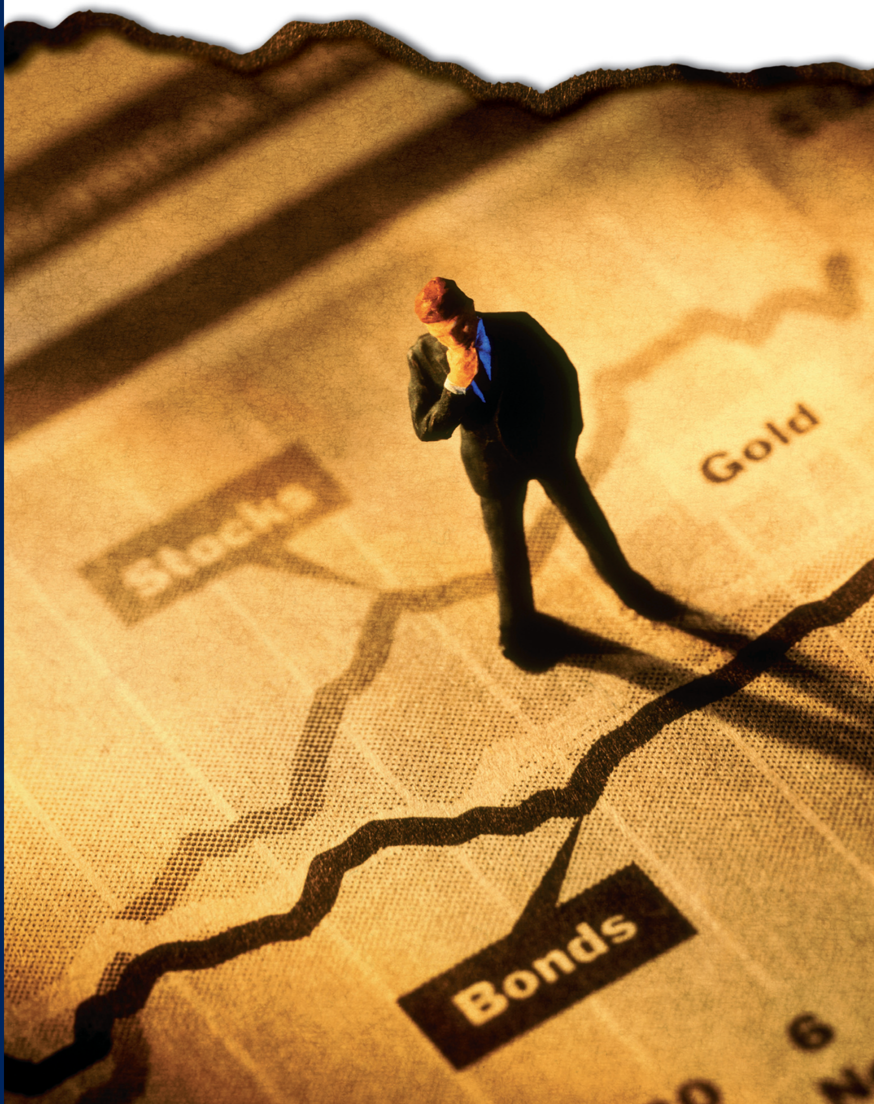
ANNA BARBARA GUTKOWSKA

Essays on the Dynamic Portfolio Choice



ESSAYS ON THE DYNAMIC PORTFOLIO CHOICE

Anna Gutkowska

Essays on the Dynamic Portfolio Choice

Essays over dynamische portfoliokeuze

Thesis

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Preface

Doing a Ph.D. is an adventure with science and with people in science involved. It is a good chance to learn, to make new friends and re-discover old ones. It is a period when the support of family is appreciated very much. Scientists, friends and family contributed to the writing of this thesis and this is the right moment to acknowledge their contribution.

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Anna Gutkowska
Rotterdam, 12 February 2006

1 Introduction

1.1 What do we know in the area of dynamic portfolio choice

This thesis concerns the subject of dynamic optimal portfolio choice both unconstrained and constrained one. By the unconstrained dynamic optimal portfolio choice we mean the trading of available financial assets in a given time interval according to the rules, which are feasible under the budget restriction and optimal according to certain criterion. This criterion in the most part of this thesis is the maximization of expected utility from terminal wealth. By the constrained dynamic optimal portfolio choice we mean the choice additionally restricted by the requirement that the terminal wealth must exceed deterministic or stochastic floor.

Merton is the unquestioned pioneer in the analysis of unconstrained dynamic optimal portfolio choice problem. Since his paper from 1971 the optimal consumption-investment problem is solved in the continuous-time framework. Merton [1971] shows that in general the optimal proportions of the risky assets in the optimal portfolio include two components. He is more explicit about it in his paper from 1973. Therein he mentions that the first term in the demand for risky assets is the demand characteristic for a single-period mean-variance maximizer. The second component is to reflect the demand arising in order to hedge against unfavourable shifts in the investment opportunity set. Hedging against such shifts is seen by Merton [1973] as demanding more of an asset whose returns are more correlated with changes in the opportunity set variable, which caused the shift.

Cox and Huang [1989] modified Merton's problem by adding non-negativity constraints on terminal wealth. Their work gave rise to the stream of literature on the constrained dynamic optimal portfolio choice known as the option-based portfolio insurance (OBPI). Inserting non-negativity restrictions significantly changes the optimal portfolio rules. An optimal constrained policy is meant as the policy, which allocates the initial wealth between the unconstrained policy as in Merton [1971, 1973] and the insurance package on the unconstrained policy and exhausts all the initial wealth. The insurance package can be thought of as consisting of a continuum of put options with zero exercise price.

In the literature there are many ways applied to avoid the difficulties of constrained portfolio choice. Probably the most common of them is to assume that the investor has power utility over the non-negative terminal wealth. For power utility function the non-negativity restriction is not binding. Hence, the problem is equivalent to the unconstrained portfolio choice. Avoidance of the difficulties in question allows the researcher to focus on another aspect of portfolio choice, namely considering increased number and more complex models for the dynamics of state variables.

When the dynamics of state variables contain stochastic terms optimal portfolio strategies consist of both speculative and hedging components as distinguished by Merton [1971, 1973]. The more state variables in the model the more hedging elements appear in the optimal portfolio. These state variables usually include stochastic interest rates, stochastic market prices of risk and stochastic price level with the latter being present when inflation is included in the model. Among many papers, which can be given here as examples Wachter [2002] maximizes expected utility from terminal wealth and from lifetime consumption when the market price of risk follows an Ornstein-Uhlenbeck process. Brennan and Xia [2002] maximize real terminal wealth and real lifetime consumption when the price level follows a diffusion process and the expected rate of inflation follows an Ornstein-Uhlenbeck process. Munk et al. [2004] consider dynamic asset allocation under mean-reverting returns, stochastic interest rates and inflation uncertainty.

When the problem is formulated in nominal terms the only observable state variable is the short-term interest rate. There are a number of papers, which deal exclusively with the subject of interest rate risk hedging in the optimal portfolio choice. As examples Sorensen [1999], Brennan and Xia [2000], Deelstra et al. [2000] and Bajeux-Besnainou et al. [2003] can be mentioned. Although these papers differ with respect to the assumed models of interest rate dynamics, their conclusions are in many respects similar. In particular they show that the interest rate risk hedging portfolio consists of the bond either zero-coupon or coupon-bearing with expiration at the investment horizon and that stock allocations are determined solely by the myopic portfolio even though stock returns can be correlated with the interest rate.

However, non-negativity constraint is not all. It is sometimes required that the terminal wealth mustn't exceed certain non-zero floor. Then, one of the ways to avoid the constrained optimal portfolio choice is to modify the arguments of power utility function, namely to derive the utility from the surplus of wealth over the floor and at the same time to add the constraint on terminal wealth to exceed the floor. In such a way the option-like component does not appear in the optimal solution. Such a route is followed in the papers of Boulier et al. [2001] and Deelstra et al. [2002, 2003] who define the utility function of the pension fund over the difference between the terminal wealth and the value of certain guarantee.

In general however, OBPI leads to the complicated patterns of portfolio policy. That is why in order to solve the problem analytically the investment opportunity set is usually assumed constant when the problem with standard utility is restricted by the non-standard constraint or vice versa. As an example Tepla [2001] considers the problem of an investor whose wealth is restricted to outperform the stochastic benchmark. She shows that this investor's optimal wealth is equal to the value of the benchmark portfolio plus a term similar to the expression for an option to exchange one risky asset for another. Nguyen and Portait [2002] analyze the problem of maximizing the quadratic utility function from terminal wealth under the wealth positiveness constraint. They show that the optimal strategies are synthetic put options on the portfolio yielding the required return with minimal second moment.

It is very tempting to reduce the asset-liability management in the financial institutions like the pension funds for instance to constrained portfolio choice problem restricted by the certain guarantee. For example, Jensen and Sorensen [2001] solve the problem of a defined contribution scheme, which is restricted by the requirement that the final wealth must be increased from today's value at least with a minimum guaranteed rate in the model with Gaussian interest rate dynamics. Also the papers of Boulier et al. [2001] and Deelstra et al. [2002] model the defined-contribution scheme. The greater popularity of this kind of scheme over its defined-benefit counterpart can be caused by the character of contributions, which in the former can be assumed a priori as a deterministic or stochastic process. In the defined-benefit scheme they are an additional decision variable.

Still a way to avoid the complexities of OBPI, but retain the advantages of portfolio insurance strategies is to apply simpler policy in this class. An example of such a simpler portfolio rule is the strategy advised by Black and Perold in 1992 and known as the constant proportion portfolio insurance (CPPI). This is a strategy, which maintains portfolio's risk exposure at a constant multiple of the excess of wealth over a floor. The paper by Black and Perold [1992] contains many interesting theoretical results. It presents the formulas for the terminal cushion of CPPI strategy in both discrete and continuous time with and without borrowing restrictions, with and without the transaction costs. Black and Perold [1992] study the behaviour of CPPI when the multiple becomes large, make the comparison between the payoffs produced by CPPI and perpetual American call options as well as derive the CPPI investment policy as a solution to the particular utility maximization problem. Recently, CPPI was compared against OBPI by Cesari and Cremonini [2003] and Bertrand and Prigent [2003].

1.2 What do we not know, although we should?

In general it is not easy to find the optimal portfolio. For instance, in the Merton's problem the methodology of dynamic programming leads to a non-linear second-order partial differential equation, which needs to be solved. At the rescue comes the approach of Ocone and Karatzas [1991] who use the generalized Clark representation formula based on the concept of Malliavin derivative to compute the optimal portfolios. However, the approach of Ocone and Karatzas [1991] requires the calculation of conditional expectations of random variables involving (also stochastic) integrals of Malliavin derivatives of state variables among others. Hence, for the more complex models of the state variables an explicit identification of this portfolio still proves to be cumbersome. For such models Detemple et al. [2003] propose to simulate these derivatives, which are shown to follow the diffusion processes and consequently to use Monte Carlo simulation to approximate the fractions of risky assets in the optimal portfolio.

As a result for many models of the state variables used in the literature the analytical formulas for the optimal portfolios remain unknown.

As we already mentioned the power utility function is assumed in a great deal of articles for its convenient properties. An exception is the article of Bajeux-Besnainou et al. [2003] where apart from the power utility also the hyperbolic absolute risk aversion (HARA) function is considered in the model with Vasicek [1977] interest rate dynamics. However, there is a lack of papers in which HARA utility is accompanied by other than this type of interest rate dynamics.

The continuous-time formulation of the optimal consumption-investment problem is convenient to work with. As a result much attention has been given to the deriving optimal hedging portfolio policies under different assumptions about the dynamics of state variables. Relatively little is known about the mechanism of hedging. This mechanism can be best investigated in the discrete time as the real trading takes place in the discrete points in time. However, such an analysis is lacking in the literature.

There are a variety of unconstrained portfolio choice problems with many states variables and a variety of constrained problems being solved in rather simplistic models of the financial market. However, there are not many papers in which constrained portfolio choice is considered in the realistic models. In particular, the analysis of problem constrained by the stochastic liability/guarantee with stochastic states variables is not commonly encountered in the literature. Even though such attempts are made their focus is on the derivation of analytical formulas, which is usually a tedious task to do. There is a lack of the empirical or simulation studies, which would indicate whether an effort of deriving new optimal policies is worth taking, in other words whether the dynamic optimal policies perform indeed better than simple portfolio strategies, such as buy-and-hold or fixed mixed. Specifically, the performance of the defined-benefit pension fund following different strategies could be treated.

As we remarked earlier, the paper of Black and Perold [1992] establish many interesting theoretical properties of CPPI strategy. In particular, they claim that the stock index volatility negatively influences the terminal cushion. However, there is a lack of empirical work that confronts the theory against the empirical evidence.

1.3 How does this thesis extend knowledge in a meaningful way?

From the plenty of models of the financial markets considered in the unconstrained optimal portfolio choice literature we focus on two of them, namely the model as in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000]. We choose these models, because the optimal portfolios formulas are given for them analytically when the investor has power utility. The former paper also deals with HARA utility investor. Optimal portfolios in these two settings can be both simulated and calculated by applying the Malliavin derivative approach or derived as Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] do. We simulate optimal portfolios for power utility in both models and confirm the convergence of the simulated portfolios to the analytically derived ones. The outcome of the simulation exercise is a nice illustration of the correspondence of alternative approaches to the portfolio choice problem. Also the derivation of optimal portfolios in the model of Bajeux-Besnainou et al. [2003] is an interesting intellectual exercise. It summarises two major results of the paper by Bajeux-Besnainou et al. [2003] on literally two pages. However, the knowledge seems to be extended most by the simulation of optimal portfolios in the model of Deelstra et al. [2000] when the investor has HARA utility. We obtain optimal portfolios in the setting, which has not been analysed in the literature yet to the best of our knowledge, even though it is not too complicated yet. All the above is done in chapter 2.

The main goal of chapter 3 is to enhance an insight into the concept of interest rate risk hedging in the context of the optimal portfolio. Our additional goal is to identify the behaviour of the discrete-time equivalents of expressions employing Malliavin derivatives of the state variables in the continuous-time optimal portfolios.

Our added value in this chapter stems first from the modelling of the movements of risky assets and state variable. In the literature devoted to the valuation of contingent claims the relevant models are those describing the dynamics of the underlying (stock, interest rate, etc.). In the kind of asset allocation problem we consider (both stock and bond as risky assets) it is necessary to model simultaneously the movement of stock, bond and interest rate from one period to another. That is why we choose to model their behaviour on the trinomial tree.

Analysis in the discrete time allows us to confirm some of the results from the continuous time. This of course does not extend our knowledge, but is a nice property. In particular we show that with an increase of the number of time steps constituents of the optimal portfolio from the tree models indeed converge to their continuous-time counterparts. Moreover, discrete-time study allows additionally investigating what happens with the hedging terms in last period of the investment as well as what is their dependence on the degree of correlation between the risky asset imperfectly correlated with interest rate and this rate. Probably the most knowledge extending result of chapter 3 is the observation that the term in the hedging portfolio formula involving the Malliavin derivative of interest rate is independent of the number of time steps in the tree models.

In chapter four we consider several dynamic portfolio strategies that can be followed by the pension schemes. Not all of these strategies suit the needs of pension schemes well. Some of them do not guarantee that scheme's stochastic liability will be met, others do not provide the hedge against adverse changes of the state variables. However, we test their performance in the models of Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] and compare it against the performance of fixed mix policies. Our added value is in the case of chapter 4 the main conclusion that the liability-constrained non-myopic strategy is not superior according to both criteria of the mean and volatility of the funding ratio and contribution rate to the liability-unconstrained and myopic policies. When the mean ratio is higher and contribution rate lower their volatility is also larger. The same can be said about the comparison of dynamic strategies against the fixed mixes.

Papers dealing with the subject of pension schemes usually focus on the defined-contribution scheme as in this kind of scheme the presence of stochastic liability, which is one of the distinguished characteristics of a defined-benefit scheme, can be neglected. In chapter 4 we propose a policy, which both provides a hedge against the risk of state variable changes and guarantees that the stochastic liability is satisfied. This strategy is to invest the wealth equal to the whole or part of the liability into the liability-replicating portfolio and the remainder as the liability-unconstrained investor with the reciprocal of the funding ratio as the weighting term. Except from proposing a new policy we also take into account the contribution rate as a new characteristic of the defined-benefit scheme.

As far as CPPI policy in chapter 5 is concerned we extend the knowledge in two major directions. On the one hand, we propose a strategy similar to CPPI, which however has the advantage that it is more commonly followed in practice by the commercial banks. We derive some of the theoretical properties of this strategy. On the other hand, we test in practice the theoretical results on CPPI, which are well established in the literature.

1.4 What are the open ends?

The models of financial markets considered in this thesis are based on the assumption of completeness. This assumption, although commonly used in the literature, is not very realistic. There is a growing stream of the literature in which the markets modelled are incomplete. More formal treatment of the incomplete case would be a nice extension of chapter 4, in which the incompleteness is treated by adding into the liability a component, which is not hedged by any of the traded assets. However, the strategies analysed are not optimal in the incomplete setting. More advanced modelling of incompleteness would switch our focus from self-financing strategies to the ones known as mean-variance and super-hedging. This would be more realistic for the reason of pensions benefits indexation by the rate of inflation in the economies where inflation-based securities are not traded on the market. For the economies with this kind of assets the problems formulated in this thesis in the nominal terms could be reformulated in real categories.

Introducing inflation into the dynamic optimal portfolio choice problem could be also beneficial for the analysis in chapters 2 and 3. Taking into account the inflation risk hedging apart from the interest rate risk hedging would be a natural extension of the study carried out in these chapters. Adding another state variable into the problem would necessitate an adjustment of the Malliavin derivative approach to the simulation of optimal portfolios.

Simulating optimal portfolios according to the Malliavin derivative method is developed in the literature for the optimal portfolio choice problem in which the optimal terminal wealth is constrained by non-negativity restriction. It would be interesting and useful from the point of view of such financial institutions as the pension funds for instance to obtain the formulas according to which the optimal portfolios can be simulated for the problem incorporating the stochastic liability constraint. This would be more realistic and general and could enrich the analysis in chapter 4.

In the literature there are arguments for the change of the pattern of portfolio value after the transaction costs are taken into account. In particular, the advantage of dynamic portfolio strategies over static strategies, if any, is often lost in the presence of transaction costs. In this thesis it is assumed that the transactions are costless. It would be useful to conduct the simulation exercise similar to the one performed in chapter 4 for the costly transactions and make the comparisons of the two outcomes.

As far as chapter 5 is concerned it could be extended by analysing more of the theoretical properties of CPPI that are derived in the literature than treated in this thesis. Certainly, a more extensive comparison of the theoretical results against empirical evidence could be of interest.

1.5 Literature review

In the literature concerning the optimal portfolio choice several phases can be distinguished. They differ mainly by the methodological approach applied. In the 50s and 60s the single-period mean-variance analysis was the leading theme. The late 60s and 70s brought the multiperiod dynamic programming techniques. With the development of option pricing methods in the 70s the martingale method started to make its way in the 80s. Because the weak point of martingale approach is the identification of portfolio policies replicating the optimal terminal wealth, the approach based on the Malliavin derivative came to the rescue in the 90s. Still in the 80s and 90s many common investment rules circulated in the literature. Some of them found their formalization.

The beginning of the modern portfolio theory is associated with the name of Markowitz. In his 1952 paper Markowitz replaces the rule that investor maximizes the discounted expected return by the rule that the investor maximizes the expected return given the maximal allowed variance or minimizes variance given the required expected return. Hence, there is trade off for which the investor can gain expected return by taking on variance or reduce variance by giving up expected return. Markowitz's rule justifies diversification of investment portfolios. Tobin [1958] uses Markowitz's mean-variance approach to explain the liquidity preference. Instead of stocks he considers monetary assets, cash as riskless asset and consols as risky assets. Tobin [1958] uses the theory of risk-avoiding behaviour to explain the inverse relationship between the demand for cash and the rate of interest as well as diversification, it is the fact that the same individual holds both cash and consols. In the case of multiple alternatives to cash he proves that the composition of non-cash assets is independent of their aggregate share of the investment balance. Tobin [1965] notices that the uncertainty about asset returns leads an investor to diversify. He distinguishes between three types of investors, namely neutral with respect to risk, the risk lovers and the risk-averters. Tobin [1965] summarizes investors' preferences in terms of the utility of return and requires the investors to maximize its expectation. The measure of risk adopted by Tobin [1965] is the standard deviation. Tobin's [1965] investor ranks portfolio on the basis of the expectation and the standard deviation, rather than on the entire probability distribution of returns. Further, Tobin [1965] focuses on the quadratic utility of returns. For such a utility for the risk-averter indifference curves are concave upward-sloping. They are concave downward-sloping for the risk-lovers. The second route followed by Tobin [1965] to preserve the general properties derived from the expected utility hypothesis is to restrict the probability distributions of outcomes to a two-parameter family for which it is enough to know just two parameters to describe the whole distribution. The normal distribution can be given here as an example. Subsequently, Tobin [1965] analyzes the risk of a portfolio consisting of two assets. The relation between the compound and constituent risks is that the diversification in the presence of negative correlation leads to a risk smaller than the weighted average of the separate risks. Hence, the minimum risk portfolio includes each asset in inverse proportion to its variance. In the two-asset case the relationship between the expected return and standard deviation is described by a hyperbola. When there are three or more basic assets the available combinations cover an area and the efficient frontier is convex. Lintner [1965] computes optimal portfolios when short sales are both permitted and prohibited. In the first case no linear or non-linear programming is required. In the second case the solution of a single quadratic programming problem is involved. Lintner [1965] shows that positive (negative) risk premiums are neither a sufficient nor a necessary condition for a stock to be held long (short). What matters is the correlation in sufficient degree with other stocks held in the portfolio. Subsequently Lintner [1965] lifts the assumption of current security prices being exogenous and under what he calls an idealized uncertainty derives a set of stable equilibrium market prices. His analysis gives good reason for seeing stocks' market values as riskless-rate present values of certainty-equivalents of random future receipts where certainty-equivalents are related to expected values by way of variances and covariances weighted by adjustment factors. It was already in 1972 that Merton gave the analytic derivation in continuous time of Markowitz's efficient portfolio frontier. He considered two cases, first when all securities are risky and second when one of the assets is riskless. For both cases Merton [1972] proves the continuous time equivalent of the mutual fund theorem. In the first case the theorem says that there are two portfolios (mutual funds) constructed from the risky assets such that all risk-averse individuals, who choose their portfolios so as to maximize utility functions dependent only on the mean and variance of their portfolios will be indifferent in choosing between portfolios from among the original assets or from these two funds. In the second case one of the funds contains only risky assets and the other only the riskless asset.

The multiperiod dynamic programming technique is a way to mitigate the weaknesses of a single-period mean-variance analysis. Mossin [1968] bridges the gap between these two approaches. An important result of Mossin [1968] is the identification of utility functions for which it is optimal for the investor to behave as if the immediate period was the last one (complete myopia) and of the functions for which it is optimal to behave as if the immediate decision was the last one (partial myopia). Optimality of partial myopia requires the risk tolerance to be linear in wealth while optimality of complete myopia demands the relative risk aversion to be constant. Samuelson [1969] defines the problem of lifetime portfolio selection in the discrete time under the assumption of no-bequest. He considers two kinds of utility functions, namely logarithmic and power ones and shows that the optimal portfolio decision in these cases is independent of wealth and of all consumption saving decisions, leading to constant fractions invested in the risky and riskless assets. Merton [1969] performs the exercise similar to the one of Samuelson [1969], but in continuous time. Merton [1969] derives the optimality equations for a two-asset and a multiasset problem when returns are generated by the arithmetic Brownian motion process. Because the system of a nonlinear partial differential equation combined with two algebraic equations is not easy to solve in general Merton [1969] focuses on the utility function with constant relative risk aversion. In this case he derives explicit solutions and shows that optimal portfolio policies are constant independent of wealth and time. The same result is achieved in the infinite horizon case, but at the cost of solving an ordinary differential equation instead of a partial differential equation. For the case of two assets and infinite-time horizon Merton [1969] assumes also constant absolute risk aversion utility and finds that instead of the proportion of wealth invested in the risky asset being constant the total dollar value of wealth is kept constant. Merton [1971] assumes a general framework in which the prices of assets are generated by Ito processes and the utility is derived from consumption with the bequest function different from zero. The fundamental partial differential equation obtained by Merton [1971] is non-linear. In case when one of the assets is risk-free Merton [1971] receives the optimal proportions of the risky assets being the sum of two terms. However, at this stage Merton [1971] does not comment on them. The general setting of Merton [1971] is specialized by presuming the log-normality of asset prices. Then he ends up with the mutual fund theorem. In order to solve the problem in a closed form Merton [1971] makes an assumption of the utility function belonging to the class whose absolute risk aversion is positive and hyperbolic in consumption (HARA class). The formula derived for the proportion of wealth invested in the risky assets shows that the demand functions are linear in wealth. In the sequel Merton [1971] introduces wages to the problem as an income generated by non-capital gains as well as gives examples of application of Poisson processes in the consumption-portfolio choice. Finally, he deals with other than geometric

Brownian motion models of assets prices and explains the portfolio selection behaviour under these alternative price expectations. Merton [1973] builds an intertemporal model of capital market. In Merton's [1973] setting asset prices follow Ito diffusions with the dynamics of the investment opportunity set driven by the Wiener processes too. The wage incomes is taken into account as well. In this framework Merton [1973] attains explicit formulas for the demand functions for assets. The expressions are sums of two components and Merton [1973] gives them more attention than in his work from [1971]. The first term is identified by Merton [1973] as the single-period mean-variance demand. The role of the second term is to hedge against adverse changes in the investment opportunity set. When this set is constant Merton's [1973] investors act as if they were single period-maximizers. Moreover, the equilibrium return relationship specified by the capital asset pricing model is obtained in continuous time. When the investment opportunity set is non-constant (e.g. stochastic interest rate) Merton [1973] arrives at the three fund theorem. Finally, Merton [1973] states that in equilibrium investors are compensated in terms of expected return for bearing the market (systematic) risk and for bearing the risk of unfavourable shifts in the investment opportunity set, which is a generalization of the security market line of the classical asset pricing model.

Option pricing theory gave rise to the introduction of the martingale method in the theory of optimal consumption-portfolio choice. However, in their seminal paper on option pricing Black and Scholes [1973] do not use the martingale approach. Instead, they take advantage of the rule that if options are correctly priced in the market it should not be possible to make sure profits by forming portfolios consisting of long position in the stock and short position in the option. The risk of such an appropriate portfolio is zero if the positions are adjusted continuously. In their formula the expected return does not appear and they show that the option is always more volatile than the stock. Black and Scholes [1973] also provide an alternative derivation of the option price by using the capital asset pricing model. The theory of option pricing was further developed by Merton [1973]. In his theory of rational option pricing Merton [1973] derives certain restrictions on the option pricing, for instance no prior exercise of American warrant, warrant's price being non-decreasing function of stock's volatility, put-call parity, etc. Further, Merton [1973] along the lines of Black and Scholes [1973] derives the formula for an option price when the interest rate is stochastic. He also gives extensive treatment of option price in case of dividend payments and exercise price changes. Harrison and Kreps [1979] are the first to consider in the continuous time option pricing theory the martingale approach in conjunction with arbitrage theory. For simple trading strategies they prove that the security market model is viable and every claim is priced by arbitrage if and only if there exists a unique equivalent martingale measure. In the case of diffusions Harrison and Kreps [1979] state the conditions for the set of equivalent martingale measures to be non-empty. Then, the price of a contingent claim equals the expected value of the discounted claim under the equivalent martingale measure. Harrison and Pliska [1981] focus on the subject of market completeness. They attempt to answer the question about the price processes which yield a complete market. They come to the conclusion that these prices must be semimartingales and that the model is complete if and only if the discounted price processes have the martingale property under the equivalent martingale measure which is the only element of the set of equivalent martingale measures. Harrison and Pliska [1981] also consider a generalization of the Black and Scholes [1973] model in which there is a bond and multiple stocks. Pliska [1986] continues in the spirit similar to Harrison and Pliska [1981] and uses stochastic calculus to model the continuous trading in optimal portfolio choice. Pliska [1986] states that the problem of optimal portfolio choice can be separated into two parts. In the first one attainable wealths are identified and the best one is chosen. In the second one the trading strategy yielding this wealth is determined. In the general set up (security prices are semimartingales) and in the complete market Pliska [1986] formulates conditions which should be satisfied by the utility function for the solution of the problem of maximizing expected utility from terminal wealth to exist. Then, for the case of exponential utility this optimal wealth is derived. An even more explicit result is presented for geometric Brownian motion as the model of assets prices. Karatzas et al. [1987] adopt the general framework in which the stock prices may fluctuate in a not necessarily Markovian fashion. In this set up they consider the maximization of utility from consumption, the maximization of utility from terminal wealth and the maximization of utility from both consumption and terminal wealth. They show that in the middle case the investor consumes nothing while in the last case there is a compromise between the two objectives of utility from the consumption and of utility from the terminal wealth maximization. The investor divides the endowment in two parts. One of them is invested to maximize the expected utility from consumption while the other maximizes the expected utility from terminal wealth. The part invested initially in consumption equals the expected value of the discounted consumption flow where the expectation is taken under the equivalent martingale measure. Thereafter, Karatzas et al. [1987] specialize the framework to the setting with constant coefficients and HARA utility. Cox and Huang [1989] consider price processes for risky securities plus their accumulated dividends as Ito processes. Their trading strategies include all simple trading policies. They impose nonnegativity constraints on the final wealth. In this setting in order to find the optimal wealth maximizing the consumption-final wealth pair they transform the problem with a dynamic budget constraint into the one with a static restriction and solve it via the Lagrange multiplier rule. Their result confirms the outcome of analogous problems solved by dynamic programming in the sense that the optimal risky asset portfolio is composed of two parts, namely a portfolio most highly correlated with state variables and a mean-variance efficient portfolio. Cox and Huang [1989] give a portfolio insurance interpretation to their results. They state that an agent can follow an unconstrained policy if he simultaneously buys an insurance package that will pay off the negative consumption and wealth as they are incurred. As a special case Cox and Huang [1989] consider the risky security gain processes being a geometric Brownian motion and various utility functions including HARA utility.

The martingale approach main drawback is that it specifies the optimal final wealth, but the computation of optimal portfolios can be quite cumbersome in certain circumstances. The way to solve this was shown in 1991 by Ocone and Karatzas who generalize the Clark representation formula under the equivalent martingale measure and apply it to calculate optimal portfolios. Expressions obtained involve the Malliavin derivatives of the market price of risk and of the interest rate as well as the expectations under the equivalent martingale measure of their also stochastic integrals. The optimal portfolios for the terminal wealth as well as consumption are shown to depend on two components as it was in the dynamic programming technique and the martingale method with the second component engaging Malliavin derivatives of state variables, namely the interest rate and market price of risk. Ocone and Karatzas [1991] accept a general framework in which assets prices are governed by Ito diffusions. Subsequently, they simplify the setting to the case of deterministic coefficients of the model to end up with a more explicit solution. Detemple, Garcia and Rindisbacher [2003] continue in a spirit similar to Ocone and Karatzas [1991], that is they employ Malliavin derivatives approach in the computation of optimal portfolios as well. However, instead of trying to derive explicit analytical formulas they propose to obtain optimal portfolios by the Monte Carlo method in which the expectations included in the Ocone and Karatzas [1991] formula are simulated. The main result of Detemple et al. [2003] is to show that Malliavin derivatives of state variables follow diffusion processes which substantially eases the numerical implementation. Detemple et al. [2003] do not limit themselves to the most commonly used power utility, but in the simulation experiment they also consider the more general HARA class. They also employ their simulation formula to the problem of investing when there are multiple assets.

Through the years dynamic portfolio strategies not stemming from any theoretical model were also present in the literature. Some of them gained relatively many proponents. As examples the stop-loss strategy, the lock-in profit strategy and the random market timing strategy can be mentioned here. It seems that the popularity of these strategies did not suffer, even though Dybvig [1988] shows that they are inefficient with the efficiency of a given policy measured by his payoff distribution pricing model. Other strategies approved by the scientific community are treated in a more formalized manner and their theoretical properties are examined. One such strategy is the constant-proportion portfolio insurance (CPPI) policy which is carefully investigated in Black and Perold [1992]. Chapter 5 is devoted to the CPPI strategy and it is in that section where we provide the reader with more literature reviews on CPPI. In this section we give a general review of literature on portfolio insurance.

The literature on portfolio insurance is concerned mainly with the effects of inclusion of the protective put options into the investment portfolio and with the equilibrium analysis of economy with the presence of portfolio insurance. One of the exceptions is the paper by Brennan and Solanki [1981] who treat the issue of optimal portfolio insurance. More specific, the paper is concerned with deriving the optimal form of portfolio insurance contract payoff function for an investor who is assumed to invest the whole of his wealth in the one contract in order to maximize his own expected utility. Two kinds of contracts are considered, the single premium and the periodic premium contract of type I and II. It is shown that the type I contract is always inefficient relative to the type II or the single premium contract. Brennan and Solanki [1981] find the investment strategy of the insurance company, which should be chosen in order for the company to make zero profit for any contract payoff function. Brennan and Schwartz [1988] analyze the class of portfolio insurance strategies under which the fraction of wealth allocated to risky assets is independent of time. They also provide an alternative notion of time invariance, which is that the value function is time invariant. Brennan and Schwartz [1988] consider a self-financing investment strategies in which wealth is divided between a reference portfolio with a value following the geometric Brownian motion and the riskless securities. They provide a sufficient condition for the portfolio insurance investment strategy to be time invariant and a necessary and sufficient condition for the value function to be invariant. Market equilibrium analysis is conducted by Brennan and Schwartz [1989]. They compare a capital market in which prices are set by a single expected utility maximizing investor with a market in which the expected utility maximizing investor is one of the agents, the other one being the portfolio insurer. They find out among others that the effect of portfolio insurers on the instantaneous market volatility increases more than proportionately with the proportion of the market subject to insurance. However, this effect is quite modest. The portfolio insurance in their framework also reduces the net interest rate. Also Grossman and Zhou [1996] in their equilibrium analysis consider both insurers and noninsurers. They characterize the optimal trading strategy in one of their lemmas, which states that if the solution to the insurer's optimization problem is attainable, then it can be achieved by synthesizing a put option with strike equal to the floor on the optimally controlled wealth obtained by ignoring the constraint. They find that the price volatility in the presence of portfolio insurers rises. As far as the optimal trading strategy of portfolio insurers is concerned they buy the risky asset if the price of risky asset goes up and sell the risky asset if the price goes down. Grossman and Vila [1989] consider a discrete-time model in which they provide a proof of the result of that "for a non-decreasing concave utility of final wealth an optimal policy with a nonnegativity constraint on wealth is a combination of the policy that would be optimal without the constraint and a policy that duplicates a European put on the final value of the unconstrained policy". The issue of changing the returns by adding protective put options into the portfolio is analyzed by Figlewski et al. [1993]. They use simulation to examine the performance of three types of protective put strategy with short-maturity options. They consider fixed strike strategy, fixed percentage strategy and the ratchet strategy. They find that for the fixed strike put strategy mean return goes down as the striking price of the put strategy is increased. At the same time the standard deviation goes down. Comparing the fixed percentage strategy with the fixed strike strategy they find that the mean return for an out-of-money fixed percentage rule is about the same, but it falls faster as the strike price is raised. The ratchet strategy has the lowest mean returns of the three at each strike, but also the lowest standard deviation. Kouwenberg and Vorst [1998] approach the dynamic portfolio insurance with the stochastic programming. They construct event trees which ensure no arbitrage. On these trees they match the moments of the underlying continuous distribution of the continuously compounded returns of the financial assets and economic variables. The portfolio insurance component is added by the introduction of put options, which mature beyond the investment horizon. The model is calibrated with a subset of one year and two year puts.

1.6 Outline of this thesis

The subject of this thesis is the dynamic portfolio choice. It consists of six chapters. Chapter 1 has an introductory character. Chapters 2 and 3 are concerned with the interest rate risk hedging in the optimal portfolio choice, respectively in the continuous and discrete time. The methodological approach applied in these chapters is the Malliavin derivative approach (chapter 2) and the martingale method in the computation of optimal portfolios (chapter 3). Chapter 4 contains the results of a simulation experiment, which is tailored to compare the dynamic portfolio strategies for the pension schemes obtained by the application of martingale method against each other and against the fixed mixes. Chapter 5 is dedicated to one of the common investment rules, namely CPPI. Chapter 6 concludes the thesis.

In chapter 2 we choose two models of the financial market known from the literature and characterized by the stochastic short-term interest rates. In these models we consider two agents, one with CRRA and one with HARA utility function. Their objective is to maximize the expected utility from the terminal wealth. We present how the martingale method can be used to solve the optimization problem in question. Subsequently, we introduce the Malliavin calculus into the optimal portfolio choice and describe how it can be applied in combination with the simulation techniques to obtain the optimal proportions of risky assets in the investment portfolio. Three out of four optimal policies considered in chapter 2 are known from the literature. Hence, in the course of simulation we can confirm the convergence of simulated portfolios to their continuous-time equivalents for the examined number of time steps. One of these policies (HARA utility, CIR [1985] model of interest rate) was not encountered in the literature we are familiar with. For this policy we investigate how the hedging terms react to the changes in the ratio of initial wealth to the subsistence level. For two of these policies (CRRA and HARA utility, Vasicek [1977] model of interest rate) we show how the analytical formulas for the optimal portfolios known from the literature can be alternatively derived by the application of Malliavin calculus.

In chapter 3 we consider the dynamics of risky assets and stochastic interest rates described by the binomial three-period, trinomial two-period and trinomial multiperiod tree models. In this discrete setting we derive the discrete-time formulas for the initial shares of risky assets in the mean-variance and hedging portfolios of CRRA investor. By doing so in the two-period and three-period models with each of the branch describing the same relative changes of state variables we are able to confirm some of the results from the continuous time and explain the mechanism of interest rate risk hedging in a more intuitive way. We present these formulas in the form as analogous as possible to their continuous-time equivalents obtained with the application of Malliavin calculus. In this way we are able to extract the expressions equivalent in the discrete time to the terms employing Malliavin derivative of the interest rate in the continuous time in order to conclude about their behaviour on the basis of multiperiod model when the number of time steps increases. We show the convergence of the discrete-time portfolios to their continuous-time equivalents when the number of time steps increases.

In chapter 4 we choose two models of financial market with stochastic interest rate the same as in chapters 2 and 3. We consider the defined-benefit pension scheme which operates on these markets and whose objective is to maximize the expected power utility from the terminal wealth. The pension fund is constrained by the stochastic liability stemming from the obligations towards the current and future retirees. In this setting we simulate the funding ratio and the contribution rate of the defined-benefit pension fund, which invests into the dynamic liability-constrained and interest rate risk hedging strategy. Our aim is to compare this strategy with mean-variance and liability-unconstrained strategies as well as fixed mixes. In the second step we examine whether the performance of dynamic strategies when the liability hedge is perfect differs substantially from the performance when the hedge is imperfect. Finally, we investigate how sensitive is the performance of dynamic strategies to the changes in scheme's characteristics such as the initial funding ratio, risk aversion coefficient, wage-to-liability ratio, pension-to-liability ratio and the allowed level of underfunding.

In chapter 5 we consider the portfolio insurance policy known in the literature as CPPI. We introduce another strategy, which we call MCPPI that is applied on a day-to-day basis by some of the commercial banks. Their major difference is the presence of an additional asset in MCPPI strategy, namely the money market account. We remind the reader some of the theoretical properties of CPPI and derive the properties of MCPPI. In the empirical part of this chapter we first compare the performance of CPPI and MCPPI strategies on the German market. Second, we examine the influence of volatility in the stock index ratio on the terminal wealth of both policies. Comparing the empirical vs. the theoretical results we investigate whether the continuous-time cushions coincide with the ones calculated from the data with daily rebalancing. As far as CPPI alone is concerned we also evaluate whether the discrete-time cushions coincide with the ones calculated from the data with trigger rebalancing. The bridge between the theoretical and empirical part of the chapter consists of the three motivating examples.

2 Continuous-time interest rate risk hedging in the optimal portfolio choice

2.1 Introduction

Since Merton [1971] the optimal consumption-portfolio choice problem in the financial economics literature is formulated in the convenient continuous-time framework. Merton [1971] solved the problem in question by the application of stochastic dynamic programming. Merton's [1971] solution shows that the demand for risky assets in the optimal portfolio consists of mean-variance and hedging components. The first component characterizes the optimal portfolio when the investment opportunity set is at most deterministic or the utility function is logarithmic, while the latter appears in a stochastic setting. In particular, stochastic interest rates, market prices of risk and inflation rates give rise to the hedging constituents. The mean-variance demand is a demand by a single period mean-variance maximizer. The hedging demand is a vehicle to hedge against unfavourable shifts in the investment opportunity set.

The stochastic dynamic programming has become a common way to solve the continuous-time consumption-portfolio choice problem. This method was applied among others by Sorensen [1999], Brennan and Xia [2000] and Munk et al. [2004]. Sorensen [1999] and Brennan and Xia [2000] solve the problem of maximizing expected constant relative risk aversion (CRRA) utility from terminal wealth. However, they differ in the model of interest rate dynamics. In Sorensen [1999] this dynamics is described by Vasicek [1977] model, while in Brennan and Xia [2000] by the Hull and White [1996] two-factor model. They show that the positions taken in stock and bond are ceteris paribus increasing functions of the expected excess returns and that the relevant hedge portfolio is the zero coupon bond with expiration at the investment horizon. For log-utility investors the hedging term vanishes. Stock allocations are determined solely by the myopic portfolio, because the hedge portfolio does not contain equity. As long as the bond has a maturity equal to the investment horizon, the optimal allocations to stock, bond and cash are independent of the horizon. The stock allocation is decreasing and the cash and bond allocations are increasing in the investor's risk aversion. Munk et al. [2004] consider dynamic asset allocation under mean-reverting returns, stochastic interest rates and inflation uncertainty. They try to explain simultaneously the Samuelson [1963] and Canner et al. [1997] puzzles. Munk et al. [2004] consider an investor with the objective to maximize the expected CRRA utility from the real terminal wealth. The excess return on the stock index as well as the nominal interest rate dynamics are described by an Ornstein-Uhlenbeck process. So is the expected rate of inflation. In the solution to the problem Munk et al. [2004] obtain three hedging terms. The optimal hedge against changes in the expected equity excess returns is obtained by investing exclusively in stocks, which is due to the perfect negative correlation between the stock process and the excess return process. The optimal hedge against changes in the interest rate is obtained by investing exclusively in the bond, which is perfectly negatively correlated with short interest rate. The inflation hedge involves both the stock and the bond.

The problem introduced by Merton [1971] with an additional non-negativity constraint on terminal wealth was solved by means of the martingale method by Cox and Huang [1989]. In contrast to the stochastic dynamic programming where the optimization is over the current consumption rate and the fractions of current wealth invested in the risky securities the approach of Cox and Huang [1989] is a two-step procedure. In the first step the distribution of optimal terminal wealth is found, while in the second stage the unique replicating portfolio is determined for which purpose the partial differential equation (PDE) needs to be solved. With the application of martingale techniques explicit portfolio rules were derived for instance by Bajeux-Besnainou et al. [2003] for the hyperbolic absolute risk aversion (HARA) utility over the terminal wealth and Vasicek [1977] interest rate dynamics, by Deelstra et al. [2000] for CRRA utility over the terminal wealth and Cox-Ingersoll-Ross (CIR) [1985] interest rate dynamics and by Munk and Sorensen [2004] for CRRA utility over the terminal wealth as well as consumption and the interest rates having Heath-Jarrow-Morton [1992] multifactor Gaussian dynamics. The hedge bond in the setting of Munk and Sorensen [2004] is the coupon bond with coupon rates equal to the certainty equivalents of optimally planned future consumption rates. The models of the financial market in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] account for the influence of stochastic interest rate on the stock prices by modelling stock returns as two-factor diffusions with one of the Brownian motions being perfectly correlated with interest rate volatility term. In both it is shown that the bond with maturity matching the investor horizon is the proper hedge bond. The models of Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] are subject of our interest in this chapter.

Though general formulas for wealth replicating portfolio exist (cf. Ocone and Karatzas [1991]) an explicit identification of this portfolio still proves to be cumbersome in more complex, and at the same time more realistic, models with more hedging terms. This is partly due to the expression of risky assets demand in the form of conditional expectations of random variables involving (also stochastic) integrals of Malliavin derivative of state variables what is shown in Ocone and Karatzas [1991]. In the recent paper Detemple et al. [2003] demonstrate that these derivatives follow diffusion processes. Consequently, they propose to use Monte Carlo simulation to approximate the expectations in question and consequently to simulate the optimal portfolio fractions of risky assets.

In this chapter we consider first a CRRA investor. We position him in two models of financial market, respectively as in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000]. Hence, we know the optimal policy an investor should follow and its decomposition into the mean-variance and hedging portfolios if he rebalances them continuously. Moreover, the selection of financial markets as in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] has the advantage of stochastic interest rate being (apart from the risky assets) the only state variable. This is because the market prices of risk are constant in the former and at most a function of interest rate in the latter. As the sole source of risk stems from the interest rate dynamics, the interest rate risk hedging component is eventually the only hedging constituent that shows up in the risky assets demand. This in turn eases the simulation of optimal portfolios in the Monte Carlo experiment. Subsequently, we consider a HARA investor in the model of the financial market as in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000]. While in the former model the literature provides us with the answer what the optimal portfolio strategy should be, in the latter model the analytical formulas are not available to the best of our knowledge even though the dynamics of interest rates is not too complex yet.

Our initial focus on CRRA investor serves well one of the goals of this chapter which is to introduce the concept of interest rate risk hedging in the continuous-time setting. The convenience of CRRA utility is connected with two aspects. First, by using CRRA utility the optimal hedging portfolio policy is not blurred by the liability hedging component, which appears with HARA utility. Second, as the optimal portfolio policy in the chosen models of financial market is known for CRRA utility we can investigate the convergence of CRRA portfolio simulated according to the guidelines of Detemple et al. [2003] and its constituents to the analytical equivalents given in the literature. Our second goal is to present how one can obtain analytical formulas for the optimal portfolios as in Bajeux-Besnainou et al. [2003] using the Malliavin calculus approach of Detemple et al. [2003] and Ocone and Karatzas [1991]. Our third objective is to simulate HARA portfolios with the hedging terms distinguished when the interest rate has the CIR [1985] dynamics and to show how the hedging terms react to the changes in the ratio of initial wealth to the subsistence level.

The chapter is organized as follows. In section 2.2 we describe the martingale approach to the valuation of contingent claims and to the optimal wealth choice. In section 2.3 we describe the partial differential equation approach to deriving the optimal portfolios and introduce the approach of Detemple et al. [2003] to simulating optimal portfolios. In section 2.4 we describe two models of financial market, as in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000]. Section 2.5 presents the solution to the optimal portfolio choice problem in the considered models of financial market derived by Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000]. It also explains in the introduced models the details of simulating optimal portfolios according to Detemple et al. [2003]. In this section we also show how the results of Bajeux-Besnainou et al. [2003] can be obtained from the Malliavin derivative approach to calculating the optimal portfolios. Section 2.6 presents the assumptions of a simulation experiment. In section 2.7 simulated CRRA portfolio rules are reported and their convergence to the continuous-time limit illustrated. Also HARA simulated portfolios and the relation between hedging and initial wealth-to-subsistence-level ratio is presented in this section. Section 2.8 concludes the chapter.

2.2 Martingale approach

2.2.1 Valuation of contingent claims

The exposition in this section is based on Duffie [2001], Pelsser [2000], Björk [1998], Baxter and Rennie [1996], Sundaram [1997] and Bajeux-Besnainou and Portait [1997]. Let us specify the model of the continuous trading economy with a trading taking place between time 0 and T. We fix a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the $\sigma-$algebra on Ω and P is the probability measure on $(\Omega, \mathcal{F})$. The probability space is constructed so that there exist standard Brownian motions. The uncertainty is resolved over $[0, T]$ according to the filtration $\{\mathcal{F}_t\}$.

There are $N+1$ marketed assets. Let their prices $S = \left(S^0, S^1, ..., S^N\right)$ be modelled as Ito processes

$$dS^i(t) = \mu^i(t)S^i(t)dt + \sigma^i(t)S^i(t)dz(t), \tag{2.1}$$

$i = 0,\ 1, ...,\ N$. Security 0 is a risk-free asset with $\mu^0(t) = r(t)$ and $\sigma^0(t) = 0$, where $r(t)$ is the short rate process. For instance, when $N = 2$ we may think of $S^1(t)$ as the price process of a bond and of $S^2(t)$ as the price process of a stock. Drifts $\mu^i(t)$ and the vectors of volatilities $\sigma^i(t)$ being the $i-$th row of volatility matrix $\sigma(t)$ are bounded adapted processes. We assume that matrix $\sigma(t)$ is invertible.

We assume that $dz(t)$ is an increment of $N-$dimensional Brownian motion. As the number of random sources equals the number of underlying traded assets excluding the risk free asset the model is complete and free of the arbitrage opportunities. An arbitrage opportunity is a self-financing trading strategy which has strictly negative initial costs and with probability 1 has a non-negative value at time T. The economy is called complete if all derivative assets are attainable in it. A derivative security is defined via its uncertain payoff at time T denoted by $X(T)$ and being an $\mathcal{F}_T-$measurable random variable with the property that the expectation of payoff for these derivatives is well-defined. The derivative is said to be attainable if we can find a self-financing trading strategy such that the value of the portfolio at time T obtained via the following of this strategy equals $X(T)$ with probability 1. A trading strategy is a predictable $N-$dimensional stochastic process of the holdings in each of the N assets at time t. The self-financing trading strategy is then called a replicating strategy. A self-financing trading strategy is a strategy that neither requires nor generates funds between time 0 and time T.

Any asset which has strictly positive prices for all $t \in [0, T]$ is called a numeraire. We choose the riskless asset to be the numeraire. We write the deflated processes $\widehat{S}^i(t)$ as

$$\widehat{S}^i(t) = S^i(t) \exp\left(-\int_0^t r(u)du\right).$$

Numeraire invariance principle says that a trading strategy is self-financing with respect to S if and only if it is self-financing with respect to $\widehat{S}$, $\widehat{S} = \left(\widehat{S}^0, \widehat{S}^1, ..., \widehat{S}^N\right)$. A trading strategy is an arbitrage with respect to S if and only if it is an arbitrage with respect to the deflated price process $\widehat{S}$.

Conditions for the financial market to be complete and arbitrage free can be also formulated in terms of the equivalent martingale measure Q. A continuous economy is free of arbitrage opportunities and complete if there exists a unique equivalent martingale measure. A probability measure Q on $(\Omega, \mathcal{F})$ is said to be equivalent to P , provided for any event A, we have $Q(A) > 0$ if and only if $P(A) > 0$. An equivalent probability measure is an equivalent martingale measure for the discounted price processes $\widehat{S}^i(t)$, $i = 1, 2, ..., N$, if $\widehat{S}^i(t)$ are martingales with respect to Q and if the Radon-Nikodym derivative $\frac{dQ}{dP}$ has finite variance.

Let $\lambda(t)$ defined as

$$\lambda(t) = \sigma^{-1}(t)\left(\mu(t) - r(t)\mathbf{1}\right) \tag{2.2}$$

be the vector of market prices of risk, where $\mathbf{1}$ is a vector of N ones. For any $\lambda(t)$ such that

$$\int_0^t \lambda^T(s)\lambda(s)ds < \infty,$$

with probability 1, the Radon-Nikodym derivative is given by

$$\frac{dQ}{dP} = \varsigma(T) = \exp\left(-\int_0^T \lambda^T(s)dz(s) - \frac{1}{2}\int_0^T \lambda^T(s)\lambda(s)ds\right).$$

Under the measure Q the process

$$z^Q(s) = z(s) + \int_0^t \lambda(s)ds \tag{2.3}$$

is a $Q-$Brownian motion. The above theorem is known as the Girsanov theorem.

It is not difficult to show that $S^i(t)$ have drifts equal to the short term interest rate under the measure Q. From formula (2.2)

$$\mu^i(t) = \sigma^i(t)\lambda(t) + r(t).$$

Substituting into formula (2.1) we obtain the price processes under P

$$dS^i(t) = \left(\sigma^i(t)\lambda(t) + r(t)\right) S^i(t)dt + \sigma^i(t)S^i(t)dz(t). \tag{2.4}$$

From (2.3)

$$dz(s) = dz^Q(s) - \lambda(t)dt. \tag{2.5}$$

Substituting (2.5) into (2.4) we end up with

$$dS^i(t) = r(t)S^i(t)dt + \sigma^i(t)S^i(t)dz^Q(s).$$

Hence, discounted price processes are martingales under Q and the measure Q defined in the Girsanov theorem is the equivalent martingale measure.

For a probability measure Q equivalent to P, the density process $\varsigma(t)$ for Q is a martingale defined by

$$\varsigma(t) = E_t\left(\frac{dQ}{dP}\right) = \exp\left(-\int_0^t \lambda^T(s)dz(s) - \frac{1}{2}\int_0^t \lambda^T(s)\lambda(s)ds\right),$$

$t \in [0, T]$. When the market price of risk is bounded the stochastic discount factor also known as the pricing kernel or state price density is defined by

$$\zeta(t) = \exp\left(-\int_0^t r(u)du\right)\varsigma(t). \tag{2.6}$$

The reciprocal of $\zeta(t)$ is shown for instance in Merton (1990) to be the value of growth-optimal portfolio when all dividends are reinvested. The growth-optimal portfolio is the portfolio that maximizes the log return on wealth and it is the optimal portfolio for an investor with log utility from terminal wealth.

As the discounted marketed assets are martingales under the measure Q then also a discounted derivative asset if attainable is a martingale under Q what we write

$$E_t^Q\left(\exp\left(-\int_0^T r(u)du\right)X(T)\right) = \left(-\int_0^t r(u)du\right)X(t).$$

Changing the measure from Q to P we have

$$E_t\left(\exp\left(-\int_0^T r(u)du\right)X(T)\varsigma(s)\right) = \left(-\int_0^t r(u)du\right)X(t)\varsigma(t). \tag{2.7}$$

From (2.7) we have

$$X(t) = \frac{E_t\left(\zeta(T)X(T)\right)}{\zeta(t)},$$

for any times t and $T > t$ and any $\mathcal{F}_T$-measurable random variable $X(T)$ such that $E^Q\left(|X(T)|\right) < \infty$, which is the price of the derivative at time t.

2.2.2 Optimal wealth choice

In the martingale approach to the optimal portfolio choice the following steps are involved. First, the dynamic problem from stochastic control approach is transformed into the static problem. Second, using the Lagrange multiplier rule the problem is solved for optimally invested wealth. Third, an optimal replicating strategy is found. In this section we deal with the first two steps.

Let us start with the stochastic control approach to the optimal investment problem in continuous time. The problem in question which is to maximize the expected utility from terminal under the budget restriction can be written as

$$\max_{(X_T,\pi)} E_t u(X(T))$$
$$\text{s.t. } \frac{dX\,(t)}{X(t)} = \left(\pi^T(t)\sigma(t)\lambda(t) + r(t)\right)dt + \pi^T(t)\sigma(t)dz(t), \tag{2.8}$$

where $\pi(t)$ is an adapted process $\pi(t) = \left(\pi(t)^1, ..., \pi(t)^N\right)^T$ defining fractions of total wealth held in the risky securities and $X\,(t)$ is the wealth process. The utility function $u(\bullet)$ is strictly concave, increasing and differentiable on $(0,\infty)$ and satisfies the Inada conditions which are $\inf_{X_T} u_{X_T}(X(T)) = 0$ and $\sup_{X_T} u_{X_T}(X(T)) = \infty$. The dynamic budget constraint can be interpreted as the total return on an asset whose price is $X(t)$. In such a case the discounted wealth process should be a martingale under the measure Q. We will verify that indeed this is the case.

Applying Ito's lemma to the process (2.1) we obtain the dynamics of state price density as follows

$$d\zeta(t) = \zeta(t)\left[-r(t)dt - \lambda^T(t)dz(t)\right].$$

Applying the product rule to $d\zeta(t)X(t)$ we have

$$\begin{aligned} d\zeta(t)X(t) &= \zeta(t)dX(t) + X(t)d\zeta(t) + d\zeta(t)dX(t) = \\ &= \zeta(t)X(t)\left(\pi(t)\sigma(t) - \lambda^T(t)\right)dz(t). \end{aligned}$$

Hence, integrating both sides leads to

$$\zeta(T)X(T) = \zeta(t)X(t) + \int_t^T \zeta(u)X(u)\left(\pi(u)\sigma(u) - \lambda^T(u)\right)dz(u).$$

Taking expectations of both sides we end up with the constraint

$$E_t\left[\zeta(T)X(T)\right] = \zeta(t)X(t).$$

Hence, in the first step we can rewrite the stochastic control problem with dynamic budget constraint as the optimization problem with static budget restrictions in the form of martingale constraints as follows

$$\max_{X_T} E_t u(X(T))$$
$$\text{s.t. } E_t\left[\zeta(T)X(T)\right] = \zeta(t)X(t), \tag{2.9}$$

where the optimization is only upon $X(T)$. Cox and Huang [1989] show that given a terminal wealth $X^*(T)$ and some initial wealth $X^*(t)$ there exists a trading strategy $\pi^*(t)$ such that $X^*(T)$ and $\pi^*(t)$ solve the stochastic control problem if and only if $X^*(T)$ solves the above static problem. In order to solve for $X^*(T)$ the Lagrangian multiplier method is used. The wealth $X^*(T)$ solves the problem (2.9) if and only if there is a Lagrange multiplier $\kappa^* > 0$ such that $X^*(T)$ solves the unconstrained problem

$$\max_{X_T} E_t u(X(T)) - \kappa E_t \left[\zeta(T)X(T) - \zeta(t)X(t)\right]$$

and satisfies the budget restriction. We write the Lagrangian as

$$\begin{aligned} L &= E_t \left[u(X(T)) - \kappa \left(\zeta(T)X(T) - \zeta(t)X(t)\right)\right] = \\ &= \int_\Omega \left[u(X(T)) - \kappa \left(\zeta(T)X(T) - \zeta(t)X(t)\right)\right] dP\left(\omega\right). \end{aligned}$$

The first-order condition for optimality of $X^*(T)$ is

$$\frac{\delta L}{\delta X(T,\omega)} = u_X(X(T)) - \kappa\zeta(T) = 0,$$

that is

$$X^*(T) = I\left(\kappa\zeta(T)\right),$$

where $I\left(\bullet\right) = u^{-1}_{X_T}(\bullet)$. Having $X^*(T)$ we may substitute it back into the budget constraint and solve for the correct Lagrange multiplier κ^*. With that multiplier we end up with the optimally invested terminal wealth.

2.3 Optimal portfolio choice

2.3.1 PDE approach

However, the terminal wealth is not all. The next step is to determine the process $X(t)$ for every $0 \leq t \leq T$ and the wealth replicating strategy. In general it is not easy to find the replicating strategies. The approach based on the Malliavin calculus presented in the papers of Karatzas and Ocone [1991] and in Detemple et al. [2003] is discussed in the next section. In this section we demonstrate the approach based on the partial differential equation (PDE). The exposition is mainly based on Campbell and Viceira [2002].

Let us denote by $Z(t)$ the reciprocal of $\zeta(t)$. Given the Markovian structure of the dynamics for $Z(t)$ and any state variable $V(t)$ the expectation from the budget constraint will be some function F of the current value of $Z(t)$ and if the process for $Z(t)$ depends on the state variable $V(t)$ it will also be a function of the current value of $V(t)$ what we write

$$X(t) = F(Z(t), V(t), t).$$

The wealth discounted with the stochastic discount factor has the martingale property

$$E_t \left[d(\zeta(t)X(t))\right] = E_t \left[d(\zeta(t)F(t))\right] = 0.$$

The above expectation implies a second-order partial differential equation for optimally invested wealth. To compute this expectation we need first to compute $d(\zeta(t)F(t))$. By the product rule

$$d(\zeta(t)F(t)) = \zeta(t)dF(t) + F(t)d\zeta(t) + d\zeta(t)dF(t). \tag{2.10}$$

Using Ito's lemma the dynamics of $dF(t)$ is given by

$$dF(t) = F_Z dZ + F_V dV + F_t dt + \frac{1}{2}F_{ZZ}\left(dZ\right)^2 + \frac{1}{2}F_{VV}\left(dV\right)^2 + F_Z F_V dZ dV, \tag{2.11}$$

where subindices denote partial derivatives. Subsequently, we substitute $dF(t)$ and $d\zeta(t)$ into equation (2.10). For simplicity let as assume that the state variable $V(t)$ follows the process

$$dV(t) = \mu^V(V,t)dt + \left(\sigma^V\right)^T(V,t)dz(t),$$

where $\sigma^V(V,t)$ is $N \times 1$ vector and $\mu^V(V,t)$ is a scalar. From Ito's lemma it can be shown that dZ follows

$$dZ(t) = Z(t)\left[\left(r(V,t) + \lambda(V,t)^T\lambda(V,t)\right)dt + \lambda^T(V,t)dz(t)\right].$$

From the substitution of $dF(t)$ and $d\zeta(t)$ into (2.10) we recognize that the condition that the drift term of $d(\zeta(t)F(t))$ equals zero writes as

$$F_Z Z r + F_V\left(\mu^V - \lambda^T\sigma^V\right) + F_t + \frac{1}{2}F_{ZZ}Z^2\lambda^T\lambda + \frac{1}{2}F_{VV}\left(\sigma^V\right)^T\sigma^V - F_Z F_V Z\lambda^T\sigma^V - Fr = 0.$$

For clearer exposition we omitted the dependence on V and t. Once we solve the above PDE we obtain $X(t)$. Then we can also solve for the optimal portfolio. To solve for optimal portfolio we simply equate the diffusion terms of the intertemporal budget constraint in (2.8) and the PDE describing the dynamics of optimally invested wealth (2.11), since both must be the same along the optimal path

$$\pi^T(t)\sigma(t)X(t)dz(t) = F_Z Z\lambda^T(V,t)dz(t) + F_V\left(\sigma^V\right)^T(V,t)dz(t).$$

Hence,

$$\pi(t) = \left(\sigma^T(t)\right)^{-1}\frac{F_Z Z\lambda(V,t)}{X(t)} + \left(\sigma^T(t)\right)^{-1}\frac{F_V\left(\sigma^V\right)^T(V,t)}{X(t)}.$$

In the above formula we can identify the first component as the myopic term in the optimal portfolio rule and the second component as the hedging term. Myopic term describes the portfolio which is optimal when the investor is a single-period mean-variance maximizer, when the investment opportunity set is at most deterministic or when the investor such as the log-utility maximizer does not hedge against changes in the investment opportunity set. Hedging constituent appears due to the long-term investment horizon, stochastic investment opportunity set and non-logarithmic utility.

2.3.2 Malliavin calculus

Solving the partial differential equation from the previous section is not always possible. Consequently, an explicit optimal trading strategy is not always found either. Instead of solving the partial differential equation, Detemple et al. [2003] in a recent paper propose to simulate the solution to the portfolio choice problem with added non-negativity constraint on the terminal wealth. The simulation is possible because the demand for risky assets is expressed as expected values of expressions detailed in Ocone and Karatzas [1991]. Theorem 1 in Detemple et al. [2003] says that the fractions invested in the risky assets $\pi(t)$ are given by

$$\pi(t) = \left(\sigma^T\right)^{-1}\left(\frac{\lambda}{R(X_t)}c(t,\lambda,r) - a(t,\lambda,r) - b(t,\lambda,r)\right), \tag{2.12}$$

where

$$\begin{aligned} a(\cdot)^T &\equiv E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\left(1-R(X_T)^{-1}\right)1_{X_T>0}\int_t^T D_t r_s ds\right), & (2.13)\\ b(\cdot)^T &\equiv E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\left(1-R(X_T)^{-1}\right)1_{X_T>0}\int_t^T \left(dz_s+\lambda_s ds\right)^T D_t\lambda_s\right), & (2.14)\\ c(\cdot) &\equiv E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\frac{R(X_t)}{R(X_T)}1_{X_T>0}\right). & (2.15)\end{aligned}$$

In the above formulas $1_{X_T>0}$ is the indicator of the event $X_T > 0$, $R(x)$ is the relative risk risk aversion coefficient defined as

$$R(x) = -\frac{u_{xx}(x)x}{u_x(x)},$$

while $\zeta_{t,T} = \frac{\zeta_T}{\zeta_t}$.

Let us generalize the framework and let $V(t)$ denote the $N-$dimensional vector of state variables. Then μ^V becomes a vector and σ^V becomes a matrix. Let $D_t r_s$ and $D_t\lambda_s$ stand for the vector and matrix respectively of Malliavin derivatives of interest rate and market prices of risk. They are given by

$$\begin{aligned} D_t\lambda_s &= \partial_2\lambda\left(s,V(s)\right)D_tV(s),\\ D_t r_s &= \partial_2 r\left(s,V(s)\right)D_tV(s),\end{aligned}$$

where the Malliavin derivative of state variable $D_tV(s) = (D_{1t}V(s), ..., D_{Nt}V(s))$ solves the linear stochastic differential equation

$$d\left(D_{kt}V(s)\right) = \partial_2\mu^V(s,V(s))D_{kt}V(s)ds + \left(\sum_{j=1}^N \partial_2\sigma_j^V(s,V(s))dz_j(s)\right)D_{kt}V(s), \tag{2.16}$$

subject to the boundary condition $\lim_{s\to t}D_{kt}V(s) = \sigma_k^V(s,V(s))$. In the above formula $\sigma_j^V(s,V(s))$ is the jth column of the matrix $\sigma^V(s,V(s))$ and $\partial_2\sigma_j^V(s,V(s))$

is the gradient with respect to $V(s)$ of $\sigma_j^V(s,V(s))$, $j=1,...,N$. The theorem shows that the Malliavin derivatives of state variables satisfy diffusion processes what implies that the simulation methods can be used to calculate the portfolio shares. The Malliavin derivatives capture the impact of an innovation in the Brownian motion $z(t)$ at time t on the state variable $V(s)$ at time s.

The first component of the portfolio (2.12) is a mean-variance term while the next two are intertemporal hedging terms. In this decomposition of replicating portfolio $\pi^{IR}(t) \equiv -\left(\sigma^T\right)^{-1} a(\cdot)$ is termed the interest rate risk hedging component and $\pi^{MPR}(t) \equiv -\left(\sigma^T\right)^{-1} b(\cdot)$ is called the market price of risk hedging constituent. These terms measure respectively the interest rate and market price of risk sensitivity to the underlying Brownian motions $z(t)$. The Malliavin derivatives differ from zero when the investment opportunity set is stochastic and then the hedging matters for the optimal asset allocation.

When the utility is the HARA function with the subsistence level $\widehat{X}$ it has the form

$$u(x) = \frac{\gamma}{1-\gamma}\left(\frac{x-\widehat{X}}{\gamma}\right)^{1-\gamma}.$$

The relative risk risk aversion coefficient for the HARA utility is equal to

$$R(x) = \frac{\gamma x}{x-\widehat{X}}. \tag{2.17}$$

Optimal terminal wealth for HARA with the subsistence level $\widehat{X}$ obtained by the application of martingale approach from section 2.2.2 is given by the formula

$$X(T) = \xi_{t,T}^{-\frac{1}{\gamma}} \frac{X_t - \widehat{X} E_t\left(\xi_{t,T}\right)}{E_t(\xi_{t,T}^{1-\frac{1}{\gamma}})} + \widehat{X}. \tag{2.18}$$

The optimal fractions of risky assets are given by formulas (2.12)-(2.15) with $1_{X_T>0}=1$, $R(X_T)$ and $R(X_t)$ given by (2.17) and $X(T)$ by (2.18).

The above formulas can be used in the numerical procedure to compute the mean-variance and hedging portfolios. The key is to simulate the M trajectories of $\zeta_{t,s}$, $H_{t,s}^r$ and $H_{t,s}^\lambda$ which according to Ito's lemma follow

$$\begin{aligned}
d\zeta_{t,s} &= -\zeta_{t,s}\left(r(s,V_s)ds + \lambda^T(s,V_s)dz(s)\right),\\
dH_{t,s}^r &= \partial_2 r(s,V_s) D_t V_s ds,\\
dH_{t,s}^\lambda &= \left(dz(s) + \lambda(s,V_s)\,ds\right)^T \partial_2\lambda(s,V_s) D_t V_s ds,
\end{aligned}$$

where

$$\begin{aligned}
H_{t,s}^r &= \int_t^s D_t r_v dv,\\
H_{t,s}^\lambda &= \int_t^s \left(dz(v) + \lambda(v,V_v)\,dv\right)^T D_t\lambda_v.
\end{aligned}$$

The simulation can be done using an Euler scheme that discretizes the time interval in N points. We can use the set of M estimates $\zeta_{t,s}^{N,i}$, $H_{t,s}^{r,N,i}$, $H_{t,s}^{\lambda,N,i}$ and $X_T^{N,i}$, $i = 1, 2, ..., M$, of $\zeta_{t,s}$, $H_{t,s}^{r}$, $H_{t,s}^{\lambda}$ and X_T to provide the estimates of the functions $a(\cdot)^{\top}$, $b(\cdot)^{\top}$ and $c(\cdot)$, which for HARA utility with the subsistence level $\widehat{X}$ are equal to

$$
\begin{aligned}
a(\cdot)^{\top} &\equiv \frac{1}{M}\sum_{i=1}^{M}\zeta_{t,T}^{N,i}\frac{X_T^{N,i}}{X_t}\left(1 - R(X_T^{N,i})^{-1}\right)H_{t,s}^{r,N,i},\\
b(\cdot)^{\top} &\equiv \frac{1}{M}\sum_{i=1}^{M}\zeta_{t,T}^{N,i}\frac{X_T^{N,i}}{X_t}\left(1 - R(X_T^{N,i})^{-1}\right)\int_t^T (dz_s + \lambda_s ds)^{\top} H_{t,s}^{\lambda,N,i},\\
c(\cdot) &\equiv \frac{1}{M}\sum_{i=1}^{M}\zeta_{t,T}^{N,i}\frac{X_T^{N,i}}{X_t}\frac{R\left(X_t\right)}{R\left(X_T^{N,i}\right)}.
\end{aligned}
$$

When the utility is the CRRA function given by

$$u(x) = \frac{1}{\gamma}x^{\gamma}$$

then the Arrow-Pratt measure of relative risk aversion is constant and equal to $R(x) = 1 - \gamma$. As the non-negativity constraint is not binding for power utility we have $1_{X_T>0} = 1$. Accounting additionally for the budget constraint results in $c(\cdot) = 1$. The expressions for $a(\cdot)^{\top}$ and $b(\cdot)^{\top}$ then simplify to

$$a(\cdot)^{\top} \equiv \frac{\gamma}{\gamma-1}E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\int_t^T D_t r_s ds\right), \tag{2.19}$$

$$b(\cdot)^{\top} \equiv \frac{\gamma}{\gamma-1}E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\int_t^T (dz_s + \lambda_s ds)^{\top} D_t\lambda_s\right). \tag{2.20}$$

For CRRA utility the terminal wealth is given by

$$X(T) = \frac{\zeta_{t,T}^{\frac{1}{\gamma-1}}X(t)}{E_t\left(\zeta_{t,T}^{\frac{\gamma}{\gamma-1}}\right)}.$$

Substituting into (2.19)-(2.20) and simplifying we end up with

$$
\begin{aligned}
a(\cdot)^{\top} &\equiv \frac{\gamma}{\gamma-1}\frac{E_t\left(\zeta_{t,T}^{\frac{\gamma}{\gamma-1}}\int_t^T D_t r_s ds\right)}{\left[E_t\left(\zeta_{t,T}^{\frac{\gamma}{\gamma-1}}\right)\right]},\\
b(\cdot)^{\top} &\equiv \frac{\gamma}{\gamma-1}\frac{E_t\left(\zeta_{t,T}^{\frac{\gamma}{\gamma-1}}\int_t^T (dz_s + \lambda_s ds)^{\top} D_t\lambda_s\right)}{\left[E_t\left(\zeta_{t,T}^{\frac{\gamma}{\gamma-1}}\right)\right]}.
\end{aligned}
$$

In case of CRRA utility the simulation of optimal portfolios can also be done using an Euler scheme that discretizes the time interval in N points. We can use the set of M estimates $\zeta_{t,s}^{N,i}$, $H_{t,s}^{r,N,i}$ and $H_{t,s}^{\lambda,N,i}$, $i = 1, 2, ..., M$, to construct the estimates $\left(\zeta_{t,s}^{N,i}\right)^{\frac{\gamma}{\gamma-1}} H_{t,s}^{r,N,i}$ and $\left(\zeta_{t,s}^{N,i}\right)^{\frac{\gamma}{\gamma-1}} H_{t,s}^{\lambda,N,i}$ of $\left(\zeta_{t,s}^{\frac{\gamma}{\gamma-1}}\right) H_{t,s}^{r}$ and $\left(\zeta_{t,s}^{\frac{\gamma}{\gamma-1}}\right) H_{t,s}^{\lambda}$. Averaging over M provides estimates of the functions $a(\cdot)^T$ and $b(\cdot)^T$ in the hedges, which for CRRA utility are equal to

$$\widehat{a}(\cdot)^T \equiv \frac{\gamma}{\gamma-1} \frac{\sum_{i=1}^{M} \left(\zeta_{t,T}^{N,i}\right)^{\frac{\gamma}{\gamma-1}} H_{t,T}^{r,N,i}}{\sum_{i=1}^{M} \left(\zeta_{t,T}^{N,i}\right)^{\frac{\gamma}{\gamma-1}}},$$

$$\widehat{b}(\cdot)^T \equiv \frac{\gamma}{\gamma-1} \frac{\sum_{i=1}^{M} \left(\zeta_{t,T}^{N,i}\right)^{\frac{\gamma}{\gamma-1}} H_{t,T}^{\lambda,N,i}}{\sum_{i=1}^{M} \left(\zeta_{t,T}^{N,i}\right)^{\frac{\gamma}{\gamma-1}}}.$$

2.4 Financial market models

We consider a financial market which is arbitrage-free, complete and frictionless. Let $z(t) = [z(t), z_r(t)]^T$, $t \geq 0$, denote standard two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Evolution of the interest rate is described by[1]

$$dr(t) = a\,(b - r(t))\,dt - \sigma_r\,(r(t))^{\eta}\,dz_r. \tag{2.21}$$

We focus on two models for the financial market. First, we choose $a, b, \sigma_r > 0$ and $\eta = 0$ which results in the Vasicek [1977] model and second we select $a, b, \sigma_r > 0$ and $\eta = \frac{1}{2}$ which yields the CIR [1985] model.

There are three assets available in the market, a stock index with price at time t equal to $S(t)$, a bond maturing at T with price $B^T(t)$, and a money market account, $B(t)$. Following Bajeux-Besnainou et al. [2003] ($\eta = 0$) and Deelstra et al. [2000] ($\eta = 0.5$) we specify their price processes as

$$\begin{aligned} dS(t) &= S(t)\left[r(t)dt + \sigma_1\,(dz + \lambda dt) + \sigma_2\,(r(t))^{\eta}\,(dz_r + \lambda_r\,(r(t))^{\eta}\,dt)\right], \\ dB_T(t) &= B_T(t)\left[r(t)dt + \sigma_T(t)\,(dz_r + \lambda_r\,(r(t))^{\eta}\,dt)\right], \\ dB(t) &= B(t)\left[r(t)dt\right]. \end{aligned}$$

When $\eta = 0$ the instantaneous volatility of the T-maturity bond is given by

$$\sigma_T(t) = \sigma_r a^{-1}\left(1 - \exp(-a(T-t))\right), \tag{2.22}$$

while when $\eta = \frac{1}{2}$ this volatility equals

$$\sigma_T(t) = \sigma_r h(T-t)\sqrt{r(t)}, \tag{2.23}$$

[1] Following Bajeux-Besnainou et al. [2003] we use $-\sigma_r$ instead of $+\sigma_r$. As noted therein this assures positive volatilities of bonds and typically observed negative correlation between interest rates and bond as well as stock prices.

with

$$\begin{aligned} h(s) &= 2\left(\exp(\delta s)-1\right)\left[\delta-(a-\sigma_r\lambda_r)+\exp(\delta s)\left(\delta+a-\sigma_r\lambda_r\right)\right]^{-1}, \\ \delta &= \sqrt{\left(a-\sigma_r\lambda_r\right)^2+2\sigma_r^2}. \end{aligned}$$

In the above formulas λ and $\lambda_r\,(r(t))^\eta$ denote stock and bond market prices of risk respectively, while σ_1 and σ_2 stand for stock index volatilities. Parameters λ, λ_r, σ_1 and σ_2 are positive constants. Let σ and λ stand for the volatility matrix and the vector of market prices of risk respectively

$$\begin{aligned} \sigma &= \begin{bmatrix} \sigma_1 & \sigma_2(r(t)^\eta) \\ 0 & \sigma_T\,(t) \end{bmatrix}, \\ \lambda &= \begin{bmatrix} \lambda \\ \lambda_r\,(r(t)^\eta) \end{bmatrix}. \end{aligned}$$

For the needs of portfolio simulation referred to in more detail in the previous section we discretize the continuous-time processes using the Euler scheme as follows: $dt=\frac{1}{n}$, $dz\tilde{\ }N\left(0,\frac{1}{n}\right)$, $dz_r\tilde{\ }N\left(0,\frac{1}{n}\right)$.

2.5 Formulas for optimal portfolios

2.5.1 Analytical

Formulas for the proportion of stock and $T-$bond in the portfolio replicating optimal final wealth for an investor with the objective to maximize expected CRRA utility from terminal wealth in the model of Bajeux-Besnainou et al. [2003] are given therein as

$$\pi(t)=\begin{bmatrix} \pi^1(t) \\ \pi^2(t) \end{bmatrix}=\begin{bmatrix} \frac{\lambda}{(1-\gamma)\sigma_1} \\ \frac{\lambda_r\sigma_1-\lambda\sigma_2}{(1-\gamma)\sigma_1\sigma_T(t)}-\frac{\gamma}{1-\gamma} \end{bmatrix},$$

where $\pi^1(t)$ denotes the fraction invested in stock and $\pi^2(t)$ is the fraction invested in bond. In the above formulas mean-variance demands, $\pi^{MV}(t)$, are calculated from

$$\pi^{MV}(t)=\frac{\left(\sigma^T\right)^{-1}\lambda}{1-\gamma}.$$

It is straightforward to obtain the mean-variance demands as

$$\pi^{MV}(t)=\begin{bmatrix} \pi^{1,MV}(t) \\ \pi^{2,MV}(t) \end{bmatrix}=\begin{bmatrix} \frac{\lambda}{(1-\gamma)\sigma_1} \\ \frac{\lambda_r\sigma_1-\lambda\sigma_2}{(1-\gamma)\sigma_1\sigma_T(t)} \end{bmatrix}.$$

Consequently, the hedging demands are given by

$$\pi^{H}(t)=\begin{bmatrix} \pi^{1,H}(t) \\ \pi^{2,H}(t) \end{bmatrix}=\begin{bmatrix} 0 \\ -\frac{\gamma}{1-\gamma} \end{bmatrix}.$$

Because in the model in question both market prices of risk are constant, hedging is needed exclusively against the interest rate risk and is performed solely by the bond.

Formulas for the proportion of stock and $T-$bond in the portfolio replicating wealth optimal for CRRA investor in the model of financial market as described in Deelstra et al. [2000] are given therein as

$$\pi(t) = \begin{bmatrix} \pi^1(t) \\ \pi^2(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{(1-\gamma)\sigma_1} \\ \frac{\lambda_r\sigma_1 - \lambda\sigma_2}{(1-\gamma)\sigma_1\sigma_r h(T-t)} + k\left(T-t, \frac{\gamma}{1-\gamma}\right) h^{-1}(T-t) \end{bmatrix},$$

where

$$k(s,c) = -c\frac{(\exp(\alpha s)-1)\left(2+\lambda_r^2(1+c)\right)}{\alpha - a - c\lambda_r\sigma_r + (\alpha + a + c\lambda_r\sigma_r)\exp(\alpha s)},$$

with

$$\alpha = \sqrt{a^2 + 2\sigma_r^2\mu}$$

and

$$\mu = -c\left(1 + \frac{\lambda_r^2}{2} - \frac{\lambda_r a}{\sigma_r}\right).$$

We calculate the mean-variance demands as

$$\pi^{MV}(t) = \begin{bmatrix} \pi^{1,MV}(t) \\ \pi^{2,MV}(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{(1-\gamma)\sigma_1} \\ \frac{\lambda_r\sigma_1 - \lambda\sigma_2}{(1-\gamma)\sigma_1\sigma_r h(T-t)} \end{bmatrix}$$

and consequently the hedging demands are

$$\pi^{H}(t) = \begin{bmatrix} \pi^{1,H}(t) \\ \pi^{2,H}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{k\left(T-t, \frac{\gamma}{1-\gamma}\right)}{h(T-t)} \end{bmatrix}.$$

In the considered model the stock market price of risk is constant and thus no hedging against it is required. In turn, bond market price of risk given by $\lambda_r\sqrt{r(t)}$ is a function of interest rate and as such the hedging against bond market price of risk reduces in fact to the hedging of interest rate risk. Though in both models of the financial market stock as well as bond returns are correlated with changes in interest rate, hedging against these changes is done solely by the asset whose volatility term is perfectly correlated with the increment of Brownian motion driving the interest rate. It results in no stock entering the hedging portfolio.

Formulas for the proportion of stock and $T-$bond in the portfolio replicating optimal final wealth for an investor with objective to maximize expected HARA utility from terminal wealth in the model of Bajeux-Besnainou et al. [2003] are given therein as

$$\pi(t) = \begin{bmatrix} \pi^1(t) \\ \pi^2(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\gamma\sigma_1}\left(1 - \frac{B_T(t)\widehat{X}}{X(t)}\right) \\ \frac{\lambda_r\sigma_1 - \lambda\sigma_2}{\gamma\sigma_1\sigma_T(t)}\left(1 - \frac{B_T(t)\widehat{X}}{X(t)}\right) + \left(1 - \frac{1}{\gamma}\right) + \frac{B_T(t)\widehat{X}}{\gamma X(t)} \end{bmatrix}.$$

In the above formulas mean-variance demands, $\pi^{MV}(t)$, are calculated from

$$\pi^{MV}(t) = \left(\sigma^T\right)^{-1} \frac{\lambda}{R(X_t)} c(t, \lambda, r).$$

It is straightforward to obtain the mean-variance demands as

$$\pi^{MV}(t) = \begin{bmatrix} \pi^{1,MV}(t) \\ \pi^{2,MV}(t) \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\gamma\sigma_1}\left(1 - \frac{B_T(t)\widehat{X}}{X(t)}\right) \\ \frac{\lambda_r\sigma_1 - \lambda\sigma_2}{\gamma\sigma_1\sigma_T(t)}\left(1 - \frac{B_T(t)\widehat{X}}{X(t)}\right) \end{bmatrix}.$$

Consequently, the hedging demands are given by

$$\pi^{H}(t) = \begin{bmatrix} \pi^{1,H}(t) \\ \pi^{2,H}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \left(1 - \frac{1}{\gamma}\right) + \frac{B_T(t)\widehat{X}}{\gamma X(t)} \end{bmatrix}.$$

Again as it was in the model with CRRA utility both market prices of risk are constant and so the hedging is needed exclusively against the interest rate risk and is performed solely by the bond. In contrast to CRRA portfolios the interest rate hedging for HARA utility with the subsistence level is combined with the liability hedging to ensure the subsistence level.

Analytical formulas for optimal portfolios in the model with HARA utility and CIR [1985] interest rate dynamics are not available yet to the best of our knowledge. It is our task to simulate them in the subsequent sections of this chapter.

2.5.2 Malliavin calculus

Instead of solving for optimal portfolios as done in Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] we can simulate the mean-variance and hedging demands according to the approach of Detemple et al. [2003]. In the models of financial markets as described in section 2.4 the simulation of hedging portfolios is done according to the following guidelines. First, the simulation of Malliavin derivatives of market price of risk and interest rate is performed. The Malliavin derivative of the stock market price of risk equals zero, $D_{1t}\lambda_{1s} = D_{2t}\lambda_{1s} = 0$, because the stock market price of risk is constant, $\lambda_{1s} = \lambda$. The bond market price of risk is a function of interest rate, $\lambda_{2s} = \lambda_r r(t)^{\eta}$. Because the interest rate, $r(t)$, is driven only by the Brownian motion $z_r(t)$ the bond market price of risk is insensitive to the perturbations in the path of $z(t)$ and consequently the Malliavin derivative of λ_{2s} with respect to $z(t)$ equals zero, $D_{1t}\lambda_{2s} = 0$. The dependence of bond market price of risk on the interest rate implies in turn from the chain rule of Malliavin calculus that $D_{2t}\lambda_{2s} = \partial_2\lambda_{2s}(s, V(s))D_{2t}r_s$. Given the kind of relation between λ_{2s} and $r(t)$ we end up with $D_{2t}\lambda_{2s} = \lambda_r\eta\left(r(t)\right)^{\eta-1} D_{2t}r_s$. Consequently, even though the bond market price of risk is not constant it is still a function of interest rate and hence the only hedging is needed against the interest rate risk. The dynamics of the Malliavin derivative of the interest rate in our models of the financial market are from diffusion (2.16) with

$$\begin{aligned} \mu^V &= a(b - r(t)), \\ \sigma^V &= -\sigma_r \left(r(t)\right)^{\eta}, \end{aligned}$$

given by

$$dD_{2t}r_s = -aD_{2t}r_s ds - \sigma_r \eta \left(r(t)\right)^{\eta-1} D_{2t}r_s dz_r,$$

with the initial condition $lim_{s\to t}D_{2t}r_s = -\sigma_r \left(r(t)\right)^{\eta}$. Summarizing, all the above implies that when the utility belongs to HARA and financial market models are as in section 2.4 the interest rate risk hedging portfolio can be written as

$$\pi^H(t) = -\frac{\left(\sigma(\cdot)^{\top}\right)^{-1}}{\gamma X_t} E_t\left(\zeta_{t,T}\left[X_T(\gamma-1)+\widehat{X}\right] MDE^T\right), \tag{2.24}$$

where the Malliavin derivative expression (MDE)

$$MDE \equiv \int_t^T [0, D_{2t}r_s]ds + \int_t^T \left(dz_s + \lambda_s ds\right)^{\top} \begin{bmatrix} 0 \\ \lambda_r \eta \left(r(t)\right)^{\eta-1} \end{bmatrix} [0, D_{2t}r_s]$$

can be rewritten as

$$MDE^T \equiv \begin{bmatrix} 0 \\ \int_t^T D_{2t}r_s ds + \int_t^T \left(dz_r + \lambda_r \left(r(t)\right)^{\eta} ds\right) \lambda_r \eta \left(r(t)\right)^{\eta-1} D_{2t}r_s \end{bmatrix}. \tag{2.25}$$

Adequately, for CRRA utility with the subsistence level the hedging demand is given by

$$\pi^H(t) = -\frac{\gamma}{\gamma-1}\left(\sigma(\cdot)^{\top}\right)^{-1} \frac{E_t\left(\zeta_{t,T}^{\frac{\gamma}{\gamma-1}} MDE^T\right)}{E_t\left(\zeta_{t,T}^{\frac{\gamma}{\gamma-1}}\right)}. \tag{2.26}$$

2.5.3 From Malliavin calculus to analytical formulas

In this section we show how the analytical formulas from section 2.5.1 in the model of Bajeux-Besnainou et al. [2003] are derived using the Malliavin calculus approach from section 2.3.2 and 2.5.2. We start with the derivation for CRRA utility.

In the model of Bajeux-Besnainou et al. [2003] the Malliavin derivative of short term interest rate satisfies the differential equation

$$dD_{2t}r_s = -aD_{2t}r_s ds,$$

with the initial condition $lim_{s\to t}D_{2t}r_s = -\sigma_r$. The solution to this equation which has the integral form

$$D_{2t}r_s = -\int_t^s aD_{2t}r_v dv,$$

is given by

$$D_{2t}r_s = -\sigma_r e^{-a(s-t)}.$$

Hence, the integral $\int_t^T D_{2t}r_s ds$ has the form

$$\int_t^T D_{2t}r_s ds = -\sigma_r \int_t^T e^{-a(s-t)} ds = \frac{\sigma_r}{a}\left(e^{-a(T-t)} - 1\right).$$

Hence,

$$\int_t^T D_{2t}r_s ds = -\frac{\sigma_r}{a}(1 - e^{-a(T-t)}),$$

which after the comparison with formula (2.22) gives

$$\int_t^T D_{2t}r_s ds = -\sigma_T(t). \tag{2.27}$$

Because the market prices of risk are constant in the model of Bajeux-Besnainou et al. [2003] the vector $\partial\lambda$ is the vector of zeros. Substituting (2.27) into formula (2.25) we end up with

$$MDE^T = \left[0,\ \int_t^T D_{2t}r_s ds\right]^T = [0,\ -\sigma_T(t)]^T.$$

Hence, the formula for the interest rate risk hedging component takes the form

$$\pi^H(t) = -\frac{\gamma}{\gamma-1}\left(\sigma(\cdot)^T\right)^{-1}\begin{bmatrix} 0 \\ -\sigma_T(t) \end{bmatrix},$$

Inverting σ^T and substituting into the above formula yields

$$\pi^H(t) = -\frac{\gamma}{\gamma-1}\frac{1}{\sigma_1\sigma_T(t)}\begin{bmatrix} \sigma_T(t) & 0 \\ -\sigma_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} 0 \\ -\sigma_T(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\gamma}{1-\gamma} \end{bmatrix}.$$

This is the same as the expression for the interest rate risk hedging component for CRRA utility from section 2.5.1.

We showed that when the utility is HARA function with the subsistence level $\widehat{X}$, the relative risk risk aversion coefficient is given by (2.17) and the optimal terminal wealth by (2.18). For such a form of the utility function the subsistence level is guaranteed and hence $1_{X_T>0} = 1$. Substituting for the relative risk aversion coefficient and $1_{X_T>0} = 1$ we obtain

$$\begin{aligned}
a(\cdot)^T &\equiv E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\left(1 - \left(\frac{\gamma X_T}{X_T - \widehat{X}}\right)^{-1}\right)\int_t^T D_t r_s ds\right), \\
b(\cdot)^T &\equiv [0,\ 0], \\
c(\cdot) &\equiv E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\frac{\left(\frac{\gamma X_t}{X_t - \widehat{X}}\right)^{-1}}{\left(\frac{\gamma X_T}{X_T - \widehat{X}}\right)^{-1}}\right).
\end{aligned}$$

After some simplifications we get

$$\begin{aligned}
a(\cdot)^{T} &= \left[0, -\sigma_T(t)\left(1-\frac{1}{\gamma}\right)E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\right) - \frac{\sigma_T(t)\widehat{X}}{\gamma X_t}E_t\left(\zeta_{t,T}\right)\right], \\
b(\cdot)^{T} &= [0,\ 0], \\
c(\cdot) &= E_t\left(\zeta_{t,T}\left(\frac{X_T-\widehat{X}}{X_t-\widehat{X}}\right)\right).
\end{aligned}$$

From the budget constraint $E_t\left(\zeta_{t,T}\frac{X_T}{X_t}\right)=1$ and from the risk-neutral valuation principle $E_t\left(\zeta_{t,T}\right)=B_T(t)$. Hence,

$$a(\cdot)^{T} = \left[0, -\sigma_T(t)\left(1-\frac{1}{\gamma}\right) - \frac{\sigma_T(t)\widehat{X}}{\gamma X_t}B_T(t)\right].$$

As a result we have

$$\pi^{MV}(t) = \left(\sigma^{T}\right)^{-1}\frac{\lambda}{R(X_t)}c(t,\lambda,r) = \begin{bmatrix} \frac{\lambda}{\sigma_1} \\ \frac{-\lambda\sigma_2+\lambda_r\sigma_1}{\sigma_1\sigma_T(t)} \end{bmatrix}\left(\frac{1}{\gamma} - \frac{\widehat{X}B_T(t)}{\gamma X_t}\right)$$

and

$$\pi^{H}(t) = -\frac{1}{\sigma_1\sigma_T(t)}\begin{bmatrix} \sigma_T(t) & 0 \\ -\sigma_2 & \sigma_1 \end{bmatrix}\begin{bmatrix} 0 \\ -\sigma_T(t)\left(1-\frac{1}{\gamma}\right) - \frac{\sigma_T(t)\widehat{X}}{\gamma X_t}B_T(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \left(1-\frac{1}{\gamma}\right) + \frac{\widehat{X}B_T(t)}{\gamma X_t} \end{bmatrix}.$$

This is the same as the expression for the interest rate risk hedging component for HARA utility from section 2.5.1.

2.6 Simulation experiment

In the experiment which results are presented in the next section we assume $T=1$ and $R(X_t)=R(X_T)=1-\gamma=2$ for CRRA utility. For HARA utility $T=1$ and $R(X_t)$ and $R(X_T)$ are given in (2.17). In the part of simulation experiment referring to HARA utility we also assume the subsistence level of 1000 and initial wealth of 1100 (HARA 1), 1300 (HARA 2) and 1500 (HARA 3). The values of remaining parameters describing financial markets from section 2.4 are given in Table 2.1. We choose the estimates for the interest rate processes from Chan et al. [1992]. The initial rate, r, is set at the mean of the sample used therein. As far as market prices of risk and volatilities of assets in the model with $\eta=0$ are concerned we follow Boulier et al. [2001]. These values imply 1-year-bond volatility of 1.83% and 1-year-bond risk premium of 0.28% at the initial date, stock volatility of 20.00% and stock risk premium of 6.00%. The correlation between stock and bond markets measured as $\rho_{S,\ r} \equiv \sigma_2\sigma_S^{-1}$ with

$\sigma_S \equiv \left(\sigma_1^2 + \sigma_2^2\right)^{0.5}$ equals 30.00%. For the model with $\eta = 0.5$ we choose σ_1 and σ_2 such that $\rho_{S,\, r} \equiv \sigma_2 \sigma_S^{-1} \sqrt{r}$ and $\sigma_S \equiv \left(\sigma_1^2 + \sigma_2^2 r\right)^{0.5}$ are as in $\eta = 0$ model. In turn, λ_r makes the initial prices of bonds with maturities $T = 1, 2, ..., 10$ years differ as little as possible in terms of mean squared error from the prices in the $\eta = 0$ model. Then, we choose λ such that the stock risk premium at initial date is as in the $\eta = 0$ model. These values imply 1-year-bond volatility of 2.15% and 1-year-bond premium of 0.15% at the initial date, stock volatility of 20% and stock premium of 6%.

Table 2.1. Parameters of the financial market

in %	Vasicek	CIR	in %	Vasicek	CIR
$a^{\blacklozenge}$	17.79	23.39	$\sigma_1^{\blacktriangle}$	19.08	19.08
$b^{\clubsuit}$	8.66	8.08	$\sigma_2^{\blacktriangle}$	6.00	23.15
$\sigma_r^{\maltese}$	2.00	8.54	$\lambda^{\blacktriangledown}$	26.64	23.21
$r^{\spadesuit}$	6.715	6.715	$\lambda_r^{\blacktriangledown}$	15.28	26.18

The numbers in Table 2.1 stand for $^{\blacklozenge}$speed of interest rate reversion, $^{\clubsuit}$mean to which the interest rate reverts, $^{\maltese}$volatility of interest rate process, $^{\spadesuit}$initial value of interest rate, $^{\blacktriangle}$volatilities of stock price process, $^{\blacktriangledown}$market prices of risk

2.7 Results

Using the formulas from section 2.5.1 we calculate the demand for risky assets in total and hedging continuous-time portfolios at the initial date for two models of the financial market described in 2.4 and for both CRRA and HARA utility except when the utility belongs to HARA and the interest rate dynamics is given by the CIR [1985] model. In this case the optimal continuous-time policy is not known to the best of our knowledge. These results are reported in the upper panels of Tables 2.2-2.5. Discretizing the continuous-time processes and using the formulas in 2.5.2 we simulate adequate portfolios by considering 3^n paths, where $n = 2,\ 3, ..., 11$. The time step $\frac{1}{n}$ as well as the number of paths are chosen to make the simulated results as much comparable as possible with the outcome from the tree model we refer to in the next chapter. Simulated portfolios are reported in the lower panels of Tables 2.2-2.5. The outcome for CRRA is displayed in Table 2.2. The results for HARA are given in 2.3-2.5.

Numbers from Table 2.2 reveal first that the total demand for bond is above the continuous-time counterpart in both models of financial markets. This difference is however much greater for $\eta = 0.5$ than $\eta = 0$ model. The surplus in relative terms diminishes in the simulated portfolios from 351.94% to 2.56% and from 61.54% to 00.16% adequately in $\eta = 0.5$ and $\eta = 0$ models. Second, as far as the hedging demand for bond is concerned it is still overestimated in the simulated portfolios. The surplus of simulated hedging demand for bond reduces in adequate models $\eta = 0.5$ and $\eta = 0$ from 259.10% to 1.88% and from 293.38% to 0.75%. In general the convergence is reasonable.

Table 2.2. Continuous-time and simulated total (π^1, π^2) and hedging ($\pi^{1,H}, \pi^{2,H}$) portfolios for CRRA

	Continuous-time portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
	.6981	2.3837	.0000	.5000	.6082	.3726	.0000	.5061
	Simulated portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
n	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
2	.6981	3.8506	.0000	1.9669	.6082	1.6840	.0000	1.8175
3	.6981	3.3856	.0000	1.5019	.6082	1.3064	.0000	1.4399
4	.6981	3.0917	.0000	1.2080	.6082	1.0152	.0000	1.1487
5	.6981	2.8921	.0000	1.0084	.6082	.8585	.0000	.9920
6	.6981	2.7484	.0000	.8647	.6082	.7252	.0000	.8587
7	.6981	2.6403	.0000	.7566	.6082	.6289	.0000	.7624
8	.6981	2.5561	.0000	.6723	.6082	.5483	.0000	.6818
9	.6981	2.4886	.0000	.6049	.6082	.4802	.0000	.6137
10	.6981	2.4334	.0000	.5497	.6082	.4279	.0000	.5614
11	.6981	2.3874	.0000	.5037	.6082	.3821	.0000	.5156

Numbers in Table 2.2 are the fractions of risky assets in total, π^k, and hedging, $\pi^{k,H}$, $k = 1, 2$, optimal CRRA portfolios at the initial date for two models of the financial market respectively with Vasicek ($\eta = 0$) and CIR ($\eta = 0.5$) dynamics.

As far as the results for HARA utility are concerned the following conclusions can be formulated. First, the convergence of simulated portfolios is reasonable in the model with Vasicek [1977] interest rate. Second, for the model with CIR [1985] interest rate dynamics the fraction of wealth invested in stock equals around 9% for HARA1, 17% for HARA 2 and 23% for HARA3. The fraction of wealth invested in bond equals around 24% for HARA1, 20% for HARA2 and 17-18% for HARA3. Hence, the higher the ratio of initial wealth to the subsistence level the less fraction of wealth is invested in bond. This means that in the institutions with certain liability to be met the importance of hedging against interest rate risk decreases with the growing financial security of the institution. As in the case of CRRA the share of stock does not contain the hedging components. Hedging demand for bond is higher than total demand implying that in the mean-variance portfolio short position is held in bond.

Table 2.3. Simulated total (π^1, π^2) and hedging $(\pi^{1,H}, \pi^{2,H})$ portfolios for HARA1

	Continuous-time portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
	.1056	1.2093	.0000	.9243	NA	NA	NA	NA
	Simulated portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
n	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
2	.1056	1.0855	.0000	.8006	.0919	.2851	.0000	.3047
3	.1056	1.0515	.0000	.7666	.0919	.2424	.0000	.2620
4	.1056	1.1028	.0000	.8178	.0919	.2353	.0000	.2550
5	.1056	1.1216	.0000	.8367	.0919	.2374	.0000	.2570
6	.1056	1.1588	.0000	.8738	.0919	.2375	.0000	.2572
7	.1056	1.1661	.0000	.8812	.0919	.2382	.0000	.2579
8	.1056	1.1760	.0000	.8910	.0919	.2396	.0000	.2592
9	.1056	1.1876	.0000	.9026	.0919	.2424	.0000	.2620
10	.1056	1.1938	.0000	.9088	.0919	.2420	.0000	.2617
11	.1056	1.1993	.0000	.9143	.0919	.2422	.0000	.2619

Table 2.4. Simulated total (π^1, π^2) and hedging $(\pi^{1,H}, \pi^{2,H})$ portfolios for HARA2

	Continuous-time portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
	.1968	1.3900	.0000	.8591	NA	NA	NA	NA
	Simulated portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
n	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
2	.1968	1.2868	.0000	.7558	.1714	.2445	.0000	.2811
3	.1968	1.2572	.0000	.7263	.1714	.2061	.0000	.2428
4	.1968	1.3003	.0000	.7693	.1714	.1996	.0000	.2362
5	.1968	1.3158	.0000	.7849	.1714	.2017	.0000	.2384
6	.1968	1.3447	.0000	.8163	.1714	.2013	.0000	.2380
7	.1968	1.3535	.0000	.8225	.1714	.2020	.0000	.2387
8	.1968	1.3617	.0000	.8308	.1714	.2032	.0000	.2399
9	.1968	1.3745	.0000	.8406	.1714	.2058	.0000	.2424
10	.1968	1.3768	.0000	.8459	.1714	.2055	.0000	.2421
11	.1968	1.3815	.0000	.8505	.1714	.2056	.0000	.2422

Table 2.5. Simulated total (π^1, π^2) and hedging ($\pi^{1,H}, \pi^{2,H}$) portfolios for HARA3

	Continuous-time portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
	.2636	1.5225	.0000	.8112	NA	NA	NA	NA
	Simulated portfolios							
	model $\eta = 0$				model $\eta = 0.5$			
n	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
2	.2636	1.4348	.0000	.7235	.2296	.2150	.0000	.2641
3	.2636	1.4085	.0000	.6971	.2296	.1796	.0000	.2287
4	.2636	1.4454	.0000	.7341	.2296	.1734	.0000	.2296
5	.2636	1.4585	.0000	.7472	.2296	.1756	.0000	.2247
6	.2636	1.4857	.0000	.7744	.2296	.1749	.0000	.2240
7	.2636	1.4911	.0000	.7798	.2296	.1755	.0000	.2246
8	.2636	1.4982	.0000	.7869	.2296	.1766	.0000	.2257
9	.2636	1.5066	.0000	.7953	.2296	.1790	.0000	.2281
10	.2636	1.5112	.0000	.7999	.2296	.1787	.0000	.2278
11	.2636	1.5152	.0000	.8039	.2296	.1787	.0000	.2279

Numbers in Tables 2.3-2.5 are the fractions of risky assets in total, π^k, and hedging, $\pi^{k,H}$, $k = 1, 2$, optimal HARA portfolios with initial wealth of 1100, 1300 and 1500 adequately at the initial date for two models of the financial market respectively with Vasicek ($\eta = 0$) and CIR ($\eta = 0.5$) dynamics.

2.8 Conclusions

In this chapter we introduced the concept of interest rate risk hedging in the optimal portfolio choice. We investigated the convergence of simulated CRRA optimal portfolios and their interest rate risk hedging components to the continuous-time limits in the models with Vasicek [1977] and CIR[1985] interest rate dynamics. We note that both total and hedging demand for bond is overestimated in the simulated portfolios in comparison with the continuous-time demand. We also simulated HARA optimal portfolios and spotted the convergence of these portfolios to the continuous-time equivalents in the model with Vasicek [1977] interest rate dynamics. As far as the model with HARA utility and CIR [1985] interest rate is concerned the analytical formulas for optimal portfolios are not available to the best of our knowledge. Hence, their simulation can give us an indication about investors' optimal policy. We find that with an increase of the ratio of initial wealth to the subsistence level the stock investment is increased, while the bond investment diminishes in both total and hedging portfolios. Hence, with an increase of the safety buffer more is invested in the most risky asset and less interest rate risk hedging is performed. Regarding HARA utility and Vasicek [1977] interest rate dynamics we also showed how to obtain the analytical formulas for optimal portfolios using the Malliavin calculus.

3 Discrete-time interest rate risk hedging in the optimal portfolio choice

3.1 Introduction

Even though the continuous-time framework of the previous chapter is more convenient to work with, the discrete-time setting is more intuitive. That is why, in order to enhance the insight into the concept of interest rate risk hedging in the context of the optimal portfolio choice we switch now to the discrete-time framework. By doing this we want to make more accessible the explanation of how the mean-variance and hedging components in the wealth replicating portfolio do their jobs. Our second goal is the identification of the behaviour of the discrete-time equivalents of expressions employing Malliavin derivatives of the state variables in continuous time. As in the preceding chapter also in this chapter an investment opportunity set is driven by the stochastic interest rate.

In our attempt to achieve the above objectives we start simple and choose the binomial three-period tree model. However, there is a price to be paid for this simplicity. By specifying the binomial tree as our model of the risky asset and of the stochastic interest rate dynamics we limit ourselves to the case of perfect correlation between the returns on the risky asset and the changes in the interest rate. That is why the second model considered, namely the trinomial two-period tree, is of higher complexity. Shifting to the trinomial model allows us to introduce the second risky asset which is imperfectly correlated with the interest rate. In this trinomial model the continuous-time dynamics of the risky assets and the interest rate are approximated as proposed in He [1990]. For his approximation He [1991] finds for a given continuous-time economy a sequence of discrete-time economies in which securities prices as well as optimal consumption and portfolio policies converge to their corresponding continuous-time limits. Because the two period model gives us only a vague idea about the behaviour of Malliavin derivative term we finally investigate the multiperiod trinomial tree model. We can say that the latter is the most general among the three models.

We proceed by deriving the discrete-time formulas for the initial fraction of wealth invested in the risky assets in the mean-variance and hedging portfolios of a constant relative risk aversion (CRRA) investor. We present them in the form as analogous as possible to the continuous-time expressions of Detemple et al. [2003] from the previous chapter employing the Malliavin derivative component. As a result in the binomial three-period model and the trinomial two-period model we attain a number of intuitive properties of the mean-variance and hedging portfolios. In the multiperiod trinomial model we find that the discrete -time equivalent of the term coming into view in the simulated hedging portfolio formula and involving the Malliavin derivative of interest rate is independent of the number of time steps. We also show that with an increase of the number of these steps the constituents of the optimal portfolio from the tree models indeed converge to their continuous-time counterparts.

The chapter is organized as follows. In section 3.2 we consider the theoretical aspects of the three-period binomial model. This section also provides the reader with an example. In section 3.3 we switch to the two-period trinomial tree. This section contains an example as well. Section 3.4 is devoted to the introduction of discrete-time policies in the multiperiod trinomial tree model. In this section the multiperiod tree example is given. Section 3.5 concludes the chapter.

3.2 Three-period binomial model

3.2.1 Theory

In the binomial model each binomial branch describes a one-period movement of the risky asset and the interest rate. It consists of one predecessor node, (i, j), from which two successor nodes, $(i+1, 2j-1)$ and $(i+1, 2j)$, emanate. We consider the risky asset whose returns are perfectly correlated with changes in the interest rate. This correlation is assumed negative and the risky asset can be given a "quasi" bond interpretation. Such a binomial branch is shown in Figure 3.1.

$$
\begin{array}{ccc}
 & & (i+1, 2j-1) \\
 & \nearrow & \\
(i, j) & & \\
 & \searrow & \\
 & & (i+1, 2j)
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & S_{1,i,j}u^1_{i,j}, r_{i,j}d^r_{i,j} \\
 & \nearrow & \\
S_{1,i,j}, r_{i,j} & & \\
 & \searrow & \\
 & & S_{1,i,j}d^1_{i,j}, r_{i,j}u^r_{i,j}
\end{array}
$$

Figure 3.1. Left panel - structure of the binomial branch, right panel - risky asset and interest rate movement

We refer to the initial node of the tree by $(0, 1)$. The initial price of the risky asset and the initial level of the interest rate are denoted respectively by $S_{1,0,1} \equiv S_1$ and $r_{0,1} \equiv r$. Other notation used in Figure 3.1 is the following: $u^1_{i,j}$, up return on the risky asset from the node (i, j) to the node $(i+1, 2j-1)$, $d^1_{i,j}$, down return on the risky asset from the node (i, j) to the node $(i+1, 2j)$, $u^r_{i,j}$, up change in the interest rate from the node (i, j) to the node $(i+1, 2j)$, $d^r_{i,j}$, down change in the interest rate from the node (i, j) to the node $(i+1, 2j-1)$. We denote by $p_{i+1,2j-1}$ the probability of reaching the node $(i+1, 2j-1)$ from the node (i, j). With reference to the wealth process we use $x_{0,1} \equiv x$ to denote the initial wealth and $x_{i,j}$ to stand for the wealth in node (i, j). The number of time steps n between 0 and maturity equals 3. The exposition below is partly based on Pliska [2002].

The investor is assumed to have a power utility function over the terminal wealth x_T, $u(x_T) = \frac{1}{\gamma}x_T^\gamma$, $\gamma < 1$, $\gamma \neq 0$ and with the relative risk aversion coefficient of $1 - \gamma$. Power utility satisfies the conditions of being a differentiable, concave and strictly increasing function. We consider the problem of maximizing the expected utility from the terminal wealth constrained by the budget restriction. The objective function is

$$Eu(x_T) = \frac{1}{\gamma} E x_T^{\gamma},$$

where $E(\cdot)$ denotes an expectation under the physical probability measure. In case of the binomial three-period model and with the notation introduced before this objective is given by

$$Eu(x_T) = \frac{1}{\gamma} \textstyle\sum_{j=1}^{2} p_{1,j} \sum_{l=0}^{1} p_{2,2j-l} \sum_{z=0}^{1} p_{3,2(2j-l)-z} x_{3,2(2j-l)-z}^{\gamma}.$$

From the valuation of the contingent claims we know that if Q is any martingale measure, then for every trading strategy Π one has

$$E_Q \frac{x_T}{B_T} = x, \tag{3.1}$$

where B_T denotes the discount factor and $E_Q(\cdot)$ is the expectation under the martingale measure Q. A martingale measure Q is a probability measure such that $Q(\omega) > 0$ for all ω and the discounted price process of the risky asset is a martingale under Q. The latter property writes

$$q_{i+1,2j-1} \frac{S_{1,i,j} u_{i,j}^{1}}{1 + \frac{r_{i,j}}{3}} + (1 - q_{i+1,2j-1}) \frac{S_{1,i,j} d_{i,j}^{1}}{1 + \frac{r_{i,j}}{3}} = S_{1,i,j},$$

where $q_{i+1,2j-1}$ denotes the martingale probability of reaching the node $(i+1, 2j-1)$ from the node (i,j) and is given by

$$q_{i+1,2j-1} = \frac{(1 + \frac{r_{i,j}}{3}) - d_{i,j}^{1}}{u_{i,j}^{1} - d_{i,j}^{1}}.$$

In order to guarantee that $q_{i+1,2j-1} \geq 0$ and $1 - q_{i+1,2j-1} \geq 0$ we choose $u_{i,j}^{1}$ and $d_{i,j}^{1}$ such that $u_{i,j}^{1} \geq 1 + \frac{r_{i,j}}{3}$ and $d_{i,j}^{1} \leq 1 + \frac{r_{i,j}}{3}$ in the example, which follows. The martingale measure Q for the multiperiod model is equal to the product of the single period probabilities along the path from the node at time zero to the node corresponding to $(3,\omega)$. As $d_{i,j}^{1} \leq 1 + \frac{r_{i,j}}{3} \leq u_{i,j}^{1}$ there exists a martingale probability measure Q and our securities market model is viable. Because this martingale measure is unique the model is also complete. If the model is complete, then we may indeed make use of (3.1) by the risk-neutral valuation principle.

In the three-period binomial model and with the riskless asset as the discount factor the condition of discounted portfolio value process being a martingale writes

$$\textstyle\sum_{j=1}^{2} \frac{q_{1,j}}{\left(1 + \frac{r}{3}\right)} \sum_{l=0}^{1} \frac{q_{2,2j-l}}{\left(1 + \frac{r_{1,j}}{3}\right)} \sum_{z=0}^{1} \frac{q_{3,2(2j-l)-z}}{\left(1 + \frac{r_{2,2j-l}}{3}\right)} x_{3,2(2j-l)-z} = x.$$

Hence, the problem we consider has the form

$$\begin{array}{rcl}\frac{1}{\gamma}\sum_{j=1}^{2} p_{1,j}\sum_{l=0}^{1} p_{2,2j-l}\sum_{z=0}^{1} p_{3,2(2j-l)-z}x_{3,2(2j-l)-z}^{\gamma} & \rightarrow & max, \\ \sum_{j=1}^{2}\frac{q_{1,j}}{\left(1+\frac{r}{3}\right)}\sum_{l=0}^{1}\frac{q_{2,2j-l}}{\left(1+\frac{r_{1,j}}{3}\right)}\sum_{z=0}^{1}\frac{q_{3,2(2j-l)-z}}{\left(1+\frac{r_{2,2j-l}}{3}\right)}x_{3,2(2j-l)-z} & = & x.\end{array}$$

The martingale approach which we use in this section to solve such a problem is a two-step procedure. First, we identify the optimal terminal wealth, that is the value of the terminal wealth which maximizes the expected utility function over the set of feasible terminal wealths. Second, we treat this optimal wealth as an attainable contingent claim and solve for the self-financing strategy that replicates the claim. A contingent claim is a random variable representing a payoff at the maturity. A contingent claim is said to be attainable or marketable if there exists some trading strategy Π, called the replicating portfolio, such that the value of the contingent claim at the maturity equals the value of the replicating portfolio. The self-financing property of the trading strategy means that no money is added to or withdrawn from the portfolio between times 0 and the maturity. Hence, any change in the portfolio's value is due to a gain or loss in the investments.

The above problem is the static optimization problem which can be solved by the Lagrange multiplier rule. We form the Lagrangian and find that the first order conditions have the form

$$p_{1,j}p_{2,2j-l}p_{3,3(3j-l)-z}x_{3,2(2j-l)-z}^{\gamma-1}+\lambda\frac{q_{1,j}}{\left(1+\frac{r}{3}\right)}\frac{q_{2,2j-l}}{\left(1+\frac{r_{1,j}}{3}\right)}\frac{q_{3,2(2j-l)-z}}{\left(1+\frac{r_{2,2j-l}}{3}\right)}=0.$$

From this condition the terminal wealth equals

$$x_{3,2(2j-l)-z}=\left[-\lambda\frac{q_{1,j}q_{2,2j-l}q_{3,2(2j-l)-z}}{\left(1+\frac{r}{3}\right)\left(1+\frac{r_{1,j}}{3}\right)\left(1+\frac{r_{2,2j-l}}{3}\right)p_{1,j}p_{2,2j-l}p_{3,2(2j-l)-z}}\right]^{\frac{1}{\gamma-1}}.$$

We introduce the symbol $\zeta_{0,3}^{j,l,z}$ to stand for

$$\zeta_{0,3}^{j,l,z}=\frac{q_{1,j}q_{2,2j-l}q_{3,2(2j-l)-z}}{\left(1+\frac{r}{3}\right)\left(1+\frac{r_{1,j}}{3}\right)\left(1+\frac{r_{2,2j-l}}{3}\right)p_{1,j}p_{2,2j-l}p_{3,2(2j-l)-z}},$$

and hence $\zeta_{0,3}^{j,l,z}$ in the discrete time plays the same role as the state price density from (2.6) in the continuous time. Then, we may write

$$x_{3,2(2j-l)-z}=\left[-\lambda\zeta_{0,3}^{j,l,z}\right]^{\frac{1}{\gamma-1}}.$$

In order to identify $[-\lambda]^{\frac{1}{\gamma-1}}$ we substitute the terminal wealth into the budget equation and we get

$$[-\lambda]^{\frac{1}{\gamma-1}}=\frac{x}{\sum_{j=1}^{2}\frac{q_{1,j}}{\left(1+\frac{r}{3}\right)}\sum_{l=0}^{1}\frac{q_{2,2j-l}}{\left(1+\frac{r_{1,j}}{3}\right)}\sum_{z=0}^{1}\frac{q_{3,2(2j-l)-z}}{\left(1+\frac{r_{2,2j-l}}{3}\right)}\left[\zeta_{0,3}^{j,l,z}\right]^{\frac{1}{\gamma-1}}},$$

which is equivalent to

$$[-\lambda]^{\frac{1}{\gamma-1}} = \frac{x}{E\left(\zeta_{0,3}^{\rho}\right)},$$

with

$$E\left(\zeta_{0,3}^{\rho}\right) = \sum_{j=1}^{2} p_{1,j} \sum_{l=0}^{1} p_{2,2j-l} \sum_{z=0}^{1} p_{3,2(2j-l)-z} \left(\zeta_{0,3}^{j,l,z}\right)^{\rho},$$

where $\rho = \frac{\gamma}{\gamma-1}$. Substituting the expression for $[-\lambda]^{\frac{1}{\gamma-1}}$ into the expression for $x_{3,2(2j-l)-z}$ we end up with the expression

$$x_{3,2(2j-l)-z} = \frac{x\left(\zeta_{0,3}^{j,l,z}\right)^{\frac{1}{\gamma-1}}}{E\left(\zeta_{0,3}^{\rho}\right)}.$$

Using the property that discounted wealth is a martingale under the equivalent martingale measure we come up with

$$x_{2,2j-l} = q_{3,2(2j-l)-1} \frac{x_{3,2(2j-l)-1}}{1+\frac{r_{2,2j-l}}{3}} + q_{3,2(2j-l)} \frac{x_{3,2(2j-l)}}{1+\frac{r_{2,2j-l}}{3}}.$$

In a similar manner one can obtain $x_{1,j}$.

The replicating trading strategy needs to satisfy in every period the following system of two constraints

$$\begin{aligned} \pi_{i,j}^{1} u_{i,j}^{1} + \left(1-\pi_{i,j}^{1}\right)\left(1+\frac{r_{i,j}}{3}\right) &= \frac{x_{i+1,2j-1}}{x_{i,j}}, \\ \pi_{i,j}^{1} d_{i,j}^{1} + \left(1-\pi_{i,j}^{1}\right)\left(1+\frac{r_{i,j}}{3}\right) &= \frac{x_{i+1,2j}}{x_{i,j}}, \end{aligned}$$

where $\pi_{i,j}^{1}$ is the optimal fraction of wealth invested in the risky asset in node (i,j).

By substituting appropriately for $x_{3,2(2j-l)-z}$, $x_{2,2j-l}$ and $x_{1,j}$ in the above system we end up with

$$\begin{aligned} \pi_{0,1}^{1} &= \left[E\left(\zeta_{0,3}^{\rho}\varsigma_{0,1}\right)\right]\left[\left(u_{0,1}^{1}-d_{0,1}^{1}\right)E\left(\zeta_{0,3}^{\rho}\right)\right]^{-1}, \\ \pi_{1,j}^{1} &= \left[E\left(\left(\zeta_{1,3}^{j}\right)^{\rho}\varsigma_{1,2}^{j}\right)\right]\left[\left(u_{1,j}^{1}-d_{1,j}^{1}\right)E\left(\zeta_{1,3}^{j}\right)^{\rho}\right]^{-1}, \\ \pi_{2,2j-l}^{1} &= \left[E\left(\left(\zeta_{2,3}^{j,l}\right)^{\rho}\varsigma_{2,3}^{j,l}\right)\right]\left[\left(u_{2,2j-l}^{1}-d_{2,2j-l}^{1}\right)E\left(\zeta_{2,3}^{j,l}\right)^{\rho}\right]^{-1}, \end{aligned}$$

where

$$\zeta_{0,3}^{\rho}\varsigma_{0,1} = \begin{cases} \left(\zeta_{0,3}^{j,l,z}\right)^{\rho}\frac{1+\frac{r}{3}}{q_{1,j}}, & \text{for } j=1,\ \forall l,\ \forall z, \\ -\left(\zeta_{0,3}^{j,l,z}\right)^{\rho}\frac{1+\frac{r}{3}}{q_{1,j}}, & \text{for } j=2,\ \forall l,\ \forall z, \end{cases}$$

$$\left(\zeta_{1,3}^{j}\right)^{\rho}\varsigma_{1,2}^{j} = \begin{cases} \left(\zeta_{1,3}^{j,l,z}\right)^{\rho}\frac{1+\frac{r_{1,j}}{3}}{q_{2,2j-l}}, & \text{for } l=1,\ \forall z, \\ -\left(\zeta_{1,3}^{j,l,z}\right)^{\rho}\frac{1+\frac{r_{1,j}}{3}}{q_{2,2j-l}}, & \text{for } l=0,\ \forall z, \end{cases}$$

$$\left(\zeta_{2,3}^{j,l}\right)^{\rho}\varsigma_{2,3}^{j,l} = \begin{cases} \left(\zeta_{2,3}^{j,l,z}\right)^{\rho}\frac{1+\frac{r_{2,2j-l}}{3}}{q_{3,2(2j-l)-z}}, & \text{for } z=1, \\ -\left(\zeta_{2,3}^{j,l,z}\right)^{\rho}\frac{1+\frac{r_{2,2j-l}}{3}}{q_{3,2(2j-l)-z}}, & \text{for } z=0, \end{cases}$$

$$\begin{aligned} E\left(\zeta_{1,3}^{j}\right)^{\rho} &= \textstyle\sum_{l=0}^{1} p_{2,2j-l}\sum_{z=0}^{1} p_{3,2(2j-l)-z}\left(\zeta_{1,3}^{j,l,z}\right)^{\rho}, \\ E\left(\zeta_{2,3}^{j,l}\right)^{\rho} &= \textstyle\sum_{z=0}^{1} p_{3,2(2j-l)-z}\left(\zeta_{2,3}^{j,l,z}\right)^{\rho}, \\ \zeta_{1,3}^{j,l,z} &= \frac{q_{2,2j-l}q_{3,2(2j-l)-z}}{\left(1+\frac{r_{1,j}}{3}\right)\left(1+\frac{r_{2,2j-l}}{3}\right)p_{2,2j-l}p_{3,2(2j-l)-z}}, \\ \zeta_{2,3}^{j,l,z} &= q_{3,2(2j-l)-z}\left[\left(1+\frac{r_{2,2j-l}}{3}\right)p_{3,2(2j-l)-z}\right]^{-1}. \end{aligned}$$

Two components of the demand for risky asset are distinguished, namely the mean-variance, $\pi_{i,j}^{MV}$, and the interest rate risk hedging, $\pi_{i,j}^{H}$, demand. We refer to the mean-variance portfolio as a portfolio which replicates the return on one-period optimal wealth. In other words, this is the optimal portfolio when the only uncertainty affecting the outcome of investment stems from the movement of the risky asset. We understand the interest rate risk hedging portfolio as a portfolio whose return when combined with the mean-variance return amounts to the return on total optimal wealth. In other words, this is the portfolio invested in a portfolio separate from the mean-variance portfolio to attain the total optimal wealth when an additional uncertainty comes from interest rate movements.

The myopic investor demanding $\pi_{i,j}^{MV}$ solves one period problems of the form

$$\begin{aligned} \frac{1}{\gamma}\textstyle\sum_{l=0}^{1} p_{i+1,2j-l}x_{i+1,2j-l}^{\gamma} &\rightarrow max, \\ \textstyle\sum_{l=0}^{1}\frac{q_{i+1,2j-l}}{1+\frac{r_{i,j}}{3}}x_{i+1,2j-l} &= x_{i,j}. \end{aligned}$$

By analogy, we derive the mean-variance portfolio to be

$$\pi_{i,j}^{1,MV} = \left[E\left(\left(\zeta_{i,i+1}^{j}\right)^{\rho}\varsigma_{i,i+1}^{1,j}\right)\right]\left[\left(u_{i,j}^{1}-d_{i,j}^{1}\right)E\left(\left(\zeta_{i,i+1}^{j}\right)^{\rho}\right)\right]^{-1},$$

where

$$\begin{array}{rcl}
\left(\zeta_{i,i+1}^{j}\right)^{\rho}\varsigma_{i,i+1}^{1,j} & = & \left\{\begin{array}{rll} & \left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\frac{1+\frac{r_{i,j}}{3}}{q_{i+1,2j-l}}, & \text{for } l=1, \\ - & \left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\frac{1+\frac{r_{i,j}}{3}}{q_{i+1,2j-l}}, & \text{for } l=0, \end{array}\right. \\
E\left(\left(\zeta_{i,i+1}^{j}\right)^{\rho}\right) & = & \sum_{l=0}^{1} p_{i+1,2j-l}\left(\zeta_{i,i+1}^{j,l}\right)^{\rho}, \\
\zeta_{i,i+1}^{j,l} & = & q_{i+1,2j-l}\left[\left(1+\frac{r_{i,j}}{3}\right)p_{i+1,2j-l}\right]^{-1}.
\end{array}$$

When $i=0$ we use $E\left(\zeta_{0,1}^{\rho}\right)$ instead of $E\left(\zeta_{0,1}^{1}\right)^{\rho}$.

We define the demand for risky asset in the hedging portfolio as a difference between the total and mean-variance demand

$$\begin{array}{rcl}
\pi_{0,1}^{1,H} & = & \pi_{0,1}^{1}-\pi_{0,1}^{1,MV}, \\
\pi_{1,j}^{1,H} & = & \pi_{1,j}^{1}-\pi_{1,j}^{1,MV}.
\end{array}$$

Because $\pi_{2,2j-l}^{1}=\pi_{2,2j-l}^{1,MV}$, $\pi_{2,2j-l}^{1,H}=0$. We derive $\pi_{0,1}^{1,H}$ and $\pi_{1,j}^{1,H}$ as

$$\begin{array}{rcl}
\pi_{0,1}^{1,H} & = & \left[E\left(\left(\zeta_{0,3}^{\rho}\right)\left(1+\frac{r}{3}\right)\left[E\left(\zeta_{0,1}^{\rho}\right)\right]^{-1}\xi_{0,1}^{1,1}\right)\right]\left[\left(u_{0,1}^{1}-d_{0,1}^{1}\right)E\left(\zeta_{0,3}^{\rho}\right)\right]^{-1}, \\
\pi_{1,j}^{1,H} & = & \left[E\left(\left(\zeta_{1,3}^{j}\right)^{\rho}\left(1+\frac{r_{1,j}}{3}\right)\left[E\left(\zeta_{1,2}^{\rho}\right)\right]^{-1}\xi_{1,2}^{1,j}\right)\right]\left[\left(u_{1,j}^{1}-d_{1,j}^{1}\right)E\left(\zeta_{1,3}^{j}\right)^{\rho}\right]^{-1},
\end{array}$$

where

$$\begin{array}{rcl}
\xi_{0,1}^{1,1} & = & \left\{\begin{array}{rll} & p_{1,2}\left(\zeta_{0,1}^{1,0}\right)^{\rho}\left(\frac{1}{q_{1,1}}+\frac{1}{q_{1,2}}\right), & \text{for } j=1,\ \forall l, \\ - & p_{1,1}\left(\zeta_{0,1}^{1,1}\right)^{\rho}\left(\frac{1}{q_{1,1}}+\frac{1}{q_{1,2}}\right), & \text{for } j=2,\ \forall l, \end{array}\right. \\
\xi_{1,2}^{1,j} & = & \left\{\begin{array}{rll} & p_{2,2j}\left(\zeta_{1,2}^{2j}\right)^{\rho}\left(\frac{1}{q_{2,2j-1}}+\frac{1}{q_{2,2j}}\right), & \text{for } l=1, \\ - & p_{2,2j-1}\left(\zeta_{1,2}^{2j-1}\right)^{\rho}\left(\frac{1}{q_{2,2j-1}}+\frac{1}{q_{2,2j}}\right), & \text{for } l=0. \end{array}\right.
\end{array}$$

3.2.2 Example

In this numerical example we assume $S_1=100$, $r=0.05$, $x=5000$, $u_{i,j}^{1}=1.2$, $d_{i,j}^{1}=0.8$, $u_{i,j}^{r}=1.5$, $d_{i,j}^{r}=0.5$, $p_{i+1,2j-1}=0.7$, $\gamma=-1$ for all (i,j).

In Table 3.1 below we report in column (3) the gross returns on the optimal wealth between (i, j) and $(i+1, 2j-1)$ (upper number) and between (i, j) and $(i+1, 2j)$ (lower number) while in columns (4)-(5) adequate returns on wealth mean-variance and hedging components. The demand for the risky asset in the replicating portfolio and its two constituents is shown in columns (6)-(8).

To interpret the results in Table 3.1 we note that the assumptions of constant gross returns on the risky asset and of the constant real probabilities imply that the drift rate, μ, and the volatility, σ, of the risky asset are constant on the tree. Consequently, the market price of risk defined by

$$\theta_{i,j} \equiv \frac{\mu - r_{i,j}}{\sigma},$$

is a linear function of the interest rate. Hence, the two hedging demands for the risky asset, that is the interest rate risk hedging demand, $\pi_{i,j}^{IR}$, and the market price of risk hedging demand, $\pi_{i,j}^{MPR}$, arise due to the same kind of risk stemmimg from the interest rate dynamics. As a result we may define the hedging portfolio $\pi_{i,j}^{H}$, $\pi_{i,j}^{H} \equiv \pi_{i,j}^{IR} + \pi_{i,j}^{MPR}$, as a portfolio which provides a hedge against the interest rate adverse movement.

Table 3.1. Portfolio returns and portfolio composition

Node	Rate	Portfolio return			Portfolio composition		
		total	MV	hedging	$\pi_{i,j}$	$\pi_{i,j}^{MV}$	$\pi_{i,j}^{H}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0, 1	.05				.8505	.8444	.0061
1, 1	.025	1.1726	1.1715	.0011	.9444	.9439	.0006
1, 2	.075	.8324	.8337	-.0013	.7518	.7442	.0075
2, 1	.0125	1.1894	1.1892	.0001	.9937	.9937	.0000
2, 2	.0375	.8116	.8117	-.0001	.8942	.8942	.0000
2, 3	.0375	1.1566	1.1552	.0013	.8942	.8942	.0000
2, 4	.1125	.8559	.8575	-.0017	.5910	.5910	.0000
3, 1		1.1988	1.1988	.0000			
3, 2		.8013	.8013	.0000			
3, 3; 3, 5		1.1802	1.1802	.0000			
3, 4; 3, 6		.8225	.8225	.0000			
3, 7		1.1335	1.1335	.0000			
3, 8		.8971	.8971	.0000			

In column (3) we report the gross returns on the optimal wealth between the preceding node and the node given in column (1), while in columns (4)-(5) adequate returns on wealth mean-variance and hedging components. The demand for the risky asset in the replicating portfolio and its two constituents is shown in columns (6)-(8) respectively.

Several intuitive findings can be deducted from the numbers in Table 3.1. First, the portfolios held in the last period (formed in nodes $(2,1)$, $(2,2)$, $(2,3)$ and $(2,4)$) contain only the mean-variance component. This is because at the beginning of the last period all the uncertainty concerning the interest rate that can influence the outcome of investment is revealed. The sole uncertainty still left is connected with the dynamics of the risky asset. But in this case the return on optimal wealth can be attained by the investment in the mean-variance portfolio. For the opposite reason, portfolios created in nodes $(0,1)$, $(1,1)$ and $(1,2)$ constitute of non-zero hedging component in addition to the mean-variance element.

As far as the initial portfolio is concerned investing in its mean-variance part guarantees the return of 1.1715 in $(1,1)$ and of .8337 in $(1,2)$. The remainder of .0011 in $(1,1)$ and of -.0013 in $(1,2)$ is attained by investing in the hedging portfolio in which the long position of .0061 is taken in the risky asset. Hence, the interest rate hedging portfolio seems to enlarge the return on the mean-variance portfolio in the node of rate's fall and to reduce it in the node of rate's rise. This can be explained as follows. The return on the riskless asset in the second period is lower when $(1,1)$ and higher when $(1,2)$ is realized than the return in the first period. Because in the mean-variance portfolio long position is held in the riskless asset in both periods this entails a decreased gain on the riskless investment in $(1,1)$ and an increased gain in $(1,2)$. Thus, as expected from the interest rate risk hedging component the return on the hedging portfolio magnifies the return on the mean-variance portfolio in $(1,1)$ and diminishes it in $(1,2)$.

Finally, the magnitude of returns on the hedging portfolios formed at the same time, but in different states differs substantially. While the returns in $(2,1)$ and $(2,2)$ on the portfolio created in $(1,1)$ are almost negligible the returns in $(2,3)$ and $(2,4)$ on the portfolio rebalanced in $(1,2)$ are significant. This observation can be explained as follows. In our model the rate between two consecutive nodes either increases or decreases by 50%. This means that between $i=0$ and $i=2$ the rate either rises by 125% when it follows $(0,1)-(1,2)-(2,4)$ path or decreases by 75% when $(0,1)-(1,1)-(2,1)$ path is followed. The remaining paths yield the rate's increase of 25%. Hence, it is not surprising that the interest rate risk hedging works more intensively in the part of the tree where bigger fluctuations of the interest rate are present.

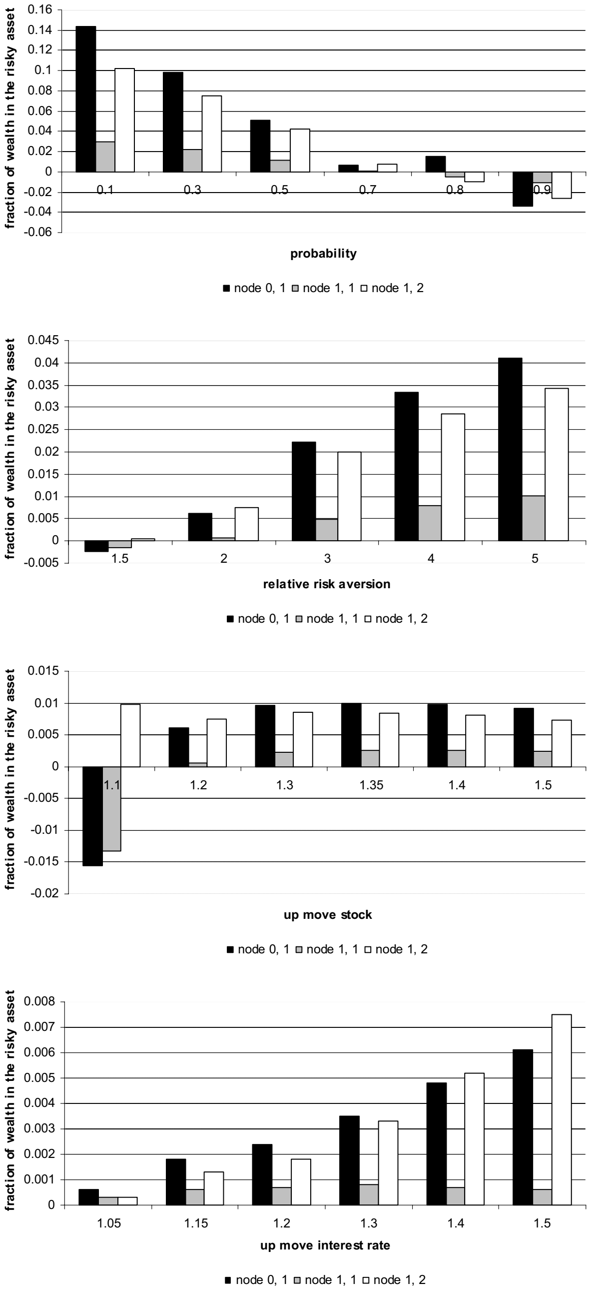


Figure 3.2 Sensitivity of the hedging portfolio composition to the probability (the first row), the risk aversion coefficient (the second row), the size of stock's up move (the third row), the size of interest rate's up move (the fourth row)

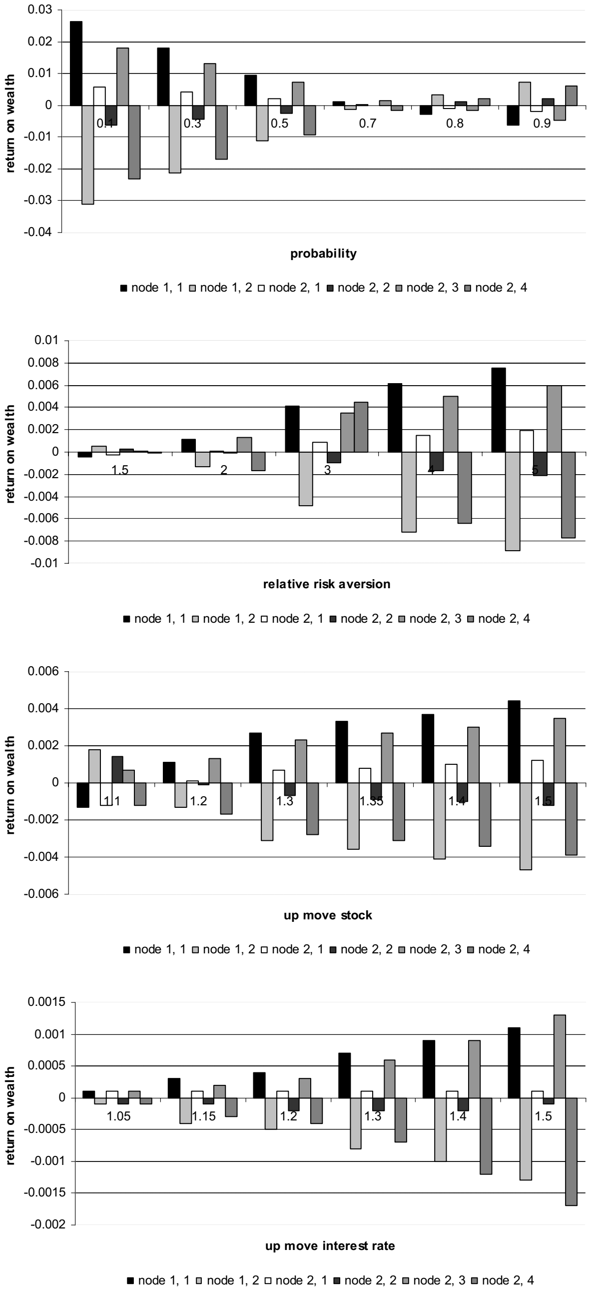


Figure 3.3 Sensitivity of the hedging portfolio return to the probability (the first row), the risk aversion coefficient (the second row), the size of stock's up move (the third row), the size of interest rate's up move (the fourth row)

We note that with the growing probability of positive stock return (and the negative change in the interest rate) the share of the risky asset in the replicating portfolio decreases and reverses sign when p is big enough. When p is not larger than .7 this share is positive and our previous findings do not alter. When p exceeds .7 short position is taken in the risky asset in the hedging portfolio. The return on this portfolio reduces the return on mean-variance portfolio in the node of rate's fall and enlarges it in the node of rate's rise. This is not counterintuitive when we observe that for large p short position is held in the riskless asset in the mean-variance portfolio in both periods. This entails decreased cost of borrowing if $(1,1)$ is realized and increased cost if $(1,2)$ happens. Thus, the return on the hedging portfolio diminishes the return on mean-variance portfolio in more favourable nodes and amplifies it in less favourable ones.

Increasing the risk aversion coefficient above 2 does not change the previous findings either. However, by decreasing it below 2 we note the negative hedging demand for the risky asset arising in nodes $(0,1)$ and $(1,1)$. As a consequence we find that the return on the hedging portfolio reduces the return on the mean-variance portfolio in the node of rate's fall and enlarges it in the node of rate's rise when the node is the successor of $(0,1)$ or $(1,1)$. The pattern here and its explanation are similar to the case of large p. For $\gamma = -0.5$ short position is taken in the riskless asset in the mean-variance portfolio in both periods. Hence, the return on the hedging portfolio diminishes the return on the mean-variance portfolio when smaller cost of borrowing is incurred and amplifies it when larger cost occurs.

An analogous situation takes place with the varying size of up movement of the risky asset price. Decreasing $u^1_{i,j}$ below 1.2 causes the negative hedging demand for risky asset in nodes $(0,1)$ and $(1,1)$. The consequences and explanation agree with those concerning the case of low risk aversion coefficient. We also examine a wide range of changes in the interest rate from $u^r_{i,j} = 1.05$ to $u^r_{i,j} = 1.5$. In all the cases the share of risky asset in the replicating portfolio is positive and increasing with $u^r_{i,j}$. Hence, altering the size of changes in the interest rate does not modify our previous findings.

3.3 Two-period trinomial model

3.3.1 Theory

In this section we switch to a two-period trinomial model. In the model in question a trinomial branch describes a one-period dynamics of two risky assets and the interest rate. It consists of one predecessor node, (i,j), from which three successor nodes, $(i+1,3j-2)$, $(i+1,3j-1)$ and $(i+1,3j)$, emanate. We consider several cases. In all of them the returns on risky asset 2 are perfectly negatively correlated with changes in the interest rate and so this asset can be still given a "quasi" bond interpretation. The returns on risky asset 1 are also negatively, but imperfectly correlated (or uncorrelated) with the interest rate and thus this asset can be seen as a stock. Switching from bi- to trinomial model allows us to increase the number of risky assets to two while remaining at the same time in complete market model. This will be best seen when we calculate the unique martingale measure probability. The trinomial branch is shown in Figure 3.4.

$$
\begin{array}{ccc}
 & & (i+1,3j-2) \\
 & \diagup & \\
(i,j) & — & (i+1,3j-1) \\
 & \diagdown & \\
 & & (i+1,3j)
\end{array}
\qquad
\begin{array}{ccc}
 & & S_{1,i,j}u^1_{i,j}, S_{2,i,j}u^2_{i,j}, r_{i,j}u^r_{i,j} \\
S_{1,i,j} & \diagup & \\
S_{2,i,j} & — & S_{1,i,j}m^1_{i,j}, S_{2,i,j}m^2_{i,j}, r_{i,j}m^r_{i,j} \\
r_{i,j} & \diagdown & \\
 & & S_{1,i,j}d^1_{i,j}, S_{2,i,j}d^2_{i,j}, r_{i,j}d^r_{i,j}
\end{array}
$$

Figure 3.4. Left panel - structure of the trinomial branch, right panel - risky assets and interest rate movement

Notation used with reference to the trinomial tree is analogous to the binomial tree case. In particular, the initial price of risky asset k writes $S_{k,0,1} \equiv S_k$, $k = 1,2$. Other notation used in Figure 3.4 is the following: $u^k_{i,j}$, the return on the risky asset k from (i,j) to $(i+1,3j-2)$, $m^k_{i,j}$, the return on the risky asset k from (i,j) to $(i+1,3j-1)$, $d^k_{i,j}$, the return on the risky asset k from (i,j) to $(i+1,3j)$, $u^r_{i,j}$, the change in the interest rate from (i,j) to $(i+1,3j-2)$, $m^r_{i,j}$, the change in the interest rate from (i,j) to $(i+1,3j-1)$, $d^r_{i,j}$, the change in the interest rate from (i,j) to $(i+1,3j)$. We denote by $p_{i+1,3j-l}$ the probability of reaching node $(i+1,3j-l)$ from node (i,j) with $l = 0,1,2$. The number of time steps n between 0 and maturity equals 2.

In this part of the chapter we derive formulas for the wealth replicating portfolio and its mean-variance and hedging components in the two-period trinomial model when the investor has CRRA utility over the terminal wealth, that is $u(x_T) = \frac{1}{\gamma}x_T^\gamma$, $\gamma < 1$, $\gamma \neq 0$, as in the previous section.

As in section 3.2 we consider the problem which compactly can be written as

$$
\begin{aligned}
Eu(x_T) &\rightarrow \max \\
E_Q \frac{x_T}{B_T} &= x,
\end{aligned}
$$

where B_T denotes the discount factor and $E_Q(\cdot)$ is the expectation under the martingale measure Q. A martingale measure Q is a probability measure such that $Q(\omega) > 0$ for all ω and the discounted price processes of the risky assets are martingales under Q. The latter property writes

$$\begin{aligned} q_{i+1,3j-2}\frac{S_{1,i,j}u^1_{i,j}}{1+\frac{r_{i,j}}{2}}+q_{i+1,3j-1}\frac{S_{1,i,j}m^1_{i,j}}{1+\frac{r_{i,j}}{2}}+\left(1-q_{i+1,3j-2}-q_{i+1,3j-1}\right)\frac{S_{1,i,j}d^1_{i,j}}{1+\frac{r_{i,j}}{2}} &= S_{1,i,j},\\ q_{i+1,3j-2}\frac{S_{2,i,j}u^2_{i,j}}{1+\frac{r_{i,j}}{2}}+q_{i+1,3j-1}\frac{S_{2,i,j}m^2_{i,j}}{1+\frac{r_{i,j}}{2}}+\left(1-q_{i+1,3j-2}-q_{i+1,3j-1}\right)\frac{S_{2,i,j}d^2_{i,j}}{1+\frac{r_{i,j}}{2}} &= S_{2,i,j}, \end{aligned}$$

where $q_{i+1,3j-l}$, $l=1,2$, denotes the martingale probability of reaching the node $(i+1,3j-l)$ from the node (i,j) and is given by

$$q_{i+1,3j-l}=\begin{cases} \frac{(1+\frac{r_{i,j}}{2})\left[-\left(m^1_{i,j}-d^1_{i,j}\right)+\left(m^2_{i,j}-d^2_{i,j}\right)\right]+\left(m^1_{i,j}d^2_{i,j}-m^2_{i,j}d^1_{i,j}\right)}{\left(u^1_{i,j}-d^1_{i,j}\right)\left(m^2_{i,j}-d^2_{i,j}\right)-\left(u^2_{i,j}-d^2_{i,j}\right)\left(m^1_{i,j}-d^1_{i,j}\right)}, & \text{for } l=2,\\[2ex] \frac{(1+\frac{r_{i,j}}{2})\left[\left(u^1_{i,j}-d^1_{i,j}\right)-\left(u^2_{i,j}-d^2_{i,j}\right)\right]+\left(d^1_{i,j}u^2_{i,j}-d^2_{i,j}u^1_{i,j}\right)}{\left(u^1_{i,j}-d^1_{i,j}\right)\left(m^2_{i,j}-d^2_{i,j}\right)-\left(u^2_{i,j}-d^2_{i,j}\right)\left(m^1_{i,j}-d^1_{i,j}\right)}, & \text{for } l=1. \end{cases}$$

In the example, which follows we choose $u^k_{i,j}$, $m^k_{i,j}$ and $d^k_{i,j}$, $k=1,2$, such that the conditions $q_{i+1,3j-1}\geq 0$, $q_{i+1,3j-2}\geq 0$ and $1-q_{i+1,3j-2}-q_{i+1,3j-1}\geq 0$ are satisfied. In the two-period trinomial model and with the riskless asset as the discount factor the condition of the discounted portfolio value process being a martingale writes

$$\sum_{j=1}^{3}\frac{q_{1,j}}{\left(1+\frac{r}{2}\right)}\sum_{l=0}^{2}\frac{q_{2,3j-l}}{\left(1+\frac{r_{1,j}}{2}\right)}x_{2,3j-l}=x.$$

Hence, the problem we consider has the form

$$\begin{aligned} \frac{1}{\gamma}\sum_{j=1}^{3}p_{1,j}\sum_{l=0}^{2}p_{2,3j-l}x^{\gamma}_{2,3j-l} &\rightarrow max,\\ \sum_{j=1}^{3}\frac{q_{1,j}}{\left(1+\frac{r}{2}\right)}\sum_{l=0}^{2}\frac{q_{2,3j-l}}{\left(1+\frac{r_{1,j}}{2}\right)}x_{2,3j-l} &= x. \end{aligned}$$

The above problem is the static optimization problem which can be solved by the Lagrange multiplier rule. We form the Lagrangian and find that the first order conditions have the form

$$\sum_{j=1}^{3}p_{1,j}\sum_{l=0}^{2}p_{2,3j-l}x^{\gamma-1}_{2,3j-l}+\lambda\sum_{j=1}^{3}\frac{q_{1,j}}{\left(1+\frac{r}{2}\right)}\sum_{l=0}^{2}\frac{q_{2,3j-l}}{\left(1+\frac{r_{1,j}}{2}\right)}=0.$$

From this condition the terminal wealth equals

$$x_{2,3j-l}=\left[-\lambda\frac{q_{1,j}q_{2,3j-l}}{\left(1+\frac{r}{2}\right)\left(1+\frac{r_{1,j}}{2}\right)p_{1,j}p_{2,3j-l}}\right]^{\frac{1}{\gamma-1}}$$

We introduce the symbol $\zeta^{j,l}_{0,2}$ to stand for

$$\zeta^{j,l}_{0,2}\equiv\left[q_{1,j}q_{2,3j-l}\right]\left[\left(1+\frac{r}{2}\right)\left(1+\frac{r_{1,j}}{2}\right)p_{1,j}p_{2,3j-l}\right]^{-1},$$

and hence $\zeta_{0,2}^{j,l}$ in the discrete time plays the same role as the state price density from (2.6) in the continuous time. Then, we may write

$$x_{2,3j-l} = \left[-\lambda\zeta_{0,2}^{j,l}\right]^{\frac{1}{\gamma-1}}.$$

In order to identify $[-\lambda]^{\frac{1}{\gamma-1}}$ we substitute the terminal wealth into the budget equation and we get

$$[-\lambda]^{\frac{1}{\gamma-1}} = \frac{x}{\sum_{j=1}^{3}\frac{q_{1,j}}{\left(1+\frac{r}{2}\right)}\sum_{l=0}^{2}\frac{q_{2,3j-l}}{\left(1+\frac{r_{1,j}}{2}\right)}\left[\zeta_{0,2}^{j,l}\right]^{\frac{1}{\gamma-1}}},$$

which is equivalent to

$$[-\lambda]^{\frac{1}{\gamma-1}} = \frac{x}{E\left(\zeta_{0,2}^{\rho}\right)},$$

with

$$E\left(\zeta_{0,2}^{\rho}\right) = \sum_{j=1}^{3} p_{1,j}\sum_{l=0}^{2} p_{2,3j-l}\left(\zeta_{0,2}^{j,l}\right)^{\rho},$$

where $\rho = \frac{\gamma}{\gamma-1}$. Substituting the expression for $[-\lambda]^{\frac{1}{\gamma-1}}$ into the expression for $x_{2,3j-l}$ we end up with the expression

$$x_{2,3j-l} = \frac{x\left(\zeta_{0,2}^{j,l}\right)^{\frac{1}{\gamma-1}}}{E\left(\zeta_{0,2}^{\rho}\right)}.$$

Using the property that the discounted wealth is a martingale under the equivalent martingale measure we come up with

$$x_{1,j} = q_{2,3j-2}\frac{x_{2,3j-2}}{1+\frac{r_{1,j}}{2}} + q_{2,3j-1}\frac{x_{2,3j-1}}{1+\frac{r_{1,j}}{2}} + \left(1-q_{2,3j-2}-q_{2,3j-1}\right)\frac{x_{2,3j}}{1+\frac{r_{1,j}}{2}}.$$

The replicating trading strategy needs to satisfy in every period the following system of three constraints

$$\begin{aligned}
\pi_{i,j}^{1}u_{i,j}^{1} + \pi_{i,j}^{2}u_{i,j}^{2} + \left(1-\pi_{i,j}^{1}-\pi_{i,j}^{2}\right)\left(1+\frac{r_{i,j}}{2}\right) &= \frac{x_{i+1,3j-2}}{x_{i,j}},\\
\pi_{i,j}^{1}m_{i,j}^{1} + \pi_{i,j}^{2}m_{i,j}^{2} + \left(1-\pi_{i,j}^{1}-\pi_{i,j}^{2}\right)\left(1+\frac{r_{i,j}}{2}\right) &= \frac{x_{i+1,3j-1}}{x_{i,j}},\\
\pi_{i,j}^{1}d_{i,j}^{1} + \pi_{i,j}^{2}d_{i,j}^{2} + \left(1-\pi_{i,j}^{1}-\pi_{i,j}^{2}\right)\left(1+\frac{r_{i,j}}{2}\right) &= \frac{x_{i+1,3j}}{x_{i,j}},
\end{aligned}$$

where $\pi_{i,j}^{1}$, $\pi_{i,j}^{2}$ are the optimal fractions of wealth invested in the risky asset 1 and 2 respectively in node (i,j).

By substituting appropriately for $x_{2,3j-l}$ and $x_{1,j}$ in the above system we end up with

$$
\begin{aligned}
\pi_{0,1}^{k} &= \left[E\left(\zeta_{0,2}^{\rho}\varsigma_{0,1}^{k,1}\right)\right]\left[\eta_{0,1}E\left(\zeta_{0,2}^{\rho}\right)\right]^{-1}, \\
\pi_{1,j}^{k} &= \left[E\left(\left(\zeta_{1,2}^{j}\right)^{\rho}\varsigma_{1,2}^{k,j}\right)\right]\left[\eta_{1,j}E\left(\left(\zeta_{1,2}^{j}\right)^{\rho}\right)\right]^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_{0,2}^{\rho}\varsigma_{0,1}^{1,1} &= \begin{cases} \left(\zeta_{0,2}^{j,l}\right)^{\rho}\left(m_{0,1}^{2}-d_{0,1}^{2}\right)\frac{1+\frac{r}{2}}{q_{1,j}}, & \text{for } j=1,\ \forall l, \\ -\left(\zeta_{0,2}^{j,l}\right)^{\rho}\left(u_{0,1}^{2}-d_{0,1}^{2}\right)\frac{1+\frac{r}{2}}{q_{1,j}}, & \text{for } j=2,\ \forall l, \\ \left(\zeta_{0,2}^{j,l}\right)^{\rho}\left(u_{0,1}^{2}-m_{0,1}^{2}\right)\frac{1+\frac{r}{2}}{q_{1,j}}, & \text{for } j=3,\ \forall l, \end{cases} \\
\zeta_{0,2}^{\rho}\varsigma_{0,1}^{2,1} &= \begin{cases} -\left(\zeta_{0,2}^{j,l}\right)^{\rho}\left(m_{0,1}^{1}-d_{0,1}^{1}\right)\frac{1+\frac{r}{2}}{q_{1,j}}, & \text{for } j=1,\ \forall l, \\ \left(\zeta_{0,2}^{j,l}\right)^{\rho}\left(u_{0,1}^{1}-d_{0,1}^{1}\right)\frac{1+\frac{r}{2}}{q_{1,j}}, & \text{for } j=2,\ \forall l, \\ -\left(\zeta_{0,2}^{j,l}\right)^{\rho}\left(u_{0,1}^{1}-m_{0,1}^{1}\right)\frac{1+\frac{r}{2}}{q_{1,j}}, & \text{for } j=3,\ \forall l, \end{cases} \\
\left(\zeta_{1,2}^{j}\right)^{\rho}\varsigma_{1,2}^{1,j} &= \begin{cases} \left(\zeta_{1,2}^{j,l}\right)^{\rho}\left(m_{1,j}^{2}-d_{1,j}^{2}\right)\frac{1+\frac{r_{1,j}}{2}}{q_{2,3j-l}}, & \text{for } l=2, \\ -\left(\zeta_{1,2}^{j,l}\right)^{\rho}\left(u_{1,j}^{2}-d_{1,j}^{2}\right)\frac{1+\frac{r_{1,j}}{2}}{q_{2,3j-l}}, & \text{for } l=1, \\ \left(\zeta_{1,2}^{j,l}\right)^{\rho}\left(u_{1,j}^{2}-m_{1,j}^{2}\right)\frac{1+\frac{r_{1,j}}{2}}{q_{2,3j-l}}, & \text{for } l=0, \end{cases} \\
\left(\zeta_{1,2}^{j}\right)^{\rho}\varsigma_{1,2}^{2,j} &= \begin{cases} -\left(\zeta_{1,2}^{j,l}\right)^{\rho}\left(m_{1,j}^{1}-d_{1,j}^{1}\right)\frac{1+\frac{r_{1,j}}{2}}{q_{2,3j-l}}, & \text{for } l=2, \\ \left(\zeta_{1,2}^{j,l}\right)^{\rho}\left(u_{1,j}^{1}-d_{1,j}^{1}\right)\frac{1+\frac{r_{1,j}}{2}}{q_{2,3j-l}}, & \text{for } l=1, \\ -\left(\zeta_{1,2}^{j,l}\right)^{\rho}\left(u_{1,j}^{1}-m_{1,j}^{1}\right)\frac{1+\frac{r_{1,j}}{2}}{q_{2,3j-l}}, & \text{for } l=0, \end{cases} \\
E\left(\left(\zeta_{1,2}^{j}\right)^{\rho}\right) &= \textstyle\sum_{l=0}^{2} p_{2,3j-l}\left(\zeta_{1,2}^{j,l}\right)^{\rho}, \\
\zeta_{1,2}^{j,l} &= q_{2,3j-l}\left[\left(1+\frac{r_{1,j}}{2}\right)p_{2,3j-l}\right]^{-1}, \\
\eta_{i,j} &= \left(d_{i,j}^{1}-m_{i,j}^{1}\right)\left(u_{i,j}^{2}-m_{i,j}^{2}\right)-\left(d_{i,j}^{2}-m_{i,j}^{2}\right)\left(u_{i,j}^{1}-m_{i,j}^{1}\right).
\end{aligned}
$$

Again, two components of the demand for risky assets are distinguished, namely the mean-variance, $\pi_{i,j}^{k,MV}$, and the interest rate risk hedging, $\pi_{i,j}^{k,H}$, demand.

The myopic investor demanding $\pi_{i,j}^{k,MV}$ solves one-period problems of the form

$$
\begin{aligned}
\frac{1}{\gamma}\textstyle\sum_{l=0}^{2} p_{i+1,3j-l}x_{i+1,3j-l}^{\gamma} &\rightarrow max, \\
\textstyle\sum_{l=0}^{2}\frac{q_{i+1,3j-l}}{1+\frac{r_{i,j}}{2}}x_{i+1,3j-l} &= x_{i,j}.
\end{aligned}
$$

By analogy to the total portfolio we derive the mean-variance portfolio to be

$$\pi_{i,j}^{k,MV} = \left[E\left(\left(\zeta_{i,i+1}^{j}\right)^{\rho}\varsigma_{i,i+1}^{k,j}\right)\right]\left[\eta_{i,j}E\left(\left(\zeta_{i,i+1}^{j}\right)^{\rho}\right)\right]^{-1},$$

where

$$\left(\zeta_{i,i+1}^{j}\right)^{\rho}\varsigma_{i,i+1}^{1,j} = \begin{cases} \left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\left(m_{i,j}^{2}-d_{i,j}^{2}\right)\frac{1+\frac{r_{i,j}}{2}}{q_{i+1,3j-l}}, & \text{for } l=2,\\ -\left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\left(u_{i,j}^{2}-d_{i,j}^{2}\right)\frac{1+\frac{r_{i,j}}{2}}{q_{i+1,3j-l}}, & \text{for } l=1,\\ \left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\left(u_{i,j}^{2}-m_{i,j}^{2}\right)\frac{1+\frac{r_{i,j}}{2}}{q_{i+1,3j-l}}, & \text{for } l=0,\end{cases}$$

$$\left(\zeta_{i,i+1}^{j}\right)^{\rho}\varsigma_{i,i+1}^{2,j} = \begin{cases} -\left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\left(m_{i,j}^{1}-d_{i,j}^{1}\right)\frac{1+\frac{r_{i,j}}{2}}{q_{i+1,3j-l}}, & \text{for } l=2,\\ \left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\left(u_{i,j}^{1}-d_{i,j}^{1}\right)\frac{1+\frac{r_{i,j}}{2}}{q_{i+1,3j-l}}, & \text{for } l=1,\\ -\left(\zeta_{i,i+1}^{j,l}\right)^{\rho}\left(u_{i,j}^{1}-m_{i,j}^{1}\right)\frac{1+\frac{r_{i,j}}{2}}{q_{i+1,3j-l}}, & \text{for } l=0,\end{cases}$$

$$E\left(\left(\zeta_{i,i+1}^{j}\right)^{\rho}\right) = \textstyle\sum_{l=0}^{2}p_{i+1,3j-l}\left(\zeta_{i,i+1}^{j,l}\right)^{\rho},$$

$$\zeta_{i,i+1}^{j,l} = q_{i+1,3j-l}\left[\left(1+\frac{r_{i,j}}{2}\right)p_{i+1,3j-l}\right]^{-1}.$$

When $i=0$ we use $E\left(\zeta_{0,1}^{\rho}\right)$ instead of $E\left(\zeta_{0,1}^{1}\right)^{\rho}$.

We define the demand for risky assets in the hedging portfolio as a difference between the total and mean-variance demands

$$\begin{aligned}\pi_{0,1}^{k,H} &= \pi_{0,1}^{k}-\pi_{0,1}^{k,MV},\\ \pi_{1,j}^{k,H} &= \pi_{1,j}^{k}-\pi_{1,j}^{k,MV}.\end{aligned}$$

Because $\pi_{1,j}^{k}=\pi_{1,j}^{k,MV}$, $\pi_{1,j}^{k,H}=0$. We derive $\pi_{0,1}^{k,H}$ as

$$\pi_{0,1}^{k,H} = \left[E\left(\zeta_{0,2}^{\rho}\left(\left(1+\frac{r}{2}\right)\left[E\left(\zeta_{0,1}^{\rho}\right)\right]^{-1}\xi_{0,1}^{k,1}\right)\right)\right]\left[\eta_{0,1}E\left(\zeta_{0,2}^{\rho}\right)\right]^{-1},$$

where

$$\xi_{0,1}^{1,1} = \begin{cases} p_{1,2}\left(\zeta_{0,1}^{1,1}\right)^{\rho}\left(\frac{m_{0,1}^{2}-d_{0,1}^{2}}{q_{1,1}}+\frac{u_{0,1}^{2}-d_{0,1}^{2}}{q_{1,2}}\right)+p_{1,3}\left(\zeta_{0,1}^{1,0}\right)^{\rho}\left(\frac{m_{0,1}^{2}-d_{0,1}^{2}}{q_{1,1}}-\frac{u_{0,1}^{2}-m_{0,1}^{2}}{q_{1,3}}\right),\\ \qquad \text{for } j=1,\ \forall l,\\ -\left[p_{1,1}\left(\zeta_{0,1}^{1,2}\right)^{\rho}\left(\frac{m_{0,1}^{2}-d_{0,1}^{2}}{q_{1,1}}+\frac{u_{0,1}^{2}-d_{0,1}^{2}}{q_{1,2}}\right)+p_{1,3}\left(\zeta_{0,1}^{1,0}\right)^{\rho}\left(\frac{u_{0,1}^{2}-d_{0,1}^{2}}{q_{1,2}}+\frac{u_{0,1}^{2}-m_{0,1}^{2}}{q_{1,3}}\right)\right],\\ \qquad \text{for } j=2,\ \forall l,\\ p_{1,1}\left(\zeta_{0,1}^{1,2}\right)^{\rho}\left(-\frac{m_{0,1}^{2}-d_{0,1}^{2}}{q_{1,1}}+\frac{u_{0,1}^{2}-m_{0,1}^{2}}{q_{1,3}}\right)+p_{1,2}\left(\zeta_{0,1}^{1,1}\right)^{\rho}\left(\frac{u_{0,1}^{2}-d_{0,1}^{2}}{q_{1,2}}+\frac{u_{0,1}^{2}-m_{0,1}^{2}}{q_{1,3}}\right),\\ \qquad \text{for } j=3,\ \forall l,\end{cases}$$

$$\xi_{0,1}^{2,1} = \begin{cases} -\left[p_{1,2}\left(\zeta_{0,1}^{1,1}\right)^{\rho}\left(\frac{m_{0,1}^{1}-d_{0,1}^{1}}{q_{1,1}}+\frac{u_{0,1}^{1}-d_{0,1}^{1}}{q_{1,2}}\right)+p_{1,3}\left(\zeta_{0,1}^{1,0}\right)^{\rho}\left(\frac{m_{0,1}^{1}-d_{0,1}^{1}}{q_{1,1}}-\frac{u_{0,1}^{1}-m_{0,1}^{1}}{q_{1,3}}\right)\right], \\ \qquad \text{for } j=1,\ \forall l, \\ p_{1,1}\left(\zeta_{0,1}^{1,2}\right)^{\rho}\left(\frac{m_{0,1}^{1}-d_{0,1}^{1}}{q_{1,1}}+\frac{u_{0,1}^{1}-d_{0,1}^{1}}{q_{1,2}}\right)+p_{1,3}\left(\zeta_{0,1}^{1,0}\right)^{\rho}\left(\frac{u_{0,1}^{1}-d_{0,1}^{1}}{q_{1,2}}+\frac{u_{0,1}^{1}-m_{0,1}^{1}}{q_{1,3}}\right), \\ \qquad \text{for } j=2,\ \forall l, \\ -\left[p_{1,1}\left(\zeta_{0,1}^{1,2}\right)^{\rho}\left(-\frac{m_{0,1}^{1}-d_{0,1}^{1}}{q_{1,1}}+\frac{u_{0,1}^{1}-m_{0,1}^{1}}{q_{1,3}}\right)+p_{1,2}\left(\zeta_{0,1}^{1,1}\right)^{\rho}\left(\frac{u_{0,1}^{1}-d_{0,1}^{1}}{q_{1,2}}+\frac{u_{0,1}^{1}-m_{0,1}^{1}}{q_{1,3}}\right)\right]. \\ \qquad \text{for } j=3,\ \forall l, \end{cases}$$

3.3.2 Example

In this example we choose the following parameters to describe the trinomial two-period tree: $S_1 = 120$, $S_2 = 100$, $r = 0.05$, $x = 5000$, $p_{1,j} = p_{2,3j-l} = \frac{1}{3}$, $j = 1,2,3$, $l = 0,1,2$, $\gamma = -1$, $u_{i,j}^{r} = 1.0766$, $m_{i,j}^{r} = 1.0300$, $d_{i,j}^{r} = 0.9834$, $u_{i,j}^{2} = 0.7902$, $m_{i,j}^{2} = 1.0500$, $d_{i,j}^{2} = 1.3098$ for all (i,j), $n = 2$. This choice of returns on the risky asset 2 and the changes in the interest rate results in the correlation of -1 between them. As far as the returns on risky asset 1 are concerned five cases are considered which yield different correlation between the returns on the risky asset 1 and the changes in the interest rate, respectively equal -.8, -.6, -.4, -.2, 0.

Table 3.2. Portfolio returns and portfolio composition

Correlation asset 1 and interest rate	Return asset 1 (1)	Portfolio return total (2)	MV (3)	hedging (4)	Portfolio composition total (5)	MV (6)	hedging (7)
-0.8	.943	1.0679	1.0683	-.0004	.6416	.6433	-.0017
	.860	.8759	.8758	.0000	.8321	.8320	.0001
	1.497	1.2830	1.2825	.0005	-.4737	-.4752	.0016
-0.6	1.052	1.0488	1.0491	-.0003	.5026	.5039	-.0013
	.780	.8925	.8925	.0000	.5368	.5367	.0001
	1.468	1.2515	1.2511	.0004	-.0394	-.0406	.0013
-0.4	1.145	1.0466	1.0468	-.0002	.3987	.3998	-.0011
	.733	.8947	.8947	.0000	.4591	.4591	.0000
	1.422	1.2477	1.2474	.0003	.1422	.1412	.0010
-0.2	1.2267	1.0505	1.0507	-.0002	.2828	.2836	-.0008
	.7081	.8910	.8910	.0000	.4442	.4442	.0000
	1.3652	1.2539	1.2537	.0002	.2729	.2722	.0008
0	1.3000	1.0590	1.0591	-.0001	.1325	.1329	-.0004
	.7000	0.8835	0.8835	.0000	.4663	.4663	.0000
	1.3000	1.2675	1.2674	.0001	.4013	.4009	.0004

We report in column 1 the returns on the risky asset 1 in nodes $(1,1)$ (upper number), $(1,2)$ (middle number) and $(1,3)$ (lower number). Columns (2)-(4) show the returns on the total replicating portfolio and its mean-variance and hedging components, realized between $i=0$ and $i=1$. Columns (5)-(7) present the composition of these portfolios. Upper number refers to the share of riskless asset, middle to the share of risky asset 1 and lower to the share of risky asset 2.

In Table 3.2 we report in column 1 the returns on the risky asset 1 in nodes $(1,1)$ (upper number), $(1,2)$ (middle number) and $(1,3)$ (lower number). Columns (2)-(4) show the returns on the total replicating portfolio and its mean-variance and hedging components, realized between $i=0$ and $i=1$. Columns (5)-(7) present the composition of these portfolios. Upper number refers to the share of riskless asset, middle to the share of risky asset 1 and lower to the share of risky asset 2. Results concerning the composition of portfolios formed when $i=1$ are skipped as the hedging constituent does not appear in them.

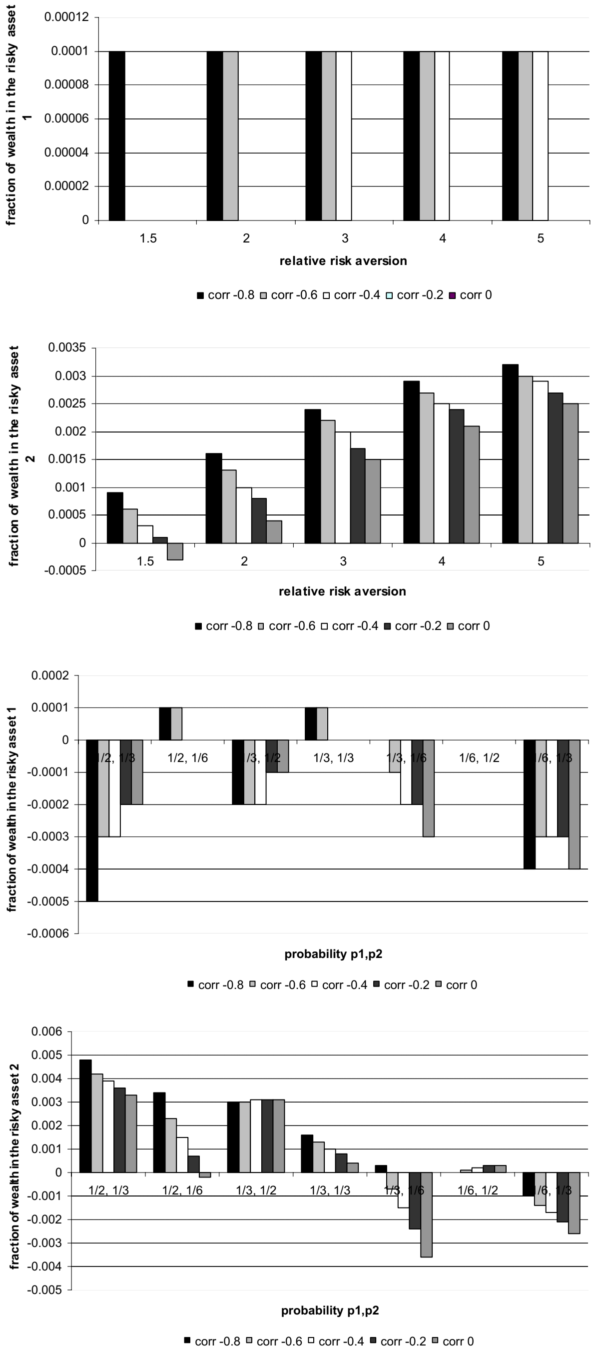
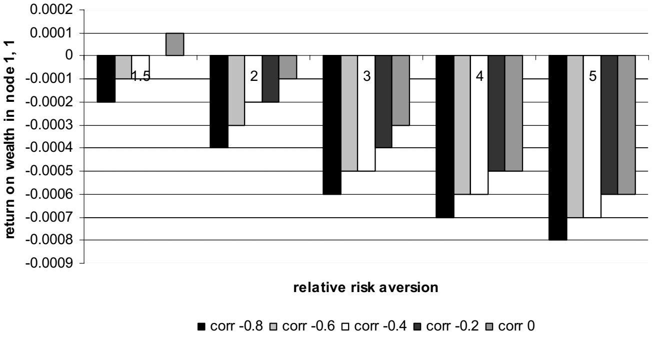
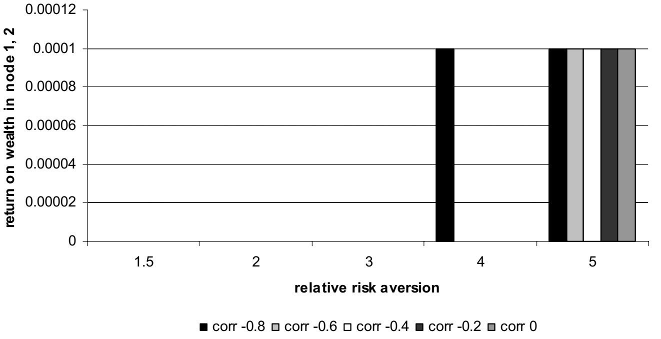


Figure 3.5 Sensitivity of the hedging portfolio composition to the risk aversion coefficient (the first and second row) and the probability (the third and fourth row)





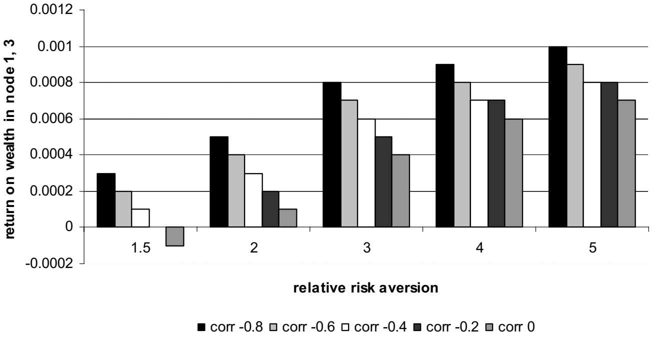


Figure 3.6 Sensitivity of the hedging portfolio return to the risk aversion coefficient

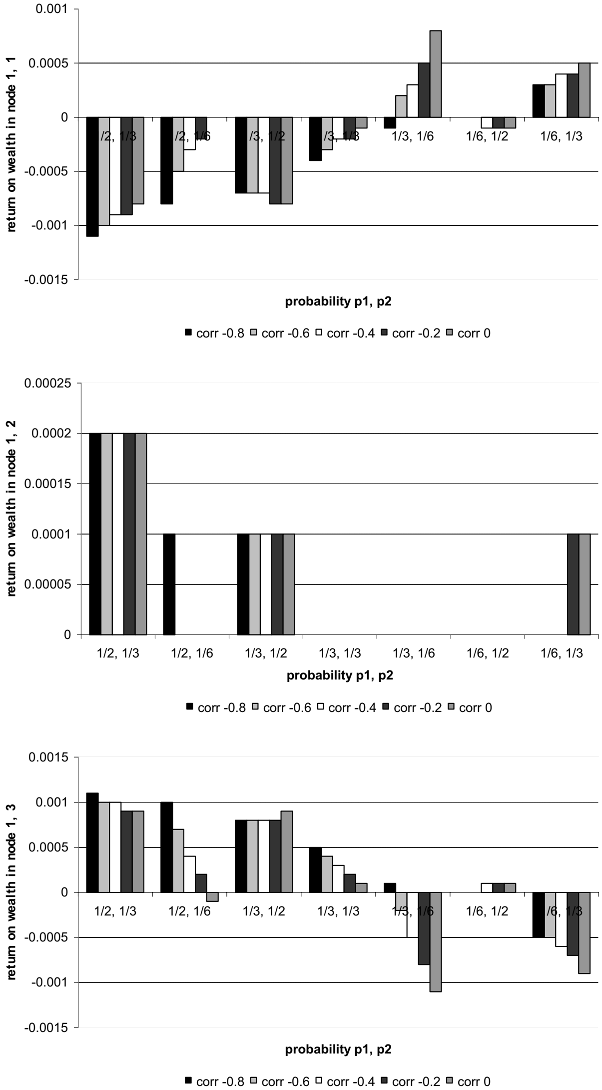


Figure 3.7 Sensitivity of the hedging portfolio return to the probability

From the figures reported in Table 3.2 the following two findings can be formulated. First, the trinomial model seems to confirm what we concluded from the binomial model, namely that the interest rate risk hedging portfolio enlarges the return on the mean-variance portfolio in the node of rate's fall and reduces it in the node of rate's biggest rise. In the node of rate's moderate increase the interest rate risk hedging does not appear. This mechanism of the interest rate risk hedging can be explained by the reasoning analogous to that in the binomial model. The return on the riskless asset in the second period is much higher when $(1,1)$, higher when $(1,2)$ and lower when $(1,3)$ is realized than the return in the first period. As the position held in the riskless asset in the mean-variance portfolio is long this implies increased gain on the riskless investment in $(1,1)$ and $(1,2)$ and decreased gain in $(1,3)$ in the second when compared with the first period. Thus, as expected from the interest rate risk hedging component the return on the hedging portfolio diminishes the return on mean-variance portfolio in $(1,1)$ and magnifies it in $(1,3)$. As the most emphasis of hedging is put on the extreme states the effect of hedging in $(1,2)$ is negligible.

Second, the composition of the hedging portfolio is dominated by risky asset 2. The proportion of the risky asset 1 in this portfolio is close to zero. Thus, the hedging is performed by the asset which is perfectly correlated with the interest rate. However, the level of correlation between the risky asset 1 and the interest rate does matter. As this correlation goes to zero fluctuations in the risky asset 1 contribute less to the total of interest rate risk. Consequently, less hedging against this risk is needed what shows up in the decreasing (absolute) magnitude of returns on the hedging portfolio and decreasing proportion of the risky asset 2 in it.

As far as the sensitivity with respect to the risk aversion coefficient in the trinomial model is concerned we note that the share of risky asset 2 in the hedging portfolio as well as the (absolute) magnitude of returns on this portfolio increases with the increasing risk aversion. More risk averse investor requires more protection against interest rate changes and so the interest rate risk hedging is the stronger the higher risk aversion. When there is a zero correlation between the risky asset 1 and the interest rate and when the risk aversion is low enough the short position is taken in the risky asset 2 in the hedging portfolio. This in turn results in the positive return on this portfolio in the node of rate's significant increase and in the negative return in the node of rate's fall. This is in contrast to our previous findings, but can be explained analogously as in the binomial model when we note that short position is held in the riskless asset in the mean-variance portfolio.

Regarding the sensitivity with respect to the probabilities of up, middle and down movement we note that when the down probability is large (0.5) short position is taken in asset 2 in the hedging portfolio. As a result the positive return on this portfolio in the node of rate's significant increase and negative return in the node of rate's fall can be spotted in contrast to our previous findings. Again we observe that in these cases short position is held in the riskless asset in the mean-variance portfolio and so the explanation follows.

3.4 Multiperiod trinomial model

3.4.1 Theory

In the multiperiod trinomial model the initial hedging demand for risky assets is still given by the formulas from the previous section with $\zeta_{0,2}$ replaced by $\zeta_{0,n}$

and $\left(1+\frac{r}{2}\right)$ by $\left(1+\frac{r}{n}\right)$. Hence, the discrete-time equivalent of the Malliavin derivative term from chapter 2 in the n-period model does not change in comparison with the two-period tree. Intuitively, with an increase of n the discrete-time hedging demand should converge to the continuous-time equivalent which from Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] is known. Instead of providing the formal proof of convergence we demonstrate it numerically in the below example.

3.4.2 Example

In the multiperiod trinomial tree model we follow He [1990] and define $\left(\widetilde{\epsilon}^1, \widetilde{\epsilon}^2\right)$ as $P\left(\widetilde{\epsilon}^1_{i,3j-2} = \frac{\sqrt{3}}{\sqrt{2}}, \widetilde{\epsilon}^2_{i,3j-2} = \frac{1}{\sqrt{2}}\right) = \frac{1}{3}$, $P\left(\widetilde{\epsilon}^1_{i,3j-1} = 0, \widetilde{\epsilon}^2_{i,3j-1} = -\frac{2}{\sqrt{2}}\right) = \frac{1}{3}$, $P\left(\widetilde{\epsilon}^1_{i,3j} = -\frac{\sqrt{3}}{\sqrt{2}}, \widetilde{\epsilon}^2_{i,3j} = \frac{1}{\sqrt{2}}\right) = \frac{1}{3}$. These random variables have the following properties $E\left(\widetilde{\epsilon}^1\right) = E\left(\widetilde{\epsilon}^2\right) = 0$, $Var\left(\widetilde{\epsilon}^1\right) = Var\left(\widetilde{\epsilon}^2\right) = 1$ and $Cov\left(\widetilde{\epsilon}^1, \widetilde{\epsilon}^2\right) = 0$ what makes the pair $\left(\widetilde{\epsilon}^1, \widetilde{\epsilon}^2\right)$ a suitable approximation of the two-dimensional standard Brownian motion on the trinomial tree. Thus, we may approximate the continuous-time interest rate process in (2.21) as

$$r_{i+1,3j-l} = r_{i,j} + \frac{a\left(b - r_{i,j}\right)}{n} - \sigma_r r^{\eta}_{i,j} \frac{\widetilde{\epsilon}^1_{i,3j-l}}{\sqrt{n}},$$

where $l = 0, 1, 2$ and $\eta = 0, \frac{1}{2}$. By analogy, we approximate the price processes of traded assets

$$\begin{aligned}
S_{i+1,3j-l} &= S_{i,j}\left[1 + \frac{r_{i,j} + \sigma_1\lambda + \sigma_2\lambda_r r^{2\eta}_{i,j}}{n} + \sigma_1 \frac{\widetilde{\epsilon}^2_{i,3j-l}}{\sqrt{n}} + \sigma_2 r^{\eta}_{i,j} \frac{\widetilde{\epsilon}^1_{i,3j-l}}{\sqrt{n}}\right], \\
B^T_{i+1,3j-l} &= B^T_{i,j}\left[1 + \frac{r_{i,j} + \sigma^T_{i,j}\lambda_r r^{\eta}_{i,j}}{n} + \sigma^T_{i,j} \frac{\widetilde{\epsilon}^1_{i,3j-l}}{\sqrt{n}}\right], \\
B_{i+1,3j-l} &= B_{i,j}\left[1 + \frac{r_{i,j}}{n}\right].
\end{aligned}$$

When $\eta = 0$ the volatility of T-bond is given by (2.22) with $T = 1$ and $t = \frac{i}{n}$ while for $\eta = \frac{1}{2}$ this volatility is defined in (2.23) with $T = 1$, $t = \frac{i}{n}$ and with $r(t)$ replaced by $r_{i,j}$. As far as the notation in the sequel is concerned $\frac{S_{i+1,3j-2}}{S_{i,j}} \equiv u^1_{i,j}$, $\frac{S_{i+1,3j-1}}{S_{i,j}} \equiv m^1_{i,j}$, $\frac{S_{i+1,3j}}{S_{i,j}} \equiv d^1_{i,j}$, $\frac{B^T_{i+1,3j-2}}{B^T_{i,j}} \equiv u^2_{i,j}$, $\frac{B^T_{i+1,3j-1}}{B^T_{i,j}} \equiv m^2_{i,j}$, $\frac{B^T_{i+1,3j}}{B^T_{i,j}} \equiv d^2_{i,j}$. In contrast to the binomial tree-period and trinomial two-period models the gross returns on risky assets, $u^k_{i,j}$, $m^k_{i,j}$, $d^k_{i,j}$, $k = 1, 2$, in the multiperiod model are not the same for each branch of the tree. The number of time steps is restricted to 11 because of the computational problems.

Table 3.3. Composition of optimal portfolios at initial date

Continuous-time portfolios

	$\eta = 0$				$\eta = 0.5$			
	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
	.6981	2.3837	.0000	.5000	.6082	.3726	.0000	.5061

Discrete-time portfolios

	$\eta = 0$				$\eta = 0.5$			
n	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$	π^1	π^2	$\pi^{1,H}$	$\pi^{2,H}$
2	.8540	2.5696	.0022	.2608	.6852	.5302	.0019	.2754
3	.8123	2.5334	.0022	.3393	.6641	.5320	.0020	.3547
4	.7907	2.5128	.0021	.3777	.6531	.5258	.0019	.3927
5	.7773	2.4990	.0019	.4006	.6463	.5187	.0017	.4150
6	.7680	2.4890	.0018	.4159	.6416	.5120	.0016	.4298
7	.7612	2.4813	.0017	.4268	.6381	.5060	.0015	.4402
8	.7559	2.4751	.0016	.4350	.6355	.5007	.0015	.4481
9	.7516	2.4700	.0015	.4415	.6333	.4959	.0014	.4542
10	.7481	2.4657	.0014	.4467	.6316	.4916	.0013	.4591
11	.7452	2.4620	.0014	.4509	.6301	.4878	.0013	.4631

The numbers in Table 3.3. stand for the fraction of initial wealth invested in the risky assets in the continuous-time (upper panel) and discrete-time (lower panel) portfolios. π^k is the fraction in the total replicating portfolio, while $\pi^{k,H}$ in the hedging component. $\eta = 0$ refers to the model with Vasicek [1977] interest rate, while $\eta = 0.5$ to the model with CIR [1985] rate.

According to the guidelines described above and for the values of parameters describing the financial markets given in section 2.6 we compute the optimal portfolios at the initial node. They are displayed in Table 3.3. Numbers from the below table reveal first that the total demand for the stock computed from the tree model is too high when compared against the continuous-time demand. This (relative) surplus decreases from 22.33% to 6.75% for the $\eta = 0$ model and from 12.65% to 3.60% for the $\eta = 0.5$ model. Also the stock demand in the hedging portfolio is in excess of the continuous-time equivalent. This overdemand in absolute terms declines from 22 to 14 and from 19 to 13 basis points adequately in $\eta = 0$ and $\eta = 0.5$ model. Second, the total demand for bond is above the continuous-time counterpart in both models of the financial market. This difference is however much greater in $\eta = 0.5$ than $\eta = 0$ model. Moreover, discrete-time results outperform the outcome of simulation from section 2.7 in terms of accuracy for n up to 9 for $\eta = 0$ and up to 8 for $\eta = 0.5$. Only for n higher than 9 (resp. 8) simulated rather than discrete-time demand is closer to the continuous-time one. The surplus in question in relative terms diminishes from 42.29% to 30.92% and from 7.80% to 3.28% in $\eta = 0.5$ and $\eta = 0$ discrete-time models respectively. Hence, there is more gain from shortening the time step and rising the number of paths in the simulation than in the tree models. Third, as far as the hedging demand for bond is concerned it is underestimated in the discrete-time models. The difference diminishes from -47.85% to -9.82% in $\eta = 0$ model and from -45.58% to -8.50% in $\eta = 0.5$ model. Again tree results are more precise than the simulated outcome for $n = 2, ..., 9$ in $\eta = 0$ model and for $n = 2, ..., 10$ in $\eta = 0.5$ model.

3.5 Conclusions

In this chapter we attempted to explain the concept of interest rate risk hedging in the optimal portfolio choice for a CRRA investor. We illustrated it in two models in which the movement of the interest rate and the risky assets was modelled on three-period binomial and two-period trinomial trees with the changes in rate and the returns on risky assets identical for every branch. Further, we approximated two continuous-time models for the financial market on the multiperiod trinomial tree and computed the composition of initial portfolio with its mean-variance and hedging constituents distinguished. We investigated the convergence of this portfolio and its hedging component to the continuous-time portfolios.

We confirmed that the portfolios held in the last period before investment horizon consist only of the mean-variance part and that the interest rate risk hedging is performed by the asset which returns are perfectly correlated with changes in the interest rate. As far as the latter is concerned we explained why with the decreasing level of correlation between the remaining asset and the interest rate less hedging against the interest rate risk is required. We found that the hedging portfolio enlarges (resp. reduces) the return on the mean-variance portfolio when the return on long (resp. short) riskless investment in the subsequent period is to fall and reduces (resp. enlarges) it when the return on long (resp. short) position is to rise. We observed that hedging is done more intensively in this part of the tree where bigger fluctuations of the interest rate are present.

From our focus on the initial demand for risky assets in the multiperiod tree models we concluded that the discrete-time equivalent of the term appearing in the simulated hedging portfolio formula and involving the Malliavin derivative of interest rate is independent of the number of time steps. By raising this number up to 11 we demonstrated the convergence of the tree portfolios to the continuous-time portfolios. We showed that the demand for risky assets in the former is in general too high in comparison with the latter except of the hedging demand for bond. Simulated portfolios from chapter 2 outperformed the tree portfolios in terms of accuracy only for the number of time steps equal to 9, 10 or 11. Though the estimates of hedging demand for bond are in the model with CIR interest rate dynamics as precise as in the model with Vasicek rate the estimates of total demand for bond in the former are less accurate than in the latter.

4 Dynamic strategies for pension schemes

4.1 Introduction

Financial institutions like banks, insurance companies and pension funds manage their assets and liabilities in an integrated fashion. The literature on asset-liability management (ALM) in these organizations is mainly dedicated to two approaches. The first one, based on simulation techniques, prevailed in the 80s. Kingsland [1982] and Winklevoss [1982] should be mentioned as examples. Models like PENSIM (Pension Simulation) or PLASM (Pension Liability and Asset Simulation Model) provide the decision maker with the simulated effects of the choice variables (including the asset mix) on items from both sides of scheme's balance sheet. The selection of the investment portfolio is based exclusively on its implications to the assets and liabilities revealed in the series of tested scenarios. It is not an uncommon practice to choose the mean-variance efficient portfolios as the candidate mixes and select from them the one with the most desirable effects on variables of interest such as the funding ratio and contribution rate (cf. MacBeth et al. [1994]). A hybrid simulation\optimisation scenario model for ALM is presented in Boender [1997].

A substantial progress, evidenced in the literature of the 90s, was made in ALM with the application of multistage stochastic programming. The Russell-Yasuda (RY) Kasai model for an insurance company is an example of its successful implementation. The essence of the stochastic programming with recourse, which was employed, is to select the decision vector, observe the random evolution of the problem and select the corrective action. The uncertainty is structured in the shape of a tree. Details on RY model can be found in Carino et al. [1994,1998], Carino and Ziemba [1998]. More on the multiperiod stochastic programming in financial institutions, in particular in pension funds, is contained in the monograph of Ziemba [2003].

In the stochastic programming models an important part is played by the description of uncertainty. The question how to generate scenarios was asked among others by Klaassen [1998] and Kouwenberg [2001]. In the former, Klaassen proposes aggregation methods used together with the financial asset-pricing models to obtain a description of the uncertainty which is arbitrage-free and consistent with the observed market prices. In the latter, Kouwenberg [2001] samples event trees fitting the mean and covariance of the return distribution to generate the coefficients of the stochastic program.

In this chapter we take yet another route. The work of Harrison and Kreps [1979] originated the martingale method in arbitrage-free pricing of contingent claims. A decade later Cox and Huang [1989] used this technique to derive consumer's/agent's optimal consumption and portfolio policy in the frictionless, complete and arbitrage-free financial markets in the presence of non-negativity constraint on the final wealth and the consumption. Following the framework of Cox and Huang [1989] numerous papers derive analytical formulas for the optimal investment strategies in various models of the financial markets, under a range of restrictions on terminal wealth and/or with different utility functions. Among papers introducing the constraints on the terminal wealth the papers by Tepla [2001] and by Nguyen and Portait [2002] can be mentioned as examples. Tepla [2001] uses the martingale approach to derive the optimal policy of an investor with CRRA utility in the model with constant investment opportunity set who is restricted by the stochastic liability assumed to be replicated by the trading strategy in the assets available on the market. Her result is that the investor's initial wealth should be allocated between the optimal unconstrained policy and a put option on unconstrained terminal wealth with a strike price equal to the value of the benchmark portfolio. Nguyen and Portait [2002] take advantage of the martingale technique to derive the optimal policy of an investor with the mean-variance preferences also in the model with constant investment opportunity set, who is restricted by the non-negativity constraint on the terminal wealth. Their result is that the investor's initial wealth should be spent to buy the put options on the minimum norm portfolio with the exercise date equal to the investment horizon and the strike price equal to the ratio of investor's risk tolerance to the number of options.

The presence of the stochastic liability is one of the distinguished characteristics of a defined-benefit scheme. In the current chapter we first assume that this liability can be exactly replicated by the portfolio of traded assets. We show that the requirement to partially or fully cover the liability implies that a scheme's investment policy is to invest the wealth equal to the whole or part of the liability into the liability-replicating portfolio and the remainder as the liability-unconstrained investor. We demonstrate that the demand for the risky assets in the liability-constrained portfolio is the average of unconstrained and liability-replicating demands weighted by the reciprocal of the funding ratio. Subsequently, we abandon the presumption of exact liability hedge by introducing a risk component which can not be spanned by the financial assets. The liability which is not necessarily perfectly hedgable on the financial market is also present in the problem considered by Rudolf and Ziemba [2004].

Among papers that apply the martingale approach to the ALM for pension schemes Sundaresan and Zapatero [1997], Boulier et al. [2001] and Deelstra et al. [2002] should be mentioned. Except for the first one the papers in question model a defined-contribution fund. In this kind of scheme contributions are agreed in advance and hence can be reasonably assumed as given deterministic or stochastic process. In contrast, in the defined-benefit scheme the amount of contributions is an additional decision variable and as such can be adequately adjusted in response to the changes in scheme's environment. This distinctive feature of defined-benefit fund is not always taken into account in the literature. For instance Sundaresan and Zapatero [1997] focus exclusively on the asset allocation policy. Also more recent study by Siegmann [2003] has the same focus. In this chapter the pension scheme's contribution policy is based on the clean-funding rule meaning that the funding ratio is restored to its initial required level at the beginning of each year. We compute the contribution rate resulting from this principle in the simulation rather than take it as exogenously given.

The aim of this chapter is to answer the following questions. First, how and whether the performance of dynamic strategies which hedge against the interest rate risk and the liability risk differs significantly from the performance of the myopic and liability-unconstrained strategies. Second, whether the dynamic strategies perform better than fixed mix strategies. Third, whether the performance of dynamic strategies, when perfect liability hedge is possible, differs substantially from the performance when the hedge is imperfect. Fourth, how sensitive is the performance of dynamic strategies to the changes in scheme's characteristics. In order to answer the above questions we run Monte Carlo simulation in which the funding ratios and contribution rates in different scenarios are computed. The funding ratio is meant to assess the impact of assets allocation on the scheme's ability to finance the liability. The contribution rate reveals the cost of participation in the scheme incurred by the sponsor. We describe the distribution of these ratios/rates in every period of the assessment interval by the first four moments.

The chapter pursues the following order. We start in section 4.2 with the reminder of financial market models considered in this chapter. In section 4.3 we state the assumptions about scheme's demographic and macroeconomic environment and their implications to the dynamics of liability. Section 4.4 exposes the interactions between the process of liability and the financial market. Section 4.5 explains the rules of assets allocation being the subject of our research. Section 4.6 states the basic rules of contribution strategy. Section 4.7 gives details on the design of the simulation experiment. Section 4.8 is devoted to the presentation of results. Section 4.9 concludes the chapter.

4.2 Financial market

We consider a financial market as in chapter 2. To remind the reader the market is arbitrage-free, complete and frictionless. We let $z(t) = [z(t), z_r(t)]^T$, $t \geq 0$, denote a standard two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Evolution of interest rate is described by[2]

$$dr(t) = a\,(b - r(t))\,dt - \sigma_r\,(r(t))^{\eta}\,dz_r.$$

We focus on two models for the financial market. First, we choose $a, b, \sigma_r > 0$ and $\eta = 0$ which results in the Vasicek [1977] model and second we select $a, b, \sigma_r > 0$ and $\eta = \frac{1}{2}$ which yields the CIR [1985] model.

There are three assets available on the market, namely a stock index with price at time t equal to $S(t)$, bond maturing at T with price $B_T(t)$, and a money

[2] Following Bajeux-Besnainou et al. [2003] we use $-\sigma_r$ instead of $+\sigma_r$. As noted therein this assures positive volatilities of bonds and typically observed negative correlation between interest rates and bond as well as stock prices.

market account, $B(t)$. Following Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] we specify their price processes as

$$\begin{aligned} dS(t) &= S(t)\left[r(t)dt + \sigma_1 (dz + \lambda dt) + \sigma_2 (r(t))^\eta (dz_r + \lambda_r (r(t))^\eta dt)\right], \\ dB_T(t) &= B_T(t)\left[r(t)dt + \sigma^T(t)(dz_r + \lambda_r (r(t))^\eta dt)\right], \\ dB(t) &= B(t)\left[r(t)dt\right]. \end{aligned}$$

When $\eta = 0$ the instantaneous volatility of the T-maturity bond is given by

$$\sigma_T(t) = \sigma_r a^{-1}(1 - \exp(-a(T-t))), \tag{1}$$

while when $\eta = \frac{1}{2}$ this volatility equals

$$\sigma_T(t) = \sigma_r h(T-t)\sqrt{r(t)}, \tag{2}$$

with

$$\begin{aligned} h(s) &= 2(\exp(\delta s) - 1)\left[\delta - (a - \sigma_r\lambda_r) + \exp(\delta s)(\delta + a - \sigma_r\lambda_r)\right]^{-1}, \quad (3) \\ \delta &= \sqrt{(a - \sigma_r\lambda_r)^2 + 2\sigma_r^2}. \end{aligned}$$

In the above formulas λ and $\lambda_r (r(t))^\eta$ denote the stock and the bond market price of risk respectively, while σ_1 and σ_2 stand for the stock index volatilities. Parameters λ, λ_r, σ_1 and σ_2 are positive constants. Let σ and λ stand for the volatility matrix and the vector of market prices of risk respectively

$$\begin{aligned} \sigma &= \begin{bmatrix} \sigma_1 & \sigma_2(r(t)^\eta) \\ 0 & \sigma_T(t) \end{bmatrix}, \\ \lambda &= \begin{bmatrix} \lambda \\ \lambda_r (r(t)^\eta) \end{bmatrix}. \end{aligned}$$

Following Bajeux-Besnainou et al. [2003] and Deelstra et al. [2000] we allow for the fourth asset, namely a bond with constant time K till maturity, $B_K(t)$. The relationship between $B_T(t)$ and $B_K(t)$ is given by

$$dB_T(t) = B_T(t)\left[\left(1 - \frac{\sigma_T(t)}{\sigma_K}\right)\frac{dB(t)}{B(t)} + \frac{\sigma_T(t)}{\sigma_K}\frac{dB_K(t)}{B_K(t)}\right].$$

Hence, we may express the portfolio strategy in terms of $S(t)$, $B_T(t)$ and $B(t)$ or $S(t)$, $B_K(t)$ and $B(t)$. In the sequel we use the latter triple with $K = T$.

4.3 Stochastic liability

In the defined-benefit scheme based on the final-salary rule pension rights of employee k at time t, $R_k(t)$, are calculated as (cf. Boender [1995])

$$R_k(t) = 0.0175 d_k(t)(W_k(t) - PP(t)).$$

In the above formula $PP(t)$ is the public pension at time t, $W_k(t)$ stands for time t wage of employee k and $d_k(t)$ denotes his length of employment. For instance if retirement takes place at t after $d_k(t) = 40$ years of work the first pension, $R_{40}(t)$, amounts to 70% of the difference between wage and public pension at this time.

We make the following strongly simplifying assumptions concerning scheme's demographic and macroeconomic environment:

(A1) Scheme's active participants differ with respect to the length of employment, but are homogenous in terms of the wage level, $W_k(t) \equiv W(t)$, $\forall k = 1, 2, ..., K$, with K denoting the number of employees in the scheme.

(A2) The scheme compensates the retirees for increase in the welfare level since the date of their retirement. Indexation takes place continuously with the rate equal to the instantaneous change in the average wage level, $\frac{dP_l(t)}{P_l(t)} = \frac{dW(t)}{W(t)}$, $\forall l = 1, 2, ..., L$, with L denoting the number of retirees in the scheme and $P_l(t)$ standing for pension benefits of retiree l at time t.

(A3) Also public pensions are indexed with the instantaneous change in the average wage level, $\frac{dPP(t)}{PP(t)} = \frac{dW(t)}{W(t)}$.

(A4) The dynamics of wages is given as[3]

$$dW(t) = W(t)\left[w(t)dt + \sigma_W dz_W(t)\right].$$

(A5) Only individuals with the full career path meant as the employment of 40 years and the retirement of 20 years constitute the scheme.

(A6) At any point in time the intensities of entering/leaving the scheme and changing the status from employee to retiree are such that no demographic fluctuations influence the total pension rights and benefits in the scheme[4].

As shown in Appendix 4A direct consequence of the above assumptions is that any instantaneous changes in the scheme's total liability, $L(t)$, stem from the dynamics of wages

$$dL(t) = L(t)\frac{dW(t)}{W(t)}. \tag{4.1}$$

Although these are strong assumptions, the results of this chapter are still interesting enough for pension scheme investment policies in general.

[3] Wage process is similar to the one assumed in Duffie and Zariphopoulou [1993] and Koo [1998] and also to that derived in Sundaresan and Zapatero [1997] and given in equation (7) therein. In Sundaresan and Zapatero [1997] the wage process is endogenous with $w(t)$ capturing the interactions between the wage contract and productivity pattern of employee. In Duffie and Zariphopoulou [1993] and Koo [1998] the wage process is exogenous with $w(t)$ constant.

[4] For example we may think of the following structure. Assume that there is exactly one active participant representing length of employment $d_k(t) = k$ with $k = 0, 1, ..., 40$. Similarly assume that there is exactly one inactive participant representing length of retirement $b_l(t) = l$ with $l = 0, 1, ..., 20$. Further, presume that there is only one new entrant and exactly one retiree leaves the scheme every period. Then in every period the number of employees and retirees and their structure with respect to the length of employment and retirement remain unchanged.

4.4 Liability and financial market

When the innovation in wages can be written as a linear function of innovations $dz(t)$ and $dz_r(t)$

$$\sigma_W dz_W(t) = \sigma_{W,1} dz(t) + \sigma_{W,2} \left(r(t)\right)^{\eta} dz_r(t),$$

where $\sigma_{W,1}$ and $\sigma_{W,2}$ are constants, the employer can perfectly hedge against changes in the labour costs by investment on the financial market. The implication to the scheme is that there exists a portfolio of stock index, bond and cash which may replicate the stochastic liability

$$\frac{dL(t)}{L(t)} = \left[h_1(t)\frac{dS(t)}{S(t)} + h_2(t)\frac{dB_K(t)}{B_K(t)} + (1 - h_1(t) - h_2(t))\frac{dB(t)}{B(t)}\right], \qquad (4.2)$$

with $h_1(t)$ and $h_2(t)$ being adequately the proportion of stock and bond in the replicating portfolio. Substituting for the asset returns and then comparing the proper terms in (4.1) and (4.2) we end up with

$$\begin{aligned} w(t) &= r(t) + \sigma_{W,1}\lambda + \sigma_{W,2}\lambda_r \left(r(t)\right)^{2\eta}, \\ h_1(t) &= \frac{\sigma_{W,1}}{\sigma_{S,1}}, & (4.3) \\ h_2(t) &= \frac{(r(t))^{\eta}}{\sigma_K}\left(\sigma_{W,2} - \frac{\sigma_{S,2}}{\sigma_{S,1}}\sigma_{W,1}\right). & (4.4) \end{aligned}$$

We say that the liability hedge is imperfect when the innovation in wages is a linear function of innovations $dz(t)$, $dz_r(t)$ and additionally $dz_u(t)$

$$\sigma_W dz_W(t) = \sigma_{W,1} dz(t) + \sigma_{W,2} \left(r(t)\right)^{\eta} dz_r(t) + \sigma_{W,3} dz_u(t), \qquad (4)$$

with $\sigma_{W,1}$, $\sigma_{W,2}$ and $\sigma_{W,3}$ constants, and with $dz_u(t)$ representing the risk which can not be hedged on the financial market. In this case the innovation in the inflation of wages can be divided into two parts. The first component $\sigma_W dz_W(t) - \sigma_{W,3} dz_u(t)$ can still be spanned by the portfolio of assets given in (4.3)-(4.4). As the second component, $\sigma_{W,3} dz_u(t)$, is unhedgable there exists no investment strategy which can guarantee that the scheme never becomes underfunded. Clearly, every portfolio policy derived under the assumption of perfect liability hedge is in this case misspecified.

4.5 Investment strategies

The liability-unconstrained nonmyopic strategy (LUNm) is a solution to the following optimization problem

$$\begin{aligned} &\max_{X_T} E_t u\left(X_T\right) \\ \text{s.t. } & E_t\left(\zeta_{t,T} X_T\right) = X_t, \\ & X_T \geq 0. \end{aligned}$$

For constant relative risk aversion (CRRA) utility, $u\left(X_T\right) = \frac{1}{\gamma} X_T^{\gamma}$, $\gamma < 1$, $\gamma \neq 0$ the terminal wealth is given by

$$X_T = \frac{\zeta_t X_t \zeta_T^{\frac{1}{\gamma-1}}}{E_t\left(\zeta_T^{\frac{\gamma}{\gamma-1}}\right)},$$

where E_t stands for the conditional expected value under the real world probability, ζ_t is defined in (2.6) and X_t stands for the wealth level at time t.

Formulas for the proportion of stock and $T-$bond in the portfolio replicating the terminal optimal wealth for an investor in the model with Vasicek [1977] interest rates are given in Bajeux-Besnainou et al. [2003]. Expressed in terms of the stock and $K-$bond they are given as

$$\pi(t) = \left[\begin{array}{c} \pi^1(t) \\ \pi^2(t) \end{array}\right] = \left[\begin{array}{c} \frac{\lambda}{(1-\gamma)\sigma_1} \\ \frac{\lambda_r\sigma_1 - \lambda\sigma_2}{(1-\gamma)\sigma_1\sigma_K} - \frac{\gamma}{1-\gamma}\frac{\sigma_T(t)}{\sigma_K} \end{array}\right], \tag{4.5}$$

where $\pi^1(t)$ denotes the fraction invested in stock and $\pi^2(t)$ is the fraction invested in bond.

Formulas for the proportion of stock and $T-$bond in the portfolio replicating the terminal optimal wealth for an investor in the model with CIR [1985] interest rates are given in Deelstra et al. [2000]. Expressed in terms of the stock and $K-$bond they are given as

$$\pi(t) = \left[\begin{array}{c} \pi^1(t) \\ \pi^2(t) \end{array}\right] = \left[\begin{array}{c} \frac{\lambda}{(1-\gamma)\sigma_1} \\ \frac{\lambda_r\sigma_1 - \lambda\sigma_2}{(1-\gamma)\sigma_1\sigma_r h(K)} + k\left(K, \frac{\gamma}{1-\gamma}\right)\left(\frac{\sigma_K}{\sigma_r\sqrt{r(t)}}\right)^{-1} \end{array}\right], \tag{4.6}$$

where

$$k\left(s, c\right) = -c\frac{\left(\exp(\alpha s) - 1\right)\left(2 + \lambda_r^2(1+c)\right)}{\alpha - a - c\lambda_r\sigma_r + \left(\alpha + a + c\lambda_r\sigma_r\right)\exp\left(\alpha s\right)},$$

with

$$\alpha = \sqrt{a^2 + 2\sigma_r^2\mu}$$

and

$$\mu = -c\left(1 + \frac{\lambda_r^2}{2} - \frac{\lambda_r a}{\sigma_r}\right).$$

In the constrained problem terminal wealth, X_T, is required to cover the fraction f, $0 < f \leq 1$, of stochastic liability. In order to find a liability-constrained nonmyopic strategy (LCNm) we define the optimization problem where utility is derived from the surplus of wealth over this fraction as follows

$$\begin{gathered}\max_{X_T} E_t u\left(X_T - fL_T\right) \\ \text{s.t. } E_t\left(\zeta_{t,T} X_T\right) = X_t, \\ X_T \geq fL_T.\end{gathered}$$

When the liability is replicable we may write

$$E_t\left(\zeta_{t,T} L_T\right) = L_t.$$

Hence, defining $Y_s \equiv X_s - fL_s$, $t \leq s \leq T$ and substituting into the above optimization problem we end up with

$$\begin{gathered}\max_{Y_T} E_t u\left(Y_T\right) \\ \text{s.t. } E_t\left(\zeta_{t,T} Y_T\right) = Y_t, \\ Y_T \geq 0,\end{gathered}$$

which is the standard unconstrained problem from 4.5.1. Because obviously

$$\begin{aligned}\frac{dX_s}{X_s} &= \frac{f dL_s + dY_s}{X_s} = \\ &= f\frac{L_s}{X_s}\frac{dL_s}{L_s} + \left(1 - f\frac{L_s}{X_s}\right)\frac{dY_s}{Y_s},\end{aligned}$$

it straightforwardly follows after substituting for $\frac{dY_s}{Y_s}$ and $\frac{dL_s}{L_s}$ that

$$\frac{dX_s}{X_s} = \widehat{\pi}_1(s)\frac{dS\left(s\right)}{S\left(s\right)} + \widehat{\pi}_2(s)\frac{dB_K\left(s\right)}{B_K\left(s\right)} + \left(1 - \widehat{\pi}_1(s) - \widehat{\pi}_2(s)\right)\frac{dB\left(s\right)}{B\left(s\right)},$$

with

$$\widehat{\pi}_k(s) = f\frac{L_s}{X_s}h_k(s) + \left(1 - f\frac{L_s}{X_s}\right)\pi_k(s),$$

$k = 1,\ 2,\ 3$, where $h_k(s)$ is given by (4.3)-(4.4) for $k = 1,\ 2$ and $\pi_k(s)$ is expressed by (4.5)-(4.6). Hence, the liability-constrained scheme investing the amount equal to fL_s in the liability replicating portfolio and the remainder, $X_s - fL_s$, as if liability unconstrained forms the portfolio in which the proportions $\widehat{\pi}_k(s)$ are the weighted average of $h_k(s)$ and $\pi_k(s)$ with the fraction f of the reciprocal of the funding ratio, $F_s^{-1} \equiv \frac{L_s}{X_s}$, as the weighting factor. When $f = 1$ and $F_s^{-1} = 1$ constrained portfolio simply replicates the liability. The bigger F_s the closer the strategy to the unconstrained policy. Analogous result for the constant interest rate model and the stochastic performance constraint is provided in Tepla [2001]. Basak [2002] provides the formulas for the case of constant interest rate and constant floor.

Portfolio demands of LUcNm policy can be decomposed into the myopic and hedging components (cf. chapter 1). The myopic demand for stock, $\pi_1^m(t)$, equals the nonmyopic demand $\pi_1(s)$. The myopic demand for bond, $\pi_2^m(t)$, in the model of Bajeux-Besnainou et al. [2003] is given by

$$\pi_2^m(s) = \frac{\lambda_r \sigma_1 - \lambda \sigma_2}{(1-\gamma)\, \sigma_1 \sigma_K}, \tag{4.7}$$

while in the model of Deelstra et al. [2000] as

$$\pi_2^m(s) = \frac{\lambda_r \sigma_1 - \lambda \sigma_2}{(1-\gamma)\, \sigma_1 \sigma_r h(K)}. \tag{4.8}$$

In the liability-constrained myopic strategy (LCM) the composition of investment portfolio is given by

$$\widehat{\pi}_k^m(s) = f \frac{L_s}{X_s} h_k(s) + \left(1 - f \frac{L_s}{X_s}\right) \pi_k^m(s),$$

meaning that a fraction of wealth is invested into the liability replicating portfolio and the remainder according to the myopic unconstrained strategy.

LCNm strategy provides both a liability and interest rate risk hedge. LUcNm strategy hedges against interest rate risk, but does not account for the presence of liability. On the other hand, a LCM strategy guarantees that the liability constraint is satisfied, but does not hedge against the interest rate. Hence, we expect the LCNm policy to result in the scheme's most favourable performance when liability can be exactly hedged. When perfect liability hedge is impossible none of the strategies can be expected to outperform the remaining ones.

The above strategies are truly dynamic. The proportion of wealth allocated according to these strategies among the assets changes from one period to another because of the interest rate risk hedging component and/or the varying funding ratio. In this section we consider fixed mixes in which rebalancing is done solely to restore the initial asset mix.

The first three fixed mixes reported in Table 4.1 are initial LCNm, LUcNm and LCM portfolio compositions. Hence, mix I serves as a basis for the assessment of LCNm strategy, while mix III of the LCM strategy. All the dynamic strategies are being compared against mix IV. This is the composition of initial portfolio in the liability-unconstrained myopic (LUcM) policy given by $\pi_i^m(t)$, $i = 1, 2$, in (4.7)-(4.8). The numbers in Table 4.1 describing the mixes are obtained for a set of parameters assumed in the simulation experiment (cf. section 4.7). As mix V we consider the common asset allocation advice, namely "the 60:40 rule". From Table 4.1 we see that some strategies are short cash, which indicates that the portfolio is leveraged.

Table 4.1. Asset allocation in fixed mix strategies

	Vasicek		Cox-Ingersoll-Ross	
	LR/LF♦	SR♣	LR/LF	SR
stock I p✠	36.03	36.03	33.95	33.95
bond I p	1.10	-8.18	-8.75	-18.09
cash I p	62.87	72.15	74.80	84.14
stock I imp♠	10.71	10.71	8.63	8.63
bond I imp	17.36	8.08	8.47	-.87
cash I imp	71.94	81.21	82.89	92.23
stock II	69.81	69.81	60.82	60.82
bond II	86.93	46.74	49.11	8.62
cash II	-56.75	-16.55	-9.93	30.55
stock III p		36.03		33.95
bond III p		-10.44		-19.63
cash III p		74.41		85.68
stock III imp		10.71		8.63
bond III imp		5.82		-3.53
cash III imp		83.47		94.90
stock IV		69.81		60.82
bond IV		36.93		-2.91
cash IV		-6.75		42.09

The numbers in Table 4.1 show the percentage proportions of assets in the investment portfolios of fixed mixes ♦LR stands for the long-rolled forward horizon, LF for the long-fixed horizon, ♣SR for the short-rolled forward horizon, ✠p refers to the case $\sigma_{W,3} = 0$, ♠imp to $\sigma_{W,3} \neq 0$.

4.6 Contribution strategy

Apart from the portfolio gains and losses pension schemes experience positive and negative cash flows from other sources. The outflows being the pension benefits paid out to the retirees, P_s, are given by equation 3A1 in Appendix 3A. The inflows due to the contributions made by the sponsors are equal to the fraction, c_s, of total wages, $K \times W_s$, earned by the employees. We make the following two assumptions

(A8) All non-portfolio cash flows are exchanged once per annum at the end of each year.

(A9) At the beginning of each year initial funding ratio, $F_* \equiv \frac{X_*}{L_*}$, is restored. Assumptions (A8) and (A9) can be written as

$$\frac{X_s - P_s + c_s\,(K \times W_s)}{L_s} = F_*,$$

with $s = 0, ..., T-1$. The above equation is the basis for specification of the contribution rate c_s in the scheme whose contribution policy is based on the clean-funding rule, which we consider in this paper.

4.7 Simulation experiment

In order to assess the performance of dynamic investment strategies from Section 4.5 we carry out Monte Carlo simulation in the models of financial market presented in Section 4.2. It involves $N = 20\ 000$ simulated paths. In the simulation continuous-time results are used after discretizing them according to the Euler scheme with $\Delta t \equiv \frac{1}{n}$, $n = 52$, what makes Δt correspond to a one-week interval. We base the evaluation of strategy on the funding ratio, F_s, and the contribution rate, c_s. In every period we characterize the distribution of the ratio/rate by the mean, standard deviation, skewness and kurtosis. We report the averages over time of the mentioned descriptive statistics. When the strategies admitting different levels of underfunding or neglecting the presence of liability are concerned the probability and mean size of underfunding are reported as well. We assume three lengths of the investment horizon, namely $T = 10$ years and $T = 1$ year rolled forward every year as well as $T = 10$ years fixed. The assessment interval equals 10 years. Because the results for 10-year fixed horizon are close to the results for 10-year rolled horizon we report in Section 4.8 only the latter.

We take the values for parameters of Vasicek and CIR interest rate process from Chan et al. [1992][5]. We use their GMM estimates obtained from June 1964-December 1989 sample of annualized one-month U.S. Treasury bill yield. The values of estimates are shown in the below table.

[5] Though Chan et al. [1992] conclude that the models of Vasicek and of CIR [1985] performed quite poorly relative to other models this can be attributed to the country effect. In a closely related study Nowman [1997] applied the Gaussian estimation to the same models and the sample from U.K. market apart from U.S. market. For U.K. data he finds that CIR model performs best with Vasicek model performing third best.

Table 4.2. Parameters of the financial market

in %	Vasicek	CIR	in %	Vasicek	CIR
$a^{\blacklozenge}$	17.79	23.39	$\sigma_1^{\blacktriangle}$	19.08	19.08
$b^{\clubsuit}$	8.66	8.08	$\sigma_2^{\blacktriangle}$	6.00	23.15
$\sigma_r^{\maltese}$	2.00	8.54	$\lambda^{\blacktriangledown}$	26.64	23.21
$r^{\spadesuit}$	6.715	6.715	$\lambda_r^{\blacktriangledown}$	15.28	26.18

The numbers in Table 4.2 stand for $^{\blacklozenge}$speed of interest rate reversion, $^{\clubsuit}$mean to which the interest rate reverts, $^{\maltese}$volatility of interest rate process, $^{\spadesuit}$initial value of interest rate, $^{\blacktriangle}$volatilities of stock price process, $^{\blacktriangledown}$market prices of risk

In the literature different values for wage volatility are assumed. For instance Bacinello [2000] takes $\sigma_W = 5\%$, Sundaresan and Zapatero [1997] set $\sigma_W = 6\%$ while in Pennacchi [1999] $\sigma_W = 1\%$ for the real wages. Most of the time in the simulation we consider $\sigma_W = 5\%$. We choose $\sigma_{W,1}$ and $\sigma_{W,2}$ to satisfy that the correlation coefficient between the wages and interest rate, $\rho_{W,\ r}$, equals -0.15 and $\sigma_W = \left(\sigma_{W,1}^2 + \sigma_{W,2}^2 r_*^{2\eta}\right)^{0.5}$, when perfect liability hedge is possible and to satisfy that $\rho_{W,\ r} = -0.15$, the correlation coefficient between the wages and stock price, $\rho_{W,\ S}$, equals -0.30 and $\sigma_W = \left(\sigma_{W,1}^2 + \sigma_{W,2}^2 r_*^{2\eta} + \sigma_{W,3}^2\right)^{0.5}$, when unhedgable component in the liability is present. The correlation of -0.15 between wages and bonds and of -0.30 between wages and stocks is reported in Boender [1995]. Resulting numbers imply that the drift rate of wages exceeds the interest rate by 1.20% in the model with Vasicek and by .95% in the model with CIR dynamics.

Table 4.3. Parameters of the wage process and the pension fund characteristics

%	V^p	CIR^p	V^{imp}	CIR^{imp}	%	Both
$\sigma_W^{\diamondsuit}$	5.00	5.00	5.00	5.00	✠ $L_* (K \times W_*)^{-1}$	210
$\sigma_{W,1}^{\clubsuit}$	4.94	4.94	-1.34	-1.34	$^{\spadesuit}P_* \times L_*^{-1}$	11.5
$\sigma_{W,2}^{\clubsuit}$	$-.75$	-2.89	$-.75$	-2.89	$^{\blacktriangle}PP_* \times W_*^{-1}$	30
$\sigma_{W,3}^{\clubsuit}$	-	-	4.46	4.46		

The superindex p refers to perfect, while imp to imperfect liability hedge in Vasicek and CIR model of interest rate and moreover $^{\diamondsuit}$total volatility of wage process, $^{\clubsuit}$components of wage volatility, ✠liability-to-wage ratio, $^{\spadesuit}$pension-to-liability ratio, $^{\blacktriangle}$public pensions-to-wage ratio

The above specification of parameters in stock, bond and wage processes imply that the liability replicating portfolio consists of 25.89% in stock in three-asset model of the financial market at initial date and of -24.65% in $K-$bond in the model with Vasicek and of -26.10% in the model with CIR dynamics respectively. In turn, in the setting with unhedgable liability the portfolio replicating its hedgable part consists of stock of -7.0% at initial date and of $K-$bond of -3.5% in the model with Vasicek and of -3.7% in the model with CIR dynamics respectively. The remainder is invested in cash. The initial financial condition of pension scheme is characterized by $F_* = 1.3$ and its CRRA preferences by relative risk aversion coefficient $1 - \gamma$ of 2. What concerns the liability and its relation to wages and pension payments (cf. the right panel of Table 4.2) it is based on the characteristics of a large Dutch pension fund provided in Boender [1995]. The ratio of public pensions to wages is close to the one reported in Turner and Wantanabe [1995] (25% therein).

In the analysis of sensitivity with respect to the initial funding ratio, risk aversion coefficient, liability-to-wage ratio, pension-to-liability ratio and the level of admitted underfunding we additionally consider, ceteris paribus, $F_* = 1.1,\ 1.5$, $\gamma = -0.5,\ -3$, $L_* (K \times W_*)^{-1} = 1.6,\ 2.6$, $P_* L_*^{-1} = 0.06,\ 0.17$ and $f = 0.7,\ 0.9$.

4.8 Results

4.8.1 Dynamic strategies with perfect liability hedge

The numbers in Table 4.4 show that following a LCNm strategy results in the "averaged" distribution of the funding ratio, which has tails fatter than normal, long right tail, mean by 0.003-0.006 higher than initial ratio and standard deviation amounting to 1.32-2.47% of the mean. A higher implies a higher volatility.

Table 4.4. Dynamic strategies with perfect liability hedge

	LCNm♦		LUcNm♣		LCM✠
	LR♠	SR▲	LR	SR	
			Vasicek interest rate		
			Funding ratio		
mean	1.3059	1.3048	1.3257	1.3210	1.3048
std. dev.	.0322	.0247	.1395	.1072	.0245
skewness	.2918	.2228	.2918	.2228	.2202
kurtosis	3.1844	3.1101	3.1844	3.1101	3.1083
			Contribution rate		
mean	.2167	.2214	.1341	.1542	.2215
std. dev.	.1019	.0778	.4414	.3372	.0771
skewness	-.4549	-.3472	-.4549	-.3472	-.3438
kurtosis	3.3936	3.2330	3.3936	3.2330	3.2293
			Underfunding		
probab.	.0000	.0000	.0103	.0021	.0000
average	.0000	.0000	.0005	.0001	.0000
			CIR interest rate		
			Funding ratio		
mean	1.3036	1.3031	1.3154	1.3133	1.3030
std. dev.	.0232	.0172	.1005	.0745	.0163
skewness	.0825	.0987	.0825	.0987	.1038
kurtosis	3.1812	3.0845	3.1812	3.0845	3.0641
			Contribution rate		
mean	.2267	.2288	.1773	.1864	.2291
std. dev.	.0725	.0531	.3142	.2301	.0509
skewness	-.1325	-.1597	-.1325	-.1597	-.1644
kurtosis	3.1893	3.1083	3.1893	3.1083	3.0949
			Underfunding		
probab.	.0000	.0000	.0032	.0001	.0000
average	.0000	.0000	.0001	.0000	.0000

The numbers in Table 4.4. are levels. ♦liability-constrained nonmyopic strategy, ♣liability-unconstrained nonmyopic strategy, ✠ liability-constrained myopic strategy, ♠long-rolled forward horizon, ▲short-rolled forward horizon

As expected we observe the trade-off between the mean of funding ratio and its volatility. Small volatility and long fat right tail of the funding ratio are required by the scheme. They assure stability of its financial position and occurrence of large ratios more often than very small ones and more frequent than normal. All the statistics in the model with Vasicek interest rate have values higher than in the model with CIR rate. These values are higher for long than short horizon except the skewness in the CIR model.

With respect to the distribution of the contribution rate its mean oscillates around 22% with standard deviation varying from ca. 23% to 47% of the mean. The distribution has tails fatter than normal with the left tail long. Relatively high and volatile contribution rate is not appreciated by the sponsor. It implies a lot of financial burden and uncertainty to the scheme's sponsor. Negative skewness and kurtosis higher than 3 of rate's distribution mean that very small contribution rates are more frequent than large ones and occur more often than normal. In particular, when negative rates happen the contributions are paid back to the sponsor. Again the trade-off between the mean and standard deviation of contribution rate is observed. All the statistics except of the mean have values higher (in absolute terms) in the model with Vasicek than in the model with CIR rate. These (absolute) values are again higher for long than short horizon except the skewness in the CIR model.

Following the LUcNm strategy results in the distributions of the funding ratio and of the contribution rate which have the same skewness and kurtosis as the LCNm distributions. In other words, the third and fourth moments of these distributions are insensitive to the liability constraint. Though the LUcNm strategy leads to around .8-1.5% increase of the mean funding ratio relative to LCNm and 19-38% decrease of the mean contribution rate it also yields a rise of standard deviation by 333%. Besides, LUcNm policy leads to non-zero probability of underfunding, which is higher for long than short term horizon strategy. The highest probability amounts to 1%. In contrast, employing LCM policy results in rather marginal decrease of mean funding ratio when compared to LCNm strategy and the rise of mean contribution rate by at most 2.21%, but at the profit of decreasing their standard deviations by 1-5% for short and 24-30% for long horizon. The LCM distributions are less peaked and have thinner tails than LCNm ones. They are less skewed in the model with Vasicek rate and more skewed in the model with CIR rate than corresponding LCNm distributions.

Averaging over time as done in Table 4.3 may lead to the loss of information on the dynamics of funding ratio and contribution rate. Figure 4.1 shows the mean, standard deviation, skewness and kurtosis (4 rows) of the funding ratio in every week of last year of the assessment period for LCNm, LUcNm and LCM (3 columns) strategies. Similar patterns are repeated in the previous years. Figure 4.2 does the same for the contribution rate in every year of the interval. The most upper curves are for long-term strategy with Vasicek interest rate, while the lowest for short-term policy with CIR rate.

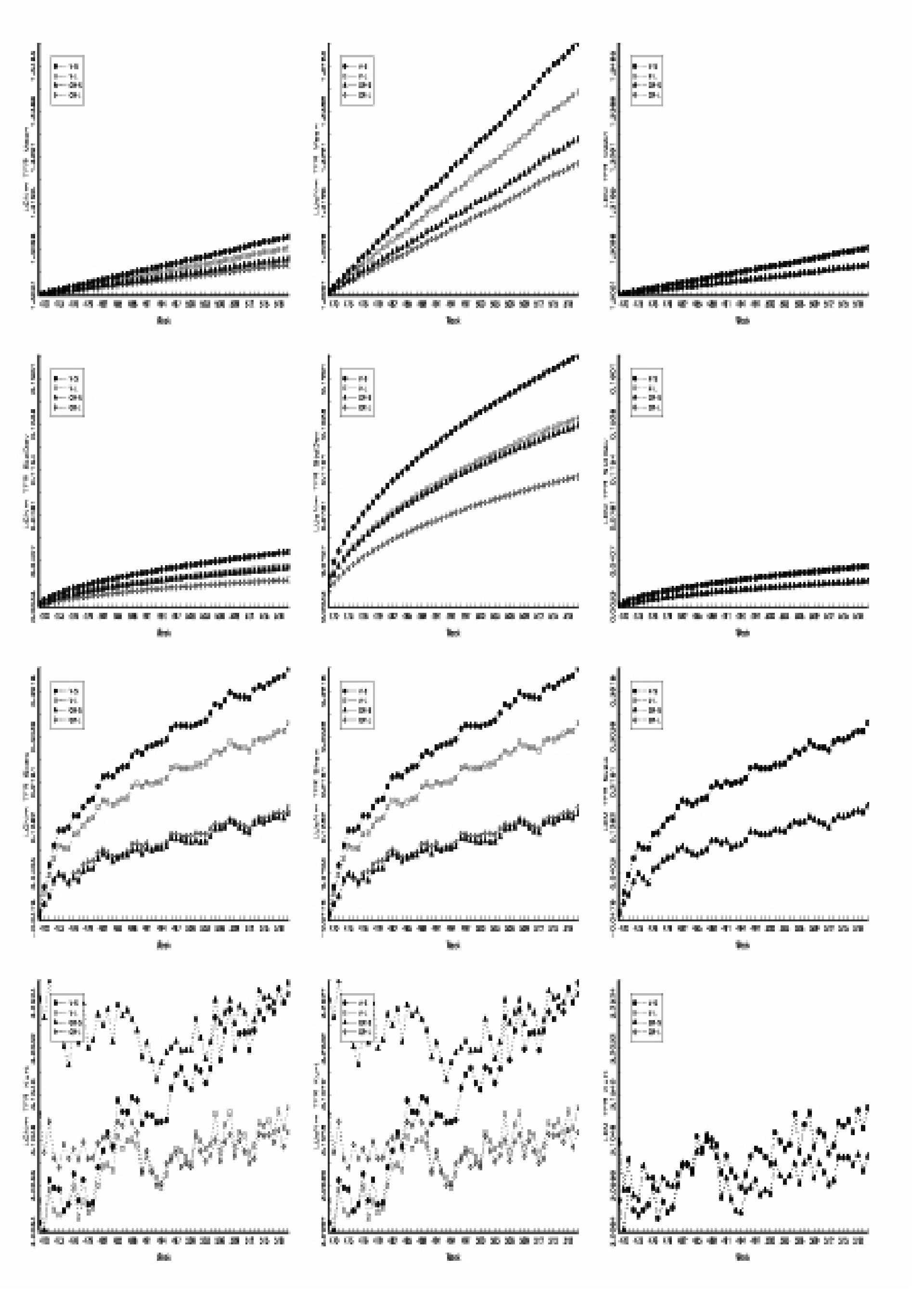


Figure 4.1. Dynamics of mean, standard deviation, skewness and kurtosis (4 rows) of the **funding ratio** in the last year of assessment interval for LCNm, LUcNm and LCM strategies (3 columns)

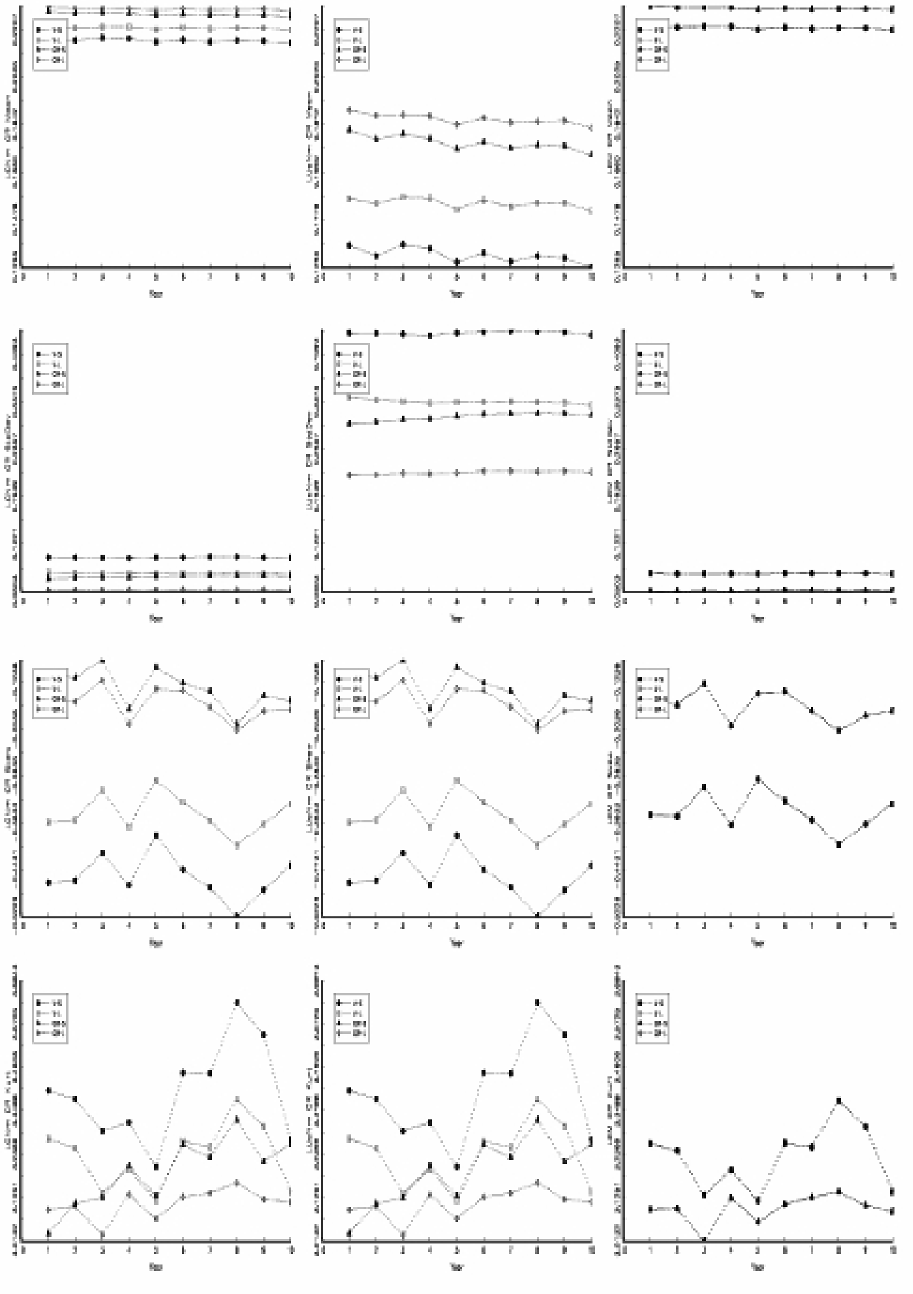


Figure 4.2. Dynamics of mean, standard deviation, skewness and kurtosis (4 rows) of the **contribution rate** in every year of assessment interval for LCNm, LUcNm and LCM strategies (3 columns)

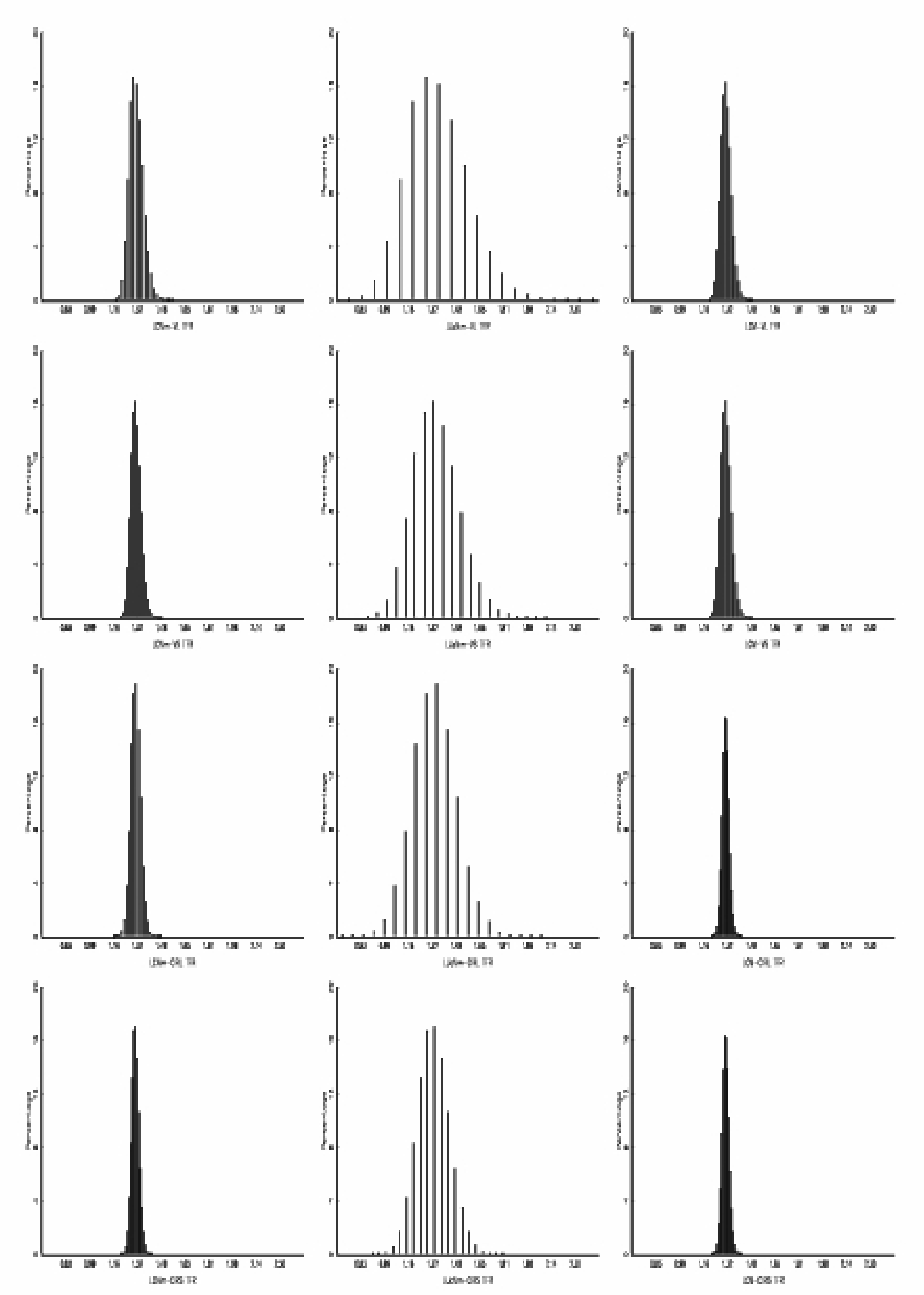


Figure 4.3. Distribution of the terminal **funding ratio** for LCNm, LUcNm and LCM strategies (3 columns) when long- and short-term strategy is followed in the Vasicek and CIR market (4 rows)

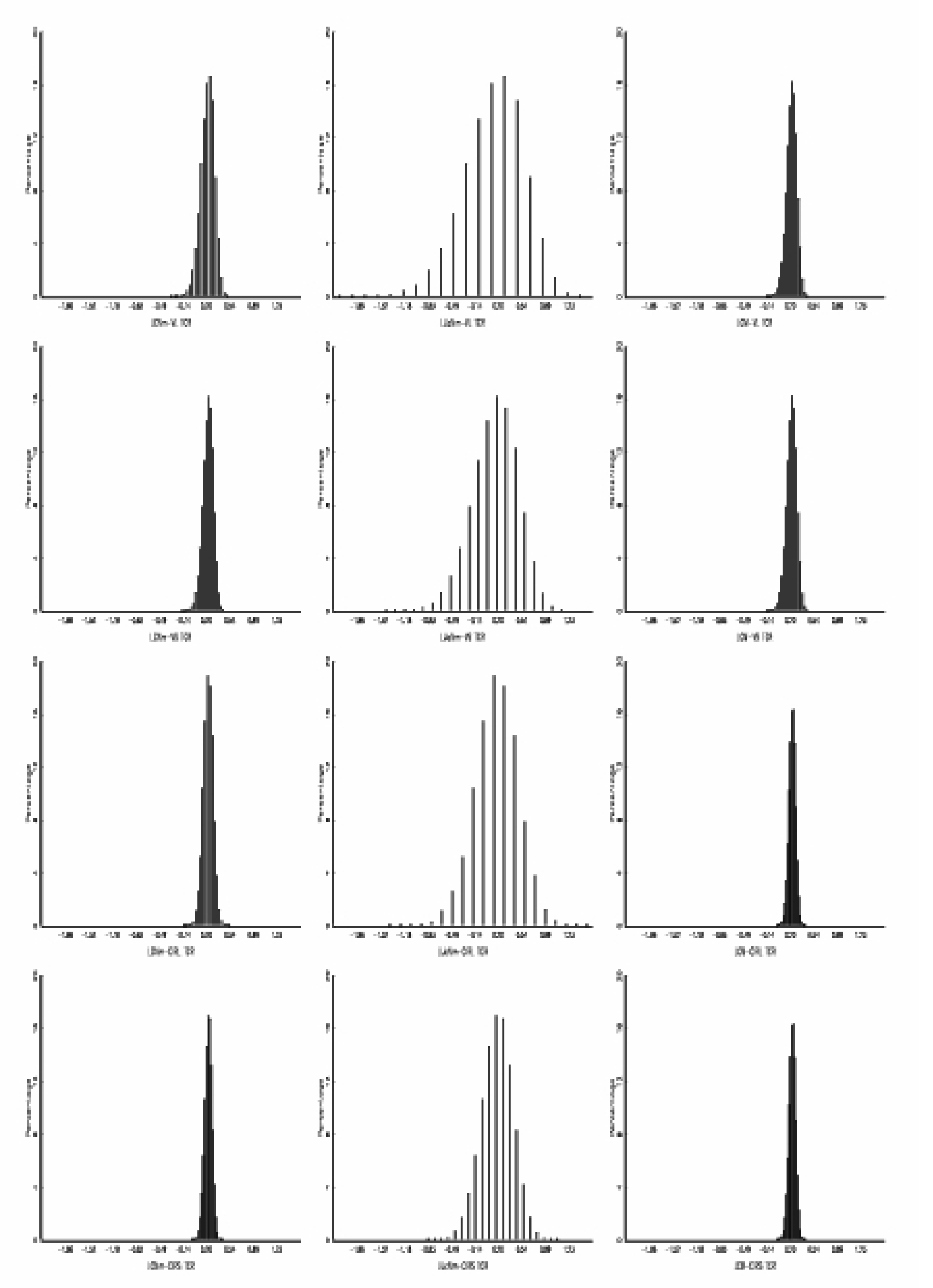


Figure 4.4. Distribution of the terminal **contribution rate** for LCNm, LUcNm and LCM strategies (3 columns) when long- and short-term strategy is followed in the Vasicek and CIR market (4 rows)

Mean of the funding ratio grows during a year. The growth is sharper for long-term strategy, for Vasicek interest rate and unconstrained strategy. This growth is accompanied by the growth of standard deviation which shows a similar pattern. Though skewness also increases with time its growth for unconstrained and constrained strategies is the same. Kurtosis does not reveal any clear pattern. In the model with Vasicek interest rate dynamics kurtosis for short term strategy grows faster than for long term and eventually achieves the same level. There are no substantial differences in the skewness and kurtosis of the funding ratio between short and long term strategy in the model with CIR dynamics.

The mean of the contribution rate during the assessment interval of 10 years is pretty stable. It is higher for the long-term strategy, for the Vasicek model and the unconstrained strategy. Standard deviation does not show rapid changes either. The pattern here is similar as for the mean. Skewness and kurtosis show more abrupt movements. Also the differences between the short and the long term strategy in the model with CIR dynamics are here bigger than for the funding ratio.

Alternatively, the terminal instead of averaged effects can be of interest. Figure 4.3 shows the distribution of the terminal funding ratio for LCNm, LUcNm and LCM (3 columns) strategies when long- and short-term strategy is followed in the models with Vasicek and CIR dynamics (4 rows). Figure 4.4 exposes the "terminal" distribution of the contribution rate. The "terminal" distributions have all the features of "averaged" distributions, namely positive asymmetry of the funding ratio and negative of the contribution rate, fat tails and standard deviation of LUcNm distribution much higher than LCNm and LCM distributions.

4.8.2 Dynamic vs. fixed mix strategies with perfect liability hedge

The performance of dynamic against fixed mix strategies in terms of funding ratio is shown in Figures 4.5-4.8 and and in terms of contribution rate in Figures 4.9-4.12. Horizontal line displays the values of descriptive statistics for dynamic strategies, while the vertical line for fixed mix strategies. The line drawn is 45 degree line. We see that dynamic strategies are not superior to fixed mix policies with respect to mean-standard deviation criterion. Liability-unconstrained strategy yields higher average funding ratio and lower average contribution rate than fixed mix policies, but also entails higher volatilities. In turn, liability-constrained strategies result in lower average funding ratio and higher average contribution rate than fixed mix policies, but at the benefit of lower volatility.

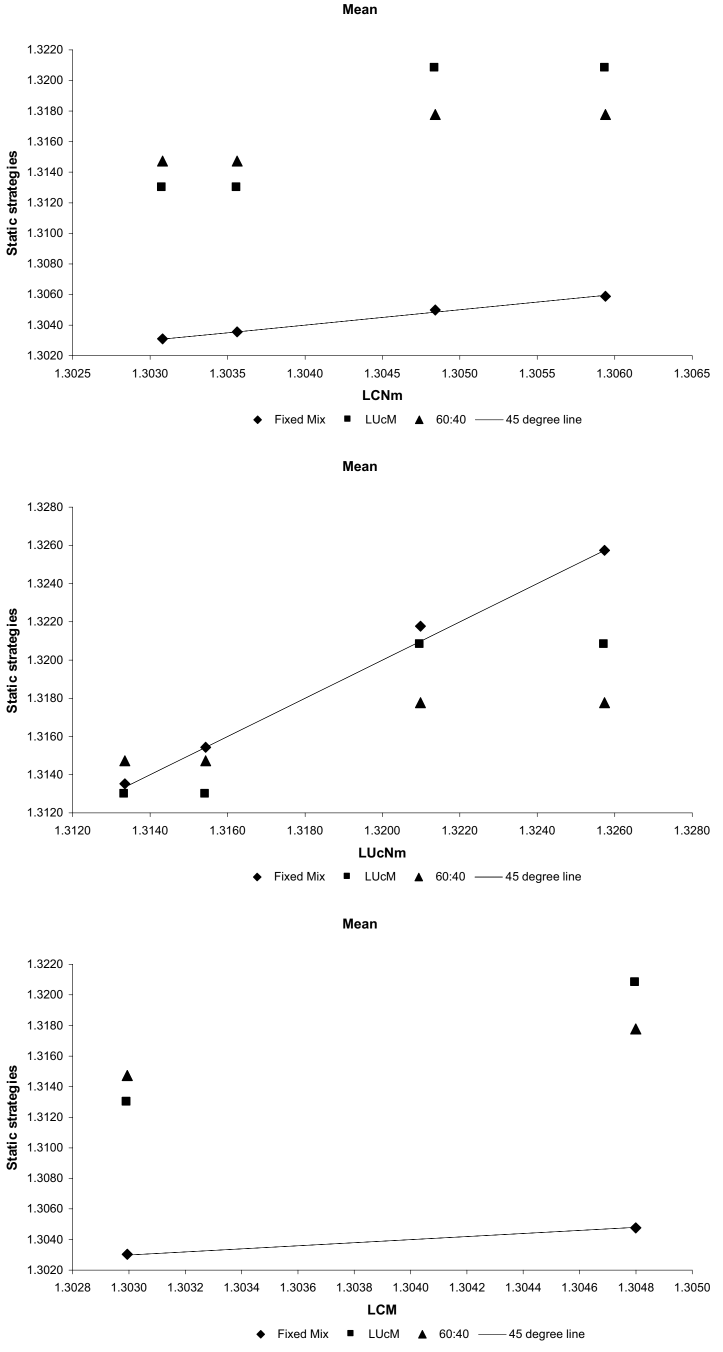


Figure 4.5. Mean of the funding ratio of dynamic vs. static strategies

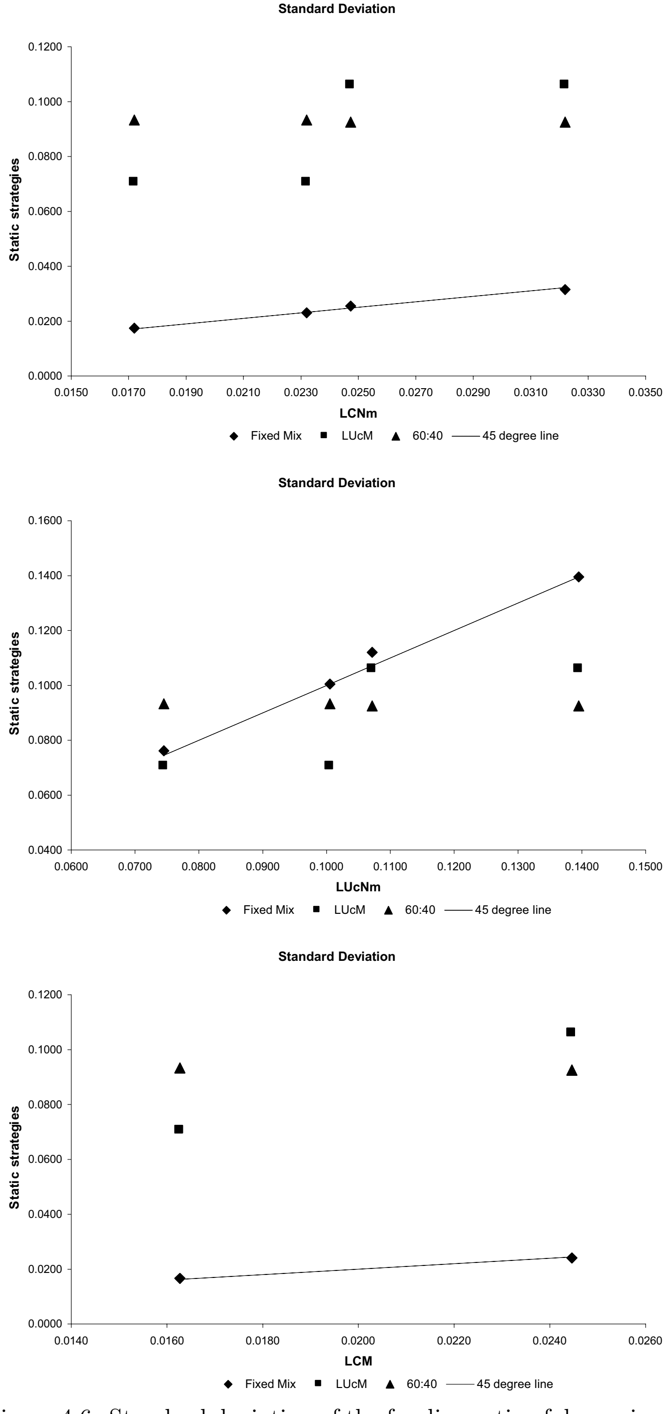
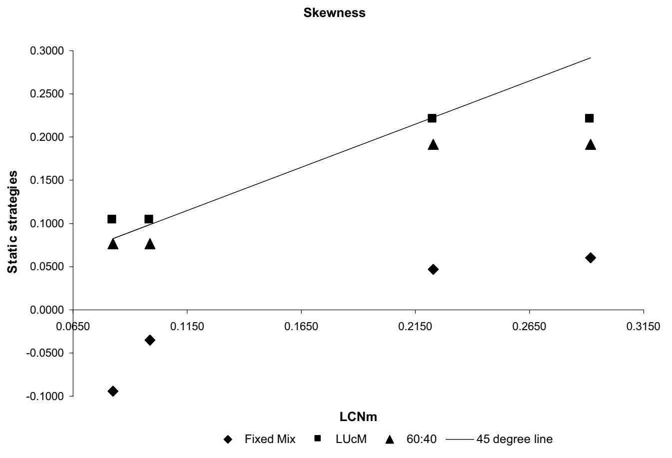
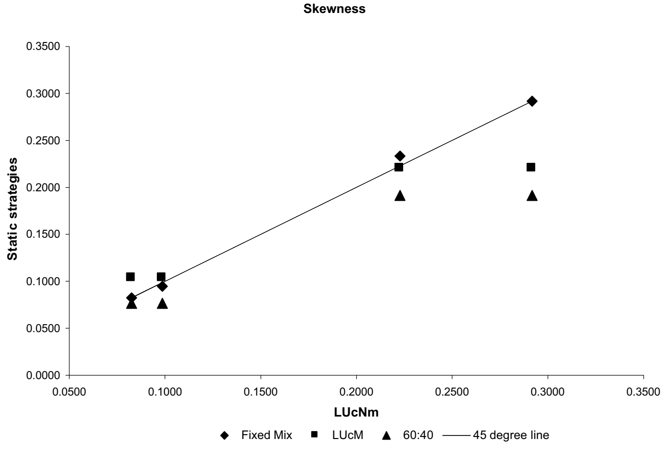


Figure 4.6. Standard deviation of the funding ratio of dynamic vs. static strategies





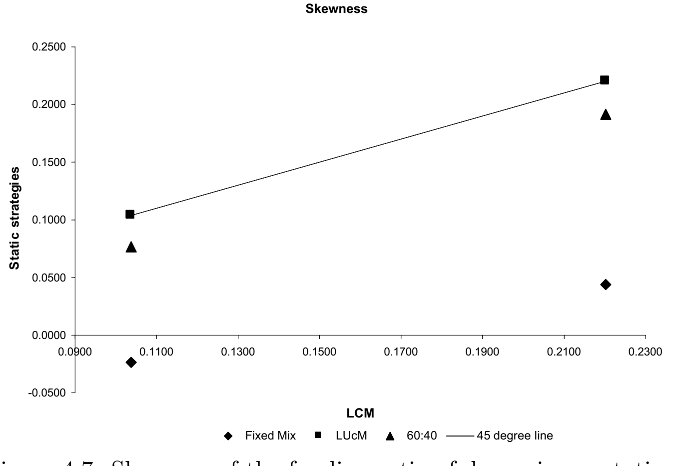


Figure 4.7. Skewness of the funding ratio of dynamic vs. static strategies

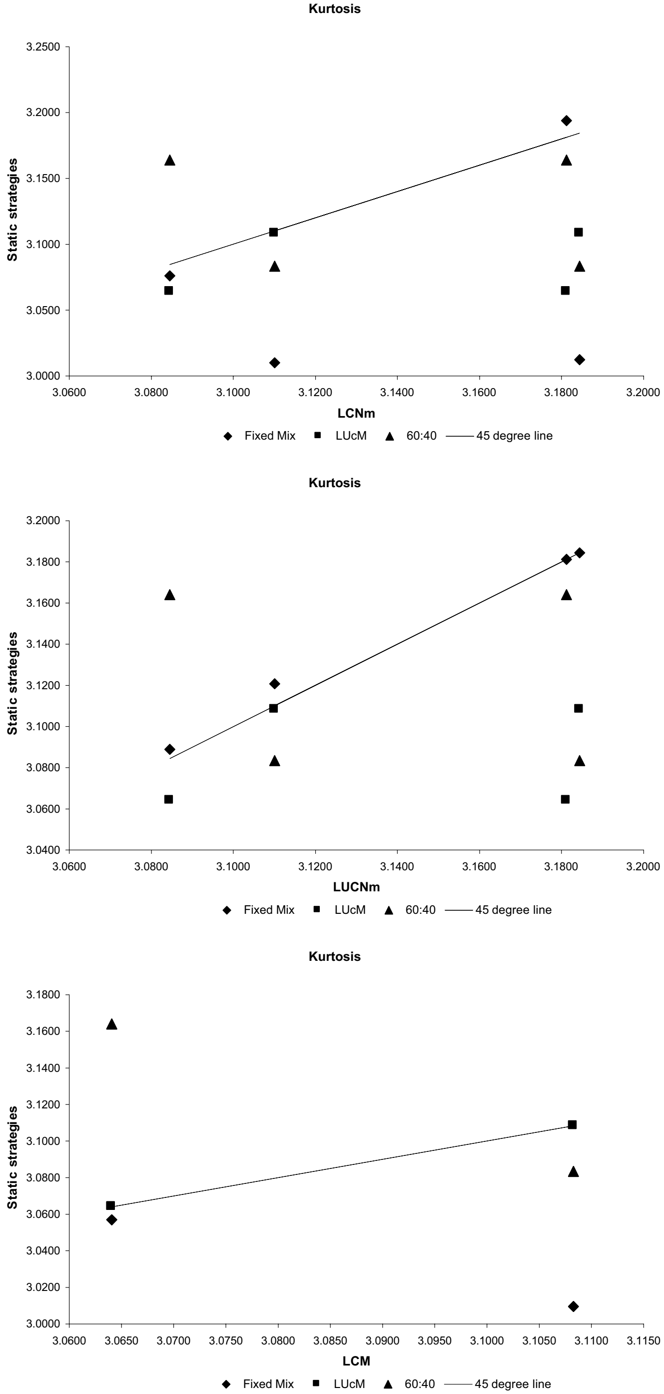


Figure 4.8. Kurtosis of the funding ratio of dynamic vs. static strategies

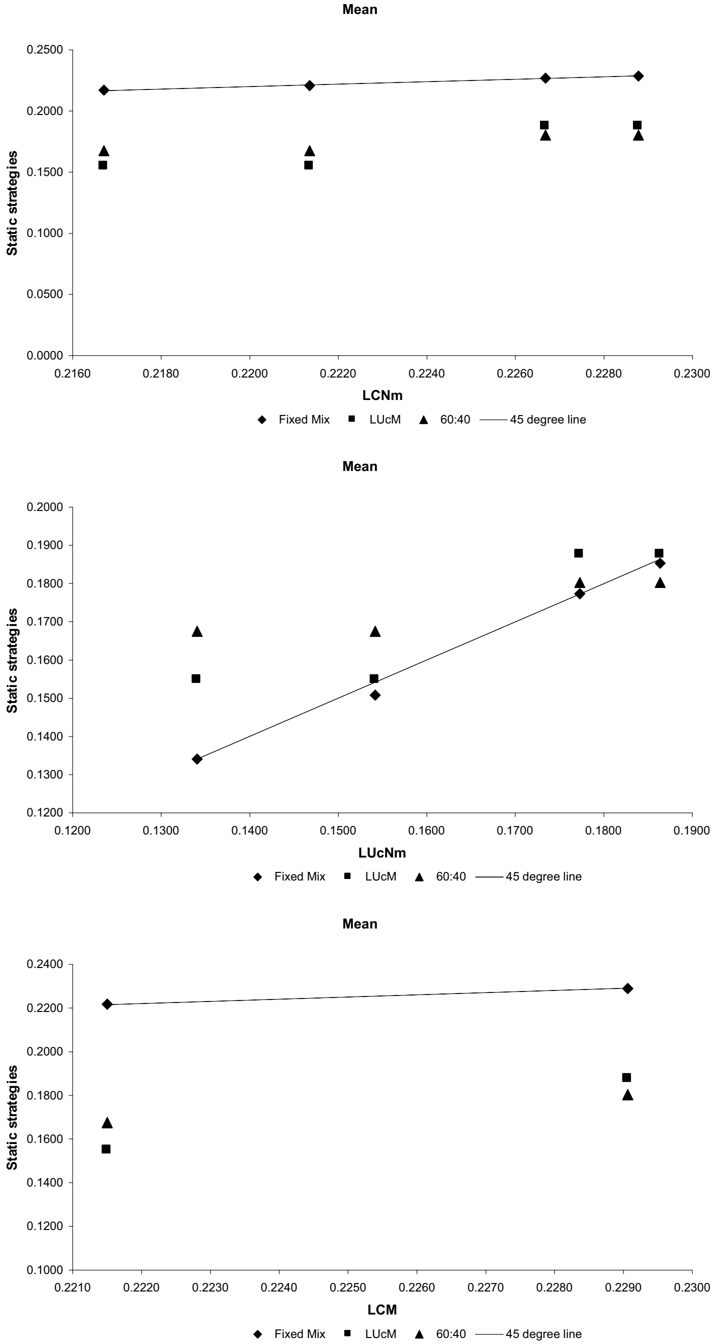


Figure 4.9. Mean of the contribution rate of dynamic vs. static strategies

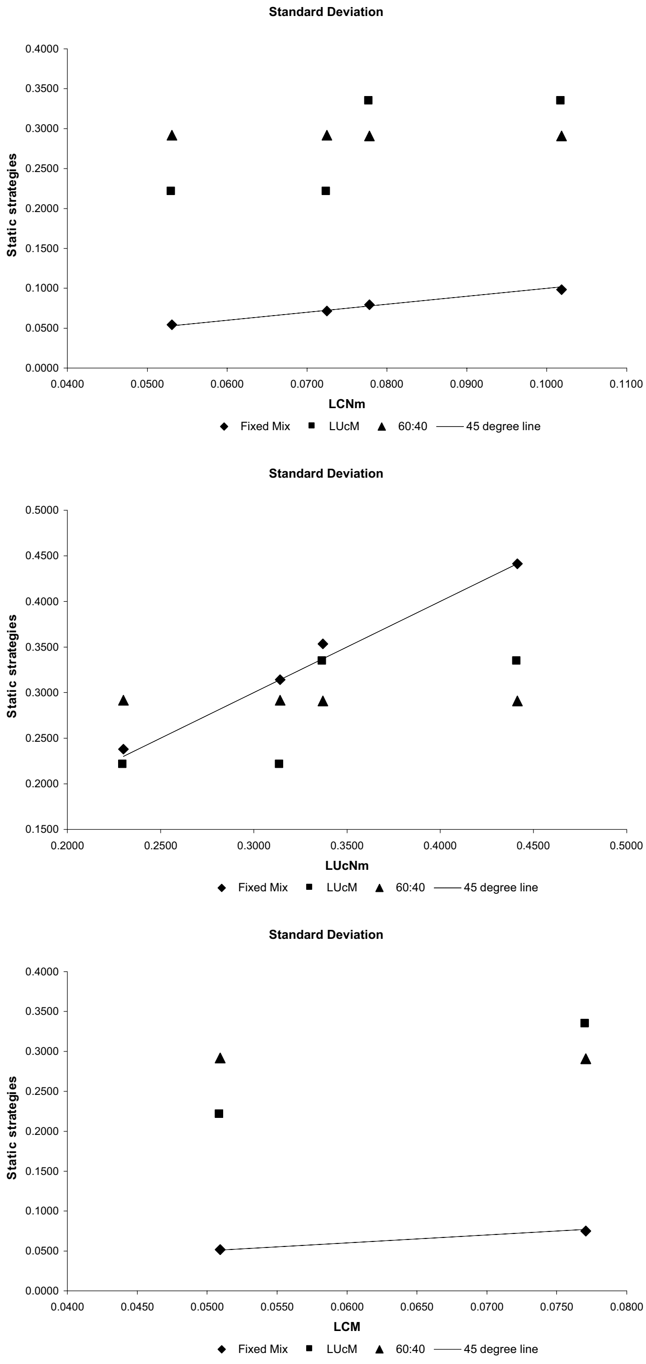


Figure 4.10. Standard deviation of the contribution rate of dynamic vs. static strategies

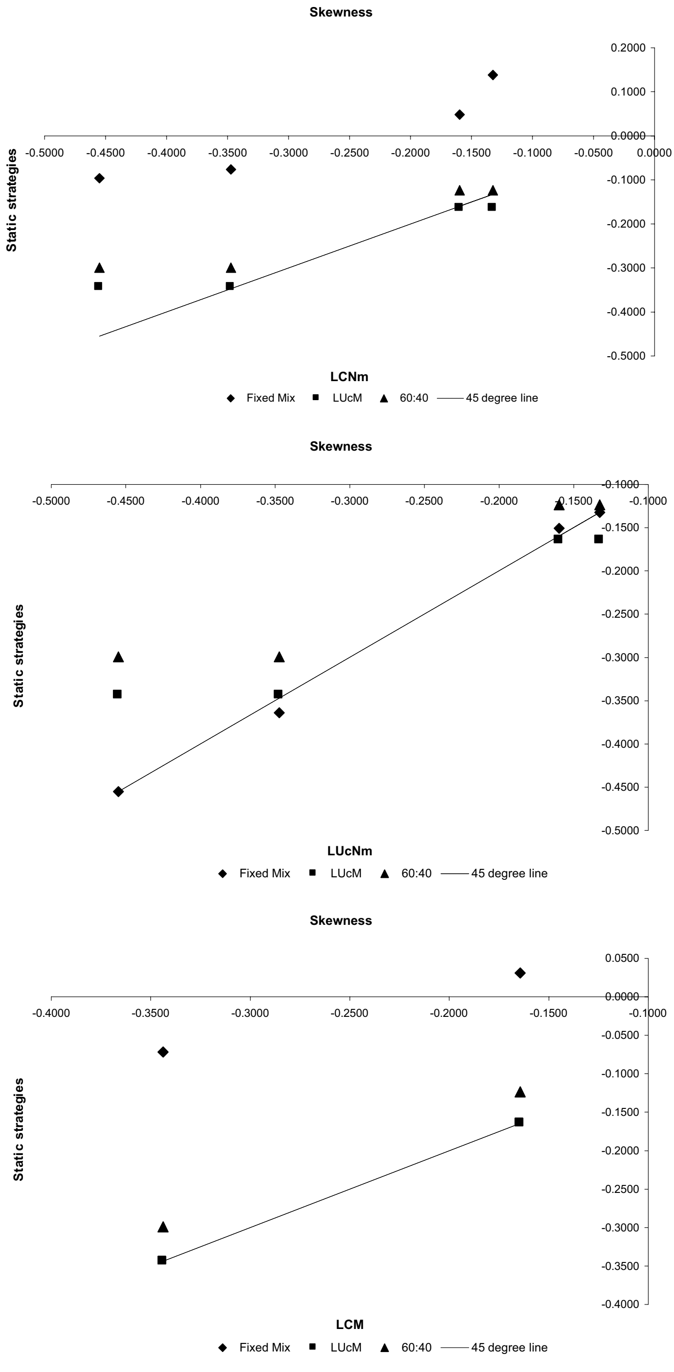


Figure 4.11. Skewness of the contribution rate of dynamic vs. static strategies

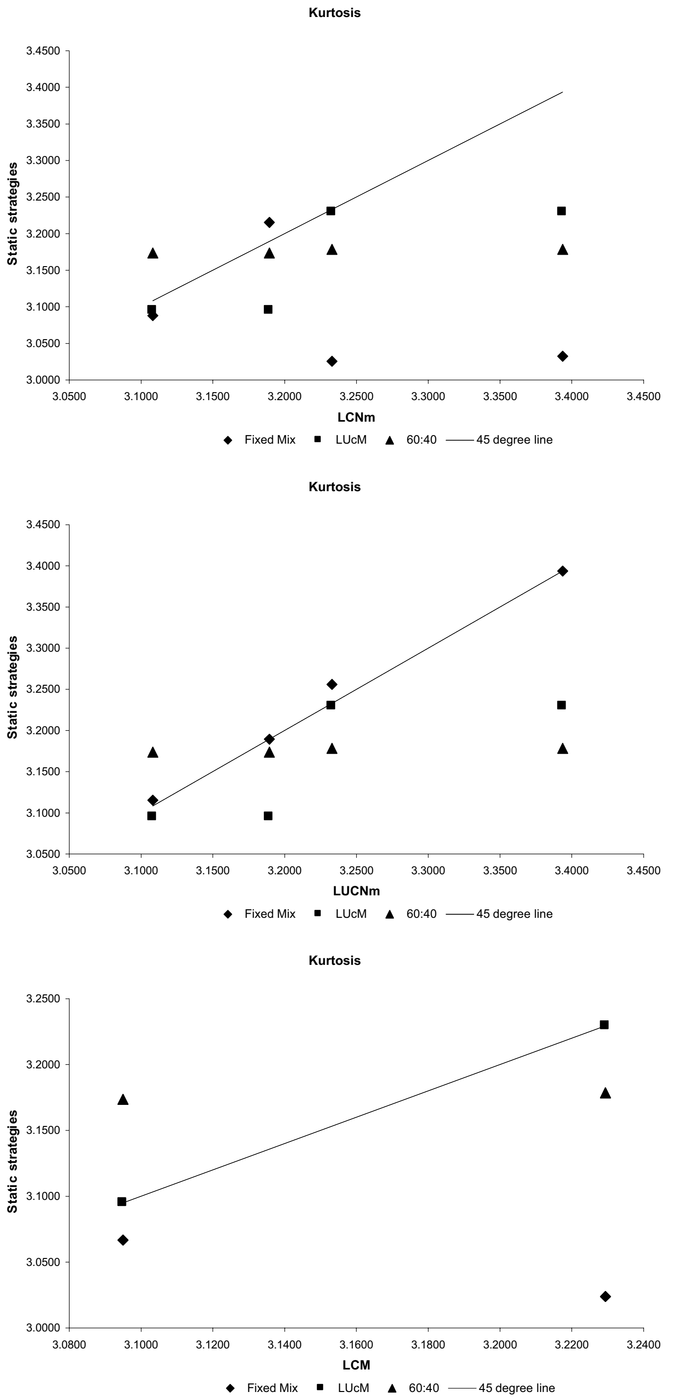


Figure 4.12. Kurtosis of the contribution rate of dynamic vs. static strategies

An advantage of the dynamic liability-constrained strategies over the fixed mixes is revealed mainly in the skewness. LCNm and LCM distributions are always more positively skewed for the funding ratio and more negatively skewed for the contribution rate. In the model with CIR dynamics fixed mix distributions have an undesirable property of negative skewness for the funding ratio and positive for the contribution rate. Also the tails of constrained distributions are in general fatter than of fixed mixes. Stronger asymmetry and peakness is not the case for LUcNm strategy, when compared against fixed mixes.

When dynamic strategies are evaluated against LUcM policy non-myopic distributions are still stronger skewed in the Vasicek model, but already less skewed in the CIR model. They remain more peaked and with fatter tails. As the liability constraint does not influence the skewness and kurtosis these moments of LCM distribution are the same as LUcM distribution. When assessed against the 60:40 rule dynamic distributions are still stronger skewed. They are also more peaked in the Vasicek model.

4.8.3 Imperfect liability hedge

The performance of dynamic strategies when there is a perfect liability hedge against the performance when the hedge is imperfect in terms of the funding ratio is shown in Figure 4.13 and in terms of the contribution rate in Figure 4.14. Darker bars refer to perfect, while lighter to imperfect hedge setting. When an unhedgable component is present in the liability the mean funding ratio is by 0.005 higher for liability-constrained and by ca. 0.0170 for liability–unconstrained strategy. The corresponding increases of its standard deviation as a percentage of mean amount to 2.7 p.p. for liability-constrained strategies and 4.3 p.p. for liability-unconstrained policies. Also skewness and kurtosis are higher in comparison with the case of perfect liability hedge. What concerns the distribution of the contribution rate its mean falls down by ca. 2.1 p.p. for constrained and by 7.2 p.p. for unconstrained policy. Its standard deviation as a percentage of mean rises by ca. 55 p.p. for constrained and 2.5-6 times for unconstrained strategy. The distribution is more negatively skewed and more peaked. Though part of the liability is unhedgable the strategy still ensures the probability of underfunding equal to zero for liability constrained strategies. However, the probability of underfunding for unconstrained strategy rises from .0032 to 0.0136 for long-term policy and from .0001 to .0068 for short-term strategy.

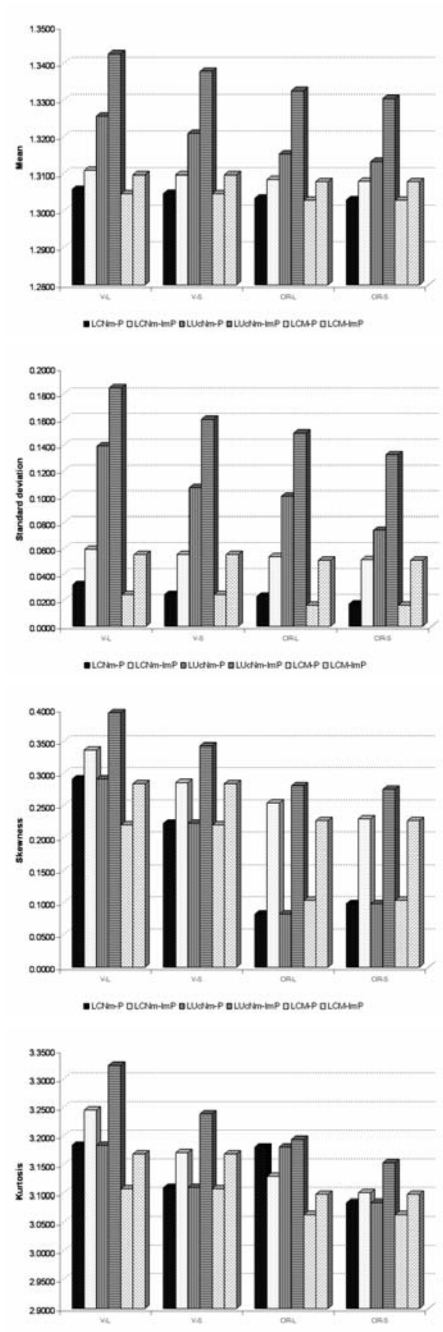


Figure 4.13. Funding ratio of dynamic strategies - perfect vs. imperfect liability hedge

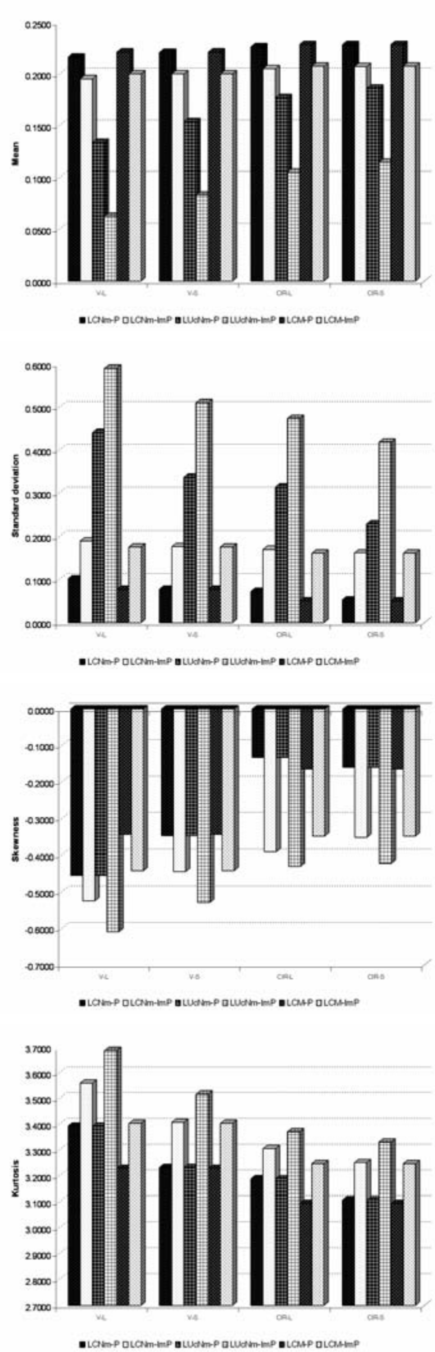


Figure 4.14. Contribution rate of dynamic strategies - perfect vs. imperfect liability hedge

4.9 Sensitivity analysis

Changes in the initial funding ratio have proportional effects on the average and standard deviation of the funding ratio, when the strategy is liability-unconstrained. The effect on the mean of liability-constrained ratio is slightly more than proportional. Change in the initial ratio by 15.38% causes a change in the mean ratio equal to 15.5-15.6%. Standard deviation changes by 2/3. Changes in the initial funding ratio have less than proportional effect on the average of contribution rate. These relative changes are reported in Table 4B.2 in Appendix 4B. They vary from ca. 3.5 to 12.5%. Standard deviation of contribution rate reacts to the changes of initial funding ratio as does the standard deviation of funding ratio. Skewness and kurtosis of both funding ratio and contribution rate are unaffected. Table 4B.3 in Appendix 4B also evidences that increasing the initial ratio decreases the probability and average of underfunding.

Increasing the relative risk aversion coefficient results in slightly lower average funding ratio and higher average contribution rate. Also kurtosis, though stronger, remains relatively weakly effected. In the Vasicek model it decreases, while in the CIR model increases with increasing risk aversion. Standard deviation and skewness of both funding ratio and contribution rate are effected the most. Both go down with the growth of aversion. The fall of standard deviation is stronger for short- than long-term policy. In the former case it is also stronger in the CIR model while in the latter in the Vasicek model. The same pattern can be spotted as far as the skewness is concerned in the Vasicek model. In the CIR model the asymmetry of both distributions for short-term horizon changes sign when risk aversion increases. Detailed results on the sensitivity to the risk aversion coefficient are given in Tables 4B.4 and 4B.5 in Appendix 4B. Table 4B.6 therein shows that increasing the risk aversion coefficient decreases the probability and average of underfunding.

Increasing the wage-to-liability ratio has no effects on the moments of the distribution of the funding ratio. It does not change the skewness and kurtosis of the contribution rate either. It does influence proportionally its average and standard deviation.

Increasing the pension-to-liability ratio has no effect on the distribution of the funding ratio. It does not change the standard deviation, skewness and kurtosis of the contribution rate either. Only the average of contribution rate reacts more than proportionally to the changes of this ratio. Detailed changes are reported in Table 4B.7 in Appendix 4B.

The changes in the level of underfunding, which is allowed in the scheme have no impact on the outcome of unconstrained strategy and on the skewness and kurtosis of the constrained funding ratio and of the constrained contribution rate. Standard deviation of both of them is effected to the same extent. Increasing the allowed level of underfunding from 0 to 10% causes the standard deviation to increase by 1/3. This sensitivity does not depend either on the dynamics of interest rate or the investment horizon. The average funding ratio increases, while the average contribution rate decreases with the growth of allowed underfunding. Detailed changes are reported in Table 4B.8 in Appendix 4B.

4.10 Conclusions

Liability-constrained dynamic portfolio strategies which we examine in this paper result in lower mean funding ratio and higher mean contribution rate than liability-unconstrained strategies at the benefit of lower volatility. The skewness and kurtosis of the ratio/rate are invariant to the liability constraint. The liability-unconstrained strategy leads to non-zero probability of underfunding, which surprisingly is higher for long- than short-term horizon strategy. In turn, following a myopic instead of a non-myopic policy results in a marginal decrease of the mean funding ratio and the rise of mean contribution rate at the profit of decreasing their standard deviations. The myopic distributions are less peaked and have thinner tails than non-myopic ones.

Utility based dynamic strategies are not superior to the fixed mix strategies. When they lead to a higher mean funding ratio and a lower contribution rate they also result in higher standard deviation. An advantage of the dynamic liability-constrained strategies over the fixed mixes is revealed mainly in the skewness. They are always more positively skewed for the funding ratio and more negatively skewed for the contribution rate. Stronger asymmetry is not the case for unconstrained strategies.

When an unhedgable component appears in the liability all the descriptive statistics are higher than corresponding statistics in the case of perfect hedge. Only the mean contribution rate is lower. Though part of the liability is unhedgable, constrained strategies still guarantee the probability of underfunding equal to zero for high enough funding ratios and cause this probability for unconstrained strategies to rise significantly.

Sensitivity analysis reveals that changes in the initial funding ratio do not effect the skewness and kurtosis of the funding ratio and the contribution rate. The effect on means and standard deviations of liability-unconstrained strategy is proportional. Changing the relative risk aversion coefficient causes standard deviation and skewness of both funding ratio and contribution rate change most. Increasing this aversion decreases the probability and average of underfunding for the unconstrained strategy. Altering the wage-to-liability ratio influences proportionally the average and standard deviation of contribution rate. Changes of pension-to-liability ratio make the average of contribution rate react more than proportionally. Allowing for underfunding in the scheme has no impact on the outcome of unconstrained strategy. It does not influence either the skewness and kurtosis of "constrained" distributions. Standard deviations are effected to the same extent.

4.11 Appendix 4A

The scheme's liability, $L(t)$, stems from two sources, namely pension rights earned by employees at time t, $L^R(t)$, and pension benefits paid out to retirees at time t, $P(t)$. Starting from the definition of $P(t)$ and accounting for (A2) and (A6) we have

$$dP(t) \equiv \sum_{l=1}^{L} dP_l(t) = \frac{dW(t)}{W(t)} \sum_{l=1}^{L} P_l(t) = P(t)\frac{dW(t)}{W(t)}. \tag{4A1}$$

To obtain the dynamics of $L^R(t)$ we start from its definition and account for the actuarial calculation of liability towards employee k at time t, $L_k^R(t)$, as follows

$$L^R(t) \equiv \sum_{k=1}^{K} L_k^R(t) = \sum_{k=1}^{K} R_k(t) \sum_{j=41-d_k(t)}^{60-d_k(t)} (1+a)^{-j}.$$

Constant a denotes the rate with which the discounting of future pension benefits is done. In the above we make use of (A5). Finally, we account for the dynamics of $R_k(t)$, which follow when we take into account assumptions (A1), (A3) and (A6) from Section 4.3, to end up with

$$\begin{aligned} dL^R(t) &\equiv \sum_{k=1}^{K} dR_k(t) \sum_{j=41-d_k(t)}^{60-d_k(t)} (1+a)^{-j} = \\ &= \frac{dW(t)}{W(t)} 0.0175\,(W(t) - PP(t)) \sum_{k=1}^{K} d_k(t) \sum_{j=41-d_k(t)}^{60-d_k(t)} (1+a)^{-j} = \\ &= \frac{dW(t)}{W(t)} \sum_{k=1}^{K} L_k^R(t) = \frac{dW(t)}{W(t)} L^R(t). \end{aligned} \tag{4A2}$$

Combining (4A1) and (4A2) we obtain total liability evolving as

$$dL(t) \equiv dL^R(t) + dP(t) = \left(L^R(t) + P(t)\right)\frac{dW(t)}{W(t)} = L(t)\frac{dW(t)}{W(t)}.$$

4.12 Appendix 4B

Table 4B.1. Dynamic strategies with imperfect liability hedge

	LCNm♦		LUcNm♣		LCM✠
	LR♠	SR▲	LR	SR	
			Vasicek interest rate		
			Funding ratio		
mean	1.3109	1.3098	1.3427	1.3379	1.3098
std. dev.	.0598	.0558	.1848	.1604	.0556
skewness	.3369	.2863	.3948	.3437	.2848
kurtosis	3.2454	3.1710	3.3240	3.2392	3.1688
			Contribution rate		
mean	.1956	.2004	.0628	.0834	.2006
std. dev.	.1902	.1767	.5912	.5109	.1763
skewness	-.5242	-.4443	-.6089	-.5290	-.4422
kurtosis	3.5605	3.4080	3.6868	3.5178	3.4040
			Underfunding		
probab.	.0000	.0000	.0268	.0157	.0000
average	.0000	.0000	.0019	.0009	.0000
			CIR interest rate		
			Funding ratio		
mean	1.3086	1.3081	1.3326	1.3305	1.3080
std. dev.	.0540	.0515	.1496	.1327	.0512
skewness	.2544	.2300	.2810	.2757	.2269
kurtosis	3.1297	3.1012	3.1944	3.1538	3.0984
			Contribution rate		
mean	.2055	.2076	.1052	.1145	.2079
std. dev.	.1706	.1623	.4741	.4188	.1615
skewness	-.3911	-.3516	-.4301	-.4219	-.3483
kurtosis	3.3060	3.2519	3.3717	3.3320	3.2480
			Underfunding		
probab.	.0000	.0000	.0136	.0068	.0000
average	.0000	.0000	.0008	.0003	.0000

The numbers in Table 4B.1. are levels. ♦liability-constrained nonmyopic strategy, ♣liability-unconstrained nonmyopic strategy, ✠ liability-constrained myopic strategy, ♠long-rolled forward horizon, ▲short-rolled forward horizon

Table 4B.2. Sensitivity of mean contribution rate to the funding ratio

	LCNm		LUcNm		LCM
	LR	SR	LR	SR	
Vasicek	7.73	6.14	12.6	8.87	6.09
CIR	4.42	3.76	5.68	4.62	3.67

The numbers in Table 4B.2. are the differences between the contribution rate when the funding ratio equals 1.1 and the contribution rate when the funding ratio equals 1.3. To obtain the differences between the contribution rate when the funding ratio equals 1.5 and the contribution rate when the funding ratio equals 1.3 place minus sign before every number.

Table 4B.3. Sensitivity of LUcNm underfunding to the funding ratio

	FR=1.1		FR=1.3		FR=1.5	
	LR	SR	LR	SR	LR	SR
			Probability in %			
Vasicek	14.07	9.12	1.00	.20	.05	.00
CIR	8.89	4.23	0.29	.01	.01	.00
			Average			
Vasicek	.0090	.0042	.0005	.0001	.0000	.0000
CIR	.0046	.0014	.0001	.0000	.0000	.0000

The numbers in the panel "Probability" is the probability of underfunding for the liability-unconstrained nonmyopic strategy when the liability hedge is perfect for different levels of initial funding ratio. The numbers in the panel "Average" is the mean size of underfunding when it occurs.

Table 4B.4. Sensitivity to the risk aversion coefficient $1-\gamma = 1.5$

	LCNm		LUcNm		LCM
	LR	SR	LR	SR	
			Vasicek interest rate		
			Funding ratio		
mean	.13	.16	.57	.69	.16
std. dev.	23.52	42.61	23.52	42.61	43.51
skewness	22.20	42.31	22.20	42.31	43.32
kurtosis	2.68	3.35	2.68	3.35	3.36
			Contribution rate		
mean	-3.35	-3.97	-23.52	-24.71	-3.99
std. dev.	24.18	43.70	24.18	43.70	44.47
skewness	21.99	41.37	21.99	41.37	42.15
kurtosis	5.50	7.00	5.50	7.00	7.03
			CIR interest rate		
			Funding ratio		
mean	.11	.12	.46	.52	.12
std. dev.	21.44	45.50	21.44	45.50	50.57
skewness	119.29	94.03	119.29	94.03	85.75
kurtosis	-1.33	.65	-1.33	.65	1.12
			Contribution rate		
mean	-2.54	-2.84	-14.08	-15.07	-2.87
std. dev.	22.11	47.45	22.11	47.45	51.32
skewness	105.56	82.34	105.56	82.34	78.84
kurtosis	1.04	2.91	1.04	2.91	3.21

The numbers in Table 4B.4. are the relative changes in percentage of the funding ratio and the contribution rate when the risk aversion coefficient decreases from 2 to 1.5.

Table 4B.5. Sensitivity to the relative risk aversion coefficient $1 - \gamma = 4$

	LCNm		LUcNm		LCM
	LR	SR	LR	SR	
			Vasicek interest rate		
			Funding ratio		
mean	-.20	-.24	-.85	-1.03	-.24
std. dev.	-25.36	-57.63	-25.36	-57.63	-59.33
skewness	-22.92	-56.06	-22.92	-56.06	-57.91
kurtosis	-2.32	-2.45	-2.32	-2.45	-2.43
			Contribution rate		
mean	4.96	5.86	34.82	36.45	5.89
std. dev.	-25.93	-58.26	-25.93	-58.26	-59.70
skewness	-24.08	-55.56	-24.08	-55.56	-56.97
kurtosis	-4.88	-5.09	-4.88	-5.09	-5.07
			CIR interest rate		
			Funding ratio		
mean	-.16	-.18	-.68	-.77	-.18
std. dev.	-12.86	-54.81	-12.86	-54.81	-66.25
skewness	-147.70	-244.82	-147.70	-244.82	-255.89
kurtosis	4.72	9.47	4.72	9.47	9.67
			Contribution rate		
mean	3.75	4.20	20.80	22.34	4.26
std. dev.	-13.52	-57.62	-13.52	-57.62	-66.51
skewness	-135.26	-217.32	-135.26	-217.32	-234.55
kurtosis	2.62	8.70	2.62	8.70	9.29

The numbers in Table 4B.5. are the relative changes in percentage of the funding ratio and the contribution rate when the risk aversion coefficient increases from 2 to 4.

Table 4B.6. Sensitivity of LUcNm underfunding to the relative risk aversion coefficient

	$1-\gamma=1.5$		$1-\gamma=2$		$1-\gamma=4$	
	LR	SR	LR	SR	LR	SR
			Probability in %			
Vasicek	2.39	1.47	1.02	.21	.21	.00
CIR	.65	.27	0.3	.01	.26	.00
			Average			
Vasicek	.0015	.0008	.0005	.0001	.0001	.0000
CIR	.0003	.0001	.0001	.0001	.0000	.0000

The numbers in the panel "Probability" are the probabilities of underfunding for the liability-unconstrained nonmyopic strategy when the liability hedge is perfect for different levels of relative risk aversion coefficient. The numbers in the panel "Average" is the mean size of underfunding when it occurs.

Table 4B.7. Sensitivity of mean contribution rate to the pension-to-liability ratio

	LCNm		LUcNm		LCM
	LR	SR	LR	SR	
Vasicek	53.29	52.18	86.10	74.89	52.14
CIR	50.94	50.48	65.06	61.94	50.42

The numbers in Table 4B.7 are the relative percentage changes in the mean contribution rate occurring when the pension-to-liability ratio increases from 11.5% to 17%. To obtain the changes when the pension-to-liability ratio decreases from 11.5% to 6% place the minus sign before each number.

Table 4B.8. Sensitivity of means to the admitted underfunding

	LCNm		LUcNm		LCM
	LR	SR	LR	SR	
			Vasicek interest rate		
			Mean funding ratio		
f=.7	.46	.37	.00	.00	.37
f=.9	.15	.12	.00	.00	.12
			Mean contribution rate		
f=.7	-11.48	-9.11	.00	.00	9.03
f=.9	-3.83	-3.04	.00	.00	3.01
			CIR interest rate		
			Mean funding ratio		
f=.7	.27	.24	.00	.00	.23
f=.9	.09	.08	.00	.00	.08
			Mean contribution rate		
f=.7	-6.54	-5.54	.00	.00	-5.40
f=.9	-2.18	-1.85	.00	.00	-1.80

The numbers in Table 4B.8 are the relative percentage changes in the mean funding ratio and the mean contribution rate occurring when the admitted level of underfunding changes from $f = 1$ to $f = 0.7$ and $f = 0.9$ respectively.

5 Portfolio insurance policies

5.1 Introduction

Portfolio insurance or constant proportion portfolio insurance strategies (CPPI) were introduced in the financial markets in the early eighties. The company Leland O'Brien and Rubinstein was among the frontrunners. The basic idea behind portfolio insurance is to invest in the stock market and at the same time buy a put option on the stock market index to prevent a blow up of the total portfolio value. However, for a sizeable investment portfolio it will be difficult to source the put option from the market. But according to the theory of Black, Scholes and Merton option payoffs can be replicated by self-financing dynamic portfolio strategies, where stocks and riskless bonds are continuously traded. Hence, instead of buying puts these dynamic strategies can be implemented, in order to achieve the same result; exposure to the stock market and at the same time downside protection. It follows from the put call parity that the portfolio insurance is also equivalent to buying a call and a riskless investment. Hence, the dynamic strategy can also be seen as replicating a call option. In replicating a call the delta goes up if the stock index goes up and one has to buy more stocks. Hence, one also has to sell stocks if the stock index decreases. We can also conclude that the strategy is a buy high sell low strategy. Next to these replicating strategies also variants called constant proportion portfolio insurance were introduced to the market. Also for these strategies stocks have to be bought if the index goes up and sold if the index goes down. By the mid eighties a significant amount of money invested in the stock markets was subject to some kind of portfolio insurance strategy.

On October 19, 1987, the Dow Jones Industrial Average fell down by over 500 points, which was equivalent to roughly 20 percent. A large number of market commentators attributed this stock market crash to the fact that too many portfolio insurance strategies were outstanding. As explained above portfolio managers following these strategies had to sell stocks when the market dropped. Their selling had a further depressing effect on stock prices and hence, they had to sell even more stocks, which again drove stock prices further down. In this way a kind of domino effect was created. Jacobs [1999] gives an excellent account of what happened during that period. The portfolio managers could not stop their strategies before they had permission of their clients, since all risks were borne by the clients. After the stock market crash the portfolio insurance strategies lost their popularity and disappeared.

Recently the interest in portfolio insurance has picked up again, however with one main difference. The strategies are now sold to investors, both retail and non-retail investors, in the form of investment notes, where money is invested according to the same principles. However, the issuer of the notes guarantees a certain final value at maturity of the notes. In a standard product the investor is guaranteed to get his initial investment back at maturity and the note issuer buys a zero coupon bond to assure the guaranteed amount at maturity. The difference between initial investment and the zero coupon bond value is the so called buffer. This buffer is leveraged up a number of times by borrowing money and then invested in e.g. the stock market. The leverage factor, which is the quotient of the stock investment and the buffer is kept within certain bounds. When stock prices drop, the buffer falls and stocks have to be sold. Since the notes are guaranteed by the issuer, the investor no longer runs gap risk. Gap risk is the risk of sudden drops in prices. E.g. if the multiplier if 5, then a drop of more than 20 percent (the gap) eats the complete buffer away. The investor still gets his money back and the issuer is liable for the losses above 20% . The portfolio insurance or CPPI strategies can all be seen as replicating options, where the option premium is not paid upfront, but paid through the hedging costs of buying high and selling low. Hence, these costs are not determined by market implied volatilities at the start, but by the realized volatilities over the period that the notes are running. These costs are borne by the investors and increasing volatility of the markets will reduce revenues of the product.

CPPI products are not only sold with investments in stocks, but currently there are variants where investments are made in foreign exchange products, commodities, credit default swaps, funds of mutual funds and even funds of hedge funds. This is the reason for our research in this chapter on the theoretical and empirical properties of portfolio insurance strategies.

The important review on asset allocation strategies (including the portfolio insurance) is the paper by Perold and Sharpe [1995]. In their review of asset allocation strategies they distinguish buy-and-hold, constant mix, constant-proportion portfolio insurance (CPPI) and option-based portfolio insurance (OBPI) strategies. The latter two guarantee in a continuous-time setting that the value of portfolio does not fall below a certain floor. An alternative classification is given by Trippi and Harriff [1991]. Formally, OBPI stems from the solution of a utility maximization problem in which the portfolio value is restricted from below. As Cox and Huang [1989] show for the case of a zero floor, the constrained portfolio is equivalent to the combination of unconstrained portfolio and a put option on it with strike price equal to the floor. Leland [1980] notices that " a portfolio of options is not equivalent to an option on a portfolio". Hence, including into the portfolio a put option on a stock index does not guarantee that the portfolio of option, stocks and bond does not fall below the floor except in the case that the portfolio is equal to the stock index.

Constant proportion portfolio insurance was introduced into the literature by Black and Jones [1987]. Theoretical properties of CPPI were developed by Black and Perold [1992] who account for the borrowing restrictions and transaction costs. CPPI is a dynamic strategy which maintains the portfolio's risk exposure at a constant multiple of the excess of wealth over a floor providing at the same time a downside protection. Recently, CPPI was compared against OBPI by Cesari and Cremonini [2003] and Bertrand and Prigent [2003]. Both papers use simulation for the purpose of this comparison. The former considers CPPI when the volatility of risky asset is stochastic. CPPI is compared against OBPI for instance in Zhu and Kavee [1988]. Further assessment of synthetic put strategies by simulation methods is given among others in Bird et al. [1988] and Bird et al. [1990].

As remarked earlier, it is believed that the portfolio insurance strategies are destabilizing the markets. However, in theoretical models of general equilibrium Basak [1995] and Basak [2002] attains the opposite result.

The aim of the current chapter is twofold. First, we introduce what we call a modified CPPI (MCPPI). MCPPI is a strategy followed by some of the commercial banks. In contrast to CPPI in which there is one active (stock index) and one reserve (bond) asset MCPPI makes use of another reserve asset (bank account). We provide some of the theoretical properties of MCPPI versus CPPI and assess their historical performance based on the German DAX-index. Second, as far as the CPPI is concerned we compare our empirical results against those obtained from the theoretical formulas contained in the paper by Black and Perold [1992].

As far as the theoretical aspects of MCPPI versus CPPI are concerned we derive formulas for the increment in exposure, cushion and exposure-to-cushion ratio since the last rebalancing. In continuous time we derive the cushion at time t when both ratios, i.e. stock-to-bond and cash-to-bond, follow geometric Brownian motions.

We find that in the German market MCPPI performs slightly better in terms of mean terminal wealth. Stock index volatility is statistically significant in explaining the terminal wealth. When the portfolio is rebalanced daily the cushion calculated from the data is very close to the cushion calculated from the theoretical continuous-time formulas for both CPPI and MCPPI. When the rebalancing takes place after there is a given move in the index ratio since the initial data/last rebalancing the empirical and theoretical cushions are similar for 1-year horizon, but strikingly different for 3- and 5-year horizons.

The chapter is organized as follows. Section 5.2 presents the asset allocation policies we consider. Section 5.3 exhibits the theoretical aspects of MCPPI and CPPI in the discrete and the continuous time. Section 5.4 provides some motivating examples which justify the empirical part of the chapter. In section 5.5 we describe the data characterizing German market. Section 5.6 provides the empirical results. Section 5.7 concludes.

5.2 Asset allocation rules

Asset allocation strategies, which are considered in this chapter include CPPI and MCPPI. In this section we describe the above strategies briefly.

5.2.1 CPPI

In a CPPI strategy the wealth is invested in the active asset (stock index) and the reserve asset (bond). Let $F(t)$ represent a required floor on the portfolio value at time t with $F(T) = F$ fixed. We assume that a $T-$bond investment guarantees the floor implying

$$F(t) = \frac{FB_T(t)}{B_T(T)} = \omega B_T(t),$$

with $\omega = FB_T^{-1}(T)$ standing for the number of bonds. Hence, the floor fluctuates with the bond price and consequently with the interest rate. Let $W(t)$ denote the wealth at time t and the cushion, $C(t)$, be defined as

$$C(t) = W(t) - F(t).$$

In a CPPI strategy the exposure, $E(t)$, i.e. the stock index investment, is usually equal to m times the cushion

$$E(t) = mC(t),$$

with $m > 1$. The remainder is invested in the zero coupon bond

$$\begin{aligned} R^B(t) &= W(t) - E(t) = \\ &= F(t) - (m-1)\,C(t). \end{aligned}$$

Long position in bond is held when $(m-1)\,C(t)$ is smaller than the floor and short when it is larger. In a discrete trading environment, the portfolio value may fall below the floor when there is a large drop in the active asset between two successive rebalancing moments. In other words, the cushion becomes negative when $C(t) + \Delta C(t) < 0$ what after substitution yields $1 + m\delta < 0$, where δ is the fractional change in the active asset. This risk is the so-called gap risk as described in the introduction. In the current environment this gap risk is taken by the issuing banks and no longer by the end investor. In the case of frequent rebalancing one might expect $1 + m\delta$ never to become negative. In the data which we use in the empirical part of the chapter this incident never happened and so the floor was never endangered.

5.2.2 MCPPI

In a MCPPI strategy the set of investment assets contains stock, bond and cash. We consider bond and cash as reserve assets. Amount equal to $F(t)$ is put in the zero-coupon bond implying $R^B(t) = F(t)$. Stock index exposure is again equal to

$$E(t) = mC(t),$$

while cash investment

$$\begin{aligned} R^C(t) &= W(t) - R^B(t) - E(t) = \\ &= (1-m)C(t). \end{aligned}$$

Because $m > 1$ the excess exposure is financed by cash borrowing. Again in a discrete trading, the portfolio value may fall below the floor when $1 + m\delta_1 + (1-m)\delta_2 < 0$, where δ_i, $i = 1, 2$, is the fractional change in the stock index, $i = 1$, and cash, $i = 2$, respectively. Again, in the data which we use in the empirical part of the chapter this never happened.

5.3 Theoretical aspects of portfolio insurance strategies

In their theoretical analysis of portfolio insurance strategies Black and Perold [1992] choose the reserve asset (bond) as numeraire. All quantities, $B_T(t)$, $S(t)$, $F(t)$, $C(t)$, $E(t)$, $W(t)$, are expressed in this numeraire. In particular, $I(t) \equiv S(t)B_T^{-1}(t)$ is called the index ratio, $B_T(t) = 1$ and $F(t) = \omega$. We choose the same numeraire when we consider MCPPI. Then, there are two ratios, $I_1(t) \equiv S(t)B_T^{-1}(t)$ and $I_2(t) \equiv M(t)B_T^{-1}(t)$. In the above $S(t)$ stands for the stock index, $B_T(t)$ for the bond price and $M(t)$ for the money market account.

5.3.1 CPPI

As in Black and Perold [1992] let δ be the fractional change in the CPPI index ratio. The formulas given in Black and Perold [1992] are

$$\Delta E(t) = \delta E(t), \tag{5.1}$$

$$\Delta C(t) = m\delta C(t), \tag{5.2}$$

$$\Delta[E(t)C^{-1}(t)] = (1-m)\delta\left(1+m\delta\right)^{-1} E(t)C^{-1}(t), \tag{5.3}$$

where Δ stands for the change since the last rebalancing. From equation (5.3) the fractional change in the exposure-to-cushion ratio is a decreasing function of δ

$$\frac{d}{d\delta}\frac{\Delta[E(t)C^{-1}(t)]}{E(t)C^{-1}(t)} = \frac{1-m}{\left(1+m\delta\right)^2} < 0,$$

meaning that any increase of δ causes this fraction to fall triggering the purchase of active asset to restore the ratio to m. Hence, CPPI is a buy high strategy.

Black and Perold [1992] assume that the portfolio is rebalanced after a fractional upward move of the underlying of size u or after a fractional downward move of size d, where $u > 0$ and $d < 0$ are chosen such that $(1+u)(1+d) = 1$. Hence, the elapsed time between two consecutive portfolio rebalancings is unknown in advance. Then, they show that after n trades the cushion is given by

$$C_n = C_0\alpha^{0.5n}\left(S_nS_0^{-1}\right)^{\gamma}, \tag{5.4}$$

where

$$
\begin{aligned}
\alpha &= (1+mu)(1+md), \\
\gamma &= 0.5\ln(1+mu)\left[(1+md)\ln(1+u)\right]^{-1}.
\end{aligned}
$$

In the continuous-time setting Black and Perold [1992] assume that the index ratio follows geometric Brownian motion with volatility σ

$$I(t) = I_0 \exp(\mu t + \sigma z(t)).$$

Then, they show that between 0 to t as $\Delta t \longrightarrow 0$

$$\frac{C(t)}{C_0} \longrightarrow \exp\left(-\frac{1}{2}\left(m^2 - m\right)\sigma^2 t\right)\left(\frac{I(t)}{I_0}\right)^m, \tag{5.5}$$

with probability 1.

Formula (5.5) allows us to focus on the role of volatility in a CPPI strategy. However, we first focus on two special cases. For $m = 0$ nothing is invested in the index and in this case $C(t) = C_0$. Since $C(t)$ is expressed in the numeraire it means that the portfolio has grown according to the risk free rate. For $m = 1$ $C(t) = C_0\frac{I(t)}{I_0}$ which indicates that we have invested only cushion in the index and hence do not have to do any rebalancing. Thus, the volatility does not influence the final result. For $m > 1$ we can interpret (5.5) as consisting of two parts: $\left(\frac{I(t)}{I_0}\right)^m$ is the increase in a value of the cushion due to the investment multiple m. The second part $\exp\left(-\frac{1}{2}\left(m^2 - m\right)\sigma^2 t\right)$ represents the reduction in the cushion due to hedging costs. This factor is smaller for larger σ, representing higher costs. $\left[1 - \exp\left(-\frac{1}{2}\left(m^2 - m\right)\sigma^2 t\right)\right]$ can be seen as the costs for replicating a kind of put option, where these hedging costs are not taken upfront. We come back to this in 5.4.2.

5.3.2 MCPPI

We next derive the fractional changes as in (5.1)-(5.3) for the MCPPI strategy.

Proposition 1 *Let δ_k be the fractional change in $I_k(t)$, $k = 1, 2$. Then, since the last rebalancing*

$$
\begin{aligned}
\Delta E(t) &= \delta_1 E(t), && \text{(5.6)} \\
\Delta C(t) &= \left[m\delta_1 + (1-m)\,\delta_2\right] C(t), && \text{(5.7)} \\
\Delta[E(t)C^{-1}(t)] &= (1-m)(\delta_1 - \delta_2)\left[1 + m\delta_1 + (1-m)\,\delta_2\right]^{-1} E(t)C^{-1}(t), && \text{(5.8)}
\end{aligned}
$$

Equation (5.7) shows that moves in $I_1(t)$ are magnified m times in the cushion and moves in $I_2(t)$ are magnified $(1-m)$ times. From equation (5.8)

$$\begin{aligned}
\frac{d}{d\delta_1}\frac{\Delta\left[E(t)C^{-1}(t)\right]}{E(t)C^{-1}(t)} &= \frac{(1-m)(1+\delta_2)}{\left(1+m\delta_1+(1-m)\,\delta_2\right)^2}, \\
\frac{d}{d\delta_2}\frac{\Delta\left[E(t)C^{-1}(t)\right]}{E(t)C^{-1}(t)} &= \frac{-(1-m)(1+\delta_1)}{\left(1+m\delta_1+(1-m)\,\delta_2\right)^2}.
\end{aligned}$$

An increase in $I_1(t)$ causes the exposure-to-cushion ratio fall for the range of $\delta_2 \in (-1, \infty)$, while an increase in $I_2(t)$ makes the ratio to rise for the same range of δ_1, ceteris paribus. In other words, in order to restore the ratio to m stock index is bought high and cash position is reduced.

As in (5.5) we derive the continuous-time cushion for MCPPI strategy.

Proposition 2 *Let the index ratios follow geometric Brownian motions with the drift rates μ_k and volatilities σ_k, $k = 1, 2$. Then,*

$$C(t) = C_0 \exp\left[\mu t + m\sigma_1 z_1(t) + (1-m)\sigma_2 z_2(t)\right],$$

where

$$\mu = m\mu_1 + (1-m)\mu_2 - \frac{1}{2}m^2\sigma_1^2 - \frac{1}{2}(1-m)^2\sigma_2^2 - (1-m)m\sigma_1\sigma_2\rho,$$

with ρ being the correlation coefficient between the standard Brownian motions $z_1(t)$ and $z_2(t)$. Represented in terms of the index ratios the cushion writes as

$$C(t) = C_0 \exp\left[\left((m^2-m)\sigma_1\sigma_2\rho - \frac{1}{2}(m^2-m)\sigma_1^2 - \frac{1}{2}(m^2-m)\sigma_2^2\right)t\right]\left[\frac{I_1(t)}{I_1(0)}\right]^m\left[\frac{I_2(t)}{I_2(0)}\right]^{1-m}. \tag{5.9}$$

5.4 Motivating examples

To illustrate our later results in an intuitive way we provide three motivating examples. The first one shows that an investor may have an incentive to implement MCPPI as well as CPPI. Our second example examines the effect of volatility on the terminal wealth of both policies, while the third one concerns the discrete-time cushion of CPPI.

5.4.1 Performance of CPPI and MCPPI

We consider simple binomial tree model of the market. In this model we assume the initial wealth of 1000, the floor at time $T = 1$ year equal to the initial wealth and the multiplier $m = 4$. The long term rate is fixed at 5%, while the short-term rate fluctuates up and down on the three-period binomial tree starting from the 5% level . When stock price increases by three steps of 5 from 100 up to 115 the rate decreases every period by 50% from 5% in the first node to .625% in the last node (path 1 the most upper on the tree). When stock prices decrease by three steps of 5 from 100 down to 85 the rate increases every period by 50% from 5% in the first node to 16.875% in the last node (path eight the most lower on the tree). The eight paths describing the movement of the stock price and the short term interest rate are given in the below tree.

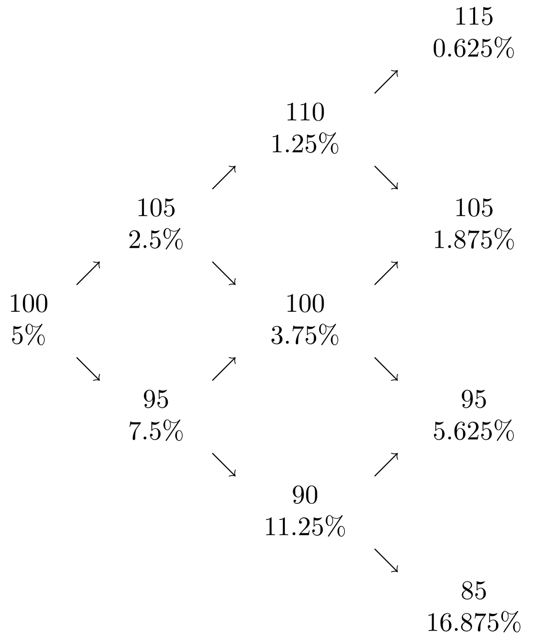


Figure 5.1. The stock index and the short interest rate movement

For the total of eight paths the terminal wealth of CPPI and MCPPI strategies are given in Table 5.1. Figures in this table show that when the short rate is below the long rate MCPPI outperforms CPPI due to the lower cost of borrowing. In the opposite case CPPI results in higher terminal wealth than MCPPI. Hence, it is worthwhile to compare CPPI and MCPPI on the empirical data concerning the stock index and interest rates. The example in question also allows two more remarks.

First, we see that paths resulting in the same stock's price at $T = 1$ year do not necessarily have the same terminal wealth. Second, both MCPPI and CPPI are buy high and sell low strategies. We illustrate the latter property by more detailed presentation of paths 1 and 8.

Table 5.1. Terminal wealth of MCPPI and CPPI strategies

Path♦	Terminal wealth	
	MCPPI	CPPI
100-105-110-115	1079.347	1072.317
100-105-110-105	1053.530	1049.074
100-105-100-105	1051.104	1048.926
100-105-100-95	1031.494	1031.902
100-95-100-105	1048.131	1048.754
100-95-100-95	1029.662	1031.790
100-95-90-95	1027.316	1031.660
100-95-90-85	1009.102	1013.648

♦Path describes the stock prices in the nodes of the tree shown in Figure 5.1.

Along path 1 the exposure before rebalancing in CPPI strategy is 195-205-235-267, while after the rebalancing it is 195-224-256-289. We spot that the exposure after the rebalancing is always larger than before meaning that additional shares of the stock index need to be bought. As the stock index increases from one period to another it means buy high. Similarly in the MCPPI strategy. The exposure before rebalancing is 195-205-240-282, while after 195-229-270-317. When stock index follows path 8 the exposure before rebalancing in CPPI is given as 195-185-139-102, while after rebalancing 195-146-108-79. The exposure after rebalancing is always smaller than before. As the stock index decreases from one period to another it means sell low. Similarly in the MCPPI strategy. The exposure before rebalancing is 195-185-133-90, while after 195-141-95-58.

5.4.2 Impact of volatility

Let us consider four markets with reversals. These reversals are given in the first column of Table 5.2. In these four cases the terminal wealth of CPPI and MCPPI strategies are also presented. We assume the long-term interest rate of 5% to calculate the growth of the bond price and the short-term interest rate of 3% to calculate the growth of the money market account.

It is easy to spot that the first two markets are characterized by the same volatility of the stock index. Also the third and fourth markets have the same volatility. Equal volatilities result in equal terminal wealth, when the investment horizon and the relations of the terminal to initial ratios are the same. This confirms the information carried out by the formulas for the continuous-time cushions in section 5.3. It is also clear that higher volatility as in the first scenarios dilute the value of the portfolio.

Table 5.2. Terminal wealth of MCPPI and CPPI strategies

Path[♦]	Terminal wealth	
	MCPPI	CPPI
100-95-100-105-100-90-100-110-100	1032.062	1029.970
100-105-100-95-100-110-100-90-100	1032.062	1029.970
100-105-105-100-95-95-100	1041.801	1039.241
100-95-95-100-105-105-100	1041.801	1039.241

[♦] Path describes the stock prices in the subsequent moments of time

5.4.3 Discrete-time cushion of CPPI

In order to illustrate formula (5.4) we construct a five-period binomial recombining tree describing the movement of the index ratio. We choose $u = 0.05$ and d satisfying $(1+u)(1+d) = 1$, $u > 0$, $d < 0$. Every period the index ratio changes by $(1+u)^{0.5}$ i.e. 1.025 or by $(1+d)^{0.5}$ i.e. 0.976. Detailed movement of the index ratio is given on the tree in Figure 5.2.

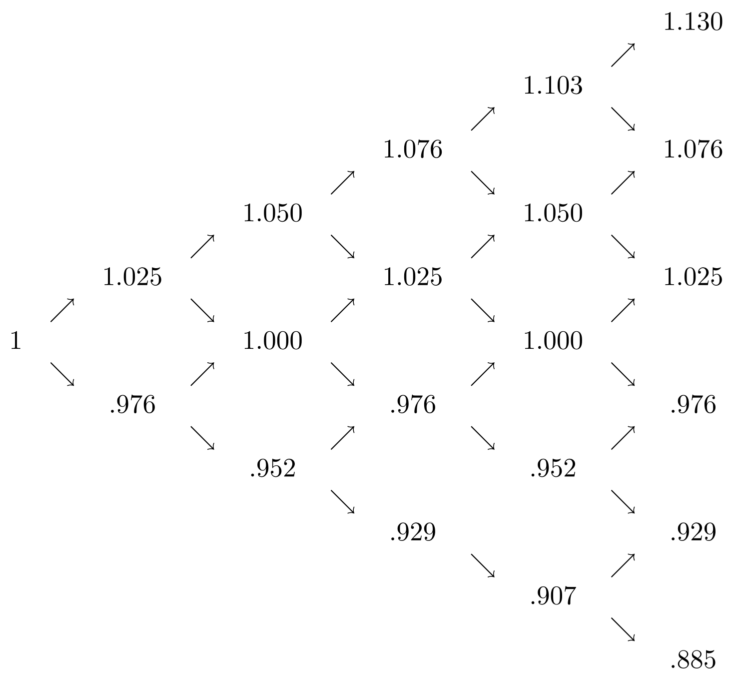


Figure 5.2. Index ratio movement

Table 5.3. The size of discrete-time cushion

Path	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	Binomial tree data♦	Formula (5.4)♣	Corrected (5.4)✠
1	1.025	1.050	1.076	1.103	1.130	1.664	1.671	1.664
2	1.025	1.050	1.076	1.103	1.076	1.368	1.373	1.368
3	1.025	1.050	1.076	1.050	1.076	1.387	1.393	1.387
4	1.025	1.050	1.076	1.050	1.025	1.140	1.144	1.140
5	1.025	1.050	1.025	1.050	1.076	1.387	1.393	1.387
6	1.025	1.050	1.025	1.050	1.025	1.140	1.144	1.140
7	1.025	1.050	1.025	1	1.025	1.123	1.127	1.123
8	1.025	1.050	1.025	1	.976	.923	.926	.923
9	1.025	1	1.025	1.050	1.076	1.387	1.393	1.387
10	1.025	1	1.025	1.050	1.025	1.140	1.144	1.140
11	1.025	1	1.025	1	1.025	1.156	1.160	1.156
12	1.025	1	1.025	1	.976	.950	.953	.950
13	1.025	1	.976	1	1.025	1.156	1.160	1.156
14	1.025	1	.976	1	.976	.950	.953	.950
15	1.025	1	.976	.952	.976	.935	.939	.935
16	1.025	1	.976	.952	.929	.769	.772	.769
17	.976	1	1.025	1.050	1.076	1.387	1.393	1.387
18	.976	1	1.025	1.050	1.025	1.140	1.144	1.140
19	.976	1	1.025	1	1.025	1.156	1.160	1.156
20	.976	1	1.025	1	.976	.950	.953	.950
21	.976	1	.976	.952	.976	.935	.939	.935
22	.976	1	.976	.952	.929	.769	.772	.769
23	.976	1	.976	1	1.025	1.156	1.160	1.156
24	.976	1	.976	1	.976	.950	.953	.950
25	.976	.952	.976	1	1.025	1.123	1.127	1.123
26	.976	.952	.976	1	.976	.935	.939	.935
27	.976	.952	.976	.952	.976	.935	.939	.935
28	.976	.952	.976	.952	.929	.769	.772	.769
29	.976	.952	.929	.952	.976	.935	.939	.935
30	.976	.952	.929	.952	.929	.769	.772	.769
31	.976	.952	.929	.907	.929	.757	.760	.757
32	.976	.952	.929	.907	.885	.623	.625	.623

Table shows the value of the cushion calculated after five time steps ♦ on the binomial tree in Figure 5.2, ♣ from the discrete-time formula in (5.4) and ✠ from the formula (5.4) corrected by the factor $\frac{1+m\delta}{(1+\delta)^{\gamma}}$

The riskless interest rate per period is r =0.02. The price process of the reserve asset is given by the formula

$$B_T(t) = B_T(0)e^{rt}$$

with $B_T(0) = 90.484$. Hence, in the five period interval the dynamics of reserve asset is as follows

$$90.484 \longrightarrow 92.312 \longrightarrow 94.176 \longrightarrow 96.079 \longrightarrow 98.020 \longrightarrow 100.000\ .$$

We use this reserve asset as the numeraire.

In total we have 32 paths the index ratio can follow. There is one path leading to 1.130 and one to 0.885. There are five paths leading to 1.076, ten to 1.025, ten to .976 and five to 0.929. All the paths are given in Table 5.3. Since we only rebalance after a $(1+u)$ or $(1+d)$ move this means that we only rebalance after the number of up moves is two more or two less than the number of down moves since the last rebalancing. In our tree there are eight paths with no rebalancing, seven with two rebalancings and the rest has one rebalancing.

For each of these paths we calculate the cushion from the binomial tree data and from the formula for the discrete-time cushion in (5.4). The results are given in Table 5.3. The Black-Perold formula only holds for nodes where there is just a rebalancing. Since there are final nodes in which no trading occurs we add in this formula an additional factor $\frac{1+m\delta}{(1+\delta)^\gamma}$ as in Black and Perold [1992], where δ is the change in the index ratio since the last trade, $d < \delta < u$. The results of our calculations are given in Table 5.3. They show that after taking into account the correction factor the terminal cushion calculated from the data agrees for all the paths with the cushion calculated from the discrete-time cushion formula (5.4).The results also show once more that the final payoffs are path dependent.

5.5 Data

Before we describe our empirical results for the portfolio insurance strategies we first describe the data that will be used. We consider the German market. The stock market is represented by the DAX index, $S(t)$. We use the DAX index since it is a reinvestment index and hence more appropriate for our test since no income is leaking. The price of the $T-$bond is calculated according to

$$B_T(t) = B_T(T)\exp(-y_{T-t}(T-t)),$$

where y_{T-t} is $T-t$-year zero rate and $B_T(T) = 100$. In order to obtain y_{T-t} when the time interval equals one day the yields on bonds with maturities equal to a number of full months are linearly interpolated. Dynamics of money market account is given by

$$\Delta M(t) \quad = \quad M(t)y_t\Delta t,$$

where y_t is the zero curve spot rate. At our disposal we have a daily data covering the period of April 29, 1997-January 1, 2004.

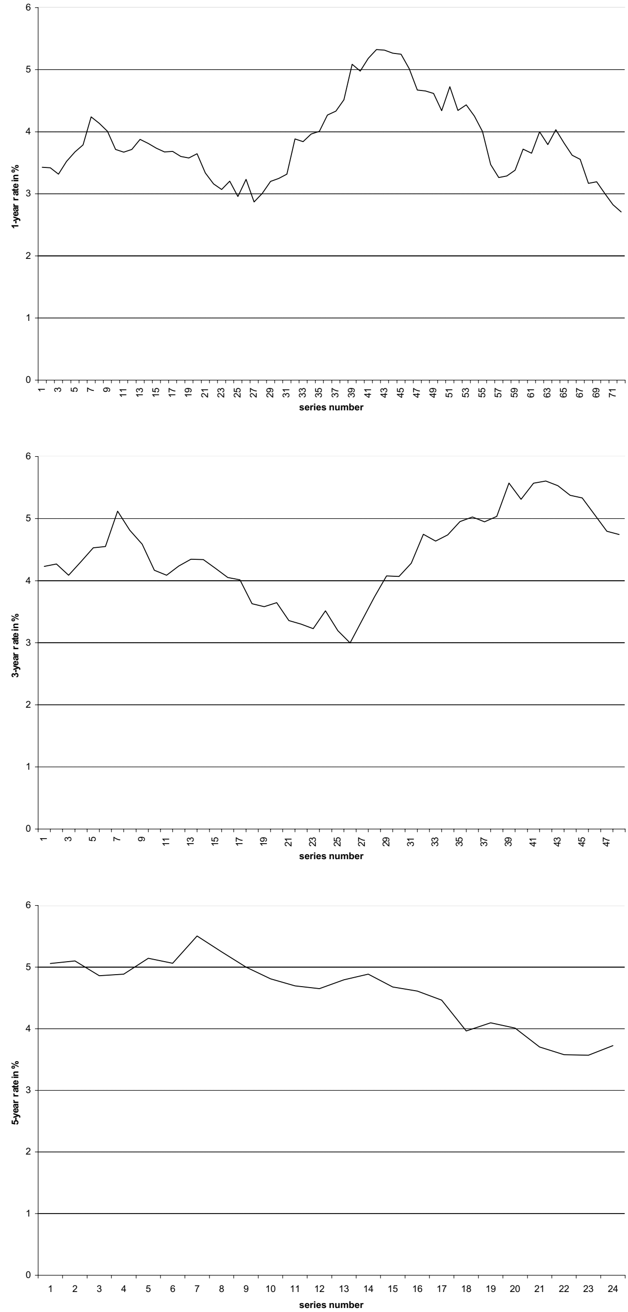


Figure 5.3. 1-year, 3-year and 5-year interest rates at the initial date

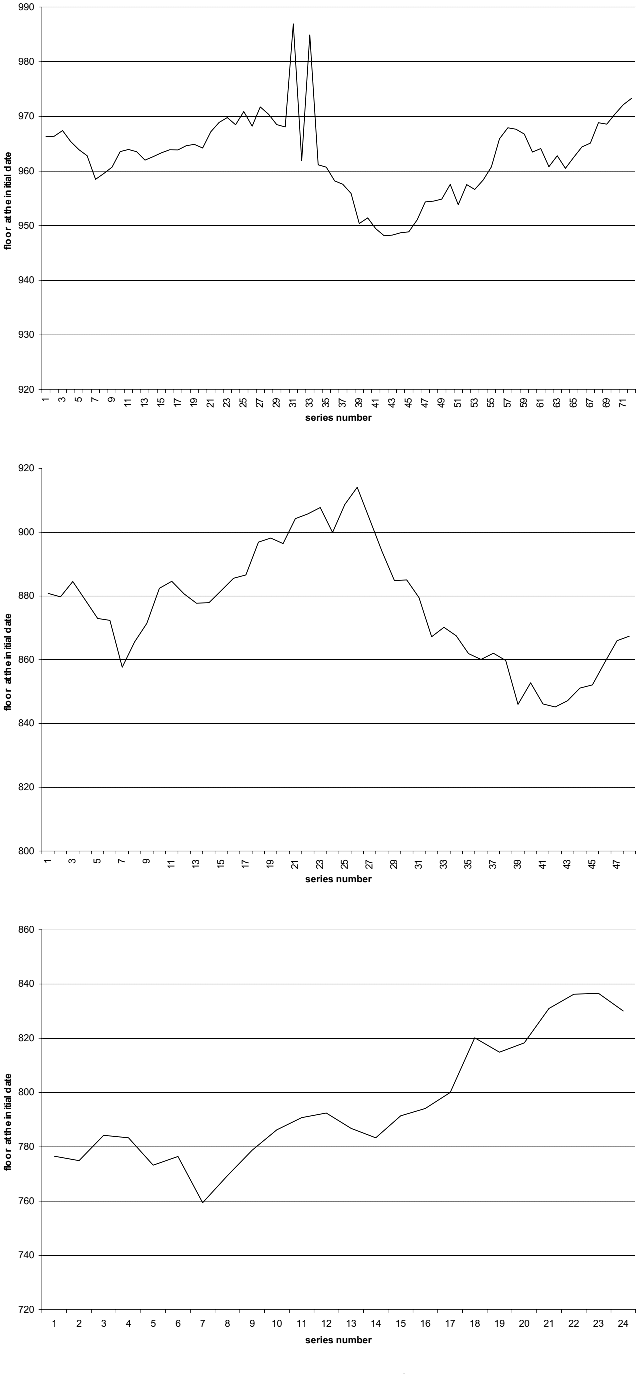


Figure 5.4. Floor at the initial date for 1-year, 3-year and 5-year horizons

We consider the strategies over periods with different length, i.e. 1-year, 3-year and 5-year horizon, rolled forward every month. We assume that one month has 21 trading days and as a result 1 year has 252 trading days. Consequently, we have 72 observations in our data set when 1-year horizon is rolled forward, 48 observations when 3-year horizon is rolled forward and 24 observations when 5-year horizon is rolled forward. The data are graphically presented in Figures 5.3-5.5. As can be seen from Figure 5.5 the DAX has moved considerably over time in our data period.

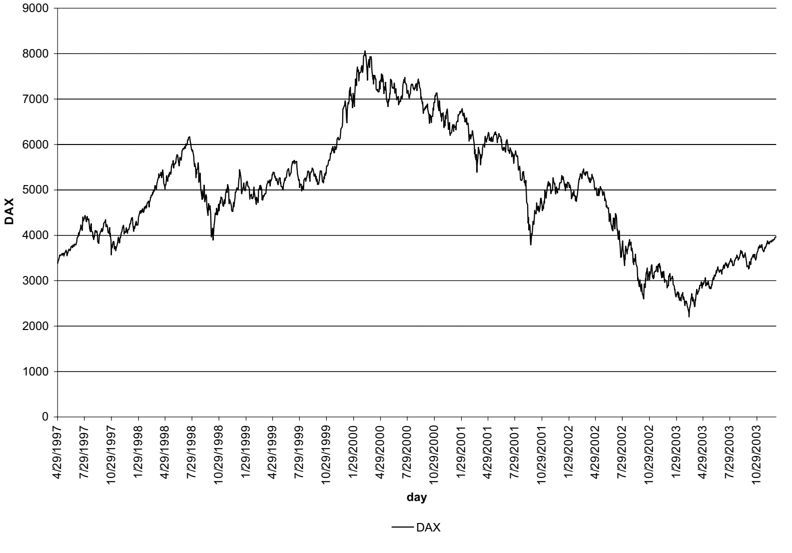


Figure 5.5. DAX index dynamics April 29, 1997-January 1, 2004.

5.6 Empirical results

In this part of the chapter we empirically compare the CPPI and MCPPI strategies and test the theoretical results. We choose the following parameters: $m = 4$, $F = W_0$, $W_0 = 1000$. In order to investigate the performance of CPPI and MCPPI on the German market we report the descriptive statistics on the terminal wealth of the investment portfolio. We also examine the impact of volatility on the terminal wealth as well as the difference between the cushions from the formulas presented earlier in this chapter and the cushions from the empirical data. In the comparison with the discrete-time cushion the portfolio is rebalanced only when the change in the index ratio since the last rebalancing/initial date exceeds 2%. In the comparison with the continuous-time cushion the portfolio is rebalanced daily.

5.6.1 Performance of CPPI and MCPPI

Descriptive statistics of the terminal wealth serving the purpose of performance comparison between CPPI and MCPPI are reported in Table 5.4. We remind the reader that the observations are not all independent since overlapping data are used. The numbers appearing in the upper panel concern 1-year sample, in the middle panel 3-year data and in the lower panel 5-year data. Each panel is divided in two. The upper part concerns a time rebalancing (daily), while lower the trigger rebalancing (when the change in the index ratio exceeds 2%).

Table 5.4. Performance comparison of CPPI and MCPPI

	Mean	Std. dev.	Skewness	Kurtosis
		1-year sample		
Strategy		Daily rebalancing♦		
MCPPI	1031.322	33.529	1.471	4.518
CPPI	1031.260	33.530	1.471	4.507
		Trigger rebalancing♣		
MCPPI	1030.629	33.053	1.499	4.608
CPPI	1030.584	33.078	1.497	4.586
		3-year sample		
		Daily rebalancing		
MCPPI	1079.130	139.491	2.665	11.285
CPPI	1072.546	126.046	2.699	11.588
		Trigger rebalancing		
MCPPI	1072.508	126.809	2.631	11.021
CPPI	1066.696	114.881	2.672	11.369
		5-year sample		
		Daily rebalancing		
MCPPI	1012.280	22.291	2.116	6.072
CPPI	1009.903	17.780	2.085	5.909
		Trigger rebalancing		
MCPPI	1010.724	19.513	2.135	6.149
CPPI	1008.804	15.920	2.108	6.002

Table 5.4. presents the descriptive statistics of the terminal wealth, ♦ refers to the rebalancing o a day-to-day basis, ♣ refers to the rebalancing when the change in the index ratio exceeds 2%

The numbers in Table 5.4 indicate that the mean wealth is bigger for MCPPI than for CPPI. MCPPI strategy is characterized by higher standard deviation than CPPI for 3- and 5-year horizon. MCPPI and CPPI have almost the same standard deviation for 1-year horizon. The skewness of MCPPI in comparison with the skewness of CPPI is almost the same for 1-year horizon, smaller for 3-year horizon and higher for 5-year horizon. As far as the kurtosis is concerned it is higher for MCPPI than CPPI for 1- and 5-year samples and lower for 3-year sample. Also on average the results on continuous rebalancing are better than those for rebalancing after a trigger event has occurred.

5.6.2 Impact of volatility

In section 5.3 we presented the formulas for the cushion of CPPI and MCPPI. Both cushions in the continuous time were shown to be a decreasing function of volatility of stock index ratio and in the case of MCPPI of volatility of cash index ratio as well. The influence of the volatility on the final wealth is nicely illustrated in Figures 5.6 through 5.8. In the top panel of each figure the final wealth per observation period is given for the four different strategies, MCPPI and CPPI with daily and triggered rebalancing. In the same graph the returns on the DAX index defined as portfolio value started with the investment of 1000 are depicted. In the bottom panel the volatilities per observation period are given. They are calculated according to the formula

$$\sigma_1 = \frac{\sqrt{\frac{1}{n-1}\sum_{t=1}^{n}\left(u_t - \overline{u}\right)^2}}{\sqrt{\frac{1}{252}}},$$

where n equals 252 for 1-year samples, 756 for 3-year samples and 1260 for 5-year samples and

$$u_t = \ln \frac{I_1(t)}{I_1(t-1)}.$$

From the figures the following remarks should be made:

(1) The final results of all strategies heavily differ per observation. Of course the initial outlay is always retrieved, but there are periods with no extra return and periods with a sizeable return, e.g. 14% over the one year horizon strategy.

(2) In most cases the four different strategies have returns that are remarkably close to each other, especially compared with the differences between observation periods.

(3) The final returns are strongly negatively correlated with realized volatility over the investment period. E.g. the peak in volatility in the 1-year sample wipes out all profits from the strategies and returns zero on the investment. The same holds for the 3-year and 5-year sample, where due to high volatility in the later years of our data there is hardly any return.

(4) The pattern of terminal wealth heavily depends on the stock index ratio return. On the downward of stock index ratio the terminal wealth approaches

the floor, while on the upward of stock index ratio extraordinary high values of the terminal wealth are realized.

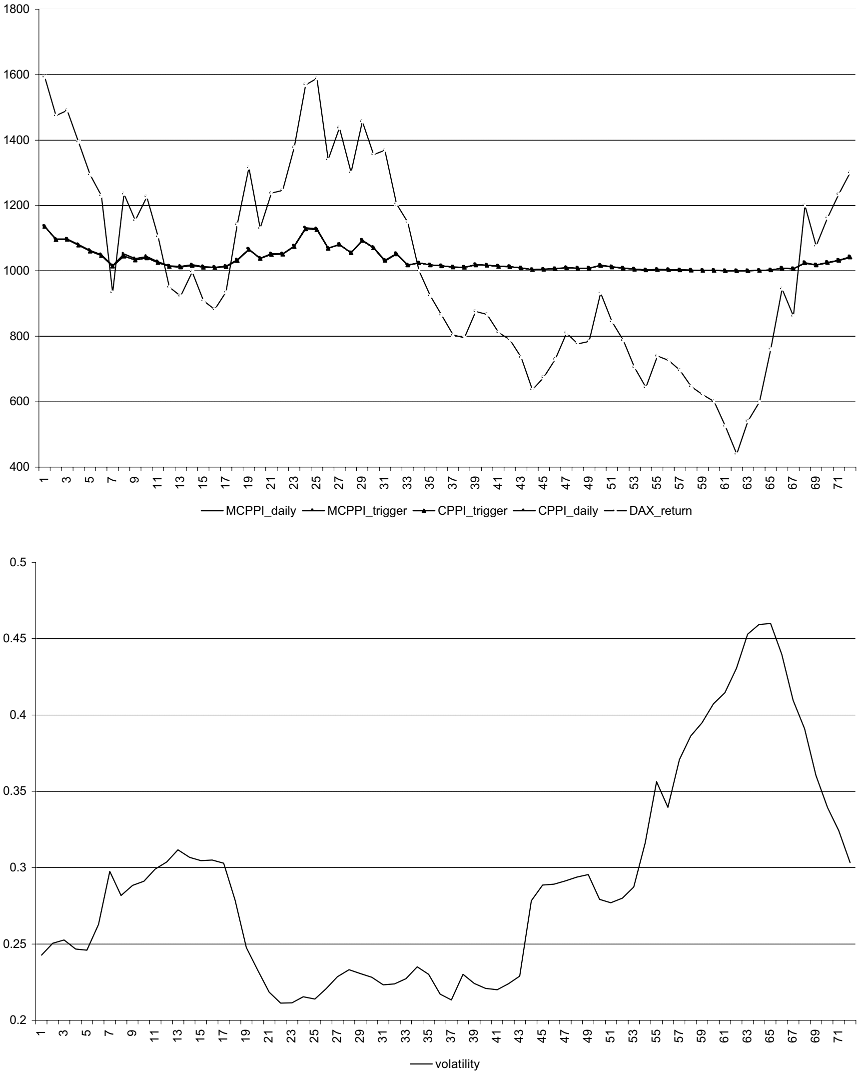


Figure 5.6. 1-year returns on the wealth of portfolio insurance strategies and returns on the DAX index both defined as portfolio value started with the investment of 1000

To illustrate our remarks above we have run linear regressions on the return of the strategies defined by $\frac{W_T - W_0}{W_0}$, using as regression variables the volatility and return of the index. The results for different strategies and periods are given in Table 5.5.

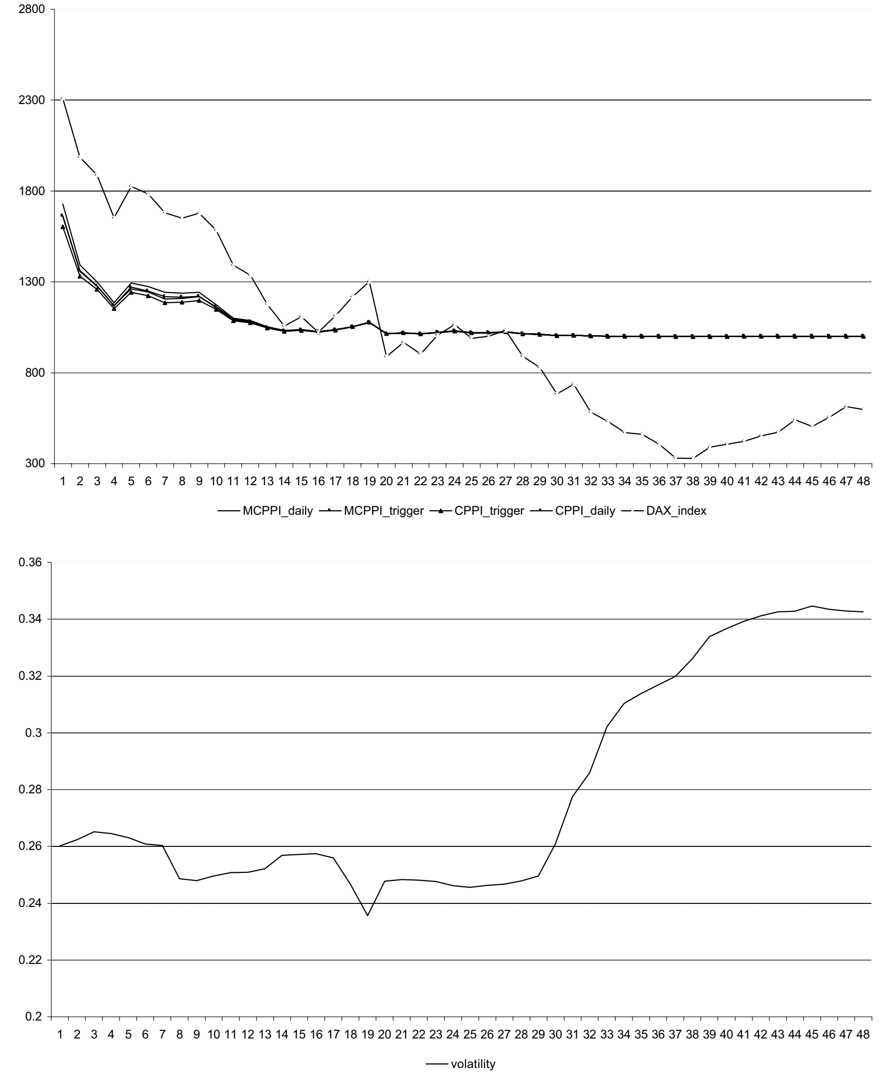


Figure 5.7. 3-year returns on the wealth of portfolio insurance strategies and returns on the DAX index both defined as portfolio value started with the investment of 1000

The results in Table 5.5. indicate that both the volatility as well as the stock index ratio returns are statistically significant at all significance levels in explaining the wealth returns. The sign of volatility coefficient is as expected from the analysis in section 5.3. The goodness of fit of the model to the data is satisfactory as measured by $R-$square and oscillates around to 75-85%.

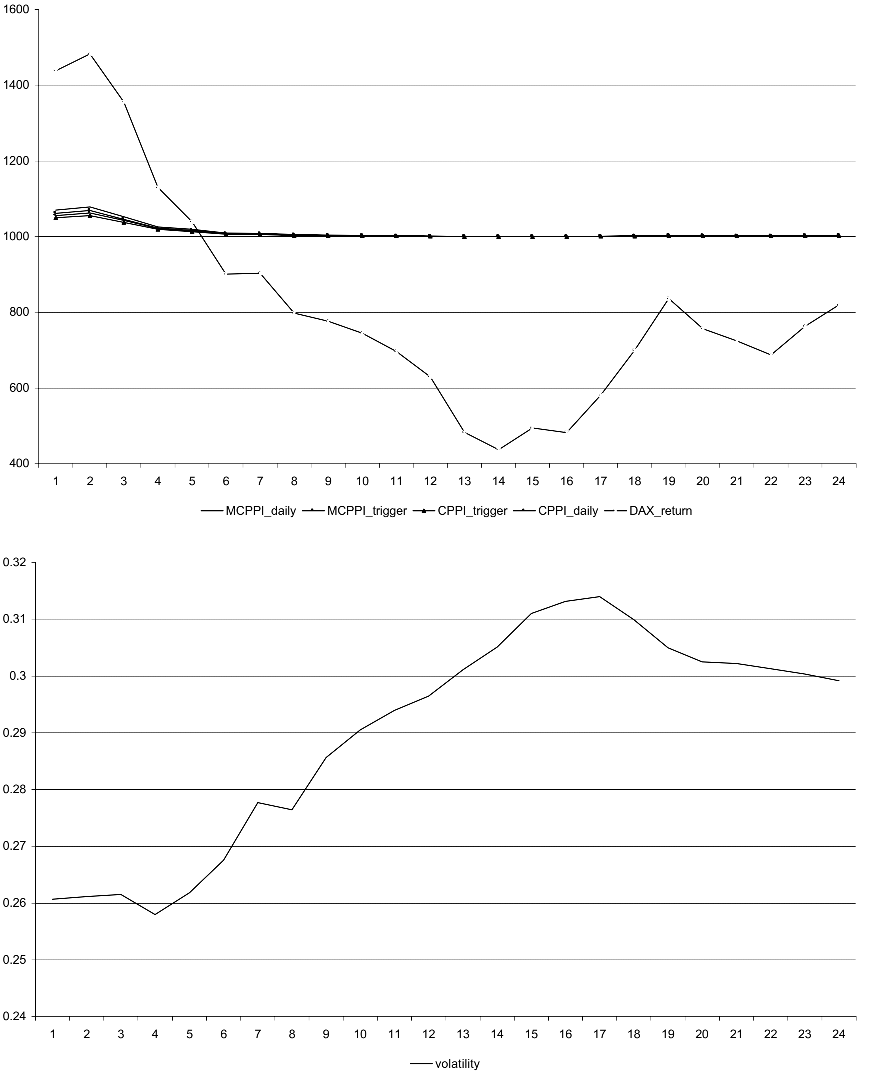


Figure 5.8. 1-year returns on the wealth of portfolio insurance strategies and returns on the DAX index both defined as portfolio value started with the investment of 1000

Table 5.5. Impact of volatility and index ratio returns on the wealth returns

Dep. var.♦	Volatility♣	Index✠	R-square♠
		1-year samples	
CPPI daily	-.140	.075	.761
CPPI trigger	-.138	.074	.758
MCPPI daily	-.139	.075	.759
MCPPI trigger	-.137	.074	.755
		3-year samples	
CPPI daily	-.346	.198	.666
CPPI trigger	-.316	.181	.671
MCPPI daily	-.382	.218	.660
MCPPI trigger	-.348	.199	.665
		5-year samples	
CPPI daily	-.110	.064	.812
CPPI trigger	-.098	.058	.808
MCPPI daily	-.138	.080	.806
MCPPI trigger	-.120	.070	.803

Table 5.5. shows the results of OLS estimation of wealth returns defined as $\frac{W_T - W_0}{W_0}$ on the volatility and stock index ratio, ♦ stands for the dependent variable, ♣ is the coefficient of volatility as independent variable, ✠ is the coefficient of stock index ratio as independent variable, ♠ stands for $R-$square of the regression

5.6.3 Continuous-time cushion

In this part of the chapter we present the results on the comparison of the continuous-time cushion given by formulas (5.5) and (5.9) respectively and the cushion calculated, when the rebalancing is done daily. From the graphs in Figures 5.9 and 5.10 we see that the cushion with daily rebalancing is very close to the continuous-time cushion. This is especially true for the 1-year samples, but also the theoretical and realized cushions are close to each other for the 3-year and 5-year samples The same holds true for the MCPPI cushions. Moreover, the cushions of MCPPI closely resemble the cushions of CPPI moving in the same manner. The OLS estimation of the cushions with daily rebalancing on the continuous-time cushions is significant at all confidence levels.

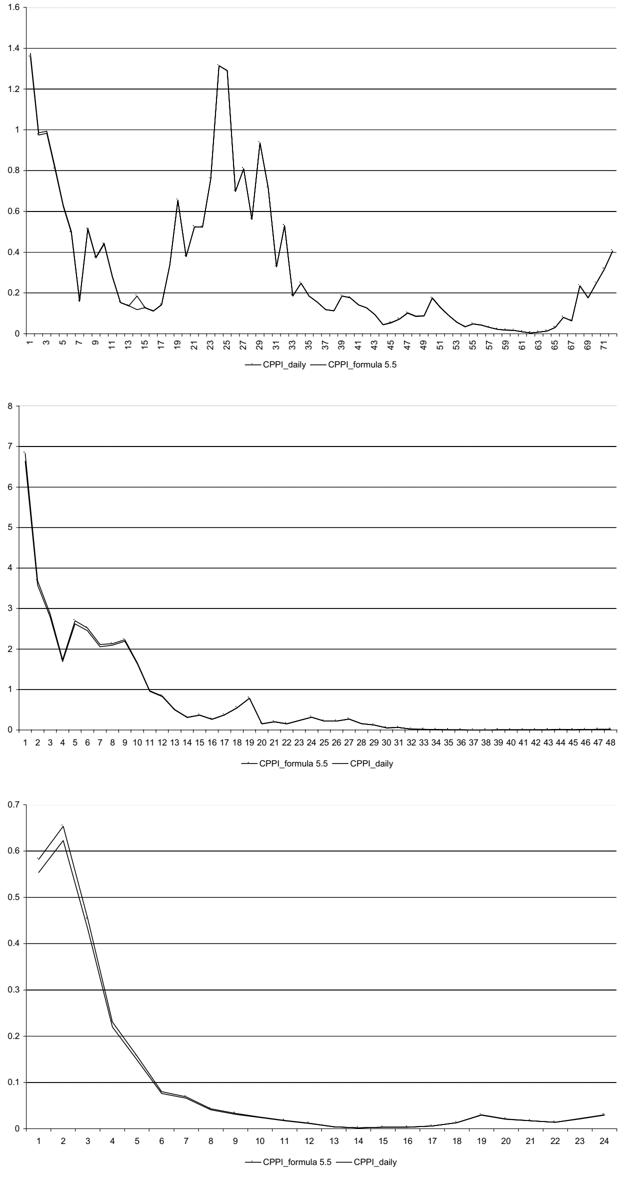


Figure 5.9 Terminal cushion of CPPI from data with daily rebalancing and formula 5.5 in 1-year, 3-year and 5-year samples

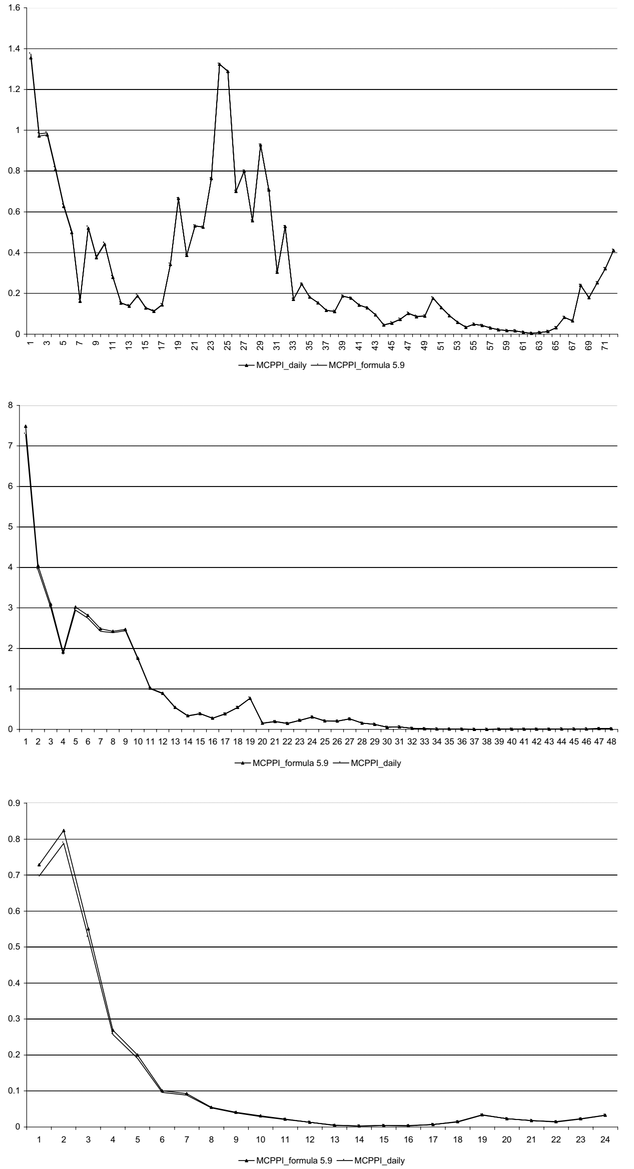


Figure 5.10 Terminal cushion of MCPPI from data with daily rebalancing and formula 5.9 in 1-year, 3-year and 5-year samples

Table 5.6. Comparison of empirical daily and theoretical continuous-time cushion

Strategy	Mean	Std. dev.	Skewness	Kurtosis
		1-year sample		
MCPPI con.♦	.316	.337	1.477	4.542
MCPPI daily♣	.315	.335	1.472	4.418
CPPI con.✠	.315	.338	1.477	4.518
CPPI daily♣	.314	.336	1.471	4.501
		3-year sample		
MCPPI con.	.811	1.432	2.679	11.372
MCPPI daily	.793	1.395	2.664	11.279
CPPI con.	.744	1.296	2.713	11.665
CPPI daily	.727	1.261	2.698	11.581
		5-year sample		
MCPPI con.	.124	.233	2.114	6.064
MCPPI daily	.130	.233	2.113	6.062
CPPI con.	.100	.178	2.084	5.904
CPPI daily	.105	.187	2.082	5.897

Table 5.6. presents the terminal cushion calculated ♦ from formula (5.9), ♣ from the empirical data with daily rebalancing, ✠ from formula (5.5)

Table 5.7 OLS estimation of the cushion with daily rebalancing on the continuous-time cushion

Indep. var♦.	Coefficient	Probability	R-square
		1-year sample	
CPPI con.	.996	.000	.999
MCPPI con.	.996	.000	1.000
		3-year sample	
CPPI con.	.974	.000	1.000
MCPPI con.	.975	.000	1.000
		5-year sample	
CPPI con.	.952	.000	1.000
MCPPI con.	.956	.000	1.000

The numbers in Table 5.7 show the results of OLS estimation, ♦ stands for the independent variable

5.6.4 Discrete-time cushion

In this subsection we compare the terminal cushions calculated from the data when the portfolio is rebalanced after the 2% move up or down in the index ratio for CPPI strategy against the terminal cushion calculated according to the formula (5.4). Figure 5.8. illustrates both the theoretical values and the actually realized values. It is clear that the theory works better for the short term strategy. The first two moments of the cushions from the data and from formula (5.4) as given in Table 5.8. are pretty different. They are the closest for 1-year horizon. The average of the cushion from formula (5.4) as well as the standard deviation overestimate the average of the cushion and its standard deviation from the data with trigger rebalancing. Their skewness and kurtosis are comparable. Both cushions are highly correlated with $R-$square coefficient of OLS regression of cushion with 2% trigger rebalancing on the cushion given by (5.4) close to 1.

Table 5.8 Comparison of discrete-time terminal cushion and the cushion with trigger rebalancing

Strategy	Mean	Std. dev.	Skewness	Kurtosis
		1-year sample		
CPPI trigger♦	.307	.331	1.496	4.584
CPPI_discrete♣	.400	.396	1.393	4.344
		3-year sample		
CPPI trigger	.669	1.149	2.671	11.363
CPPI_discrete	1.515	2.672	2.751	11.780
		5-year sample		
CPPI trigger	.089	.159	2.108	6.004
CPPI_discrete	.399	.643	2.103	6.014

Table 5.8. presents the descriptive statistics of the terminal cushions ♦ calculated from the data when there is a 2% trigger rebalancing, ♣ calculated from formula (5.4)

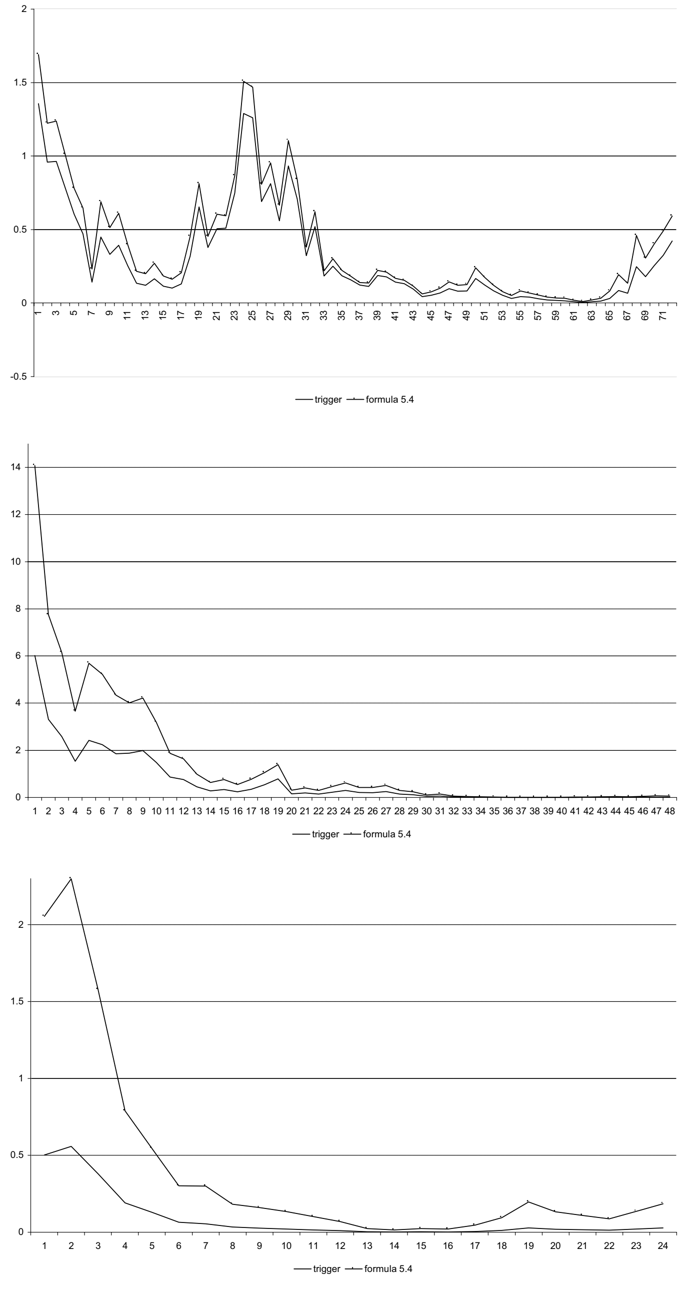


Figure 5.8. Terminal cushion of CPPI from data with 2% trigger rebalancing and formula (5.4) in 1-year, 3-year and 5-year samples respectively

Table 5.9 OLS estimation of the cushion calculated from the data with 2% trigger rebalancing as dependent variable with the cushion calculated from (5.4) as independent variable

	Coefficient	Probability	R-square
1-year sample♦	.798	.000	.985
3-year sample♣	.433	.000	.998
5-year sample✠	.241	.000	.996

The numbers in Table 5.9. are the results of regression estimation for ♦ 1-year sample, ♣ 3-year sample, ✠ 5-year sample

The comparison of the results for the continuous-time cushion and for the discrete-time cushion undoubtedly indicates that the theoretical values are closer to the actually realized values if we daily rebalance the portfolio.

5.7 Conclusions

In the theoretical part of the paper we compare the CPPI and MCPPI strategies in terms of the increment in exposure, cushion and exposure-to-cushion ratio in the discrete time. In the continuous time we focus on the comparison of cushions.

We find that for MCPPI strategy the fractional change in the index ratio is magnified m times in the cushion, while the fractional change in the cash ratio is magnified $1 - m$ times. The fractional change in the exposure-to-cushion ratio is a decreasing function of the fractional change in the index ratio when the fractional change in the cash ratio is bigger than -1. For the same range of the fractional change in the index ratio the exposure-to-cushion ratio is an increasing function of the fractional change in the cash ratio. We discover that the continuous-time cushion is a decreasing function of volatility of the index ratio and the cash ratio when the correlation coefficient is smaller than the ratio of volatilities of the index ratio to the cash ratio and the cash ratio to the index ratio respectively.

The bridge between the theoretical and empirical part of the chapter consists of three motivating examples, which illustrate the issues of the performance of portfolio-insurance strategies, the impact of volatility on the terminal wealth and the size of the discrete-time cushion of CPPI policy.

In the empirical part of this chapter first we compare the descriptive statistics of terminal wealth of CPPI and MCPPI strategies on the German market. Second, we examine the influence of volatility in the stock index on the terminal wealth when CPPI and MCPPI policies are followed. Third, we investigate whether the continuous-time cushions in (5.5) and (5.9) coincide with the ones calculated from the data with daily rebalancing. Finally, we focus on CPPI strategy and calculate discrete-time cushions and evaluate them against cushions obtained from the data with rebalancing caused by a 2% trigger.

Our findings are the following. As far as the terminal wealth is concerned the averages for the same time horizon of MCPPI are slightly higher than of CPPI. Also the mean of terminal wealth with continuous rebalancing is higher than the mean with rebalancing after the trigger. The terminal wealth is significantly dependent on the stock index volatility. The final returns are strongly negatively correlated with realized volatility over the investment period. Concerning the cushion we conclude that the cushion with daily rebalancing is very close to the continuous-time cushion especially in the 1-year samples. Also the pattern in which the cushion of MCPPI changes closely resembles the pattern of CPPI cushion dynamics. Comparison of the terminal cushion with 2% trigger rebalancing against the discrete-time cushion in (5.4) of CPPI results in the finding that the first two moments of the cushions are pretty different.

5.8 Appendix A

Below we derive the proof of formula (5.4) according to the guidelines of Black and Perold [1992]. We know that

$$\begin{aligned}
\alpha &= (1+mu)\,(1+md)\,, \\
1 &= (1+u)(1+d), \\
S &= S_0(1+u)^i(1+d)^j, \\
C &= C_0(1+mu)^i(1+md)^j, \\
n &= i+j.
\end{aligned}$$

Let us take the logarithms

$$\begin{aligned}
\ln\frac{S}{S_0} &= i\ln(1+u)+j\ln(1+d), \\
\ln\frac{C}{C_0} &= i\ln(1+mu)+j\ln(1+md).
\end{aligned}$$

Let us substitute $n-i$ for j. Then,

$$\begin{aligned}
\ln\frac{S}{S_0} &= i\ln\frac{1+u}{1+d}+n\ln(1+d), \\
\ln\frac{C}{C_0} &= i\ln\frac{1+mu}{1+md}+n\ln(1+md).
\end{aligned}$$

Let us substitute $\frac{1}{1+u}$ for $1+d$ in $\ln\frac{S}{S_0}$. Then,

$$\ln\frac{S}{S_0}=(2i-n)\ln\left(1+u\right),$$

and

$$i=\frac{1}{2}\left(\frac{\ln\frac{S}{S_0}}{\ln\left(1+u\right)}+n\right).$$

Substituting for i in $\ln\frac{C}{C_0}$ we have

$$\begin{aligned}
\ln\frac{C}{C_0} &= \frac{1}{2}\left(\frac{\ln\frac{S}{S_0}}{\ln(1+u)}+n\right)\ln\frac{1+mu}{1+md}+n\ln(1+md)= \\
&= \ln\left(\frac{S}{S_0}\right)^{\frac{1}{2}\frac{\ln\frac{1+mu}{1+md}}{\ln(1+u)}}+\frac{1}{2}n\ln\frac{1+mu}{1+md}+n\ln(1+md)= \\
&= \ln\left(\frac{S}{S_0}\right)^{\gamma}+\frac{1}{2}n\ln\frac{\alpha}{(1+md)^2}+n\ln(1+md)= \\
&= \ln\left(\frac{S}{S_0}\right)^{\gamma}+\frac{1}{2}n\ln\alpha-\frac{1}{2}n\ln(1+md)^2+n\ln(1+md)= \\
&= \ln\left(\frac{S}{S_0}\right)^{\gamma}+\ln\alpha^{\frac{1}{2}n}-n\ln(1+md)+n\ln(1+md)= \\
&= \ln\left(\frac{S}{S_0}\right)^{\gamma}+\ln\alpha^{\frac{1}{2}n}= \\
&= \ln\left(\left(\frac{S}{S_0}\right)^{\gamma}\alpha^{\frac{1}{2}n}\right),
\end{aligned}$$

where we denoted by γ the expression $\frac{1}{2}\frac{\ln\frac{1+mu}{1+md}}{\ln(1+u)}$ and substituted $\frac{\alpha}{1+md}$ for $1+mu$. Skipping logarithms on both sides yields the final result

$$C=C_0\alpha^{\frac{1}{2}n}\left(\frac{S}{S_0}\right)^{\gamma}.$$

5.9 Appendix B

Below we derive the formulas in Proposition 1. The first equality is obvious. The second equality follows from

$$\begin{aligned}
\Delta C(t) &= C(t+\Delta t)-C(t)= \\
&= W(t+\Delta t)-F(t+\Delta t)-C(t)= \\
&= E(t)(1+\delta_1)+\omega+R^c(t)(1+\delta_2)-\omega-C(t)= \\
&= mC(t)(1+\delta_1)+(1-m)C(t)(1+\delta_2)-C(t)= \\
&= [m\delta_1+(1-m)\,\delta_2]\,C(t).
\end{aligned}$$

The proof of the third equality is the following

$$
\begin{aligned}
\Delta[E(t)C^{-1}(t)] &= E(t+\Delta t)C^{-1}(t+\Delta t) - E(t)C^{-1}(t) = \\
&= (E(t) + \Delta E(t))\,(C(t) + \Delta C(t))^{-1} - E(t)C^{-1}(t) = \\
&= E(t)(1+\delta_1)\,[C(t)\,(1 + m\delta_1 + (1-m)\,\delta_2)]^{-1} - E(t)C^{-1}(t) = \\
&= E(t)C^{-1}(t)\left[(1+\delta_1)\,[1 + m\delta_1 + (1-m)\,\delta_2]^{-1} - 1\right] = \\
&= E(t)C^{-1}(t)\left[(1-m)\,(\delta_1 - \delta_2)\,[1 + m\delta_1 + (1-m)\,\delta_2]^{-1}\right].
\end{aligned}
$$

5.10 Appendix C

Below we derive the result in Proposition 2. The increment of portfolio value is given by

$$dW(t) = dE(t) + dF(t) + dR^c(t).$$

Choosing bond as numeraire we have

$$
\begin{aligned}
dE(t) &= \delta_1 E(t), \\
dF(t) &= 0, \\
dR^c(t) &= \delta_2 R^c(t).
\end{aligned}
$$

Substituting for $E(t)$ and $R^c(t)$ we have

$$
\begin{aligned}
dE(t) &= \delta_1 m C(t), \\
dF(t) &= 0, \\
dR^c(t) &= \delta_2 (1-m) C(t),
\end{aligned}
$$

and consequently

$$dW(t) = [m\delta_1 + (1-m)\delta_2]\,C(t) = dC(t).$$

Replacing δ_k, $k = 1, 2$, by $\frac{dI_k(t)}{I_k(t)}$ we have

$$dC(t) = [(m\mu_1 + (1-m)\mu_2)\,dt + (m\sigma_1 dz_1(t) + (1-m)\sigma_2 dz_2(t))]\,C(t).$$

Finally, using Ito's lemma we end up with the result. Representation in terms of index ratios follows after the substitution from $I_k(t)$, $k = 1, 2$.

6 Conclusions

Our findings from chapter 2 are as expected. With the growth of the number of time steps constant relative risk aversion (CRRA) simulated portfolios converge to the optimal continuous-time portfolios, which are known from the literature. Also for hyperbolic absolute risk aversion (HARA) simulated portfolios and Vasicek [1977] interest rate dynamics we do observe the convergence. For HARA utility and Cox-Ingersoll-Ross (CIR) [1985] dynamics of interest rate the optimal continuous-time portfolios are not available in the literature. That is why we cannot state the convergence of our simulated portfolios in this case. However, the results obtained are stable enough to conclude that the higher the ratio of initial wealth to the subsistence level the more is invested in the most risky asset and the less interest rate risk hedging is performed with the bond.

What we note in chapter 3 is that with an increase of the number of time steps the constituents of optimal portfolio from the tree models converge to their continuous-time counterparts, but at the same time the discrete-time equivalent of Malliavin derivative expression from the simulated portfolios is independent to the number of time steps. Moreover, portfolios from the tree models outperform the simulated portfolios in terms of accuracy when the number of time steps is small enough. Our finding is that portfolios held in the last period contain only the mean-variance component. This is intuitive because in the last period there is no uncertainty concerning the interest rate, which can be relevant to the outcome of investment. Another intuitive result is that the interest rate hedging portfolio enlarges the return on the mean-variance portfolio in the node of rate's fall and reduces it in the node of rate's rise when long position is held in the riskless asset. We also show that the interest rate risk hedging works more intensively in this part of the tree where bigger fluctuations of interest rate are present. The hedging is performed by asset, which is perfectly correlated with interest rate, namely the zero-coupon T-maturity bond. As the correlation between the risky asset imperfectly correlated with the interest rate and the interest rate goes to zero the changes in this asset dynamics add less to the total of interest rate risk what has the consequence of decreasing (absolute) magnitude of returns on the hedging portfolio.

For all the strategies inspected in chapter 4 we note the trade off between the mean and standard deviation of the funding ratio as well as of the contribution rate. We further observe that the liability constraint does not influence the skewness and kurtosis of the funding ratio and of the contribution rate. Our finding is that the liability-unconstrained strategy has much higher, while myopic strategy much lower standard deviation of the funding ratio and of the contribution rate than liability-constrained interest rate risk hedging policy. We conclude that the dynamic strategies are not superior to fixed mixes with respect to mean-standard deviation criterion. The benefit of following the dynamic liability-constrained strategies over the fixed mixes is exposed mainly in the skewness. All descriptive statistics grow when there is imperfect liability hedge. As far as the sensitivity of results to the initial funding ratio is concerned the standard deviations of the funding ratio and the contribution rate change more than proportionally. With the growth of risk aversion the standard deviation and skewness of these two characteristics of the pension fund are affected the most and both go down. Altering the wage-to-liability ratio does change proportionally the average and the standard deviation of the contribution rate and does not change other statistics. Only the average of contribution rate reacts more than proportionally to the changes of the pension-to-liability ratio. Only the means and standard deviations of the constrained strategies change with the change in the level of underfunding.

In chapter 5 we consider constant proportion portfolio insurance (CPPI) and modified CPPI (MCPPI) strategies. As far as the theoretical results on MCPPI are concerned we find that for the strategy considered in the discrete time the fractional change in the index ratio is magnified m times in the cushion, while the fractional change in the cash ratio is magnified $1-m$ times with m being a fixed multiplier. In turn, the continuous-time cushion is a decreasing function of volatility of the index ratio and of the cash ratio when the correlation coefficient is smaller than the ratio of volatilities of the index ratio to the cash ratio and the cash ratio to the index ratio respectively.

Our empirical findings are the following. As far as the terminal wealth is concerned its averages for the same time horizon of MCPPI are slightly higher than of CPPI. The terminal wealth is significantly dependent on the stock index volatility. Concerning the cushion we conclude that the cushion with daily rebalancing is very close to the continuous-time cushion. Also the pattern in which the cushion of MCPPI changes closely resembles the pattern of CPPI cushion dynamics. Comparison of the terminal cushion with trigger rebalancing against the discrete-time cushion of CPPI results in the finding that the first two moments of the cushions are pretty different.

Our main conclusions from this thesis can be summarized as follows. The dynamic optimal portfolios simulated as well as computed on the tree converge to their continuous-time equivalents in the investigated models of the financial market with the stochastic interest rate. If the utility function takes into account the subsistence level than the higher the ratio of initial wealth to this level the more is invested in the most risky asset and the less interest rate risk hedging is performed with the bond. If a dynamic liability-constrained portfolio policy is followed by the pension scheme on the market with stochastic interest rate it is not superior to the fixed mix strategies. There is always a trade off between the mean and standard deviation of the funding ratio as well as the contribution rate of the examined portfolio policies. An alternative to the option-based portfolio insurance can be provided by the constant-proportion portfolio insurance such as CPPI or MCPPI. However, the terminal wealth generated by them is strongly dependent on the stock index ratio volatility.

Nederlandse samenvatting (Summary in Dutch)

Het onderwerp van dit proefschrift is dynamische portfoliokeuze. In de eerste twee hoofdstukken behandelen we het probleem dat vaak bij portfoliokeuze zonder beperkingen een rol speelt, namelijk afdekking (hedging) van ongunstige veranderingen in de toestandsvariabelen. In deze context concentreren we ons op het afdekken van het renterisico. We analyseren dit probleem zowel in continue als in discrete tijd door het toepassen van de martingale methode voor optimale portfoliokeuze, gecombineerd met Malliavin calculus. In de continue tijd geven we een alternatieve afleiding voor het berekenen van het optimale portfoliobeleid in het gekozen model en we bevestigen dat de optimale portfoliokeuze die door simulaties met behulp van Malliavincalculus verkregen wordt, naar de continue-tijd limieten convergeren. In de discrete tijd bevestigen we een aantal eigenschappen van optimale portfolio's die uit de continue analyse bekend zijn. Eigenschappen die in continue tijd niet voldoende intuïtief zijn worden in discrete tijd in meer detail geanalyseerd. De laatste twee hoofdstukken zijn gewijd aan portfoliokeuze onder restricties. We onderzoeken het probleem van asset en liability management in een defined-benefit pensioenstelsel met onzekere verplichtingen. Met behulp van simulaties vergelijken we de performance van een aantal dynamische portfoliostrategieën en fixed mixes in zowel volledige als onvolledige financiële markten. We concluderen dat dynamische strategieën niet beter zijn dan vaste portefeuilles. Het laatste deel van dit proefschrift is een empirisch onderzoek van constant proportion portfolio insurance (CPPI) strategieen, waarvan de theoretische eigenschappen goed in de literatuur bekend zijn. We vergelijken de theorie met empirische resultaten en CPPI met de alternatieve strategie van portfolioverzekering.

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Essays on the Dynamic Portfolio Choice

This thesis is concerned with dynamic optimal portfolio choice both unconstrained and constrained. The unconstrained dynamic optimal portfolio choice is meant in this thesis as trading the available financial assets in a given time interval according to the rules, which are feasible under the budget restriction and optimal according to certain criterion. This criterion in this thesis is the maximization of expected utility from terminal wealth. The problem of constrained dynamic optimal portfolio choice is additionally restricted by the requirement that the terminal wealth must exceed zero, given constant, or deterministic/stochastic liability/guarantee. Unconstrained portfolio choice is treated in the first two chapters of the thesis, while the next two are devoted to the constrained choice. In the unconstrained portfolio choice context we consider the phenomenon of interest rate risk hedging. It is meant as complementary to mean-variance investment into the risky assets in order to protect the portfolio against the adverse changes in the short-term interest rate. We analyse the problem of interest rate risk hedging in both continuous and discrete time. We approach it with the martingale methodology supplemented by the Malliavin calculus. In the constrained portfolio choice framework we deal with the problem of asset and liability management (ALM) of the pension schemes as well as the common portfolio insurance policy known as the constant proportion portfolio insurance (CPPI). The ALM research is the simulation experiment tailored to investigate the performance of dynamic portfolio strategies in the chosen models of financial markets. The CPPI research is the analysis of empirical data, which is to answer the question of superiority of CPPI over an alternative as well as the correspondence between theoretical and empirical properties of CPPI.

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