T-t)} \sin(k\pi \frac{x}{l}) \left(\frac{(\frac{\mu}{\sigma^2}+\alpha)\sin(k\pi\frac{y}{l}) - \frac{k\pi}{l}\cos(k\pi\frac{y}{l})}{(\frac{\mu}{\sigma^2}+\alpha)^2 + \frac{k^2\pi^2}{l^2}} \right). \tag{31}$$

The value of the double knock-out call can now be expressed as

$$V_{\text{DKOC}}(t) = e^{-r_d(T-t)} \Big(L\big(Q(1,l) - Q(1,d)\big) - K\big(Q(0,l) - Q(0,d)\big) \Big). \tag{32}$$

The value of a double knock-out put is given by

$$\begin{aligned} V_{\text{DKOP}}(t) &= e^{-r_d(T-t)} \int_0^d (K - Le^y) p(t,x;T,y)\, dy \\ &= e^{-r_d(T-t)} \left(K \int_0^d p(t,x;T,y)\, dy - L \int_0^d e^y p(t,x;T,y)\, dy \right), \end{aligned} \tag{33}$$

[3] An alternative method of deriving the expressions given in this section, is to invert the Laplace transform of the option price given in (2.11) of Geman and Yor (1996) using a contour integration. This would involve a tedious calculation, which we avoid in the derivation of this section.

which can be expressed as

$$V_{\mathrm{DKOP}}(t) = e^{-r_d(T-t)}\Big(K\big(Q(0,d) - Q(0,0)\big) - L\big(Q(1,d) - Q(1,0)\big)\Big). \tag{34}$$

The derivation given above, also holds for options which have a payoff wich depends on $S(T)^\alpha$. Normal call and put payoffs have $\alpha = 1$. However, a so-called "bull/bear" contract has a payoff of $\max\{0, \frac{S(T)-K}{S(T)}\} = \max\{0, 1 - KS(T)^{-1}\}$, which can be valued in our framework with $\alpha = -1$.

4.4. Knock-in options

Thus far, we have only considered knock-out options. However, we can also consider double-barrier knock-in options. A knock-in option can be viewed as a "rebate-at-hit" option. However, at the time one of the barriers is hit, not a constant amount is payed, but a payoff equal to a standard Black-Scholes (1973) formula. For example, a double barrier knock-in option with knocks in a call if the upper barrier is hit first, or knocks in a put of the lower barrier is hit first has a value given by

$$V_{\mathrm{DKI}}(t) = \int_t^T e^{-r_d(s-t)}\mathbf{C}(s;U)g^+(t,x;s)\,ds + \int_t^T e^{-r_d(s-t)}\mathbf{P}(s;L)g^-(t,x;s)\,ds, \tag{35}$$

where $\mathbf{C}(s;U)$ denotes the (Black-Scholes) value of a call-option at time s, with spot-price $S(s) = U$, and $\mathbf{P}(s;L)$ denotes the put option. The integrals above cannot be solved analytically, but using a numerical integration routine it is straightforward to obtain an accurate value for the one-dimensional integrals.

5. Numerical Implementation

If we want to use the formulas given above to calculate prices of double barrier options, we have to truncate the infinite sums to a finite number of terms. Fortunately, the $\exp\{-\lambda_k(s-t)\}$ terms decline very fast to zero.

To determine the number of terms needed, we propose the following. Set ϵ to a small number, say $\epsilon = 10^{-10}$. We can then find the number k, for which $\exp\{-\lambda_k(s-t)\} < \epsilon$. This number is given by

$$k > \sqrt{\frac{-2\frac{\log\epsilon}{s-t} - \frac{\mu^2}{\sigma^2}}{\frac{\pi^2\sigma^2}{l^2}}}. \tag{36}$$

If we set k^* to the smallest integer that satisfies the inequality, we truncate our sums at k^*. The error we now make is of order ϵ.

To assess the validity of our implementation, we have compiled the following tables. In these tables, we compare the value of a double knock-out call with the results presented in Table 3.1 of Kunitomo and Ikeda (1992). Their paper deals with the valuation of double barrier options with (exponentially) curves boundaries, but the "b" columns in their table correspond to flat barriers. However, they consider only knock-out options with no rebate.

To make a comparison in the case where rebates are paid out, we have also implemented a finite difference method (Crank-Nicholson) to solve the Black-Scholes (1973) partial differential equation numerically, with boundary conditions K_U at U and K_L at L.

The results are presented in Table 1 (options with maturity equal to one month) and in Table 2 (options with maturity equal to half a year). The columns "KI" denote the results obtained by Kunitomo-Ikeda. The columns "Ana" denote our analytical results, and the columns "FD" denote the values obtained from the finite difference method.

It is clear from the tables, that for options with no rebate, our results are exactly equal to the results obtained by Kunitomo-Ikeda. It is also clear, that even for a 1000 by 1000 grid, the finite difference method is still not pricing the double barrier options correctly. However, the pricing errors are relatively small. Geman and Yor (1996) already pointed out the problems of using Monte Carlo methods for pricing double barrier options. From the results we have shown it will be clear that also finite difference methods converge quite slowly when pricing double barrier options. Hence, the analytical results obtained

also provide a good benchmark for assessing the accuracy of finite difference methods in this case.

For options that do pay a rebate, we see that our analytical results are close to the results obtained by the finite difference method. The differences in prices can readily be explained by the inaccuracy of the finite difference method.

6. Conclusions

In this paper we have provided valuation formulas for a wide range of double-barrier knock-out and knock-in options. We derived Laplace transforms which we inverted analytically using contour integration. With the analytical expressions obtained, we can efficiently calculate values for double-barrier options, without having to resort to numerical inversion methods.

To our knowledge this has been one of the first applications of contour integration to an option pricing problem. Given the power of this approach, we think that many more applications will follow.

Table 1.

Value of Double Knock-Out Call Option if $T - t = \frac{1}{12}$

$(S = 1000, r_d = 0.05, r_f = 0.00, K = 1000)$

			No Rebate			Rebate at Hit*	
	U	L	KI	Ana	FD†	Ana	FD†
$\sigma = 0.2$	1500	500	25.12	25.12	24.57	25.12	24.57
	1200	800	24.76	24.76	24.69	25.12	25.05
	1050	950	2.15	2.15	2.15	22.29	22.27
$\sigma = 0.3$	1500	500	36.58	36.58	36.04	36.59	36.04
	1200	800	29.45	29.45	29.40	36.55	36.48
	1050	950	0.27	0.27	0.27	25.14	25.12
$\sigma = 0.4$	1500	500	47.85	47.85	47.31	48.05	47.51
	1200	800	25.84	25.84	25.82	47.88	47.80
	1050	950	0.02	0.02	0.01	25.34	25.32

* Rebate is equal to intrinsic value of option at barrier.

† Prices calculated on a 1000 by 1000 grid.

Table 2.

Value of Double Knock-Out Call Option if $T - t = \frac{1}{2}$

$(S = 1000, r_d = 0.05, r_f = 0.00, K = 1000)$

			No Rebate			Rebate at Hit*	
	U	L	KI	Ana	FD†	Ana	FD†
	1500	500	66.13	66.13	65.58	68.87	68.26
$\sigma = 0.2$	1200	800	22.08	22.08	22.08	66.49	66.42
	1050	950	0.00	0.00	0.00	26.48	26.45
	1500	500	67.88	67.88	67.59	95.97	95.38
$\sigma = 0.3$	1200	800	9.26	9.26	9.27	86.54	86.47
	1050	950	0.00	0.00	0.00	25.66	25.63
	1500	500	53.35	53.35	53.24	122.46	121.87
$\sigma = 0.4$	1200	800	3.14	3.14	3.14	97.57	97.50
	1050	950	0.00	0.00	-0.01	25.37	25.34

* Rebate is equal to intrinsic value of option at barrier.

† Prices calculated on a 1000 by 1000 grid.

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