# A Family of Humped Volatility Structures.

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**Abstract**

Recent empirical studies on interest rate derivatives have shown that the volatility structure of interest rates is frequently humped. Mercurio and Moraleda (1996) and Moraleda and Vorst (1996a) have modelled interest rate dynamics in such a way that humped volatility structures are possible and yet analytical formulas for European options on discount bonds are derived. However, both models are Gaussian, and hence interest rates may become negative. In this paper we propose a family of interest rate models where (i) humped volatility structures are possible; (ii) the interest rate volatility may depend on the level of the interest rates themselves; and (iii) the valuation of interest rate derivative securities can be accomplished through recombining lattices. The second item implies that a number of probability distributions are possible for the yield curve dynamics, and some of them ensure that interest rates remain positive. We propose, for instance, models of the type of the proportional Ritchken and Sankarasubramanian (1995) and the Black and Karasinski (1991) model. To gain the computational tractability (iii), we show how to embed all models in this paper in either the Ritchken and Sankarasubramanian (1995) or the Hull and White (1990, 1994) class of models.

## 1 Introduction.

After two decades of intense research, a full agreement on an interest rate model for pricing derivative securities has not yet been obtained. Moreover, numerous models proposed in the literature either fail because they are unrealistic, or because they are computationally very time consuming. The two major stylized facts that have been detected by recent empirical studies are: (i) the volatility of the interest rate dynamics seems to depend on the level of

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the interest rates themselves (see, e.g., Chan et al. (1994) and Amin and Morton (1994)) and (ii) humped volatility structures are typically found both in yield curve dynamics and when calibrating the models from real option data. A humped volatility structure has increasing volatilities for longer maturities in the short end of the yield curve, while it exhibits decreasing volatilities in the long end. Examples are Kahn (1991), Heath, Jarrow, Morton and Spindel (1992) and Moraleda and Vorst (1996b). Kahn (1991) studied the historical volatility functions that best explain the fluctuation of the US Treasury rates over the period 1980-1990. Heath et al. (1992) detected implied volatility pattern similar to that of Kahn (1991) for Libor cap data. Finally, Moraleda and Vorst (1996b) estimated model parameters using time series of Spanish Government bonds and found humped volatility shapes in 90% of the sample periods analyzed. All these authors empirically studied the yield curve dynamics using models that explicitly allow for humped volatility structures.

On the other hand, also indirect evidence for humped volatility structures has been found. For instance, Amin and Morton (1994) estimated the implied volatility of the yield curve from Eurodollar futures options data. They tested six relevant examples in the Heath, Jarrow and Morton (1992) framework and, among them, they studied the exponentially decaying volatility model whose volatility is given by $\sigma \exp[-\lambda(T-t)]$. They found an estimated value of the exponential coefficient, $\lambda$, that is negative on average, making positive the overall sign of the exponent of the volatility. Amin and Morton concluded that the volatility is humped, and that the Eurodollar data shows the upwards sloping side of the hump. Pelsser (1995) obtained remarkably similar results using US-dollar caps and floors data. Among others, he studied the extended-Vasicek Hull and White (1990) model and also obtained negative estimates for the mean reverting coefficient. These negative estimates are significant and take place for “prolonged” periods in his sample.

It is clear, however, that negative values for the mean reversion coefficient in the volatility structure of the above mentioned models are not plausible for all maturities. This would lead in fact to inconsistencies such as the explosion of future interest rates. Such possibility could be precluded by restricting these parameters to be non-negative. But this would remove the mean-reverting effect in the yield curve dynamics whereas, on the grounds of economic theory, there are “compelling” arguments for the mean reversion of interest rates (see e.g. Hull (1993), section 15.10).

The purpose of this paper is to model the mean-reverting effect of interest rates by allowing for humped shapes in their volatility structure and by combining the empirical evidence previously discussed with computational tractability. To this end, we consider models for which either the Hull and White (1990, 1993) tree or the Li, Rithcken and Sankarasubramanian (1995) algorithm can be used. More precisely, we propose a general class of models such that:

- Humped volatility structures are possible;
- The interest rate volatility might depend on the level of interest rates.
- The valuation of interest rate derivatives securities can be accomplished through

recombining lattices.

We contribute to the literature by presenting some models that capture these three features at one time. However, all the models we propose in this paper are based on the assumption of a non-stationary volatility.[1] This is because, by generalizing a previous result by Moraleda and Vorst (1996a), we prove that humped and stationary volatilities are incompatible, if the computational tractability is not to be lost.[2] This is unfortunate since strong empirical support has been found for both. For example, Amin and Morton (1994) concluding their study claimed that "the implied volatility series is stationary and mean-reverting irrespective of the models (six) used".

The paper is organized as follows. Section 2 presents the models and discusses the implications of humped but non stationary volatility models versus stationary though monotone volatilities. Section 3 shows how to implement the Ritchken and Sankarasubramanian (1995) and the Hull and White (1993, 1994, 1995) algorithms for our models. Section 4 compares the models through simulation results, while section 5 concludes and summarizes the paper.

# 2 The model

Our target in this paper is to develop models that combine humped shapes in the volatility structure of interest rates with computational tractability. For the latter feature we can use the Li, Ritchken and Sankarasubramanian (1995) [LRS] lattice which applies to a class of volatilities identified by Ritkchen and Sankarasubramanian (1995) [RS] that leads to a Markov model for future interest rates with respect to two state variables. Alternatively, we can use the Hull and White (1993, 1994, 1995) [HW] tree. To our knowledge, these are, in fact, the only available algorithms for approximating yield curve dynamics in recombining lattices. We show below that both procedures allow for a number of specifications of the instantaneous spot rate volatility, whereas there are some restriction on the possible functions relating the volatility of all forward rates to the spot rate volatility.

In both the RS and HW frameworks, we propose a family of humped volatility models.[3]

## 2.1 Humped volatility models in the RS framework

Consider a continuous-time economy where bonds are traded for all maturities and markets are frictionless. Denote by $P(t,T)$ the price at time $t$ of a pure discount bond that pays \$1 at time $T$, with $0 \le t \le T$, and assume that $P(t,T) > 0$ for all $t \in [0,T]$. The instantaneous forward rate at time $t$ for a maturity $T$ is defined by $f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$,

[1] The volatility is said to be stationary when its structure at the current time is fully specified by the time to maturity and the current instantaneous spot rate.

[2] By computationally tractable, we consider a model whose evolution in time can be made Markovian at most with respect to two state variables (see Ritchken and Sankarasubramanian (1995)).

[3] Some of the models proposed in these two settings are actually equivalent.

so that

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

Following Heath, Jarrow and Morton (1992) [HJM], the evolution of the instantaneous forward rate for a fixed maturity $T$ is modelled by the diffusion

$$df(t,T) = \eta(t,T)dt + \sigma_{HJM}(t,T)dW(t), \tag{1}$$

with $f(0,T)$ given and deterministic, and where $\eta(t,T)$ and $\sigma_{HJM}(t,T)$ are stochastic processes whose values are known at time $t$ (adapted processes), and $W(t)$ is a Brownian motion.

As shown by Heath, Jarrow and Morton (1992), the assumption of no-arbitrage implies the existence of an equivalent martingale measure under which

$$\eta(t,T) = \sigma_{HJM}(t,T)\int_t^T \sigma_{HJM}(t,u)du. \tag{2}$$

As follows from (1) and (2), the stochastic evolution of the forward rates in a risk-adjusted world is fully characterized by the specification of the initial forward rate curve, $f(0,T)$, and the volatility function, $\sigma_{HJM}(t,T)$. The volatility function plays a key role in the analysis. Its exogenous specification uniquely determines the drift term of the process by the no arbitrage argument. The initial forward rate curve is observable in the market at any moment in time and it is therefore given exogenously.

The instantaneous spot rate at time $t$, $r(t)$, is by definition equal to $f(t,t)$ so that

$$r(t) = f(0,t) + \int_0^t \sigma_{HJM}(u,t)\left(\int_u^t \sigma_{HJM}(u,y)dy\right)du + \int_0^t \sigma_{HJM}(u,t)d\tilde{W}(u), \tag{3}$$

where $\tilde{W}(u)$ denotes a Brownian motion under the equivalent martingale measure.

Unfortunately, $f(t,T)$ and $r(t)$ are typically path-dependent for a general choice of $\sigma_{HJM}(t,T)$. This makes computations very slow when not unfeasible. Recently, however, Ritchken and Sankarasubramanian (1995) have identified a class of volatility structures within the HJM paradigm for which the evolution of the term structure is Markovian with respect to two state variables. This basically means that the path-dependence in the spot rate dynamics can be somehow eliminated. The RS class of volatilities within the HJM framework is defined by

$$\begin{aligned} \sigma_{RS}(t,T) &= \sigma_S(t)h(t,T), \\ h(t,T) &= e^{-\int_t^T \kappa(x)dx}, \end{aligned} \tag{4}$$

where $\sigma_S(t)$ is a stochastic process whose value is known at time $t$ and $\kappa(x)$ is a deterministic function. The function $\sigma_S(t)$ denotes the instantaneous spot rate volatility, while $h(t,T)$ relates all forward rate volatilities to the spot rate volatility. As a particular case, Ritchken and Sankarasubramanian (1995) set $\kappa(x) = \kappa$ in (4), being $\kappa$ a positive constant. In this case, $h(t,T)$ is a monotone decreasing function of the time-to-maturity $(T-t)$ and is given by $h(t,T)=\exp[-\lambda(T-t)]$. Thus a mean-reverting effect in the yield curve dynamics is modelled through the function $h(t,T)$. However, in this case, the forward rate volatility $\sigma_{HJM}(t,T)$ is a strictly decreasing function of $(T-t)$. This is not in agreement with the empirical evidence previously reported. In fact, calibrating such a model leads, in many cases, to negative estimates of $\kappa$, which is implausible for all interest rate maturities. The aim of this section is to propose more realistic models for which these inconsistencies can be avoided while keeping the computational tractability of the RS class of models.

The first attempt to model humped volatility structures in the HJM framework is due to Mercurio and Moraleda (1996). They proposed a model where the forward rate volatility is given by

$$\sigma_{MM}(t,T)=\sigma[1+\gamma(T-t)]\exp[-\lambda(T-t)], \tag{5}$$

where $\sigma$, $\gamma$ and $\lambda$ are non-negative constants. The specification (5) has the following properties: (a) it provides a humped volatility structure for any $\gamma>\lambda$; (b) it is stationary, in that it depends only on the difference $T-t$; (c) it leads to analytical formulas for European options on discount bonds; (d) it generalizes all the existing specifications of the volatility for which analytical formulas for European options on discount bonds have been derived in the HJM framework (i.e., the constant and exponentially decaying volatilities).

Unfortunately, the volatility (5) does not belong to the RS class. Hence, American style options can only be priced through non-recombining lattices.

An alternative humped volatility model that overcomes the previous drawback has been recently proposed by Moraleda and Vorst (1996a). In this model, the instantaneous forward rate volatility is given by

$$\sigma_{MV}(t,T)=\sigma\frac{1+\gamma T}{1+\gamma t}\exp[-\lambda(T-t)], \tag{6}$$

where $\sigma$, $\gamma$ and $\lambda$ are non-negative constants.

This choice fulfills the properties (a), (c) and (d) of the model proposed by Mercurio and Moraleda (1996). In particular, for any given $t$, the function $\sigma_{MV}(t,x)$, $x\geq t$, has a humped graph for $\gamma>\lambda$ and $t<\frac{\gamma-\lambda}{\gamma\lambda}$. Moreover, the volatility (6) is in the RS class and, therefore, recombining trees can be used for pricing derivative securities. Notice, in fact, that (6) can be written in the form (4) by setting $\sigma_S(t)=\sigma$ and $\kappa(x)=\lambda-\frac{\gamma}{1+\gamma x}$. A negative feature of (6) is that it is non-stationary. However, the exponentially decaying term of (6) is indeed stationary and, moreover, the first order Taylor approximation in $\gamma$ of the term accounting for the upward side of the hump, i.e. $\frac{1+\gamma T}{1+\gamma t}$, is also a function that only depends on the time to maturity $T-t$.

The non-stationarity of (6) is not actually surprising. In fact, Moraleda and Vorst (1996a) proved that it is not possible to find in the RS class a humped and stationary volatility for which American-style options can be priced through recombining trees with at most two state variables. This statement, originally proved under the restriction that the volatility function $\sigma_{HJM}(t,T)$ is deterministic, also applies to a broader class of volatilities.

This is explained in the following proposition whose proof is a straightforward extension to that of Proposition 2 in Moraleda and Vorst (1996a).

**Proposition 1.** *If the instantaneous forward rate volatility $\sigma_{HJM}(t,T)$ in (1) is restricted to belong to the RS class (4), it is not possible to choose $\kappa(x)$ in such a way that the corresponding $h(t,T)$ is both humped and stationary.*

As follows from Proposition 1, the three desirable properties for the function $h(t,T)$ (i.e., humped shape, stationarity and leading to a computationally tractable model) can not be fulfilled within one model. Computational tractability is essential for a model that can be used in real time. As a consequence, a choice between a stationary mean reversion and a humped shape in the volatility structure must be made. In this paper, we have chosen to abandon the stationarity feature in order to allow for humped shapes, as was done by Moraleda and Vorst (1996a) through the specification (6). However, their model can be generalized in the choice for the instantaneous spot rate volatility. In fact, they chose $\sigma_S(t)$ to be a constant. This has the following drawbacks: interest rates can become negative with positive probability and the instantaneous spot rate volatility $\sigma_S(t)$ is not proportional to the spot rate itself as suggested, e.g., by Chan et al. (1994) and Amin and Morton (1994). To tackle these issues, in this subsection we model the family of forward rates volatilities (4) as follows

$$\begin{aligned} \sigma_S(t) &= \hat{\sigma}(r(t)) = \sigma[r(t)]^{\rho}, \\ \kappa(x) &= \lambda - \frac{\gamma}{1+\gamma x}, \end{aligned} \tag{7}$$

so that

$$h(t,T) = \frac{1+\gamma T}{1+\gamma t}\exp[-\lambda(T-t)] \tag{8}$$

with $\sigma$, $\rho$, $\gamma$ and $\lambda$ non negative constants. Notice that for $\rho = 0$ we get the Moraleda and Vorst (1996a) specification.

Our model is as general as that of Ritchken and Sankarasubramanian (1995), as far as the instantaneous spot rate volatility is concerned. As such, different probability distributions for the forward rates are possible for different choices of $\rho$.

Moreover, for $\rho > 0$, the instantaneous spot interest rate volatility is proportional to the level of the spot interest rate. Unlike the particular case considered by Ritchken and Sankarasubramanian (1995), however, we add the possibility of humped shapes in the volatility structure of interest rates. This is due to the choice of $\kappa(x)$ in (7).

The function $h(t,T)$ in (8) is a two-variable function with the following features: (i) it is non negative for $\sigma$, $\rho$, $\gamma$ and $\lambda$ non negative constants; (ii) it has a local maximum for $t = T = \frac{\gamma-\lambda}{\gamma\lambda}$ if $\gamma > \lambda > 0$; (iii) it goes to zero as $T$ goes to infinity, for each fixed $t > 0$.

In Figure 1, we plot the evolution of $h(t,T)$ for $T \geq t$ as we move in $t$. In particular, we set $\gamma = 0.4$, $\lambda = 0.2$ and $t \in \{0, 0.25, 0.5, 1, 2.5, 10\}$. For $t \leq 1$, the plotted graphs have similar humped shapes (solid lines in Figure 1). This is a good feature since most of the options traded in the financial markets have a maturity of less than one year. Notice in fact that we hardly change the initial volatility structure along the life of these options and their possible mispricing will likely be rather small. For very long time-to-maturity options the evolution of $h(t,T)$ is different (dotted lines in Figure 1). Nevertheless, we always end up with a decaying volatility structure as that of stationary models corresponding to setting $\kappa(x) = \kappa$.

## 2.2 Humped volatility models in the HW framework

While Heath, Jarrow and Morton (1992) and Ritchken and Sankarasubramanian (1995) modeled the dynamics of the entire term structure of forward rates, Hull and White (1990) focused on the instantaneous spot rate. They proposed a model whose dynamics, under risk neutrality, is given by the difussion

$$dr(t) = [\theta(t) - \beta(t)r(t)]\,dt + \sigma(t)dW(t), \tag{9}$$

with $r(0)$ given and deterministic, where $\beta(t)$ and $\sigma(t)$ are deterministic functions and $W(t)$ is a Brownian motion. The function $\theta(t)$ is chosen to exogenously fit the initial yield curve, as shown by Hull and White (1993, 1994) and Rogers (1994). The process (9) implies that interest rates are normally distributed and that explicit formulae for discount bond prices and European option prices are available.

Starting from the model in (9), Hull and White (1993, 1994) developed an algorithm for efficiently computing option prices through a recombining trinomial tree. Hull and White (1994) also showed that their algorithm can be applied to more general processes of the form

$$\begin{cases} dx(t) = [\theta(t) - \beta(t)x(t)]\,dt + \sigma_{HW}(t)dW(t) \\ r(t) = g(t, x(t)) \end{cases} \tag{10}$$

where $x$ is an underlying process, and the spot interest rate $r$ is determined from $x$ through some function $g$. The pair $(\sigma_{HW}(t), \beta(t))$ is referred to as the volatility function in the HW model.

By appropriately choosing $g$, a number of known models satisfy the representation (10). It is obvious that by setting $g(t,x) = x$ for each $(t,x)$, we obtain the extended-Vasicek version of the Hull and White (1990) model (9). The Pelsser (1996) model is obtained by setting $g(t,x) = x^2$, for each $(t,x)$. The Black and Karasinski model (1991) corresponds to the choise $g(t,x) = e^x$, for each $(t,x)$.

Hull and White (1995) argued that there should not be more than one time varying parameter in one-factor models of the class (10) and this should be used to fit the initial term structure. In particular, they set $\sigma_{HW} = \sigma$ and $\beta(t) = \beta$, with $\sigma$ and $\beta$ positive constants, and used $\theta(t)$ for incorporating the initial yield curve into the model. By doing so, they modeled a strictly decaying mean reversion for the process $x$. In fact, for $T > t$,

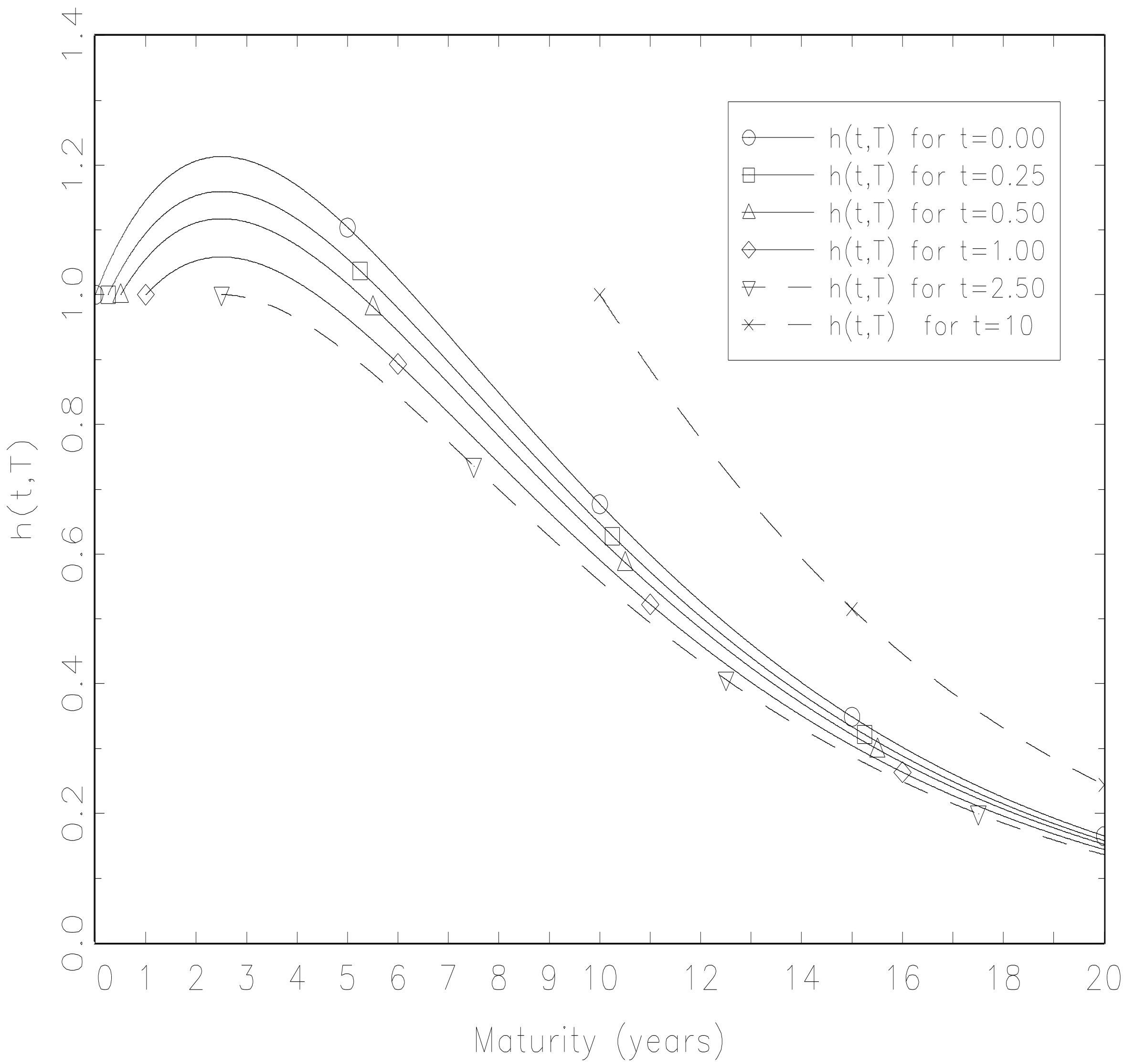


**Figure 1: Functions $h(t,T)$ as in (7), where $\gamma = 0.4$, $\lambda = 0.2$ and $t \in \{0, 0.25, 0.5, 1, 2.5, 10\}$**

the conditional distribution of $x(T)$ given $x(t)$ is normal with variance

$$Var(x(T)|x(t)) = \int_t^T \sigma^2 e^{-2\beta(T-u)} du = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta(T-t)}\right), \tag{11}$$

$$\frac{d(Var_t(x))}{dT} = \sigma^2 e^{-2\beta(T-t)} \tag{12}$$

Thus for a given $t$ and $\beta > 0$, the variance rate[4] of $x(T)$ conditional to $x(t)$ decreases with $T$. Moreover, by increasing $T$, the variance rate tends to zero and the process $x$ tends to its long-run average. In other words, we have a straightforward mean reverting effect for the interest rates.

However, several empirical works, as those reported in the introduction, show that typically $\beta < 0$ when calibrating this model from real data. Besides being unrealistic, this also leads to the explosion of interest rates in the future. As already noticed, negative estimates for $\beta$ seem to suggest that the interest rate volatility first increases and then decreases with $T$, i.e. that its shape is actually humped.

To model humped volatility structures in the HW framework (10), the following result by Moraleda and Vorst (1996a) is very useful:

**Proposition 2.** *If the instantaneous spot rate volatility is deterministic, then the HJM model (1) and (2) restricted to the RS class (4) of volatilities with $\rho = 0$ is equivalent to the HW model (10) with $g(t,x) = x$, for each $(t,x)$. Thus, the HJM class of volatilities that can be equivalently modeled in the HW framework is given by*

$$\sigma_{RS}(t,T) = \sigma_S(t) e^{-\int_t^T \kappa(x)dx}, \tag{13}$$

*with $\sigma_S(t)$ and $\kappa(x)$ deterministic functions. Moreover, the equivalence between both approaches is established through the following relations:*

$$\begin{aligned} \sigma_{HW}(t) &= \sigma_S(t) \\ \beta(t) &= \kappa(t). \end{aligned} \tag{14}$$

The previous proposition not only states that the Gaussian examples in both the RS and the HW frameworks are perfectly equivalent, but also shows how to create a one-to-one correspondence between the volatility coefficients in these two settings. We can therefore introduce humped volatility structures in the Gaussian HW model (10), where $g(t,x) = x$ for each $(t,x)$, by choosing $\beta(t)$ exactly in the same way as $\kappa(x)$ in the previous subsection. Moreover, under this choice for $\beta(t)$ and setting e.g. $g(t,x) = x^2$ or $g(t,x) = e^x$ in (10), we can also introduce humped volatilities in other well-known models where interest rates are always positive. This is similar to what we have previously done under the RS framework.

[4] We define the variance rate as the derivative of the variance in $T$, i.e. $\sigma^2 \exp[-2\beta(T-t)]$

In this subsection, therefore, we propose the following class of models

$$\begin{cases} d\hat{x}(t) = [\theta(t) - \beta(t)\hat{x}(t)]\,dt + \sigma dW(t) \\ \beta(t) = \lambda - \frac{\gamma}{1+\gamma t} \\ r(t) = g(t, \hat{x}(t)) \end{cases} \tag{15}$$

where $\hat{x}$ is an underlying Gaussian process, $g$ is some real function of two real variables, $\sigma$, $\lambda$ and $\gamma$ are non-negative constants and $\theta(t)$ is an arbitrary deterministic function.

If $g(t,x) = x$ for each $(t,x)$, the process for $r$ exactly matches the process $\hat{x}$. This leads to a restricted version of the extended-Vasicek HW (1990) model or, equivalently, to an extended version of the HW (1994) model. In this case, due to Proposition 2, the process $r$ is perfectly equivalent to the RS process (7) with $\rho = 0$. Hence, analytical formulas for the prices of discount bonds and European options on discount bonds are available. However, for each $t > 0$, $r(t)$ can become negative with positive probability, due to its normal distribution. To avoid the unfortunate feature of negative interest rates, we can set $g(t,x) = x^2$ for each $(t,x)$. In this case $r(t)$ is always positive since zero becomes a *reflecting barrier*. This choice leads to an extended version of the square Gaussian model by Pelsser (1996), where the possibility of humped shapes in the volatility structure has been added through the $\beta(t)$ in (15). Notice that the RS spot rate process with $\rho = 0.5$ is similar to the process $r = \hat{x}^2$ in that both involve the term $\sqrt{r}$ in the specification of their dynamics. However, the equivalence between the RS and HW frameworks reported in Proposition 2 no longer holds.

In fact, the distribution of the interest rates under the *square root* RS process (3) with (7) and $\rho = 0.5$ is unknown, whereas the interest rates under the process $g(t,\hat{x}) = \hat{x}^2$ in (15) are $\chi$-square distributed. A smoother way to avoid negative interest rates and matching any possible initial yield curve is achieved by setting $g(t,x) = e^x$ for each $(t,x)$. Contrary to the previous case of a square Gaussian model, this choice keeps interest rates away from zero rather than imposing a reflecting barrier at that level. By doing so, we obtain a restricted version of the Black and Karansinski (1991) model where humped volatilities are explicitly modelled. Since interest rates are now lognormally distributed, we have fatter tails in the probability distribution of $r(t)$ than those of the previous models. This is quite appealing given the empirical findings by Chan et al. (1994) and Amin and Morton (1994). However, this model completely loses the analytical tractability of the previous two.[5] Notice also that this model is similar to the RS model (3) with (7) and $\rho = 1$. However, there are two major differences. First, the distribution of the interest rates under the RS model with $\rho = 1$ is unknown. Second, under this RS model analytical prices for discount bonds can be obtained.

In this subsection we have seen that we can explicitly model humped volatility structures in the HW framework by properly choosing $\beta(t)$ in (15). As in the previous case of the RS framework, however, the mean reversion of the process $\hat{x}$ is no longer stationary. Therefore we face again the problem of the choice between humped volatility structures

[5] In the following section, we will outline how to price numerically derivative securities under this model.

and a stationary mean reversion. Notice, in fact, that due to the equivalence provided by Proposition 2, Proposition 1 also holds for the process $\hat{x}$. For a complete treatment of this issue we then refer to the previous subsection.

# 3 Lattice approximation: relevant examples

In this section we explain how the models previously proposed can be used for pricing interest rate derivative securities. In particular, we will point out the cases where analytical formulas can be derived.[6] For the general case, however, numerical procedures must be implemented. In the sequel, we will briefly outline the algorithms developed by Li, Ritchken and Sankarasubramanian (1995) and by Hull and White (1993, 1994, 1995). For the models considered in this paper, we can use at least one of these procedures.

## 3.1 The LRS two-state variable algorithm

Ritchken and Sankarasubramanian (1995) have shown that the spot rate dynamics restricted to their class (4) of volatilities can be written as follows

$$\begin{aligned} dr(t) &= \mu(r,t)dt + \sigma_S(t)dW(t), \\ d\phi(t) &= \left[\sigma_S^2(t) - 2\kappa(t)\phi(t)\right] dt, \end{aligned} \tag{16}$$

with

$$\mu(r,t) = \kappa(t)[f(0,t) - r(t)] + \phi(t) + \frac{d}{dt}f(0,t). \tag{17}$$

Ritchken and Sankarasubramanian (1995) have also shown that, for choices of the volatility as in (4), the bond prices can be computed analytically. Precisely, the price at time $t$ of a bond with maturity $T$, $t \in [0,T]$, is given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\Lambda(t,T)(r(t)-f(0,t)) - \frac{1}{2}\Lambda^2(t,T)\phi(t)}, \tag{18}$$

with

$$\begin{aligned} \Lambda(t,T) &= \int_t^T h(t,u)du \\ \phi(t) &= \int_0^t \sigma_{RS}^2(u,t)du \end{aligned} \tag{19}$$

Moreover, the yield curve dynamics described by (16) can be discretised in a Markovian (or recombining) lattice in terms of the two variables $r(t)$ and $\phi(t)$. Li, Ritchken and Sankarasubramanian (1995) have developed an efficient lattice to approximate the

[6] In these cases, we will either provide these formulas or refer to the papers where they can be found.

processes (16). Their algorithm is outlined in this subsection for general choices of the parameter $\rho$ in (7).[7] We also show how this procedure applies to our models as presented in the previous section.

Following Li, Ritchken and Sankarasubramanian (1995), we consider the following transformation that yields a process with constant volatility

$$Y(t) = \int \frac{1}{\hat{\sigma}(x)} dx \bigg|_{x=r(t)} \tag{20}$$

where the right-hand side denotes a primitive of $\hat{\sigma}(x)$ calculated in $x = r(t)$ and where the constant in the primitive is set to be zero. Hence,

$$Y(t) = \bar{Y}(r(t)) := \begin{cases} \frac{1}{\sigma} \ln(r(t)) & \text{if } \rho = 1 \\ \frac{1}{\sigma(1-\rho)} [r(t)]^{1-\rho} & \text{if } 0 \leq \rho < 1 \end{cases} \tag{21}$$

with the function $Y$ defined in a suitable domain $\mathcal{D}_\rho$ depending on the value of $\rho$.

Denoting by $x = \varphi(y)$ the inverse function of $y = \bar{Y}(x)$ on $\mathcal{D}_\rho$, we have

$$\varphi(y) = \begin{cases} e^{\sigma y} & \text{if } \rho = 1 \\ (\sigma(1-\rho)y)^{1/(1-\rho)} & \text{if } 0 \leq \rho < 1 \end{cases} \tag{22}$$

Application of Ito's Lemma and straightforward algebra show that

$$dY(t) = m(Y, \phi, t)dt + dW(t)$$
$$d\phi(t) = [\hat{\sigma}(r(t)) - 2\kappa(t)\phi(t)]\, dt$$

where

$$m(Y, \phi, t) = \frac{\kappa(t)(f(0,t) - \varphi(Y(t))) + \phi(t) + \frac{df}{dt}(0,t)}{\sigma[\varphi(Y(t))]^\rho} - \frac{\sigma\rho}{2[\varphi(Y(t))]^{1-\rho}} \tag{23}$$

for any $0 \leq \rho \leq 1$.

The lattice which provides a discretised approximation for $r(t)$ is constructed as follows. Suppose that the given time horizon is divided into intervals of equal length $\Delta t$ and that at the beginning $t$ of some time increment the state variables are $y$ and $\phi$. Then, in the next time period, the variables move to either $(y^+, \phi^*)$ or to $(y^-, \phi^*)$ where

$$y^+ = y^a + (J(y, \phi) + 1)\sqrt{\Delta t},$$
$$y^- = y^a + (J(y, \phi) - 1)\sqrt{\Delta t},$$
$$\phi^* = \phi + [\hat{\sigma}^2(\varphi(y)) - 2\kappa(t)\phi]\Delta t$$

[7] Li, Ritchken and Sankarasubramanian (1995) provided explicit expressions of some quantities we will consider in the sequel only for the cases $\rho = 0.5$ and $\rho = 1$. Their approach however is quite general.

The function $J$ is defined as follows. Set $Z(y,\phi) = \text{int}\left[m(y,\phi,t)\sqrt{\Delta t}\right]$, where $\text{int}[x]$ denotes the largest integer smaller or equal than the real $x$. Then

$$J(y,\phi) = \begin{cases} |Z(y,\phi)| & \text{if } Z(y,\phi) \text{ is even} \\ Z(y,\phi)+1 & \text{otherwise.} \end{cases}$$

This choice ensures that

$$y^+ \geq y + m(y,\phi,t) \geq y^- \tag{24}$$

and hence that the probabilities of moving from one state to another one in the lattice lie always in the interval $[0,1]$. Once the LRS tree has been built, derivative prices can be calculated backwards. We refer to Li, Ritchken and Sankarasubramanian (1995) for a detailed description of how these calculations are performed.

We consider three relevant examples. They are obtained by setting $\rho$ in (7) equal to $0, 0.5, 1$. In the case $\rho = 0$, the volatility is deterministic, and the model has a great deal of analytical tractability. Thus, for example, analytical formulas for European options on discount bonds can be obtained. However, interest rates are normally distributed and negative rates in the future can occur with positive probability. Choosing $\rho$ to be either 0.5 or 1 precludes this unfortunate feature. Yet, we lose the previous analytical tractability and approximating procedures are the only possible tool to evaluate interest rate derivatives.

### 3.1.1 Gaussian example

This is obtained by setting $\rho = 0$. We get the Moraleda and Vorst (1996a) model given by

$$\sigma_{MV}(t,T) = \sigma \frac{\gamma T + 1}{\gamma t + 1} e^{-\lambda(T-t)}, \tag{25}$$

with $\sigma$ a positive constant and $\gamma$ and $\lambda$ non-negative constants. This model is analytically very tractable. In fact, Moraleda and Vorst (1996a) derived analytical formulas for European options on discount bonds. However, American options and other contingent claims can not be priced analytically. We can use instead the LRS algorithm. We show now how this algorithm is simplified in this case.

As already seen, the volatility in (25) can be nested into the RS class by setting

$$\sigma_S(t) = \sigma,$$
$$\kappa(x) = \lambda - \frac{\gamma}{1+\gamma x},$$

so that $h(t,T) = \frac{1+\gamma T}{1+\gamma t} \exp[-\lambda(T-t)]$. For this model, the values of $\Lambda(t,T)$ and $\phi(t)$ in bond price formula (18) are given by

$$\Lambda(t,T) = \frac{1}{\lambda^2(\gamma t+1)} \left[(\gamma\lambda t + \gamma + \lambda) - (\gamma\lambda T + \gamma + \lambda)e^{-\lambda(T-t)}\right], \tag{26}$$

and

$$\begin{aligned}\phi(t) =&\sigma^2 \int_0^t \left(\frac{1+\gamma t}{1+\gamma u}\right)^2 e^{-2\lambda(t-u)} du \\ =&\frac{\sigma^2(1+\gamma t)}{\gamma^2}\left[2\lambda Ei\left(\frac{2\lambda(1+\gamma t)}{\gamma}\right) e^{\frac{-2\lambda(1+\gamma t)}{\gamma}}(1+\gamma t) - \gamma\right] \\ &- \frac{\sigma^2(1+\gamma t)^2}{\gamma^2}\left[2\lambda Ei\left(\frac{2\lambda}{\gamma}\right) e^{\frac{-2\lambda(1+\gamma t)}{\gamma}} - \gamma e^{-2\lambda t}\right],\end{aligned} \tag{27}$$

where $Ei$ denotes the exponential integral function[8]

$$Ei(z) = \int_{-\infty}^{z} \frac{e^t}{t} dt. \tag{28}$$

The process for the spot rate is then given by

$$dr(t) = \left[\left(\lambda - \frac{\gamma}{1+\gamma t}\right)(f(0,t) - r(t)) + \phi(t)\right] dt + \sigma dW(t). \tag{29}$$

with $\phi(t)$ given by (27).

Contrary to the general LRS model which is a two variable model, this case reduces to a single variable model, namely $r(t)$. This is because the deterministic volatility function in (25) implies that the second state variable, $\phi(t)$, becomes deterministic.

As explained above, Li, Ritchken and Sankarasubramanian (1995) proposed a change of variable in order to obtain a process with constant volatility. However, this is no longer required in this case, since the instantaneous spot rate volatility is already constant. The discretised lattice approximation for $r(t)$ can then be simplified as follows. Suppose that at the start of some time increment the approximating variable is $r^a$. Over the next time period the variable moves to either $r^{a+}$ or to $r^{a-}$ whose values are given by

$$\begin{aligned} r^{a+} &= r^a + \sigma\left[(J+1)\sqrt{\Delta t}\right], \\ r^{a-} &= r^a + \sigma\left[(J-1)\sqrt{\Delta t}\right], \end{aligned}$$

where

$$J = \begin{cases} |Z| & \text{if } Z \text{ is even} \\ Z+1 & \text{otherwise,} \end{cases}$$

with $Z = \text{int}\left[\frac{\mu(r^a,t)\sqrt{\Delta t}}{\sigma}\right]$, where $\mu(.)$ denotes the drift of the process (29). The remaining procedure closely follows that of LRS.

[8] Moraleda and Vorst (1996a) provide an approximation of this integral function of any desired accuracy.

### 3.1.2 "Square Root" example

Consider now $\rho = 0.5$. This implies modeling the evolution of the spot rates in such a way that interest rates are positive with probability 1 as long as the initial value of $r(t)$ is positive. In particular, for $r(t) > 0$, $Y(t) = 2\sqrt{r(t)}/\sigma$, $\varphi(y) = \frac{\sigma^2 y^2}{4}$ and $m(Y, \phi, t)$ in (23) reduces to

$$m(Y,\phi,t) = \frac{\kappa(t)[f(0,t) - \frac{1}{4}\sigma^2(Y(t))^2] + \phi(t) + \frac{df}{dt}(0,t)}{\frac{1}{2}\sigma^2 Y(t)} - \frac{1}{2Y(t)} \tag{30}$$

with

$$\kappa(t) = \lambda - \frac{\gamma}{1+\gamma t}. \tag{31}$$

Notice that by setting $\rho = 0.5$ we have a volatility structure that depends on $\sqrt{r}$. This is a similar structure to models such as Cox, Ingersoll and Ross (1985) and Pelsser (1996). However, the latter processes imply a $\chi$-square distribution for the interest rates, while the probability distribution for the interest rates under the RS model with $\rho = 0.5$ is unknown. On the other hand, the RS model with $\rho = 0.5$ leads to a very simple analytical formula for the bond prices and it is, moreover, capable to fit any positive initial term structure of interest rates.

### 3.1.3 "Proportional" example

Consider now $\rho = 1$. This leads to a spot rate process for which interest rates are always positive with probability 1, if the initial value of $r(t)$ is positive. In particular, for $r(t) > 0$, $Y(t) = \ln[r(t)]/\sigma$, $\varphi(y) = e^{\sigma y)}$ and $m(Y, \phi, t)$ in (23) reduces to

$$m(Y,\phi,t) = \frac{\kappa(t)[f(0,t) - e^{\sigma Y(t)}] + \phi(t) + \frac{df}{dt}(0,t)}{\sigma e^{\sigma Y(t)}} - \frac{1}{2}\sigma \tag{32}$$

with

$$\kappa(t) = \lambda - \frac{\gamma}{1+\gamma t} \tag{33}$$

A seminal work by Chan et el (1992) concluded that one-factor interest rate models should allow the volatility to fluctuate according to the level of the interest rates. The *proportional* model presented here allows for this property. Unfortunately, the probability distribution for the interest rates under this model is unknown.

## 3.2 The HW numerical procedure

Hull and White (1993, 1994) constructed a Markovian trinomial tree approximating the process $x$ in (10) with $g(x,t) = x$, and $\beta(t) = \beta$ and $\sigma_{HW} = \sigma$. Recently, Hull and White

(1995) have extended their approach to the general form of $x$ in (10) with all parameters being deterministic functions. We can use this extension for our class of models (15).

In the sequel, we explain how the HW procedure applies to our process $\hat{x}$ in (15), and hence to all the models we consider in the HW framework.

We first must build a preliminary tree for $\hat{x}$, setting $\theta(t) = 0$ and the initial value of $\hat{x}$ equal to 0. Assume that the value of $\hat{x}$ at the $j$-th node at time $t_i$ is $\hat{x}_{i,j}$. The mean and standard deviation of $\hat{x}$ at time $t_{i+1}$ conditional on $\hat{x} = \hat{x}_{i,j}$ at time $t_i$ are approximately $\hat{x}_{i,j} + M_i\hat{x}_{i,j}$ and $\sqrt{V_i}$ respectively, where

$$M_i = e^{-\beta(t_i)\Delta t} - 1$$

$$V_i = \frac{\sigma^2}{2\beta(t_i)}\left(1 - e^{-2\beta(t_i)\Delta t}\right)$$

The tree branching procedure is established as follows. From $\hat{x}_{i,j}$, the variable moves either to $\hat{x}_{i+1,k-1}$, or to $\hat{x}_{i+1,k}$ or to $\hat{x}_{i+1,k+1}$, where $k$ is chosen so that $\hat{x}_{i+1,k}$ is as close as possible to $\hat{x}_{i,j} + M_i\hat{x}_{i,j}\Delta t$.

The associated probabilities for this branching are given by

$$\begin{aligned} p_u(i,j) &= \frac{\sigma^2\Delta t}{2\Delta\hat{x}_i^2} + \frac{\eta^2}{2\Delta\hat{x}_i^2} + \frac{\eta}{2\Delta\hat{x}_i} \\ p_m(i,j) &= 1 - \frac{\sigma^2\Delta t}{\Delta\hat{x}_i^2} - \frac{\eta^2}{\Delta\hat{x}_i^2} \\ p_d(i,j) &= \frac{\sigma^2\Delta t}{2\Delta\hat{x}_i^2} + \frac{\eta^2}{2\Delta\hat{x}_i^2} - \frac{\eta}{2\Delta\hat{x}_i} \end{aligned} \tag{34}$$

where $p_u(i,j)$, $p_m(i,j)$ and $p_d(i,j)$ respectively denote the up, middle and down branching probabilities at the $j$-th node at time $t_i$, and where

$$\eta = \hat{x}_{i,j}M_i + (j-k)\Delta\hat{x}_i, \tag{35}$$

with $k$ to be determined at each node in the tree as previously explained.

The branching probabilities should always lie in the interval $[0,1]$. This can be ensured by choosing $\Delta\hat{x}_i$ in such a way that, at each node in the tree, $\hat{x}_{i+1,k-1}$ and $\hat{x}_{i+1,k+1}$ bracket the expected value of $\hat{x}$ in the next time interval conditional to $\hat{x}_{i,j}$. Hull and White (1993) showed that $\Delta\hat{x}_i = \sqrt{3V_i}$ is an appropiate choice.

The procedure described so far applies to the process $\hat{x}$ with $\theta(t) = 0$. The tree should be now displaced in order to add the proper drift to the model. For doing so, Hull and White (1993) devised a forward induction procedure such that the tree prices all discount bonds consistently with the term structure of interest rates as observed in the market.[9]

[9]Notice that the initial yield curve is added to the tree through the discount bond prices produced by the model. However, bond prices are functions of the process $r$ rather than $\hat{x}$. As a consequence, from this point on, the algorithm will be different for different choices of $g(t,\hat{x})$.

The displacement $\alpha_m$ in the tree at period $m$ is computed by solving

$$\psi(\alpha_m) = 0 \tag{36}$$

with

$$\psi(\alpha_m) = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp\left[-g(\alpha_m + j\Delta\hat{x}_m)\Delta t\right] - P(0, m+1) \tag{37}$$

where $P(0, m+1)$ denotes the price at the initial time 0 of a discount bond with maturity at time $t_{m+1}$, $n_m$ is the number of nodes on each side of the central node at the $m$-th time step in the tree, and $Q_{i,j}$ is defined as the present value of a security that pays off 1 if node $(i, j)$ is reached and zero otherwise.[10]

Equation (36) can be analitically solved in $\alpha_m$ if, for example, $g(t, \hat{x}) = \hat{x}$. In general, however, $\alpha_m$ must be computed numerically by using, for instance, the Newton-Raphson procedure. Notice that if $g$ is differentiable, the first derivative of the function $\psi$ is analitycally given by

$$\psi'(\alpha_m) = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp\left[-g(\alpha_m + j\Delta\hat{x}_m)\Delta t\right] \left(-g'(\alpha_m + j\Delta\hat{x}_m)\Delta t\right) \tag{38}$$

Similarly to what we did in the previous subsection under the RS framework, we next consider three relevant examples obtained by setting $g(t, \hat{x}) = \hat{x}$, $g(t, \hat{x}) = \hat{x}^2$ and $g(t, \hat{x}) = e^{\hat{x}}$.

### 3.2.1 Gaussian example

By setting $g(t, \hat{x}) = \hat{x}$ in (15), one gets the Gaussian model that was first considered by Moraleda and Vorst (1996a). This model is analytically very tractable. In fact, Moraleda and Vorst (1996a) proved (see Proposition 2) that this model is exactly equivalent to the model in (7) with $\rho = 0$. Equivalent analytical formulas for discount bonds as those provided in section 3.1.1 can be derived under the notation used in the HW setting. Moraleda and Vorst (1996a) showed that the price at time $t$ of a discount bond with maturity $T$ is given by:

$$P(t, T) = A(t, T)e^{-B(t,T)r}, \tag{39}$$

with

$$B(t, T) = \frac{1}{\lambda^2(1 + \gamma t)}\left[(\gamma\lambda t + \gamma + \lambda) - (\gamma\lambda T + \gamma + \lambda)e^{-\lambda(T-t)}\right]$$

[10] See also Hull and White (1994).

and

$$
\begin{aligned}
A(t,T) =& \frac{P(0,T)}{P(0,t)} \exp\left\{F(0,t)B(t,T) - \frac{1}{2}B^2(t,T)\right. \\
& \left[\frac{\sigma^2(1+\gamma t)}{\gamma^2}\left(2\lambda Ei\left(\frac{2\lambda(1+\gamma t)}{\gamma}\right)e^{\frac{-2\lambda(1+\gamma t)}{\gamma}}(1+\gamma t) - \gamma\right)\right. \\
& \left.\left. - \frac{\sigma^2(1+\gamma t)^2}{\gamma^2}\left(2\lambda e^{\frac{-2\lambda(1+\gamma t)}{\gamma}} Ei\left(\frac{2\lambda}{\gamma}\right) - \gamma e^{-2\lambda t}\right)\right]\right\},
\end{aligned}
$$

where $Ei$ again denotes the exponential integral function. As expected, the bond price in (39) is exactly the same as (18) with (26) and (27).

The computation of $\alpha_m$ in (36) can also be derived analytically as follows

$$
\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta\hat{x}_m\Delta t} - \ln P(0,m+1)}{\Delta t} \tag{40}
$$

The valuation of derivative securities through the HW tree under this model can then be easily accomplished.

### 3.2.2 $\chi$-square example

Consider now $g(t,\hat{x}) = \hat{x}^2$ in (15). This yields to an extension of the model proposed by Pelsser (1996) in that humped volatility are explicitly modelled. This case is quite appealing. In fact, interest rates are positive with probability one since they are $\chi$-square distributed. Unfortunately, bond prices can not be computed analytically. One possibility is to compute them through the tree itself. In this case, the displacements $\alpha$ in the HW tree must also be computed numerically by solving equation (36). However, this choice compels us to expand the tree up to the maturity of the underlying bond. Another possibility is to extend Pelsser (1996)'s procedure to our case where $\beta(t)$ is a deterministic function rather than a constant.[11] In this case, the tree must be expanded only up to the maturity of the option, since the discount bonds prices can be computed numerically according to the following procedure.

Denote by A(t,T) the solution of the Riccati equation

$$
\frac{\partial y}{\partial t}(t,T) - 2\beta(t)y(t,T) - 2\sigma^2 y^2(t,T) + 1 = 0 \tag{41}
$$

with boundary condition $y(T,T) = 0$, and define

$$
B(t,T) := e^{-\int_t^T [\beta(s) + 2\sigma^2 A(s,T)]ds}, \tag{42}
$$

[11] As seen, Pelsser (1996) considered the case where the $g(x) := x^2$, $\beta(t) = \beta$ and $\sigma_{HW} = \sigma$ in (10). He derives analytical formulas for bond and option prices that, unfortunately, can no longer be derived in our case.

$$C(t,T) := B(t,T)\int_t^T 2\theta(s)B(s,T)ds, \tag{43}$$

$$D(t,T) := \int_t^T \left[\tfrac{1}{2}\sigma^2 C^2(s,T) - \sigma^2 A(s,T) - \theta^2(s)\right] ds, \tag{44}$$

$$M(t,T,r) := B(t,T)r - \int_t^T \sigma^2 B(s,T)C(s,T)ds, \tag{45}$$

$$S(t,T) := \int_t^T \sigma^2 B^2(s,T)ds, \tag{46}$$

where $\theta(t)$, chosen to fit the initial term structure, satisfies the integral equation

$$\theta(t) = \sqrt{f(0,t) - S(0,t)} + \int_0^t \sigma^2 B(s,t)C(s,t)ds. \tag{47}$$

Then the discount bond prices are given by

$$P(t,T,r(t)) = \exp\{D(t,T) - C(t,T)r(t) - A(t,T)r^2(t)\}, \tag{48}$$

where $T$ is the underlying bond maturity.

A disadvantage of computing discount bond prices according to (48) is that if $f(0,t) < S(0,t)$, the model cannot be fitted to the initial term structure.

#### 3.2.3 Log-normal example

Black and Karansinski (1991) considered the case where the function $g(t,x)$ in (10) is given by $g(t,x) := e^x$, so that interest rates are lognormally distributed. This is very interesting since interest rates are kept away from zero in a smooth fashion. Moreover, the interest rate volatility is made proportional to the level of the interest rates themselves as suggested by empirical evidence. However, this model lacks the analytical tractability of the previous models.

We now consider a restricted version of the Black and Karansinski (1991) model by setting $g(t,\hat{x}) = e^{\hat{x}}$ in (15). This model still lacks analytical tractability, and numerical procedures must be used. In particular, one can use the binomial tree developed by Black and Karansinski (1991). One can also implement the HW tree outlined before. In either case, a greater computational effort than for previous models is expected since the tree should be built up to the maturity of the underlying asset. By using the HW algorithm one can reduce this effort by enlarging the time step of the tree after the option maturity. This is explained in Hull and White (1995).

**Table 1**
**Comparison of the RS Proportional and HW Lognormal Models**
**Sensitivity of European put option prices to $\sigma$**

| | X=0.5185 | | X=0.5685 | | X=0.6185 | |
|---|---|---|---|---|---|---|
| $\sigma$ | RS-Pr | HW-Ln | RS-Pr | HW-Ln | RS-Pr | HW-Ln |
| 0.10 | 0.0002 | 0.0011 | 0.0077 | 0.0116 | 0.0430 | 0.0437 |
| 0.15 | 0.0014 | 0.0041 | 0.0117 | 0.0173 | 0.0435 | 0.0462 |
| 0.20 | 0.0038 | 0.0081 | 0.0158 | 0.0227 | 0.0447 | 0.0498 |
| 0.25 | 0.0072 | 0.0123 | 0.0197 | 0.0279 | 0.0462 | 0.0537 |
| 0.30 | 0.0094 | 0.0167 | 0.0221 | 0.0328 | 0.0467 | 0.0577 |

The table shows prices of 3-year European put options on 10 years discount bond. The yield curve is given by $r(T) = 0.08 - 0.05\exp(-0.18T)$. The coefficient of the models are set $\lambda = 0.1$, $\sigma \in \{0.10, 0.15, 0.20, 0.25, 0.30\}$ and $\gamma = 0$ for the RS-Pr in (7) with $\rho = 1$. For the HW-Ln model, we also fix $\lambda = 0.1$, $\sigma \in \{0.10, 0.15, 0.20, 0.25, 0.30\}$ and $\gamma = 0$ in (15) with $g(t, \hat{x}) = e^{\hat{x}}$.

# 4 Simulation results

In the previous sections we have proposed a number of models where humped volatility structures are modeled. This has been accomplished within the both the RS and HW frameworks. As shown above, similar models in the different frameworks are obtained. However, the models are not perfectly equivalent unless we consider the Gaussian example in each framework. To investigate the differences of the models we use in this section simulation studies.

First of all, we compare a relevant example of the RS and the HW class of models as they were originally proposed. In particular, we consider the "proportional" example by RS, which is obtained by setting $\rho = 1$ and $\gamma = 0$ in (7) and a restricted version of the original Black and Karasinski (1991) model given by choosing $g(t, \hat{x}) = e^{\hat{x}}$ and setting $\sigma_{HW} = \sigma$ and $\gamma = 0$ in (15). These models, that do not allow for humped volatilities since $\gamma = 0$ in (7) and $\beta(t) = \lambda$ in (15), are denoted in Tables 1 and 2 by RS-Pr and HW-Ln, respectively. In these Tables, some simulations of European put option prices are provided. The maturity of the option is three years and they are written on a zero-coupon bond with ten years to maturity. The initial term structure of interest rates is assumed to be $r(T) = 0.08 - 0.05\exp(-0.18T)$. For fixed $\lambda = 0.1$ and $\sigma \in \{0.10, 0.15, 0.20, 0.25, 0.30\}$, Table 1 reveals that the prices produced by the HW-Ln model are indeed more sensitive to changes in the parameter $\sigma$. Similarly pattern can be found in Table 2 where values for the same options for fixed $\sigma = 0.25$ and $\lambda \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$ are reported.

Table 3 provides some simulation of European put option values produced by humped volatility models proposed in this paper. The notation in Table 3 reads as follows. MV stands for the Moraleda and Vorst (1996a) model as given by (6) or alternatively by (15) with $g(t, \hat{x}) = \hat{x}$. As Proposition 2 states both models are perfectly equivalent and the option values produces by both of them are indeed identical. MM-RS-Sq and MM-RS-

**Table 2**
**Comparison of the RS Proportional and HW Lognormal Models**
**Sensitivity of European put option prices to $\lambda$**

| | X=0.5185 | | X=0.5685 | | X=0.6185 | |
|---|---|---|---|---|---|---|
| $\lambda$ | RS-Pr | HW-Ln | RS-Pr | HW-Ln | RS-Pr | HW-Ln |
| 0.05 | 0.0107 | 0.0184 | 0.0241 | 0.0348 | 0.0490 | 0.0594 |
| 0.1 | 0.0072 | 0.0123 | 0.0197 | 0.0279 | 0.0462 | 0.0537 |
| 0.15 | 0.0047 | 0.0080 | 0.0162 | 0.0226 | 0.0444 | 0.0496 |
| 0.2 | 0.0029 | 0.0050 | 0.0135 | 0.0185 | 0.0435 | 0.0468 |
| 0.25 | 0.0017 | 0.0029 | 0.0113 | 0.0153 | 0.0431 | 0.0451 |

The table shows prices of 3-year European put options on 10 years discount bond. The yield curve is given by $r(T) = 0.08 - 0.05\exp(-0.18T)$. The coefficient of the models are set $\sigma = 0.25$, $\lambda \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$ and $\gamma = 0$ for the RS-Pr in (7) with $\rho = 1$. For the HW-Ln model, we also fix $\sigma = 0.25$, $\lambda \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$ and $\gamma = 0$ in (15) with $g(t, \hat{x}) = e^{\hat{x}}$.

Pr denotes our models in the RS framework as given by (7) with $\rho = 0.5$ and $\rho = 1$, respectively. Finally, our models in the HW framework given by (15) with $g(t, \hat{x}) = \hat{x}^2$ and $g(t, \hat{x}) = e^{\hat{x}}$ are denoted, respectively, by MM-HW-Sq and MM-HW-Ln. In all cases in Table 3 the underlying asset is a zero-coupon bond with maturity $T = 10$ years. Three options maturities are considered, i.e. $\tau = 1, 3, 5$ years. For each maturity we provide prices for at-the-money puts since the strike price $X$ is set at the current forward price of a discount bond with maturity $T - \tau$ years. The initial term structure of interest rates is assumed to be $r(T) = 0.08 - 0.05\exp(-0.18T)$. The model parameters values are set to be $\lambda = 0.2$ for all models. The instantaneous interest rate volatility coefficient is set such that the unconditional instantaneous interest rate volatility is approximately equal for all models. In particular, we set $\sigma_{MV} = 0.02$, $\sigma_{MM-RS-Sq} = \sigma_{MM-HW-Sq} = 0.07$, and $\sigma_{MM-RS-Pr} = \sigma_{MM-HW-Ln} = 0.25$. Finally, the parameter allowing for humped shapes in the models is set to be $\gamma \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ in all models.

# 5 Conclusions

In this paper, we have proposed a class of models that allow for humped shapes in the volatility structure of interest rates and thus is consistent with the latest empirical findings. This class is broad enough to possess a number of desirable properties. In fact, interest rates can be modeled to be always positive and derivative securities can be priced through recombining trees. More precisely, we have proposed a family of humped volatility models in both the HW and RS frameworks. The models in these two frameworks share common features and are perfectly equivalent in the Gaussian cases. Our models in the RS setting have a great analytical tractability since bond price formulas can be derived, while, in the HW framework, only our Gaussian example has this property. On the other hand, all the

**Table 3**
**Comparison of the Models for At-The-Money Options**

| | $\gamma$ | MV | MM-RS-Sq | MM-RS-Pr | MM-HW-Sq | MM-HW-Ln |
|---|---|---|---|---|---|---|
| | 0 | 0.0148 | 0.0101 | 0.0071 | 0.0241 | 0.0125 |
| | 0.2 | 0.0247 | 0.0168 | 0.0118 | 0.0381 | 0.0213 |
| $\tau = 1$ year | 0.4 | 0.0331 | 0.0223 | 0.0156 | 0.0486 | 0.0285 |
| X=0.5071 | 0.6 | 0.0405 | 0.0271 | 0.0189 | 0.0571 | 0.0347 |
| | 0.8 | 0.0471 | 0.0314 | 0.0218 | 0.0638 | 0.0400 |
| | 1 | 0.0531 | 0.0351 | 0.0244 | 0.0693 | 0.0447 |
| | $\gamma$ | MV | MM-RS-Sq | MM-RS-Pr | MM-HW-Sq | MM-HW-Ln |
| | 0 | 0.0194 | 0.0156 | 0.0135 | 0.0339 | 0.0185 |
| | 0.2 | 0.0313 | 0.0246 | 0.0206 | 0.0513 | 0.0297 |
| $\tau = 3$ years | 0.4 | 0.0396 | 0.0306 | 0.0248 | 0.0617 | 0.0371 |
| X=0.5685 | 0.6 | 0.0462 | 0.0351 | 0.0277 | 0.0690 | 0.0427 |
| | 0.8 | 0.0518 | 0.0388 | 0.0300 | 0.0743 | 0.0473 |
| | 1 | 0.0567 | 0.0419 | 0.0318 | 0.0784 | 0.0512 |
| | $\gamma$ | MV | MM-RS-Sq | MM-RS-Pr | MM-HW-Sq | MM-HW-Ln |
| | 0 | 0.0181 | 0.0162 | 0.0146 | 0.0328 | 0.0183 |
| | 0.2 | 0.0276 | 0.0238 | 0.0199 | 0.0468 | 0.0275 |
| $\tau = 5$ years | 0.4 | 0.0335 | 0.0281 | 0.0223 | 0.0540 | 0.0328 |
| X=0.6577 | 0.6 | 0.0380 | 0.0312 | 0.0239 | 0.0587 | 0.0366 |
| | 0.8 | 0.0417 | 0.0336 | 0.0251 | 0.0620 | 0.0397 |
| | 1 | 0.0449 | 0.0356 | 0.0260 | 0.0646 | 0.0422 |

The table shows prices of European put options on 10 years discount bond. The yield curve is given by $r(T) = 0.08 - 0.05\exp(-0.18T)$. The coefficient $\lambda$ is set $\lambda = 0.2$ for all models. The instantaneous interest rate volatility coefficient is set $\sigma_{MV} = 0.02$, $\sigma_{MM-RS-Sq} = \sigma_{MM-HW-Sq} = 0.07$, and $\sigma_{MM-RS-Pr} = \sigma_{MM-HW-Ln} = 0.25$. The unconditional volatility for the instantaneous interest rate is thus approximately the same for all models. Finally $\gamma \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

models we have considered in the HW setting imply a well-known probability distribution for the short term interest rate.

A negative feature of all the models we have proposed in this paper, is that the mean-reversion of the interest rates is non-stationary. We have shown, however, that the three desirable properties of the function relating the volatility of all forward rates (humped shapes, stationarity and leading to a computational tractable model) are not obtainable all together. Moreover, we have motivated why one should drop the stationarity feature in order to allow for the other two properties. In particular, by using our models in this paper, one can avoid theoretical inconsistencies such as those reported in the introduction. We address, therefore, calibrating problems that have been systematically found for traditional stationary mean-reverting processes for the interest rates.

Further research should test the models proposed in this paper with real option data. In particular, empirical studies are needed to quantify our loss of the stationarity property in the mean-reversion of the interest rates.

# References


[1] K.I. Amin, and A.J. Morton, (1994): "Implied Volatility Functions in Arbitrage-Free Term Structure Models", Journal of Financial Economics, 35, 141-180.

[2] F. Black and P. Karasinski (1991): "Bond and Option Pricing when Short Rates are Lognormal", Financial Analysts Journal, 47, 52-59.

[3] K.C. Chan, G.A. Karolyi, F.A. Longstaff and A.B. Sanders (1992): "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate", Journal of Finance, 47, 1209-1227.

[4] J.C. Cox, J.E. Ingersoll, and S.A. Ross, (1985): "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-408.

[5] D. Heath, R. Jarrow, and A.J. Morton, (1992): "Bond Pricing and the Term Structure of Interest Rates : a New Methodology for Contingent Claims Valuation". Econometrica, 60, 77-105.

[6] D. Heath, R. Jarrow, A.J. Morton, and M. Spindel, (1992): "Easier Done Than Said", RISK, 5, 77-80.

[7] T.S.Y. Ho, and S.B. Lee, (1986): "Term Structure Movements and Pricing Interest Rate Contingent Claims". Journal of Finance, 41, 1011-1029.

[8] J. Hull, and A. White, (1990): "Pricing Interest Rates Derivative Securities", Review of Financial Studies, 3, 573-592.

[9] J. Hull, and A. White, (1993): "One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities", Journal of Financial and Quantitative Analysis, 28, 235-254.

[10] J. Hull, and A. White, (1994): "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models", Journal of Derivatives, Fall 1994, 7-16.

[11] J. Hull, and A. White, (1995): "Using Hull-White Interest Rate Trees", Journal of Derivatives, Spring 1996, 26-36. December 1995.

[12] R. Kahn, (1991): "Fixed Income Risk Modelling". The Handbook of Fixed Income Securities, Third edition, edited by F. Fabozzi, 1307-1319.

[13] A. Li, P. Ritchken, L. Sankarasubramanian (1995): "Lattice models for pricing American interest rate claims", Journal of Finance, 1, 719-737.

[14] F. Mercurio, J.M. Moraleda(1996): "An Analytically Tractable Interest Rate Model with Humped Volatility", report 9619/A, Econometric Institute, Erasmus University Rotterdam, The Netherlands.

[15] J.M. Moraleda, A.C.F. Vorst (1996a): "Pricing American Interest Rate Claims with Humped Volatility Models", Erasmus University Rotterdam, The Netherlands.

[16] J.M. Moraleda, A.C.F. Vorst (1996b): "Pricing Derivative Securities with One-Factor Yield Curve Models: An Empirical Investigation", Erasmus University Rotterdam, The Netherlands.

[17] A.A.J. Pelsser, (1996): "Efficient Methods for Valuing and Managing Interest Rate and other Derivative Securities". Ph. D. Dissertation, Erasmus University Rotterdam.

[18] P. Ritchken, L. Sankarasubramanian (1995): "Volatility structures of forward rates and the dynamics of the term structure", Mathematical Finance 5, 55-72.

[19] L. Rogers (1994): "Which Model for Term-Structure of Interest Rate Should One Use?", To appear in Mathematical Finance: Proceedings of IMA Workshops, June 1993, ed, D. Duffie and I. Karatzas, Springer.

[20] O. Vasicek, (1977): "An Equilibrium Characterization of the Term Structure". Journal of Financial Economics, 5, 177-188.