Axiomatic Characterization of the Median and Antimedian Function on a Complete Graph minus a Matching

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Abstract

A median (antimedian) of a profile of vertices on a graph G is a vertex that minimizes (maximizes) the sum of the distances to the elements in the profile. The median (antimedian) function has as output the set of medians (antimedians) of a profile. It is one of the basic models for the location of a desirable (obnoxious) facility in a network. The median function is well studied. For instance it has been characterized axiomatically by three simple axioms on median graphs. The median function behaves nicely on many classes of graphs. In contrast the antimedian function does not have a nice behavior on most classes. So a nice axiomatic characterization may not be expected. In this paper an axiomatic characterization is obtained for the median and antimedian function on complete graphs minus a matching.

Keywords: median, antimedian, consensus function, consistency, cocktail-party graph, complete graph, consensus axiom

1 Introduction

Facility location problems in discrete location theory deal with functions that find an appropriate location for a common facility or resource in a discrete network. The main objective is to minimize the cost of accessing a facility or sharing a resource in the network. Typical problems of this kind that have been studied extensively are: (i) The *median problem*: finding a vertex that minimizes the distance sum to the clients. (ii) The *mean problem*: finding a vertex that minimizes the sum of the squares of the distances to the clients. (iii) The *center problem*: minimizing the maximum distance to the clients. The first two problems can be used to model finding the optimal location for a distribution center. The last problem can be used to model finding the optimal location for a fire station. The *antimedian problem* is a different type of location problem in which the facility is of obnoxious nature (i.e. the clients want to have it as far away as possible), for example a garbage dump. In this case minimizing the 'cost' is equivalent to maximizing the distance sum.

A consensus function is used to model the problem of achieving consensus amongst agents or clients in a rational way. The input of the consensus function is information on the clients and the output concerns the issue on which consensus should be reached. To guarantee the rationality of the process, the consensus function satisfies certain rules called 'consensus axioms'. Such axioms should be appealing and simple. Of course, this depends on the consensus function. A function with nice properties might be characterized by simple axioms. But a function that behaves badly might need more complicated or less appealing axioms. K. Arrow initiated the study of the axiomatics of consensus functions in his seminal paper [1] of 1951. For more references in this area see [2], [3], [18].

Location problems can also be viewed as consensus problems. Then one wants to characterize the location function by a set of axioms that are as nice and simple as possible. Holzman [11] was the first to study location functions from this perspective. His focus was on the mean function on a tree network (the continuous variant of a tree, where internal points of edges are also allowed as location). Then Vohra [28] characterized the median function axiomatically on tree networks. The discrete case was first dealt with by McMorris, Mulder & Roberts [17]: the median function on cube-free median graphs was characterized using three simple and appealing axioms, see below. The mean function on trees (discrete case) was first characterized by McMorris, Mulder & Ortega [15], [16]. The center function on trees has been characterized by McMorris, Roberts & Wang [19], see also [25]. Recently the median function has been characterized on hypercubes and median graphs by Mulder & Novick [23], [24] using the same three simple axioms as in [17]. In the case of the median function and the center function all axioms satisfy the criterion of being appealing and natural at first sight. The characterizations for the mean function are more complex than those for the median function or the center function. But except for one complex axiom they still satisfy the criterion of being simple and appealing. All above results for the center function and the mean function so far are on trees. The characterization for the median function is on a much wider class, viz. that of median graphs. The reason for this is the very nice behavior of the median function on these graphs. For more information on median graphs see e.g. [12], [21], [22].

We focus on the characterization of two location functions: the median function and the antimedian function. The antimedian function maximizes the sum of the distances to the clients, see e.g. [20], [4], [5], [6], [7], [26]. The differences between these two functions are quite striking. A first inspection of the antimedian function already shows that, even on trees, it does not behave nicely at all, let alone on arbitrary graphs. Only on special classes, such as paths, hypercubes and complete graphs, does it seem to have a nice behavior. The axiomatization of the antimedian function on hypercubes and paths is well studied in [8]. The median and antimedian function on cocktail-party graphs and antimedian function on complete graphs are characterized in [9]. In this paper we generalize this characterization to complete graphs minus a matching. Cocktail-party graphs and complete graphs are special cases in this graph class. A cocktail-party graph is a complete graph of even order minus a perfect matching.

In Section 2 we set the stage. In Section 3 we characterize the median function on complete graphs minus a matching by a set of four axioms. In Section 4 we characterize the antimedian function on the same graph by another set of five axioms. For the case of complete graphs only four axioms are needed. Except for anonymity, we have examples to show that in these characterizations each axiom is independent from the others. For axiomatic characterization of the median function on complete graphs we refer to [27],

[14].

2 Preliminaries

Let $G = (V, E)$ be a finite, connected, simple graph with vertex set V and edge set E. The distance $d(u, v)$ between u, v in G is the length of a shortest u, v-path. The interval $I(u, v)$ between two vertices u and v in G consists of all vertices on shortest u, v-paths, that is:

$$I(u, v) = \{x \mid d(u, x) + d(x, v) = d(u, v)\}.$$

A profile π of length $k = |\pi|$ on G is a non-empty sequence $\pi = (x_1, x_2, \ldots, x_k)$ of vertices of V with repetitions allowed. We define V^* to be the set of all profiles of finite length on V. We call $x_1, x_2, \ldots, x_k$ the *elements* of the profile. A *vertex* of π is a vertex that occurs as an element in π. By $\{\pi\}$ we denote the set of all vertices of π. Note that a vertex may occur more than once as element in π. If we say that x is an *element* of π, then we mean an element in a certain position, say $x = x_j$ in the j-th position. A *subprofile* of π is just a non-empty subsequence of π. The concatenation of profiles π and ρ is denoted by $\pi\rho$. The profile consisting of the concatenation of m copies of π is denoted by π^m. Let π be a profile on G. A vertex in with highest occurrence in π is called a *plurality vertex* of π. We denote the set of plurality vertices of π by $Pl(\pi)$.

A *consensus function* on G is a function $F : V^* \to 2^V - \emptyset$ that gives a non-empty subset of V as output for each profile on G. For convenience, we write $F(x_1, \ldots, x_k)$ instead of $F((x_1, \ldots, x_k))$, for any function F defined on profiles, but will keep the brackets where needed.

The *remoteness* of a vertex v to profile π is defined as

$$r(v, \pi) = \sum_{i=1}^{k} d(x_i, v).$$

A vertex minimizing $r(v, \pi)$ is called a *median* of the profile. The set of all medians of π is the *median set* of π and is denoted by $M(\pi)$. A vertex maximizing $r(v, \pi)$ is called an *antimedian* of the profile. The set of all antimedians of π is the *antimedian set* of π and is denoted by $AM(\pi)$. We can also think of M and AM as functions from V^* to $2^V - \emptyset$, and then call them the *median function* and *antimedian function*. Note that we have $M(x) = \{x\}$, and $M(x, y) = I(x, y)$. Moreover, if $I(u, v) \cap I(v, w) \cap I(w, u) \neq \emptyset$, then

$$M(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u).$$

The median function has been studied extensively, especially on median graphs. A median graph is defined by the property that $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$, for any three vertices u, v, w. Equivalently, a median graph is a graph such that any profile of length 3 has a unique median. See e.g. [21], [12], [22] for a rich structure theory on median graphs. There are nice axiomatic characterizations for the median function on median graphs, see e.g. [17], [18], [24]. Three simple and natural axioms suffice for the characterization of the median function in this case. We present these here. The first and third axiom are defined without any reference to metric.

(A) Anonymity: $F(\pi) = F(x_{\chi(1)}, x_{\chi(2)}, \ldots, x_{\chi(k)})$, for any profile $\pi = (x_1, x_2, \ldots, x_k)$ on V and for any permutation χ of $\{1, 2, \ldots, k\}$.

(B) Betweenness: $F(u, v) = I(u, v)$, for all u, v in V.

(C) Consistency: If $F(\pi) \cap F(\rho) \neq \emptyset$, for profiles π and ρ, then $F(\pi\rho) = F(\pi) \cap F(\rho)$.

Clearly, the median function satisfies axioms (A) and (B) on any graph. It is probably part of folklore that the median function also satisfies (C). Anyway, a proof of this can be found in [17].

A *matching* is a set $\mathtt{M}$ of edges in a graph G such that no two edges in $\mathtt{M}$ have a common end vertex. Two vertices joined by an edge in $\mathtt{M}$ are called *mates* in G. If v has a mate in G, then we denote this mate by $\tilde{v}$. Vertices that do not have a mate, i.e. are not incident with an edge in $\mathtt{M}$, are called *mateless*. A matching is perfect if every vertex has a mate (is incident with an edge in the matching), so that there are no mateless vertices.

Our focus in this paper is on graphs obtained from the complete graph K_n by deleting a matching $\mathtt{M}$. We denote this graph by $K_n - \mathtt{M}$. If $\mathtt{M}$ is an empty set, then we get the complete graph K_n. Note that K_n consists of mateless vertices only. If n is even and $\mathtt{M}$ is a perfect matching we get *cocktail party graph*. In this case, we have $n \geq 4$ since the graphs need to be connected.

Let v be a vertex with a mate $\tilde{v}$. We call the profile $(v, \tilde{v})$ a *mating pair*. We call a profile a *mating profile* if it is (a permutation of) the concatenation of a set of mating pairs. A *mateless profile* is a profile that consists solely of mateless vertices. A *mate-free profile* is a profile that contains at least one vertex of some pair of mates, but does not contain both vertices of any mating pair.

Let π be a profile that is not mateless. The profile $\tilde{\pi}$ is obtained from π by removing all mateless vertices from π and taking mates of all other vertices in π. Since π is not mateless, we get a nonempty profile. Note that π is a mating profile if and only if $\tilde{\pi}$ is a permutation of π.

The following lemma is obvious, but quite helpful in the sequel.

Lemma 1 *Let $\pi = (v, \tilde{v})$ be a mating pair in $K_n - \mathtt{M}$ with vertex set V. Then $r(u, \pi) = 2$, for all u in V.*

An immediate consequence of this lemma is that we can compute the median and antimedian function quite simply for mating profiles.

Corollary 2 *Let π be a mating profile on $K_n - \mathtt{M}$ with vertex set V. Then $M(\pi) = V = AM(\pi)$.*

Now let π be a profile on $K_n - \mathtt{M}$ that is not a mating profile. Assume that π contains two elements that form a pair of mates, say $v, \tilde{v}$. Let π' be the profile obtained from π by removing the two elements v and $\tilde{v}$. Note that π' is nonempty. Consider the remoteness of any vertex u with respect to π and π'. Then, by Lemma 1, u minimizes (maximizes) $r(u, \pi)$ if and only if it minimizes (maximizes) $r(u, \pi')$. So we have $M(\pi) = M(\pi')$

and $AM(\pi) = AM(\pi')$. Hence, in computing the median set or antimedian set of π, we can delete any pair of mates. Thus a nonempty subprofile ρ remains that does not contain both vertices of a mating pair. Moreover $M(\pi) = M(\rho)$ and $AM(\pi) = AM(\rho)$. Therefore we only need to compute $M(\rho)$ and $AM(\rho)$ for mate-free and mateless profiles. See Lemma 4 for M, and Lemmata 9 and 10 for AM.

3 Axiomatic Characterization of the Median Function on K_n minus a Matching

On the complete graph K_n, the median function is just the plurality function: $M(\pi) = Pl(\pi)$. As early as 1991, Roberts [27] gave various nice characterizations of this plurality function. Recently McMorris et al. [14] presented some new characterizations from the view point of what they call the ABC-problem on graphs: determine and characterize all consensus functions that satisfy the three axioms (A), (B) and (C).

In this section we characterize the median function on $K_n - \mathtt{M}$ with vertex set V, for nonempty matchings $\mathtt{M}$. By the closing observations of Section 2, we only need to compute $M(\rho)$ for mate-free and mateless profiles, which is taken care of in the next two lemmata. They are the motivation for the two additional axioms that we need for our characterization of the median function, besides (A) and (C). Lemma 3 is a trivial consequence of Lemma 1.

Lemma 3 *Let F be the median function defined on the vertex set V of $K_n - \mathtt{M}$. Then $F(v, \tilde{v}) = V$, for any mating pair $(v, \tilde{v})$.*

The next lemma is also simple.

Lemma 4 *Let F be the median function defined on the vertex set V of $K_n - \mathtt{M}$. Then $F(\pi) = Pl(\pi)$, for all mate-free and mateless profiles π.*

Proof. Let $\pi = (x_1, x_2, \ldots, x_k)$ be a mate-free profile or a mateless profile. Let $\{\pi\} = \{y_1, y_2, \ldots, y_\ell\}$, and let f_j be the number of occurrences of y_j in π. Then, for any vertex w outside the profile π, we have $d(w, y_j) \geq 1$, for each vertex y_j in π. Write $f = \sum_{j=1}^{\ell} f_j$. So we have $r(w, \pi) \geq f$.

Let u be any vertex in π. Then we have $d(u, x_i) = 1$, for any $x_i \neq u$. Clearly $r(u, \pi) = f - f_j$, for $u = y_j$. So the vertices that minimize remoteness are all in π. Note that $r(u, \pi) = f - f_j$ is minimum when f_j is maximum. So the vertices that minimize remoteness are precisely those that occur most often in π. □ □ □

By Lemma 3 and Lemma 4, the median function on $K_n - \mathtt{M}$ satisfies the following two axioms. Note that Restricted Plurality is a special case of the Plurality axiom in [13].

Mating Pairs (*Mates*)**:** $F(v, \tilde{v}) = V$, for all mating pairs $(v, \tilde{v})$ in V.

Restricted Plurality (*RP*)**:** $F(\pi) = Pl(\pi)$, for all mate-free and mateless profiles π.

Let F be any consensus function satisfying the axioms (*Mates*) and (*RP*). Note that for any mating pair $(v, \tilde{v})$ in $K_n - \mathtt{M}$, we have $I(v, \tilde{v}) = V$. Now consider any other profile $\pi = (u, v)$ such that $u \neq v, \tilde{v}$. Then, clearly, u and v are adjacent. Hence axiom (*RP*) implies $F(\pi) = \{u, v\} = I(u, v)$. Finally, we have $F(u, u) = Pl(u, u) = \{u\} = I(u, u)$. So F on $K_n - \mathtt{M}$ satisfies (B) also.

Theorem 5 *Let F be a consensus function on $K_n - \mathtt{M}$ with vertex set V. Then F is the median function if and only if F satisfies axioms (A), (C), (Mates) and (RP).*

Proof. The median function satisfies (A) by definition. By Lemma 1, it satisfies (*Mates*), and by Lemma 3, it also satisfies (*RP*). Above we observed already that it is probably part of folklore that it satisfies (C), but a simple and straightforward proof can be found in [17].

To prove the converse, let F be a function that satisfies the four axioms. Take any profile π. If it contains a pair of mates $v, \tilde{v}$, then we can permute π such that v and $\tilde{v}$ are moved to the front two positions, thus getting the profile $(v, \tilde{v})\rho$, where ρ is the subprofile of π obtained by deleting the elements v and $\tilde{v}$ from their respective positions. By (*Mates*), we have $F(v, \tilde{v}) = V$. So $F(v, \tilde{v}) \cap F(\rho) \neq \emptyset$. Hence, by ($C$), we have $F((v, \tilde{v})\rho) = F(v, \tilde{v}) \cap F(\rho) = F(\rho)$. Finally, by ($A$), we have $F(\pi) = F((v, \tilde{v})\rho) = F(\rho)$. We can repeat this process until we end up with a subprofile σ of π that is either a mating pair or mate-free or mateless. In the latter two cases, we have $F(\pi) = F(\sigma)$. From axiom (*RP*), it follows that $F(\sigma) = Pl(\sigma) = M(\sigma) = M(\pi)$. If σ is a mating pair, then we have $F(\sigma) = V = F(\pi) = M(\pi)$. $\square\ \square\ \square$

For any axiomatic characterization, we want to know whether the axioms involved are independent. We present some examples. In all cases F is a consensus function on $K_n - \mathtt{M}$ with vertex set V having at least 4 vertices. Recall that $\mathtt{M}$ is nonempty.

Example 6 [(*Mates*) excluded]
Define the function F on $K_n - \mathtt{M}$ by $F(\pi) = Pl(\pi)$, for all profiles π. It is straightforward to check that F satisfies (A), (C) and (*RP*). Take a mating pair $(v, \tilde{v})$. Since $F(v, \tilde{v}) = \{v, \tilde{v}\} \neq V$, the function F does not satisfy (*Mates*).

Example 7 [(*RP*) excluded]
Define the function F on $K_n - \mathtt{M}$ by $F(\pi) = V$, for all profiles π. Obviously, F satisfies axioms (A), (C) and (*Mates*). Take any two adjacent vertices u and v. Then $F(u, v) = V \neq \{u, v\} = Pl(u, v)$. So F does not satisfy (*RP*).

Example 8 [(C) excluded]
Define the function F on $K_n - \mathtt{M}$ by
($c1$): $F(v, \tilde{v}) = V$, for all mating pairs $(v, \tilde{v})$ in V,
($c2$): $F(\pi) = Pl(\pi)$, for all profiles π that are not a mating pair.
Clearly, F satisfies (A), (*Mates*) and (*RP*). Take a mating pair $(v, \tilde{v})$ and a vertex u distinct from $v, \tilde{v}$, and let $\pi = (u, v, \tilde{v})$. Then, by ($c2$) and ($c1$), we have

$$F(\pi) = Pl(\pi) = \{u, v, \tilde{v}\} \neq \{u\} = \{u\} \cap V = F(u) \cap F(v, \tilde{v}).$$

So F does not satisfy (C).

The case of (A) seems to be different. This is not a trivial axiom: see [13], for a case where the presence or absence of anonymity has deep non-trivial repercussions. We do not yet have an example that shows independence of (A). On the other hand one would not expect that it follows from the other axioms. So we leave it as an open problem here.

4 Axiomatic Characterization of the Antimedian Function on K_n minus a Matching

First we present two Lemmata, by which we can compute $AM(\pi)$ on $K_n - \mathtt{M}$ with vertex set V. Let π be a profile. We denote by $Low(\pi)$ the set of vertices with lowest occurrence in π including zero occurence. By definition, if $\{\pi\}$ is a proper subset of V, then $Low(\pi) = V - \{\pi\}$. Note that the following lemma holds for all profiles when $\mathtt{M}$ is the empty matching, that is, when the graph is a complete graph.

Lemma 9 *Let F be the antimedian function on $K_n - \mathtt{M}$ with vertex set V, and let π be a mateless profile. Then $F(\pi) = Low(\pi)$.*

Proof. Let $\pi = (x_1, x_2, \ldots, x_k)$ be a mateless profile. Let $\{\pi\} = \{y_1, y_2, \ldots, y_\ell\}$, and let f_j be the number of occurrences of y_j in π. For each vertex y_j in π we have $r(y_j, \pi) = k - f_j < k$. If there is a vertex w outside π, then $r(w, \pi) = k$. So, if $\{\pi\} \neq V$, then we have $F(\pi) = V - \{\pi\} = Low(\pi)$. And if $\{\pi\} = V$, then also we have $F(\pi) = Low(\pi)$. □ □ □

The next lemma applies only to the case where $\mathtt{M}$ is a nonempty matching.

Lemma 10 *Let F be the antimedian function on $K_n - \mathtt{M}$ with $\mathtt{M} \neq \emptyset$, and let π be a mate-free profile. Then $F(\pi) = Pl(\tilde{\pi})$.*

Proof. Let $\pi = (x_1, x_2, \ldots, x_k)$ be a mate-free profile. Let $\{\pi\} = \{y_1, y_2, \ldots, y_\ell\}$, and let f_j be the number of occurrences of y_j in π. Let x be any vertex in π, say $x = y_i$. Then $r(x, \pi) = k - f_i$. Let w be any vertex outside π that has no mate in π. Then $r(w, \pi) = k$. Let z be a vertex that has a mate in π, say y_j. Then $r(z, \pi) = k - f_j + 2f_j = k + f_j$. The mates of the vertices in $Pl(\tilde{\pi})$ are precisely those for which this value $k + f_j$ is maximized. So $F(\pi) = Pl(\tilde{\pi})$. □ □ □

Using the observations at the end of Section 2 about profiles containing a mating pair, we can now easily compute $AM(\pi)$ for any profile on $K_n - \mathtt{M}$, regardless whether $\mathtt{M}$ is empty or not.

We need the following three axioms, in which F is a consensus function on $K_n - \mathtt{M}$ with vertex set $V = \{v_1, v_2, \ \ldots \ , v_n\}$. Here V has a specific ordering, and the specific profile $\pi = (v_1, v_2, \ \ldots \ , v_n)$ is called the *full profile.* The Fullness axiom was already introduced in [13].

Mate-Freeness (MF)**:** $F(\pi) = Pl(\tilde{\pi})$, for all mate-free profiles π.

Complementation $(Compl)$**:** $F(x) = V - \{x\}$, for each mateless vertex $x \in V$.

Fullness $(Full)$**:** $F(v_1, v_2, \ldots, v_n) = V$, for the full profile.

The complete graph is a special case. We deal with this case first. Here we need axiom $(Full)$, since the value of $F(v_1, v_2, \ldots, v_n)$ cannot be determined using the other axioms. Since all profiles are mateless, axioms $(Mates)$ and (MF) do not apply.

Theorem 11 *Let F be a consensus function on K_n with $n > 1$. Then F is the antimedian function if and only if F satisfies (A), (C), $(Compl)$ and $(Full)$.*

Proof. Clearly the antimedian function satisfies the four axioms.

Conversely, let F satisfy the four axioms. Take a profile $\pi = (x_1, x_2, \ldots, x_k)$. If $\{\pi\}$ is a proper subset of V, then we can write π as the concatenation of the singleton profiles $(x_1), (x_2), \ldots, (x_k)$. By $(Compl)$, the intersection of the sets $F(x_1), F(x_2), \ldots, F(x_k)$ equals $V - \{\pi\}$, and by (C) we are done. If all vertices of V occur exactly m times in π with $m > 0$, then, due to (A), we can write $\pi = (v_1, v_2, \ldots, v_n)^m$, and we are done by (C) and $(Full)$.

Now let π be any other profile. Then there is a number $m > 0$ such that some but not all vertices occur exactly m times in π whereas the other vertices occur more than m times in π. Due to (A), we can write $\pi = \pi'(v_1, v_2, \ldots, v_n)^m$, where π' is the profile obtained by removing those vertices of π that occur exactly m times. Hence $\{\pi'\}$ is a proper subset of V, and by the above argument we have $F(\pi') = Low(\pi') = Low(\pi)$. By $(Full)$ and(C), we have $F(\pi) = F(\pi') \cap V = Low(\pi) = AM(\pi)$. □ □ □

Again in this case we do not yet have an example that shows whether (A) is independent from the other axioms. The examples below show the independence of the other three axioms. In the examples F is a consensus function on $K_n - \mathtt{M}$ with $\mathtt{M} \neq \emptyset$ and $n \geq 3$.

Example 12 [$(Compl)$ excluded]
Let F be defined by $F(\pi) = V$ for all profiles.
Then it fails $(Compl)$ but trivially satisfies the other axioms.

Example 13 [(C) excluded]
Let F be defined by
$(k1)$ $F(x) = V - \{x\}$, for any $x \in V$,
$(k2)$ $F(\pi) = V$, for any profile π of length at least 2.
Then F fails (C) but trivially satisfies the other axioms.

Example 14 [$(Full)$ excluded]
Fix a vertex p in V. Let F be defined by
$(k3)$ $F(\pi) = \{p\}$, for any π with $\{\pi\} = V$,
$(k4)$ $F(\pi) = V - \{\pi\}$, for any π with $\{\pi\} \neq V$.

Clearly F satisfies (A) and $(Compl)$. By $(k3)$ function F fails $(Full)$. It remains to check consistency. So let π and ρ be two profiles. If $\{\pi\} = V = \{\rho\}$, then

$$F(\pi) = F(\rho) = F(\pi\rho) = \{p\},$$

and we are done. If $\{\pi\} = V$ and $\{\rho\} \neq V$, then $F(\pi) \cap F(\rho) \neq \emptyset$ only if ρ does not contain p. In this case it again follows that

$$F(\pi) = F(\pi) \cap F(\rho) = \{p\} = F(\pi\rho).$$

Finally, let $\{\pi\}$ and $\{\rho\}$ both be proper subsets of V. Then $F(\pi) = V - \{\pi\}$ and $F(\rho) = V - \{\rho\}$. These two sets have a non-empty intersection if and only if $\{\pi\} \cup \{\rho\} = \{\pi\rho\}$ is a proper subset of V. Again we have $F(\pi\rho) = F(\pi) \cap F(\rho)$.

In the case that $\mathtt{M}$ is a nonempty matching, axiom $(Full)$ does not apply. Now we need both $(Mates)$ and (MF).

Theorem 15 *Let F be a consensus function on $K_n - \mathtt{M}$ with vertex set V and $\mathtt{M}$ a nonempty matching. Then F is the antimedian function if and only if F satisfies axioms (A), (C), $(Mates)$, (MF) and $(Compl)$.*

Proof. Let F be the antimedian function. Then F satisfies all the above five axioms.

Let F be a function that satisfies the five axioms. Take any profile π. If it contains a pair of mates $v, \tilde{v}$, then we can permute π such that v and $\tilde{v}$ are moved to the front two positions, thus getting the profile $(v, \tilde{v})\rho$, where ρ is the subprofile of π obtained by deleting the elements v and $\tilde{v}$ from their respective positions. By $(Mates)$, we have $F(v, \tilde{v}) = V$. So $F(v, \tilde{v}) \cap F(\rho) \neq \emptyset$. Hence, by (C), we have $F((v, \tilde{v})\rho) = F(v, \tilde{v}) \cap F(\rho) = F(\rho)$. Finally, by (A), we have $F(\pi) = F(v, \tilde{v}) \cap F(\rho) = F(\rho)$. We can repeat this process until we end up with a subprofile σ of π that is either a mating pair or mate-free or mateless. In the latter two cases, we have $F(\pi) = F(\sigma)$. From axiom (MF) it follows that $F(\sigma) = Pl(\tilde{\sigma}) = AM(\sigma) = AM(\pi)$, when σ is a mate-free profile. From axiom $(Compl)$ and (C) we have $F(\sigma) = W_\sigma = V - \{\sigma\} = AM(\sigma) = AM(\pi)$, when σ is a mateless profile. If σ is a mating pair, then we have $F(\sigma) = V = AM(\sigma) = AM(\pi)$. This completes the proof. □ □ □

Note that a cocktail party graph does not have mateless vertices. So in this case $(Compl)$ is an empty axiom and can be deleted from the set of characterizing axioms. We formulate this case as a separate theorem. It is an immediate corollary of the previous theorem.

Theorem 16 *Let F be a consensus function on a cocktail-party graph with vertex set V. Then F is the antimedian function if and only if F satisfies axioms (A), (C), $(Mates)$ and (MF).*

Again we study the independence of the axioms. Let V be the vertex set of $K_n - \mathtt{M}$ with $n \geq 4$ and $\mathtt{M}$ a nonempty matching.

Example 17 [(*Mates*) excluded]
Define the function F on $K_n - \mathtt{M}$ by
$(a1)$: $F(\pi) = Low(\pi)$, for all mateless profiles π,
$(a2)$: $F(\pi) = Pl(\tilde{\pi})$, for all other profiles π.
It is straightforward to check that F satisfies (A), (C), (MF) and $(Compl)$. Since $F(v, \tilde{v}) = \{\tilde{v}, v\} \neq V$, for any mating pair $(v, \tilde{v})$, the function F does not satisfy $(Mates)$.

Example 18 [(MF) excluded]
Define the function F on $K_n - \mathtt{M}$ by
$(b1)$: $F(v, \tilde{v}) = V$, for any mating pair $(v, \tilde{v})$,
$(b2)$: $F(\pi) = Low(\rho)$, where ρ is the profile obtained by removing mating pairs, for all other profiles π.
Then F satisfies (A), (C), $(Mates)$ and $(Compl)$, but not (MF).

Example 19 [$(Compl)$ excluded]
Define the function F on $K_n - \mathtt{M}$ by
$(c1)$: $F(x) = V$, for all mateless vertex x,
$(c2)$: $F(\pi) = V$, for mating profiles π,
$(c3)$: $F(\pi) = Pl(\tilde{\pi})$ for all mate-free profiles π,
$(c4)$: for all other profiles π let $F(\pi) = F(\pi_1) \cap F(\pi_2) \cap F(\pi_3)$, where π_1 is the subprofile of π with all mateless vertices in π, and π_2 the subrpofile with all mate-pairs in π, and π_3 is the mate-free subprofile with remaining vertices in π .
Then it is obvious that F satisfies (A), (C), $(Mates)$ and (MF), but not $(Compl)$.

Example 20 [(C) excluded]
Define the function F on $K_n - \mathtt{M}$ by
$(d1)$: $F(\pi) = Low(\pi)$, for all mateless profiles π,
$(d2)$: $F(\pi) = Pl(\tilde{\pi})$ for all mate-free profiles π,
$(d3)$: $F(\pi) = V$, for all other profiles π.
Clearly, F satisfies (A), $(Mates)$ and (MF) and $(Compl)$. Now, consider the following profiles:

$\pi_1 = (u, v)$ where u is a vertex of a mating pair $(u, \tilde{u})$ and $v \neq u, \tilde{u}$,

$\pi_2 = (u, \tilde{u})$.

By $(d3)$, we have $F(\pi_2) = V = F(\pi_1\pi_2)$. By $(d2)$, we have $F(\pi_1) = \{\tilde{u}, \tilde{v}\}$ or $\{\tilde{u}\}$ depending on whether v has a mate or not. Now $F(\pi_1) \cap F(\pi_2) = F(\pi_1) \neq V = F(\pi_1\pi_2)$. Therefore (C) fails.

Again we do not have an example to show the independency of (A). We leave this as an open problem. But in this case we want to elaborate a little more on our trials to find an example.

Open Problem. Is (A) independent?
Note that for independence of (A) the ordering of the elements in a profile is essential. We want to split π into subprofiles to get a grip on $F(\pi)$. The only subprofiles of π

that we consider are those containing consecutive elements of π. So, if we say that π' is a subprofile of π, then it is assumed that π' consists of consecutive elements of π. As above, a profile is of length at least 1, but for our purposes here, a subprofile now may be empty. We use the convention that ρ^m is the empty subprofile if $m = 0$. We set $\tau = (v_1, v_2, \ldots, v_n)$ to be the profile containing each vertex once and in a *preferred ordering* of V. Any profile π can be written as

$$\pi = \tau^{m_0}\pi_1\tau^{m_1}\pi_2 \ \ldots \ \pi_{r-1}\tau^{m_{r-1}}\pi_r\tau^{m_r} \tag{1}$$

such that

(i) $\pi_1, \pi_2, \ldots, \pi_r$ are non-empty subprofiles of π that do *not* contain τ,

(ii) $m_1, m_2, \ldots, m_{r-1} > 0$ and $m_0, m_r \geq 0$.

We call this the *standard form* of π. If π does not contain τ, then we take $r = 1$ and $m_0 = m_1 = 0$. If $\pi = \tau^m$ for some $m > 0$, then we take $r = 0$ and $m_0 = m$. Note that, if $m_j > 0$ for some j, then $\{\pi\} = V$. Hence, if $\{\pi\} \neq V$, then π is the standard form of π.

We distinguish two types of profiles.

Type α: $\pi = \tau^{m_0}\pi_1\tau^{m_1}\pi_2 \ \ldots \ \pi_{r-1}\tau^{m_{r-1}}\pi_r\tau^{m_r}$ with $\cup_{1\leq j\leq n} \ \{\pi_j\} \neq V$.

Type β: $\pi = \tau^{m_0}\pi_1\tau^{m_1}\pi_2 \ \ldots \ \pi_{r-1}\tau^{m_{r-1}}\pi_r\tau^{m_r}$ with $\cup_{1\leq j\leq n} \ \{\pi_j\} = V$.

If π is of Type α, then either $\pi = \tau^m$ for some $m > 0$, in which case

$$AM(\pi) = V, \tag{2}$$

or $\pi = \tau^{m_0}\pi_1\tau^{m_1}\pi_2 \ \ldots \ \pi_{r-1}\tau^{m_{r-1}}\pi_r\tau^{m_r}$ for some $r \geq 1$, in which case

$$AM(\pi) = V - [\cup_{j=1}^{n} \ \{\pi_j\}] = \cap_{j=1}^{n} \ [V - \{\pi_j\}]. \tag{3}$$

Let F be any consensus function satisfying (C), $(Full)$ and $(Compl)$. Then it follows that we have $F(\pi) = AM(\pi)$, for any profile π of type A. So the only way to differ from AM is on profiles of Type β.

Let π and ρ be two profiles of Type α. If the concatenated profile $\pi\rho$ is also of Type α, then it is straightforward to check that (C) holds. If $\pi\rho$ is of Type β, then it is straightforward to check that $F(\pi) \cap F(\rho) = \emptyset$, so this does not affect (C). Problems may arise when π or ρ are of Type β. Let π and ρ be two profiles with $F(\pi) \cap F(\rho) \neq \emptyset$. We write both in standard form:

$$\pi = \tau^{m_0}\pi_1\tau^{m_1}\pi_2 \ \ldots \ \pi_{r-1}\tau^{m_{r-1}}\pi_r\tau^{m_r}, \tag{4}$$
$$\rho = \tau^{n_0}\rho_1\tau^{n_1}\rho_2 \ \ldots \ \rho_{s-1}\tau^{n_{s-1}}\rho_s\tau^{n_s}. \tag{5}$$

Now consider the case that $m_r = 0$ and $\pi_r = (v_1, v_2, \ldots, v_t)$ and $n_0 = 0$ and $\rho_1 = (v_{t+1}, \ldots, v_n)$, for some t with $1 \leq t < n$. If we concatenate π and ρ and consider the standard form of $\pi\rho$, then $\pi_r\rho_1 = \tau$ and π_r and ρ_1 'disappear'. So they do not count when we want to determine the type of $\pi\rho$. So $\pi\rho$ might be of Type α, whereas at least one of π and ρ is of Type β. This makes it difficult to assign values to profiles of type β. So far we could not overcome this road block. On the other hand, one does not expect that one can deduce anonymity from the other axioms.

5 Concluding Remarks

The median and antimedian functions both satisfy *Anonymity* and *Consistency* on any metric space. In this paper we characterized both functions on the complete graph minus a matching. In the case of the median function we required the matching to be nonempty. In the case of the antimedian function we also presented a first characterization for the case of complete graphs. These results can be viewed as a generalization of the results in [9]. In all cases we presented examples to show that the axioms are independent, except for the case of anonymity. It seems that a similar situation arises here as in the situation of the characterization of the median function on median graphs, see [13]. There it was shown that the presence or absence of anonymity has non-trivial repercussions. So we leave the independence of anonymity in our characterizations as an open problem.

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