Two-echelon supply chain coordination under information asymmetry with multiple types

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Abstract

We analyse a principal-agent contracting model with asymmetric information between a supplier and a retailer. Both the supplier and the retailer have the classical non-linear economic ordering cost functions consisting of ordering and holding costs. We assume that the retailer has the market power to enforce any order quantity. Furthermore, the retailer has private holding costs. The supplier wants to minimise his expected costs by offering a menu of contracts with side payments as an incentive mechanism. We consider a general number of discrete single-dimensional retailer types with type-dependent default options.

A natural and common model formulation is non-convex, but we present an equivalent convex formulation. Hence, the contracting model can be solved efficiently for a general number of retailer types. We also derive structural properties of the optimal menu of contracts. In particular, we completely characterise the optimum for two retailer types and provide a minimal list of candidate contracts for three types. Finally, we prove a sufficient condition to guarantee unique contracts in the optimal solution for a general number of retailer types.

Keywords: economic order quantity, mechanism design, asymmetric information, hidden convexity

1 Introduction

We consider the classical 2-echelon Economic Order Quantity (EOQ) setting with a supplier and a retailer. Both the supplier and the retailer act as fully rational individualistic entities (or agents) that want to minimise their own costs. It is well-known that such individualistic viewpoints are suboptimal for the entire supply chain. This loss of efficiency is often called the price of anarchy, see for example Perakis and Roels 2007. We assume that the supply chain uses a pull ordering strategy, i.e., the retailer places orders at the supplier. Therefore, the retailer's ordering policy is optimal for herself. The supplier can decrease his costs by somehow persuading the retailer to change to a different ordering policy.

One way the supplier can do so is by offering a contract to the retailer that typically includes a side payment or discounts. If the contract is accepted by the retailer, the costs for the entire supply chain decrease and the resulting profit is divided between the two parties as agreed upon in the contract. Being selfish, the supplier wants the largest possible share of this profit. Depending on the type of contract, it is non-trivial to determine a contract that maximises the supplier's profit and that is accepted by the retailer.

The complexity of the matter is increased significantly if the retailer has private information that is not shared with the supplier. For example, the retailer's cost structure can be undisclosed.

Furthermore, private information typically leads to inefficiencies for the supply chain, see for example Inderfurth et al. 2013. This partial cooperation between the supplier and the retailer leads to a principal-agent optimisation problem with asymmetric information.

In the case that the retailer holds private information, the supplier can use mechanism design or incentive theory to improve his situation, see Laffont and Martimort 2002. That is, he presents a menu of contracts for the retailer to choose from. We focus on constructing the optimal menu of contracts that minimises the supplier's expected costs, provided that the retailer is not worse off by choosing one of these contracts.

1.1 Contracting model

To further specify the considered optimisation problem, we need to introduce the economical setting. The retailer faces external demand with constant rate $d \in \mathbb{R}_{>0}$, which must be satisfied immediately, i.e., there is no backlogging. Placing an order at the supplier has a fixed ordering cost of $f \in \mathbb{R}_{>0}$ for the retailer. Delivery of the products is assumed to be instantaneous (no lead times). Furthermore, the retailer has inventory holding cost of $h \in \mathbb{R}_{>0}$ per product and time unit.

Since we assume that the retailer minimises her own costs, she places an order if and only if her inventory is depleted (the zero-inventory property). An order quantity of $x \in \mathbb{R}_{>0}$ products leads to an average holding cost per time unit of $\frac{1}{2}hx$ and an average ordering cost of $df\frac{1}{x}$. In total, the average costs per time unit for the retailer is given by

$$\phi_R(x) = df\tfrac{1}{x} + \tfrac{1}{2}hx,$$

which is minimised by ordering the well-known economic order quantity $x_R^* = \sqrt{2df/h}$ (see for example Banerjee 1986). The minimal costs are $\phi_R^* = \phi_R(x_R^*) = \sqrt{2dfh}$.

The cost structure of the supplier is similar: the supplier has a fixed setup cost $F \in \mathbb{R}_{>0}$ to handle an order and inventory holding cost $H \in \mathbb{R}_{>0}$. Production takes place with constant rate $p \in \mathbb{R}_{\geq d}$. To minimise his own costs, the supplier produces according to a just-in-time lot-for-lot policy.

Per time unit the supplier has average holding costs of $\frac{1}{2}H\frac{d}{p}x$ and average setup costs of $dF\frac{1}{x}$. This leads to a total cost for the supplier of

$$\phi_S(x) = dF\tfrac{1}{x} + \tfrac{1}{2}H\tfrac{d}{p}x,$$

which is minimised if the order quantity is $x_S^* = \sqrt{2Fp/H}$.

The supplier and retailer both have their own optimal order quantity and either policy is suboptimal for the entire supply chain (unless $x_R^* = x_S^*$). From the perspective of the supply chain, the supplier and retailer can cooperate to lower the total joint costs. The joint costs are given by

$$\phi_J(x) = d(f+F)\tfrac{1}{x} + \tfrac{1}{2}\left(h + H\tfrac{d}{p}\right)x,$$

with optimal joint order quantity $x_J^* = \sqrt{2d(f+F)/(h+H\frac{d}{p})}$. It is not difficult to verify that x_J^* always lies between x_R^* and x_S^* (see Lemma A.1). Therefore, lower joint costs can be achieved by deviating from the individually optimal order quantities. Whether such coordination takes place depends on further assumptions on power relations and market options.

As mentioned before, we assume that both the supplier and the retailer behave rationally and want to minimise their own costs. Furthermore, assume that the retailer has the market power to enforce any order quantity on the supplier. Consequently, the retailer chooses her own optimal order quantity x_R^* by default, called the default ordering policy or default option. By using incentive mechanisms, the supplier can persuade the retailer to deviate from the default policy. We analyse using a side payment $z \in \mathbb{R}$ as an incentive mechanism for cooperation. Note that in the literature side payments are sometimes called quantity discounts. The pair (x, z) of an order quantity x and a side payment z is called a contract.

The presented contract (x, z) must be constructed such that the retailer is not worse off than with her default option: $\phi_R(x) - z \leq \phi_R^*$. This condition is called the Individual Rationality (IR) constraint or participation constraint. If the offered contract leads to the same costs for the retailer as her default option, we assume that the retailer is indifferent and that the supplier can convince the retailer to choose the contract preferred by the supplier. By assumption, the supplier can do so without any additional costs. Hence, the retailer always accepts the presented contract if it satisfies the IR constraint.

If the supplier has complete information of the supply chain, it is straightforward to determine that the optimal contract offers the joint order quantity $x = x_J^*$ and minimal side payment $z = \phi_R(x_J^*) - \phi_R^*$. The resulting ordering policy leads to perfect supply chain coordination: it is optimal for the entire supply chain, as if there is a central decision maker.

However, we study the case that the retailer has private information on her cost structure: either the ordering cost or the holding cost is private (but not both). We consider the case that the supplier is uncertain about the retailer's holding cost, which is without loss of generality as will be shown in Section 2.1. The supplier has narrowed the retailer's real holding cost down to one of $K \in \mathbb{N}$ possible scenarios. Each scenario corresponds to a so-called retailer type. Type $k \in \mathcal{K} = \{1, \dots, K\}$ has cost function

$$\phi_R^k(x) = df\tfrac{1}{x} + \tfrac{1}{2}h_k x,$$

where $0 < h_1 < h_2 < \cdots < h_{K-1} < h_K$ are the possible holding costs. We assume that the default option depends on the type. As such, we add the index $k \in \mathcal{K}$ to our notation to discern between retailer types. For example, for type $k \in \mathcal{K}$ the default order quantity is $x_R^{k*} = \sqrt{2df/h_k}$ with corresponding costs $\phi_R^{k*} = \phi_R^k(x_R^{k*})$.

The supplier designs a menu of K contracts for the retailer to choose from, one for each retailer type. For each type $k \in \mathcal{K}$ the supplier constructs a contract (x_k, z_k) that is individually rational for that specific type, similar to before. However, the retailer can lie about her type and choose any of the presented contracts. This situation is also called a contracting or screening game in the literature, see Laffont and Martimort 2002.

Furthermore, the supplier assigns an objective weight $\omega_k \in \mathbb{R}_{>0}$ to each type $k \in \mathcal{K}$, indicating its likelihood, and minimises his expected costs. Without loss of generality, ω is a probability distribution ($\sum_{k \in \mathcal{K}} \omega_k = 1$), but this is not required for the model and our results.

This leads to the following non-linear optimisation problem:

$$\begin{aligned}
\min \quad & \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(\tilde{x}_k) + \tilde{z}_k \right), && (1.1)\\
\text{s.t.} \quad & \phi_R^k(x_k) - z_k \leq \phi_R^{k*}, \qquad \forall\, k \in \mathcal{K}, && (1.2)\\
& (\tilde{x}_k, \tilde{z}_k) \in \{(x_1, z_1), \dots, (x_K, z_K)\}, \qquad \forall\, k \in \mathcal{K}, && (1.3)\\
& \phi_R^k(\tilde{x}_k) - \tilde{z}_k \leq \phi_R^k(x_l) - z_l, \qquad \forall\, k, l \in \mathcal{K}, && (1.4)\\
& x_k \geq 0, \qquad \forall\, k \in \mathcal{K}.
\end{aligned}$$

The designed contracts (x_k, z_k) must satisfy the IR constraints (1.2). The pair $(\tilde{x}_k, \tilde{z}_k)$ denotes the chosen contract by retailer type $k \in \mathcal{K}$, which must be one of the presented contracts, see constraints (1.3). The retailer chooses the most beneficial contract for herself by possibly lying, which is enforced by constraints (1.4). The supplier's objective is to minimise his expected costs including side payment, see (1.1).

Consider an optimal solution to the non-linear problem and suppose that the retailer lies about her true type. By relabelling the presented contracts, we can construct another optimal solution for which the retailer will never lie about her type, i.e., $(\tilde{x}_k, \tilde{z}_k) = (x_k, z_k)$ for all $k \in \mathcal{K}$. This is also known as the revelation principle.

For example, suppose the retailer type $i \in \mathcal{K}$ lies being type $j \in \mathcal{K}$. This implies that $(\tilde{x}_i, \tilde{z}_i) = (x_j, z_j)$ and in particular

$$\phi_R^i(x_j) - z_j = \phi_R^i(\tilde{x}_i) - \tilde{z}_i \overset{(1.4)}{\leq} \phi_R^i(x_i) - z_i \overset{(1.2)}{\leq} \phi_R^{i*}.$$

So, contract (x_j, z_j) is individually rational for type i. Relabelling or redefining (x_i, z_i) to be equal to (x_j, z_j) leads to an equivalent feasible solution where type i does not lie.

A direct consequence is that we can use the following equivalent simpler non-linear model:

$$\begin{aligned}
\min \quad & \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right), & & \\
\text{s.t.} \quad & \phi_R^k(x_k) - z_k \leq \phi_R^{k*}, & \forall\, k \in \mathcal{K}, & \quad (1.5)\\
& \phi_R^k(x_k) - z_k \leq \phi_R^k(x_l) - z_l, & \forall\, k, l \in \mathcal{K}, & \quad (1.6)\\
& x_k \geq 0, & \forall\, k \in \mathcal{K}. &
\end{aligned}$$

We call this simpler model the default contracting model. Here, (1.6) are the Incentive Compatibility (IC) constraints to prevent types from lying. These enable us to implicitly set $(\tilde{x}_k, \tilde{z}_k) = (x_k, z_k)$ and drop the choice of contracts completely from the model. Note that the menu of contracts with $(x_k, z_k) = (x_R^{k*}, 0)$ for all $k \in \mathcal{K}$ is a feasible solution.

1.2 Connection to the literature

Similar models have been studied in the literature and there are many variations. One variation is to consider a continuous range of retailer types such as in Corbett and De Groote 2000; Corbett, Zhou et al. 2004. Pinar 2015 analyses the model with structurally different cost functions. In Cakanyildirim et al. 2012 the roles of the supplier and retailer are swapped: the supplier has private information and the retailer designs a menu of contracts. We focus on literature that closely relates to our model, see also Table 1 for a comparison.

In this paper, we assume that only one cost parameter of the retailer is private, which leads to so-called single-dimensional types. Pishchulov and Richter 2016 analyse the same setting, but with two-dimensional retailer types. That is, both the ordering cost and the holding cost are uncertain. Their research provides a complete analysis of the model in Sucky 2006, who considers the same problem. Both use optimality conditions to determine a list of candidates for the optimal solution. However, the analysis is restricted to only two retailer types, whereas we consider a general number of types, albeit single-dimensional types. From our results we see different qualitative properties of the optimal solution for two types versus more than two types.

Li et al. 2012 incorporate a controllable lead time into the contracting model. The retailer has additional safety stock proportional to the square root of the lead-time demand. Only two retailer types are considered. The two types are two-dimensional, but the type with low costs has lower ordering and holding costs than the type with high costs.

In Voigt and Inderfurth 2011 the supplier's setup cost is an additional decision variable in the contracting model. The supplier has to decide whether to lower his setup cost at the cost of lost investment opportunities. Furthermore, the supplier has no holding costs and the retailer no ordering costs. Besides the differences in cost functions, their model assumes the same default option for all retailer types. To our knowledge, Voigt and Inderfurth 2011 is the only paper with a related model that considers a general number of retailer types, although the authors do assume a certain condition on the distribution of the retailer types.

Another model similar to ours is discussed in Zissis et al. 2015, but there are only two retailer types. Furthermore, the supplier has no holding costs, which reduces the number of optimal menus of contracts that can occur. Since we analyse the case for two types in detail, our results generalise their derived structural properties of the optimal menu of contracts.

In light of the previous references, we emphasise that the inclusion of both ordering/setup costs and holding costs for the retailer and supplier results in structurally different optimal menus of contracts. This is because both involved parties have a finite individually optimal order quantity. Deviating from that quantity leads to higher costs. This is not true if only one type of cost (ordering or holding) is included, since then the individually optimal order quantity is either zero or infinity. Furthermore, in the literature it is common to assume that the supplier prefers a larger order quantity than the retailer. We do not make this assumption and therefore also provide insight into contracts when the supplier prefers smaller order quantities.

Paper	**Supplier's costs:**		**Retailer's costs:**		**Number of types:**		**Type-dependent**	**Dimension of type:**	
	Setup	**Holding**	**Ordering**	**Holding**	**Two**	**Multiple**	**default option**	**One**	**Two**
Sucky 2006	✓	✓	✓	✓	✓		✓	✓	✓
Voigt and Inderfurth 2011	✓			✓	✓	✓		✓	
Li et al. 2012	✓	✓	✓	✓	✓		✓	✓	✓
Zissis et al. 2015	✓		✓	✓	✓		✓	✓	
Pishchulov and Richter 2016	✓	✓	✓	✓	✓		✓	✓	✓
Our paper	✓	✓	✓	✓	✓	✓	✓	✓	

Table 1: Comparison of related literature.

1.3 Contribution

We consider a principal-agent contracting model with asymmetric information under the EOQ setting. Our model distinguishes itself from the literature by having a general number of retailer types with type-dependent default options. Furthermore, the supplier and the retailer types have both ordering/setup costs and holding costs. Consequently, a typical analysis using optimality conditions is complex and does not appear to lead to a generalisable solution method.

Our main contributions are as follows. First, we show that this non-convex model has a hidden convexity, which is achieved by a change of decision variables. Hence, in practice we can numerically solve our model to optimality for a general number of retailer types using various efficient techniques. Second, we determine structural properties of the optimal solution for a general number of retailer types. The analysis shows significant differences in the structure of optimal menus of contracts for two types compared to more than two types. Third, we prove a sufficient condition to guarantee unique contracts in the optimal solution. We provide counterexamples when this condition is omitted.

In particular, we use the structural properties to analyse the difference between two and three retailer types. To do so, we analytically solve the model for these two cases. We provide a complete characterisation of the optimal solution for the case with two retailer types. The derived closed-form formulas of the optimal solution are not only simpler than those found in related literature, they also show additional structure of the solution. For the specific case of three retailer types we did not find any results in the literature. We give a minimal list of candidate contracts for the optimal solution.

The remainder is organised as follows. In Section 2 we present an alternative model which shows the hidden convexity and leads to an efficient solution method. We continue with structural properties of the contracting model in Section 3. In Section 4 we discuss the optimal menus of contracts for two and three retailer types, where we give examples of each occurring optimal menu. The derivations of the optimal contracts for two and three types are given in Appendices A and B. We end with a general discussion of our results in Section 5.

2 Efficient solution method

In this section we show that the contracting problem can be solved efficiently. This insight becomes apparent after a change of decision variables of the contracting model. Before we give the details, we prove that for single-dimensional retailer types we can assume without loss of generality that the retailer's holding cost is uncertain. Consequently, we can efficiently solve two kinds of contracting models.

2.1 Equivalence when one cost parameter is uncertain

Consider a contracting problem where all retailer types instead have the same holding cost h, but different ordering costs f_k. We can transform any such problem to an equivalent contracting problem where all types have the same ordering cost $\hat{f}$, but different holding costs $\hat{h}_k$.

The transformation is as follows. For arbitrary $\hat{d} \in \mathbb{R}_{>0}$ and $\hat{p} \in \mathbb{R}_{\geq \hat{d}}$, define the following parameters:

$$\hat{\omega}_k = \omega_k, \qquad \hat{H} = 2(dF)\frac{\hat{p}}{\hat{d}}, \qquad \hat{F} = (\tfrac{1}{2}H\tfrac{d}{p})\frac{1}{\hat{d}}, \qquad \hat{f} = (\tfrac{1}{2}h)\frac{1}{\hat{d}}, \qquad \hat{h}_k = 2(df_k).$$

These parameters are well-defined and result in a contracting problem instance where all retailer types have the same ordering cost, instead of the same holding cost. To distinguish the instances, let $\hat{S}$ be the supplier and $\hat{R}$ the retailer for the newly constructed problem. We claim that both instances are equivalent, i.e., both have the same optimal objective value and there is a bijection between the optimal solutions.

To show any equivalence between instances, the important expressions of the contracting model are: ϕ_S, ϕ_R^k, and ϕ_R^{k*}. Consider any order quantity $x_k \in \mathbb{R}_{>0}$ and set $\hat{x}_k = 1/x_k$, leading to the expressions:

$$\begin{aligned}
\phi_S(x_k) &= dF\frac{1}{x_k} + \frac{1}{2}H\frac{d}{p}x_k = \frac{1}{2}H\frac{d}{p}\frac{1}{\hat{x}_k} + dF\hat{x}_k = \hat{d}\hat{F}\frac{1}{\hat{x}_k} + \frac{1}{2}\hat{H}\frac{\hat{d}}{\hat{p}}\hat{x}_k = \phi_{\hat{S}}(\hat{x}_k),\\
\phi_R^k(x_k) &= df_k\frac{1}{x_k} + \frac{1}{2}hx_k = df_k\hat{x}_k + \frac{1}{2}h\frac{1}{\hat{x}_k} = \hat{d}\hat{f}\frac{1}{\hat{x}_k} + \frac{1}{2}\hat{h}_k\hat{x}_k = \phi_{\hat{R}}^k(\hat{x}_k),\\
\phi_R^{k*} &= \sqrt{2df_kh} = \sqrt{2\hat{d}\hat{f}\hat{h}_k} = \phi_{\hat{R}}^{k*},
\end{aligned}$$

where the equalities follow by definition. Thus, any pair (x, z) is a feasible solution for the original instance if and only if $(1/x, z)$ is feasible for the newly constructed instance. Moreover, the objective values of the two instances are equal.

To conclude, the qualitative properties of the contracting model with one uncertain cost parameter are irrespective of which cost parameter (ordering or holding cost) is uncertain.

2.2 Alternative convex model

The contracting model is not convex, since the IC constraints state:

$$df\left(\frac{1}{x_k}-\frac{1}{x_l}\right)+\frac{1}{2}(h_k-h_l)x_k+z_l-z_k\leq 0, \qquad \forall\, k,l\in\mathcal{K}.$$

Here, the term $-1/x_l$ is not convex in the decision variables. Non-convex optimisation problems are generally difficult to solve, but we show that this is not the case for our problem. We reveal a hidden convexity of our problem by changing the perspective from side payments to so-called 'information rents'.

An alternative contracting model can be obtained by rescaling the side payments as follows. The individual rationality constraints imply that $z_k \geq \phi_R^k(x_k)-\phi_R^{k*}\geq 0$. As such, it is natural to interpret the value $\phi_R^k(x_k)-\phi_R^{k*}$ as the minimum side payment that always has to be paid to satisfy the IR constraint. We introduce a new variable y_k which denotes the additional side payment required by the IC constraints:

$$y_k = z_k-(\phi_R^k(x_k)-\phi_R^{k*})\geq 0.$$

This variable is also known as the information rent for type k. Substituting $z_k = y_k+\phi_R^k(x_k)-\phi_R^{k*}$ in the default contracting model leads to:

$$\begin{aligned}
\min \quad & \sum_{k\in\mathcal{K}}\omega_k\left(\phi_S(x_k)+\phi_R^k(x_k)+y_k-\phi_R^{k*}\right), && \\
\text{s.t.} \quad & y_k\geq 0, && \forall\, k\in\mathcal{K}, \qquad (2.1)\\
& y_l-y_k+\phi_R^l(x_l)-\phi_R^k(x_l)\leq\phi_R^{l*}-\phi_R^{k*}, && \forall\, k,l\in\mathcal{K}, \qquad (2.2)\\
& x_k\geq 0, && \forall\, k\in\mathcal{K}.
\end{aligned}$$

So, (2.1) are the IR constraints and (2.2) are the IC constraints. We call the new model the alternative contracting model to differentiate it from the earlier defined default model. By definition of y_k, there is a bijection between the feasible region of the alternative model and that of the default model. Furthermore, the corresponding objective values are the same. Hence, we can solve the default model by solving the alternative model and vice versa.

Although both models are equivalent in the sense mentioned above, there is one significant difference. Notice that the non-linear terms in (2.2) cancel out if we expand the cost functions:

$$y_l-y_k+\frac{1}{2}(h_l-h_k)x_l = y_l-y_k+\phi_R^l(x_l)-\phi_R^k(x_l)\leq\phi_R^{l*}-\phi_R^{k*}.$$

Thus, all constraints of the alternative model are linear in the decision variables. Since the objective function is convex, we conclude that the alternative model is convex. Moreover, the feasible solution $x_k = x_R^{k*}$ and $y_k=\epsilon\in\mathbb{R}_{>0}$ for all $k\in\mathcal{K}$ is a Slater point, i.e., strictly feasible. It is well-known that a convex model with differentiable functions and Slater points can be solved efficiently using scalable methods such as interior-point or cutting-plane methods (see Bertsekas 2015; Boyd and Vandenberghe 2004). This conclusion is stated in Theorem 2.1.

Theorem 2.1. *The contracting model can be solved efficiently via the alternative model.*

Proof. The proof is given in the above discussion. □

Remark 2.2. Recalling the results from Section 2.1, we note that the contracting model with single-dimensional types can be solved efficiently. If both the ordering cost f and the holding cost h are private information, we have two-dimensional retailer types specified by cost parameters (f_k, h_k). In this case, both the default model and the alternative model fall in the category of Difference of Convex functions (DC) programming. In the literature, there exist good numerical methods to find local optima of DC models, see Horst et al. 1991; Pham Dinh and Le Thi 2014. However, to guarantee global optimality such methods need to be incorporated into for example a Branch-and-Bound procedure. ♦

To conclude, in practice we can determine optimal solutions of our problem numerically. We have implemented a cutting-plane algorithm using Gurobi as Linear Programming solver. Typical computational times are less than a second for one hundred types on a standard desktop computer. However, it is worthwhile to further analyse the model theoretically. In the following sections we determine qualitative properties of the optimal menu of contracts and in some cases even provide closed-form solutions. The used model (default or alternative) has no significant effect on the results. Hence, we present all results using the default model and place remarks where needed for the alternative model.

3 Structural properties

We continue with additional properties of the contracting model and its optimal solutions. These results hold for a general number of retailer types. In particular, the model is connected to a one-to-all shortest path problem in a certain directed graph. This allows us to use the theory of the shortest path problem and have a different view of the contracting model. Furthermore, we use the well-known Karush-Kuhn-Tucker conditions to determine structures in the optimal solution. In the end, we derive a sufficient condition to guarantee unique contracts in the optimal solution. Moreover, the analysis leads to a minimal list of menus of contracts for two and three retailer types which contains the optimal solution. These are discussed in Section 4.

3.1 Shortest path interpretation

A closer look into the structure of the IR and IC constraints shows a connection with a dual shortest path interpretation. For given fixed quantities x_k, constraints (1.5) and (1.6) can be seen as the dual constraints of a shortest path problem. To be specific, for given x_k the whole model is equivalent to the dual of a specific minimum cost flow formulation for the one-to-all shortest path problem. A similar connection to shortest paths has been described in Vohra 2012.

Consider the directed graph $G = (\mathcal{V}, \mathcal{A})$ with nodes $\mathcal{V} = \{s\} \cup \mathcal{K}$ and directed arcs $\mathcal{A} = \{(s,k) : k \in \mathcal{K}\} \cup \{(k,l) : k,l \in \mathcal{K}, k \neq l\}$. That is, G is the complete graph of K retailer nodes with a source added. See Figure 1 for an example. We call such a graph an IRIC graph, which stands for Individual Rationality and Incentive Compatibility graph for reasons to become apparent.

The lengths (or costs) of the arcs are:

- arc (s,k) with $k \in \mathcal{K}$ has length $\phi_R^{k*} - \phi_R^k(x_k)$,
- arc (k,l) with $k,l \in \mathcal{K}$, $k \neq l$, has length $\phi_R^l(x_k) - \phi_R^l(x_l)$.

Finally, node s has supply $\sum_{k \in \mathcal{K}} \omega_k$ and each retailer node $k \in \mathcal{K}$ has demand ω_k. There are no capacity restrictions on the arcs. Consequently, flow will be sent along shortest paths in the optimal solution of the flow formulation. Hence, we see this flow formulation as a one-to-all shortest path representation.

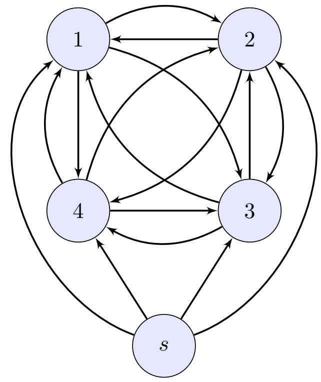


Figure 1: IRIC graph for $K = 4$ retailer types.

It is useful to mention some well-known properties of the dual flow formulation, see also Ahuja et al. 1993. Consider the optimal solution (x, z) of the contracting model. The value $-z_k$ is equal to the length of the shortest (s, k)-path. Moreover, Strong Duality implies that the IRIC graph contains a negative cycle if and only if the dual is infeasible. In such cases there exist no side payments that will satisfy the IC constraints for the considered order quantities x_k. Thus, the IC constraints can be satisfied if and only if the corresponding IRIC graph has no negative cycles.

In the non-degenerate case, the set of all used arcs in the optimal shortest paths from s to the other nodes forms a spanning tree in the IRIC graph. In the degenerate case, this does not hold, but the optimal solution can be modified such that the used arcs form a spanning tree again. In particular, if the set contains cycles, these cycles must have length 0.

From the complementary slackness conditions it follows that if arc (i, j) is in the spanning tree, then the corresponding constraint in the dual is satisfied with equality. For example, if arc (s, k) is part of the shortest path tree, then the IR constraint for type k is tight. If arc (k, l) is used, with $k, l \in \mathcal{K}$, then type l wants to pretend to be type k. That is, the IC constraint $\phi^l_R(x_l) - z_l \le \phi^l_R(x_k) - z_k$ is satisfied with equality.

Due to the bijection between retailer types and retailer type nodes, and the bijection between arcs and the IR and IC constraints, we often interchange interpretation and terminology. For example, we can refer to outgoing arcs out of a retailer type, referring to the outgoing arcs of the corresponding node in the graph. These insights explain why we call the graph the IRIC graph.

Remark 3.1. We note that the same results hold for the alternative model, with the exception that the arc lengths are given by:

- arc (s, k) with $k \in \mathcal{K}$ has length 0,
- arc (k, l) with $k, l \in \mathcal{K}$, $k \neq l$, has length $\phi^l_R(x_k) - \phi^k_R(x_k) + \phi^{k*}_R - \phi^{l*}_R$. ♦

Remark 3.2. Due to personal taste, one can prefer a *longest* path formulation instead. The arc lengths are somewhat easier to remember directly from the model by rewriting the IR and IC constraints to:

$$z_k \ge \underbrace{\phi^k_R(x_k) - \phi^{k*}_R}_{\text{arc length }(s,k)}, \qquad\qquad z_k \ge z_l + \underbrace{\left(\phi^k_R(x_k) - \phi^k_R(x_l)\right)}_{\text{arc length }(l,k)}.$$

In the optimal solution, z_k is the length of the longest path from node s to node k when using these arc lengths. Naturally, the longest and shortest path formulations are equivalent. ♦

3.2 Adjacent retailer types

Since the types are ordered such that $h_1 < h_2 < \cdots < h_K$, there is a sense of adjacent or neighbouring types. We define the neighbours of type $k \in \mathcal{K}$ to be the types $k-1$ and $k+1$, where types 1 and K have only one neighbour. The adjacency of types plays an important role as we will see.

Intuitively, one would expect that in an optimal solution a type with higher holding cost gets offered a lower order quantity (i.e., more frequent orderings) to prevent too high inventory costs. Lemma 3.3 shows that this intuition is mathematically correct.

Lemma 3.3. *Any feasible menu of contracts satisfies $x_1 \geq x_2 \geq \cdots \geq x_K$.*

Proof. Consider a feasible menu of contracts (x, z). From the shortest path interpretation in Section 3.1 we know that no negative cycles exist in the corresponding IRIC graph. In particular, any 2-cycle in the IRIC graph has non-negative length. Without loss of generality, consider $i, j \in \mathcal{K}$ with $h_i < h_j$ and consider the length of the 2-cycle between nodes i and j, which must be non-negative:

$$\begin{aligned}
& & \left(\phi_R^i(x_j) - \phi_R^i(x_i)\right) + \left(\phi_R^j(x_i) - \phi_R^j(x_j)\right) &\geq 0 \\
&\iff & \tfrac{1}{2}(h_j - h_i)(x_i - x_j) &\geq 0 \\
&\overset{(h_i<h_j)}{\iff} & x_i - x_j &\geq 0.
\end{aligned}$$

Hence, $x_i \geq x_j$ must hold in any feasible solution. □

The ordering (or monotonicity) in the order quantities is a common property of contracting models, see for example Laffont and Martimort 2002; Vohra 2012. However, there is no monotonicity in the side payments (see Section 4 for examples).

A consequence of Lemma 3.3 is that adjacent retailer types follow both from the holding costs and from the (feasible) order quantities. In fact, using this result we can restrict the incentive compatibility constraints to take only the neighbouring types into account, without changing the feasible region. See Lemma 3.4 for the result. We call these constraints the adjacent IC constraints.

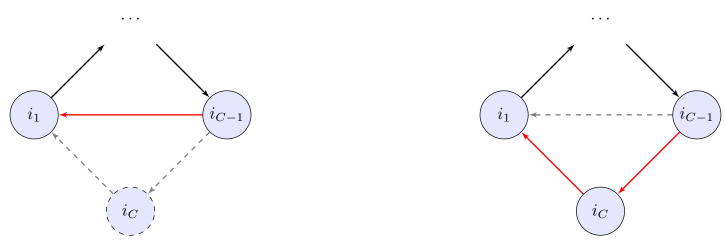
Lemma 3.4. *The adjacent incentive compatibility constraints are sufficient to ensure general incentive compatibility.*

Proof. Let (x, z) be the optimal menu of contracts when we *only* use the adjacent IC constraints, instead of all general IC constraints. Consider a cycle $\mathcal{C} = (i_1, \ldots, i_C)$ of unique retailer nodes in the IRIC graph corresponding to (x, z). We prove that any such cycle has non-negative length, implying that *all* general IC constraints are satisfied. The proof is by induction on the cardinality of $\mathcal{C}$.

If $C = 2$, then the adjacent IC constraints enforce that the cycle length is non-negative. Therefore, let $C > 2$ and without loss of generality, assume that type i_C has the greatest holding cost. By induction, the cycle $(i_1, \ldots, i_{C-1})$ has non-negative length. We compare the difference in length between the two cycles, see also Figure 2:

$$\begin{aligned}
&\left(\left(\phi_R^{i_C}(x_{i_{C-1}}) - \phi_R^{i_C}(x_{i_C})\right) + \left(\phi_R^{i_1}(x_{i_C}) - \phi_R^{i_1}(x_{i_1})\right) \right) - \left(\phi_R^{i_1}(x_{i_{C-1}}) - \phi_R^{i_1}(x_{i_1})\right) \\
&\quad = \phi_R^{i_C}(x_{i_{C-1}}) - \phi_R^{i_C}(x_{i_C}) + \phi_R^{i_1}(x_{i_C}) - \phi_R^{i_1}(x_{i_{C-1}}) \\
&\quad = \tfrac{1}{2}(h_{i_C} - h_{i_1})(x_{i_{C-1}} - x_{i_C}) \geq 0.
\end{aligned}$$

The inequality follows from our assumptions on the holding costs ($h_{i_C} > h_{i_1}$) and Lemma 3.3. Thus, $\mathcal{C}$ must have non-negative length as well. Consequently, all IC constraints hold without explicitly incorporating the corresponding IC constraints in the optimisation model. To conclude, (x, z) is also optimal for the complete contracting model with all general IC constraints. □



(a) Smaller cycle $(i_1, \dots, i_{C-1})$. (b) Larger cycle $\mathcal{C} = (i_1, \dots, i_C)$.

Figure 2: Relevant arcs in the induction proof of Lemma 3.4.

We can use Lemma 3.4 to prove that order quantities satisfying $x_1 \geq x_2 \geq \dots \geq x_K > 0$ can always be extended to a feasible menu of contracts (x, z), see Corollary 3.5. Therefore, we sometimes call such order quantities feasible for the contracting model.

Corollary 3.5. *For given order quantities satisfying $x_1 \geq x_2 \geq \dots \geq x_K > 0$, it is feasible and optimal to determine the side payments via the shortest path interpretation.*

Proof. From Lemma 3.4 it follows that for feasibility we only need to determine side payments such that the adjacent IC constraints are satisfied. From the shortest path interpretation, we know that side payments satisfying the adjacent IC constraints exist if and only if 2-cycles in the corresponding graph have non-negative length. Now consider arbitrary $i, j \in \mathcal{K}$ with $h_i < h_j$. The proof of Lemma 3.3 shows that the 2-cycle between i and j has non-negative length if and only if $x_i \geq x_j$, which holds by assumption.

Hence, we can determine feasible side payments by solving a one-to-all shortest path problem as described in Section 3.1. Furthermore, this leads to the best possible feasible side payments with respect to the given order quantities. □

3.3 KKT conditions

Since the contracting model consists of continuously differentiable functions with a continuous domain, there are well-known necessary conditions for optimality and even sufficient optimality conditions in certain cases. Using these conditions we can design candidate solutions for further inspection. This allows us to analytically investigate properties of the optimal menu of contracts. In the following sections we use the Karush-Kuhn-Tucker (KKT) optimality conditions to do so.

First of all, we point out a subtle issue regarding KKT conditions. The default contracting model is non-convex. As such, the *general* KKT conditions, also known as Fritz-John conditions, (see Brinkhuis and Tikhomirov 2005; John 1948), are necessary for the optimal solution. We need regularity conditions to be able to use the *standard* KKT conditions (Karush 1939; Kuhn and Tucker 1951), such as the Mangasarian-Fromovitz constraint qualification.

However, with a slight detour we can ignore this issue. We have an equivalent convex model with a Slater point, namely the alternative contracting model of Section 2.2. Thus, the standard KKT conditions are necessary and sufficient for the alternative model. Both models lead to the same general KKT conditions, from which we conclude that the standard KKT conditions are also necessary and sufficient for the default model.

The KKT conditions lead to a (large) set of candidate solutions. These solutions will be called KKT menus and their contracts KKT contracts. The optimal solution of our model is the best KKT menu. However, in general determining this set will be intractable due to its size. Therefore, we analyse our problem to exclude certain KKT menus. Unfortunately, we do not end up with a tractable solution approach for a general number of retailer types. Hence, this KKT approach seems unsuccessful to provide a generalisable solution method, but the analysis will nevertheless provide additional insight in optimal menus of contracts.

With the above mentioned remarks in mind, we determine the general KKT conditions for the contracting model. Using Lemma 3.4 we only incorporate the adjacent IC constraints in our model. The Lagrangian function with Lagrange multipliers $\kappa \in \mathbb{R}_{\geq 0}$, $\lambda, \nu \in \mathbb{R}^K_{\geq 0}$, and $\mu \in \mathbb{R}^{2K-2}_{\geq 0}$ is given by:

$$\begin{aligned} L(x, z, \kappa, \lambda, \mu, \nu) = \kappa \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right) &+ \sum_{k \in \mathcal{K}} \lambda_k \left(\phi_R^k(x_k) - z_k - \phi_R^{k*}\right) \\ &+ \sum_{k \in \mathcal{K} \setminus \{1\}} \mu_{k-1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k-1}) + z_{k-1}\right) \\ &+ \sum_{k \in \mathcal{K} \setminus \{K\}} \mu_{k+1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k+1}) + z_{k+1}\right) \\ &+ \sum_{k \in \mathcal{K}} \nu_k(-x_k). \end{aligned}$$

We deliberately choose this order of the indices of μ and will explain in Section 3.4 why this notation is useful.

The KKT conditions consist of primal and dual feasibility, complementary slackness, and stationarity constraints. The dual feasibility constraints require all multipliers to be non-negative with the additional condition that not all multipliers are zero. The complementary slackness constraints are:

$$\begin{aligned} \lambda_k \left(\phi_R^k(x_k) - z_k - \phi_R^{k*}\right) &= 0, && \forall\, k \in \mathcal{K}, \\ \mu_{k-1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k-1}) + z_{k-1}\right) &= 0, && \forall\, k \in \mathcal{K} \setminus \{1\}, \\ \mu_{k+1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k+1}) + z_{k+1}\right) &= 0, && \forall\, k \in \mathcal{K} \setminus \{K\}, \\ \nu_k x_k &= 0, && \forall\, k \in \mathcal{K}. \end{aligned} \tag{3.1}$$

Since $x_k = 0$ is never optimal, it follows from (3.1) that $\nu_k = 0$ for all $k \in \mathcal{K}$. Therefore, we set $\nu_k = 0$ and ignore related terms completely. Likewise, we argued above that the standard KKT conditions hold, implying that $\kappa = 1$. Thus, we ignore this multiplier as well.

For each $k \in \mathcal{K}$, the stationarity constraints with respect to x_k are:

$$\begin{aligned} \omega_k \frac{\mathrm{d}\phi_S}{\mathrm{d}x}(x_k) + \lambda_k \frac{\mathrm{d}\phi_R^k}{\mathrm{d}x}(x_k) + (\mu_{k-1,k} + \mu_{k+1,k}) \frac{\mathrm{d}\phi_R^k}{\mathrm{d}x}(x_k) \\ - \mu_{k,k-1} \frac{\mathrm{d}\phi_R^{k-1}}{\mathrm{d}x}(x_k) - \mu_{k,k+1} \frac{\mathrm{d}\phi_R^{k+1}}{\mathrm{d}x}(x_k) = 0, \end{aligned} \tag{3.2}$$

and with respect to z_k:

$$\omega_k - \lambda_k - (\mu_{k-1,k} + \mu_{k+1,k}) + (\mu_{k,k-1} + \mu_{k,k+1}) = 0, \tag{3.3}$$

where all ill-defined multipliers with out of bound indices are set to zero. We can simplify the stationarity constraints by substituting (3.3) in (3.2):

$$\omega_k \left(-\frac{d(f+F)}{x_k^2} + \frac{1}{2}\left(h_k + H\tfrac{d}{p}\right)\right) + \frac{1}{2}\mu_{k,k-1}(h_k - h_{k-1}) + \frac{1}{2}\mu_{k,k+1}(h_k - h_{k+1}) = 0. \tag{3.4}$$

To conclude, the KKT conditions consist of the primal and dual feasibility constraints, complementary slackness constraints, and stationarity constraints (3.3) and (3.4).

Remark 3.6. The KKT conditions for the alternative model directly give (3.3) and (3.4). ♦

3.4 KKT graph

As mentioned before, we only use adjacent IC constraints (Lemma 3.4). The shortest path interpretation of Section 3.1 still holds and the corresponding Adjacent IRIC graph is shown in Figure 3. Notice that the order of indices of μ corresponds nicely to the Adjacent IRIC graph. If $\mu_{lk} > 0$, then the equality $\phi_R^k(x_k) - z_k = \phi_R^k(x_l) - z_l$ must hold by the KKT complementary slackness conditions. Hence, arc (l,k) is used by the shortest paths, as discussed in Section 3.1. The same holds for λ_k, constraint $\phi_R^k(x_k) - z_k \leq \phi_R^{k*}$, and arc (s,k). Consequently, we have bijections between multipliers λ (or μ), the IR (or IC) constraints, and certain arcs in the Adjacent IRIC graph. As such, we can refer to the multiplier of an arc in the Adjacent IRIC graph.

Thus, the strictly positive multipliers indicate which arcs are for certain part of shortest paths in the IRIC graph. Unfortunately, there could be arcs in a shortest path for which the multiplier is zero, as degenerate cases may occur.

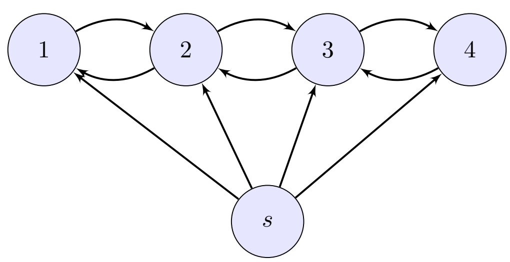


Figure 3: Adjacent IRIC graph for $K = 4$ types.

Each KKT menu can be identified by the subset of multipliers which are strictly positive. We can visualise the contract in the Adjacent IRIC graph by only considering the arcs for which the corresponding multipliers are strictly positive. That is, we have a directed graph $\hat{G} = (\mathcal{V}, \hat{\mathcal{A}})$ with $\mathcal{V} = \{s\} \cup \mathcal{K}$ and arcs

- (s,k) with $k \in \mathcal{K}$ if $\lambda_k > 0$,
- $(k,k-1)$ with $k \in \mathcal{K} \setminus \{1\}$ if $\mu_{k,k-1} > 0$,
- $(k,k+1)$ with $k \in \mathcal{K} \setminus \{K\}$ if $\mu_{k,k+1} > 0$.

We call this graph the KKT graph. In the results to come, we often use the term 'connected component' of the KKT graph. To avoid confusion, a subset $\mathcal{S} \subseteq \mathcal{V}$ is a connected component if between each pair of nodes in $\mathcal{S}$ there exists an *undirected* path in the graph. Furthermore, a node is called isolated if it has no (in- or outgoing) arcs.

The KKT graph allows for easy-to-draw names of KKT menus. We call arc (s, k) the Up arc for retailer type $k \in \mathcal{K}$, arc $(k, k+1)$ the Right arc, and arc $(k, k-1)$ the Left arc. The name of a KKT menu is simply a list of the Up, Right, and Left arcs of the corresponding KKT graph. For example, KKT menu 1Right2UpLeft3UpLeftRight4x is shown in Figure 4.

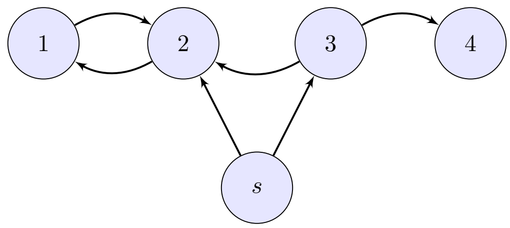


Figure 4: KKT graph for 1Right2UpLeft3UpLeftRight4x.

3.5 Properties of optimal contracts

The result that only adjacent IC constraints need to be taken into account greatly reduces the number of possible KKT menus to consider. We continue to analyse which cases can also be excluded from consideration, i.e., which combinations of strictly positive multipliers (or which KKT graphs) can occur. We start with Lemma 3.7, which shows an explicit connection to shortest paths.

Lemma 3.7. *Every retailer node $k \in \mathcal{K}$ must be reachable from source node s in the KKT graph.*

Proof. First, suppose $k \in \mathcal{K}$ has no ingoing arcs, i.e., $\lambda_k = \mu_{k-1,k} = \mu_{k+1,k} = 0$. From (3.3) we have:

$$\omega_k + \mu_{k,k-1} + \mu_{k,k+1} = 0 \qquad \Longrightarrow \qquad \mu_{k,k-1} + \mu_{k,k+1} < 0.$$

This contradicts the fact that all multipliers are non-negative. Hence, any node in $\mathcal{K}$ must have an ingoing arc.

Second, let $\mathcal{S} = \{i, i+1, \ldots, j-1, j\} \subseteq \mathcal{K}$ be an arbitrary maximal connected subset of retailer nodes with $\lambda_k = 0$ for all $k \in \mathcal{S}$. That is, no node in $\mathcal{S}$ is directly reachable from node s. Adding up (3.3) for all $k \in \mathcal{S}$ results in:

$$\sum_{k \in \mathcal{S}} \omega_k - \mu_{i-1,i} - \mu_{j+1,j} + \mu_{i,i-1} + \mu_{j,j+1} = 0. \tag{3.5}$$

Notice that all internal arcs of $\mathcal{S}$ cancel out. Furthermore, by maximality of the subset, all remaining multipliers in (3.5) must be zero. This leads to a contradiction, since $\omega_k > 0$ for all $k \in \mathcal{K}$. To conclude, every maximal connected component is reachable from s.

Finally, by iteratively using that each node has an ingoing arc we can conclude that every node must be reachable from node s. □

Notice that this is a stronger property than the fact that the side payments follow from shortest paths. Shortest paths imply that each node is reachable from s using only arcs for which the corresponding constraint is tight. As weak complementary slackness may hold, tightness does not automatically imply that the corresponding multiplier is strictly positive. However, a strictly positive multiplier does imply tightness of the constraint. This result allows us to discard certain combinations of multipliers, significantly reducing the number of options.

The next lemma describes a general pattern (a 'T-pattern') that will never occur in the optimal solution.

Lemma 3.8. *There exist no $k \in \mathcal{K} \setminus \{1, K\}$ such that the constraints corresponding to arcs (s,k), $(k,k-1)$, and $(k,k+1)$ are satisfied with equality.*

Proof. Let $i,j,k \in \mathcal{K}$, $i < k < j$, be such that the constraints corresponding to arcs (s,k), (k,i), and (k,j) are satisfied with equality. Consequently, we have:

$$\begin{aligned} &\phi_R^k(x_k) - z_k = \phi_R^{k*}, && \\ &\phi_R^i(x_i) - z_i = \phi_R^i(x_k) - z_k, && \phi_R^j(x_j) - z_j = \phi_R^j(x_k) - z_k, \\ &\phi_R^i(x_i) - z_i \le \phi_R^{i*}, && \phi_R^j(x_j) - z_j \le \phi_R^{j*}. \end{aligned}$$

Combining these relations leads to the following:

$$\begin{aligned} \phi_R^{i*} &\ge \phi_R^i(x_i) - z_i = \phi_R^i(x_k) - z_k = \phi_R^i(x_k) - \phi_R^k(x_k) + \phi_R^{k*} \\ &= \tfrac{1}{2}(h_i - h_k)x_k + \phi_R^{k*}, \\ \phi_R^{j*} &\ge \phi_R^j(x_j) - z_j = \phi_R^j(x_k) - z_k = \phi_R^j(x_k) - \phi_R^k(x_k) + \phi_R^{k*} \\ &= \tfrac{1}{2}(h_j - h_k)x_k + \phi_R^{k*}. \end{aligned}$$

Rewriting these results gives:

$$\frac{\phi_R^{i*} - \phi_R^{k*}}{h_i - h_k} \le \tfrac{1}{2}x_k \le \frac{\phi_R^{j*} - \phi_R^{k*}}{h_j - h_k}.$$

Recall that $\phi_R^{l*} = \sqrt{2dfh_l}$ for all $l \in \mathcal{K}$. Thus, we arrive at the following inequality:

$$\begin{aligned} \frac{\sqrt{h_k} - \sqrt{h_i}}{h_k - h_i} \le \frac{\sqrt{h_j} - \sqrt{h_k}}{h_j - h_k} \quad &\iff \quad \frac{1}{\sqrt{h_k} + \sqrt{h_i}} \le \frac{1}{\sqrt{h_j} + \sqrt{h_k}} \\ &\iff \quad \sqrt{h_i} \ge \sqrt{h_j}. \end{aligned}$$

The first inequality compares two slopes between three points on the square root curve. Such an inequality never holds for $h_i < h_k < h_j$, as the equivalent inequality shows. □

Corollary 3.9. *A retailer node directly connected to node s in the KKT graph has at most one outgoing arc.*

Proof. Suppose a node $k \in \mathcal{K}$ directly connected to s has more outgoing arcs. The direct connection to node s implies $\lambda_k > 0$. Furthermore, the outgoing arcs must be $(k,k-1)$ and $(k,k+1)$, so $\mu_{k,k-1}, \mu_{k,k+1} > 0$. The KKT complementary slackness conditions imply that the corresponding constraints are tight, violating Lemma 3.8. □

Corollary 3.9 implies that the graph in Figure 4 is never a valid KKT graph, since node 3 has Up, Left, and Right arcs (violating the corollary).

3.5.1 Cycles are restrictive

As a result of Lemma 3.4, the only cycles of interest are 2-cycles between adjacent nodes. The next lemma and corollary show that 2-cycles lead to having the same contract, also called 'bunching' in the literature.

Lemma 3.10. *Both incentive compatibility constraints between (adjacent) retailer types i and j are tight if and only if $x_i = x_j$. Furthermore, if $x_i = x_j$ then $z_i = z_j$ must hold.*

Proof. First, suppose the order quantities for types $i, j \in \mathcal{K}$ are the same. The incentive compatibility constraints state:

$$\begin{aligned} \phi_R^i(x_i) - z_i \leq \phi_R^i(x_j) - z_j = \phi_R^i(x_i) - z_j \quad &\iff \quad z_i \geq z_j, \\ \phi_R^j(x_j) - z_j \leq \phi_R^j(x_i) - z_i = \phi_R^j(x_j) - z_i \quad &\iff \quad z_j \geq z_i. \end{aligned}$$

Thus, $z_i = z_j$ must hold and both contracts are the same. Consequently, substituting $z_i = z_j$ shows that both incentive compatibility constraints are tight.

Second, suppose both incentive compatibility constraints between i and j are tight:

$$\phi_R^i(x_i) - z_i = \phi_R^i(x_j) - z_j, \qquad \phi_R^j(x_j) - z_j = \phi_R^j(x_i) - z_i.$$

Combining both equalities leads to:

$$\phi_R^i(x_i) - \phi_R^j(x_i) = \phi_R^i(x_j) - \phi_R^j(x_j) \iff \tfrac{1}{2}(h_i - h_j)x_i = \tfrac{1}{2}(h_i - h_j)x_j \iff x_i = x_j.$$

The first equivalence follows from having the same ordering cost f and the last equivalence from $h_i \neq h_j$. As proved above, $x_i = x_j$ implies that $z_i = z_j$. Thus, the contracts for types i and j are the same. □

Corollary 3.11. *Types part of a 2-cycle in the KKT graph have the same contract.*

Proof. Being part of a 2-cycle in the KKT graph means that $\mu_{kl}, \mu_{lk} > 0$ for some adjacent $k, l \in \mathcal{K}$. From complementary slackness it follows that both incentive compatibility constraints between types k and l must be tight. Lemma 3.10 implies that $x_k = x_l$ and $z_k = z_l$. □

The KKT conditions become more restrictive if certain types have the same order quantity, as it introduces additional dependency between the decision variables. Using this fact, we can exclude more cases from consideration, see Lemma 3.12.

Lemma 3.12. *In the KKT graph, a maximal subset of retailer nodes connected with only 2-cycles must have at least one ingoing arc (possibly from node s) and exactly one outgoing arc.*

Proof. The statement that at least one ingoing arc exists follows directly from Lemma 3.7. We prove the statement for the outgoing arcs by contradiction.

Let $\mathcal{S} = \{i, i+1, \ldots, j-1, j\} \subseteq \mathcal{K}$ be such a maximal subset and suppose that $\mathcal{S}$ as a whole has no outgoings arcs. By Corollary 3.11, all $k \in \mathcal{S}$ get the same contract, say order quantity x. The stationarity conditions state that

$$\omega_k \left(-d(f+F)\frac{1}{x^2} + \frac{1}{2}\left(h_k + H\tfrac{d}{p}\right)\right) + \underbrace{\frac{1}{2}\mu_{k,k-1}(h_k - h_{k-1})}_{>0} + \underbrace{\frac{1}{2}\mu_{k,k+1}(h_k - h_{k+1})}_{<0} = 0.$$

By assumption, $\mu_{i,i-1} = 0$ or non-existent (if $i = 1$). Likewise, $\mu_{j,j+1} = 0$ or non-existent (if $j = K$). The stationarity constraint for type i requires that:

$$\left(-d(f+F)\frac{1}{x^2} + \frac{1}{2}\left(h_i + H\tfrac{d}{p}\right)\right) > 0.$$

This implies that the similar term in the stationarity constraint for j is also strictly positive, as $h_i < h_j$. However, the resulting constraint only contains strictly positive terms, which is infeasible. Thus, $\mathcal{S}$ as a whole has at least one outgoing arc.

Suppose $\mathcal{S}$ as a whole has two outgoing arcs, i.e., $\mu_{i,i-1}, \mu_{j,j+1} > 0$. By Lemma 3.7, all nodes in $\mathcal{S}$ must be reachable from s via arcs with strictly positive multipliers. Corollary 3.9 implies that $\lambda_i, \ldots, \lambda_j = 0$, otherwise the solution is infeasible. Therefore, $\mu_{i-1,i} > 0$ and/or $\mu_{j+1,j} > 0$ must hold, but this contradicts the maximality of $\mathcal{S}$. To conclude, $\mathcal{S}$ has exactly one outgoing arc. □

For example, the graph in Figure 4 is not a valid KKT graph, since nodes 1 and 2 form a 2-cycle but do not have an outgoing arc.

A direct consequence is that for two retailer types KKT graphs with 2-cycles are not valid KKT graphs. For more than two retailer types 2-cycles in the optimal solution can actually occur, see Section 4. This implies that types can get the same contract in the optimal solution. We return to this issue in Section 3.6.

3.5.2 The joint order quantity

If a retailer node $k \in \mathcal{K}$ has no outgoing arcs in the KKT graph it is straightforward to determine that $x_k = x_J^{k*}$ must hold. The next lemma shows that this is an if-and-only-if relation.

Lemma 3.13. *In the optimal solution, $x_k = x_J^{k*}$ if and only if node $k \in \mathcal{K}$ has no outgoing arcs in the KKT graph.*

Proof. First, suppose node $k \in \mathcal{K}$ has no outgoing arcs in the KKT graph, i.e., $\mu_{k,k-1} = \mu_{k,k+1} = 0$. The KKT stationarity condition (3.4) requires that:

$$\omega_k \left(-d(f+F)\frac{1}{x_k^2} + \frac{1}{2}\left(h_k + H\tfrac{d}{p}\right)\right) = 0 \qquad \Longrightarrow \qquad x_k = x_J^{k*},$$

which proves one direction of the lemma.

Second, suppose that $x_k = x_J^{k*}$. Again using the KKT stationarity condition (3.4), we get:

$$\mu_{k,k-1}(h_k - h_{k-1}) + \mu_{k,k+1}(h_k - h_{k+1}) = 0.$$

Since $h_{k-1} < h_k < h_{k+1}$, either $\mu_{k,k-1}, \mu_{k,k+1} > 0$ (node k has two outgoing arcs) or $\mu_{k,k-1} = \mu_{k,k+1} = 0$ (node k has no outgoing arcs). In the latter case we are done. Therefore, suppose $\mu_{k,k-1}, \mu_{k,k+1} > 0$. We discern two cases.

Case I: node k is not part of a 2-cycle, i.e., $\mu_{k-1,k} = \mu_{k+1,k} = 0$. By Lemma 3.7 node k must be reachable from node s, implying $\lambda_k > 0$. This case is infeasible, see Corollary 3.9.

Case II: node k is part of a 2-cycle. Let k be part of the maximal subset $\mathcal{S} = \{i, i+1, \ldots, k, \ldots, j-1, j\} \subseteq \mathcal{K}$ of retailer nodes connected with 2-cycles. Recall that from Corollary 3.11 we know that all types in $\mathcal{S}$ have the same contract. Furthermore, by Lemma 3.12 either $\mu_{i,i-1} > 0$ or $\mu_{j,j+1} > 0$ (but not both).

Consider the case that $\mu_{i,i-1} > 0$, and thus $\mu_{j,j+1} = 0$ and $j > k$. The KKT stationarity conditions state:

$$\omega_j \left(-d(f+F)\frac{1}{x_j^2} + \frac{1}{2}\left(h_j + H\tfrac{d}{p}\right)\right) + \frac{1}{2}\mu_{j,j-1}(h_j - h_{j-1}) = 0.$$

Since $\mu_{j,j-1}(h_j - h_{j-1}) > 0$, it must hold that $x_j < x_J^{j*}$. Since $x_k = x_j$, we have the required contradiction:

$$x_J^{k*} = x_k = x_j < x_J^{j*} < x_J^{k*}.$$

The other case, $\mu_{i,i-1} = 0$ and $\mu_{j,j+1} > 0$, is similar and is omitted.

To conclude, if $x_k = x_J^{k*}$ node k must have no outgoing arcs in the KKT graph, which completes the proof. □

Our last result for this section, Lemma 3.14, states that at least one type is assigned the joint order quantity in the optimal solution.

Lemma 3.14. *In the optimal solution the order quantity for at least one retailer type is the joint order quantity. Moreover, the total costs for at least one retailer type equals its default costs.*

Proof. The result that there exists a retailer type with the same total costs as its default option follows directly from Lemma 3.7. That is, there exists a $k \in \mathcal{K}$ such that $\lambda_k > 0$. Hence, $\phi_R^k(x_k) - z_k = \phi_R^{k*}$ by complementary slackness.

By combining Lemmas 3.12 and 3.13, we can prove the other claim as follows. Suppose each retailer node has an outgoing arc in the KKT graph. Thus, arc $(1, 2)$ from type 1 to type 2 exists in the graph. If arc $(2, 3)$ is in the graph, we continue to type 3. If type 2 has only outgoing arc $(2, 1)$, then type 2 forms a 2-cycle with type 1. By Lemma 3.12 arc $(2, 3)$ must exist as well, a contradiction. Repeat this argument until we reach type K. Since node K also has an outgoing arc, a 2-cycle with type $K - 1$ is formed. Again by Lemma 3.12, this cycle must have an outgoing arc, namely arc $(K - 1, K - 2)$. Repeat this argument until we reach type 1. Hence, all retailer nodes are part of 2-cycles which contradicts Lemma 3.12.

Thus, there exists at least one type with no outgoing arcs. Lemma 3.13 states that this retailer type is assigned the joint order quantity in the optimal solution. □

3.6 Uniqueness of contracts

Additional assumptions are needed in order to guarantee that the optimal menu of contracts uniquely identifies each type, i.e., that each contract in the menu is unique. Suppose that the supplier only has bounds for the retailer's holding cost, $h \in [h^{LB}, h^{UB}]$, and due to the lack of information assumes a uniform distribution for h. We discretise the uniform distribution using $K \in \mathbb{N}$ equidistant points: $h \in \{h_1, \ldots, h_K\}$ with $h_{k+1} = h_k + \delta$ for some appropriate step $\delta \in \mathbb{R}_{>0}$. Furthermore, the uniform distribution implies that all weights are equal, $\omega_k = \omega$ for all $k \in \mathcal{K}$.

The assumptions lead to the following KKT stationarity conditions for $k \in \mathcal{K}$:

$$\left(-\frac{2d(f+F)}{x_k^2} + H\frac{d}{p}\right) + h_k + \delta(\mu_{k,k-1} - \mu_{k,k+1}) = 0, \tag{3.6}$$

$$1 - \lambda_k - \mu_{k-1,k} - \mu_{k+1,k} + \mu_{k,k-1} + \mu_{k,k+1} = 0, \tag{3.7}$$

where all ill-defined multipliers with out of bound indices are set to zero. Notice that without loss of generality we set $\omega_k = 1$ in the KKT conditions by uniformly rescaling all multipliers.

It turns out that uniformity on types and equidistant holding costs is sufficient to guarantee *a priori* to obtain an optimal menu with unique contracts, see Theorem 3.15. Be aware that the exclusion of 2-cycles of arcs with strictly positive multipliers does not automatically imply that all contracts are unique, at least not without improving the result of Corollary 3.11.

Theorem 3.15. *Assume uniformity on types and equidistant holding costs. In the optimal solution, all contracts are unique.*

Proof. For all types $k \in \mathcal{K}$, let $\omega_k = \omega$ and $h_{k+1} = h_k + \delta$ for some $\delta \in \mathbb{R}_{>0}$. First, realise that if $x_k = x_l$ for some $l > k + 1$, then all intermediate types also have the same order quantity: $x_k = x_{k+1} = \cdots = x_{l-1} = x_l$. This follows from the ordering of the order quantities (Lemma 3.3). Second, if $x_k = x_l$ then automatically $z_k = z_l$ must hold to be feasible (Lemma 3.10). So, both contracts are exactly the same.

Assume that there are contracts for retailer types that are the same, else there is nothing to prove. Let $\mathcal{S} = \{i, i + 1, \ldots, j - 1, j\} \subseteq K$ with $i < j$ be a maximal set of types with the same contract. Note that Lemma 3.8 implies that $\lambda_{i+1}, \ldots, \lambda_{j-1} = 0$. We have to distinguish two cases based on the KKT multipliers.

Case I: $\mu_{i,i-1} > 0$. We have three direct implications: $\lambda_i = 0$ (by Lemma 3.8), $\mu_{i-1,i} = 0$ (by maximality of $\mathcal{S}$ and Corollary 3.11), and thus $\mu_{i+1,i} > 0$ (by Lemma 3.7). Furthermore, since all nodes in $\mathcal{S}$ must be reachable (Lemma 3.7), we have that $\mu_{i+1,i}, \ldots, \mu_{j,j-1} > 0$. Finally, we can conclude that $\mu_{j,j+1} = 0$ with a simple argument by contradiction using the maximality of $\mathcal{S}$ or Lemmas 3.7 and 3.8.

Now we can derive two contradictory equations. The first equation is as follows. Since $\lambda_i, \ldots, \lambda_{j-1} = 0$, the sum of the corresponding KKT conditions (3.7) from i to $j-1$ is equal to

$$(j-i-1) + \mu_{i,i-1} - \mu_{j,j-1} + \mu_{j-1,j} = 0.$$

For the second equation we need to consider the KKT conditions (3.6) for type i and j. As $x_i = x_j$ both conditions have a common part, hence the difference must be equal:

$$h_i + \delta(\mu_{i,i-1} - \mu_{i,i+1}) = h_j + \delta\mu_{j,j-1} \qquad \Longleftrightarrow \qquad \mu_{i,i-1} - \mu_{i,i+1} = (j-i) + \mu_{j,j-1}.$$

Finally, both equations combined state that

$$-\mu_{i,i+1} = 2(j-i) - 1 + \mu_{j-1,j} \geq 1 + \mu_{j-1,j} > 0.$$

This contradicts that $\mu_{i+1,i}$ is non-negative.

Case II: $\mu_{i,i-1} = 0$. The KKT stationarity for type i simplifies to

$$\left(-\frac{2d(f+F)}{x_i^2} + H\tfrac{d}{p}\right) + h_i = \delta\mu_{i,i+1} \geq 0.$$

Therefore, for any $k \in \mathcal{S}$, $k > i$, it must hold that $\mu_{k,k+1} > 0$, since $x_k = x_i$ and from the above inequality:

$$0 = \left(-\frac{2d(f+F)}{x_k^2} + H\tfrac{d}{p}\right) + h_k + \delta(\mu_{k,k-1} - \mu_{k,k+1}) \geq (h_k - h_i) + \delta(\mu_{k,k-1} - \mu_{k,k+1}).$$

Consequently, $\lambda_j = 0$ by Lemma 3.8, and $\mu_{j+1,j} = 0$ by maximality of $\mathcal{S}$ and Corollary 3.11.

As in Case I, we derive two contradictory equations. The sum of the corresponding KKT conditions (3.7) from $i+1$ to j is equal to

$$(j-i-1) + \mu_{i+1,i} - \mu_{i,i+1} + \mu_{j,j+1} = 0.$$

The KKT conditions (3.6) for type i and j lead to:

$$h_i - \delta\mu_{i,i+1} = h_j + \delta(\mu_{j,j-1} - \mu_{j,j+1}) \qquad \Longleftrightarrow \qquad -\mu_{i,i+1} = (j-i) + \mu_{j,j-1} - \mu_{j,j+1}.$$

These two equations give a contradiction:

$$0 = 2(j-i) - 1 + \mu_{j,j-1} + \mu_{i+1,i} \geq 1.$$

To conclude, a menu with non-unique contracts between retailer types is never optimal, irrespective of the actual value of δ. This shows that uniformity of types and equidistant holding costs are sufficient for unique contracts. □

Corollary 3.16. *If we assume uniformity on types and equidistant holding costs, the KKT graph has no cycles.*

Proof. Suppose the KKT graph has a 2-cycle (which are the only cycles possible). Corollary 3.11 implies that those two types have the same contract, which contradicts Theorem 3.15. ☐

If we remove one of the two assumptions, there are instances where the optimal menu does not have unique contracts. Table 2 provides such examples where the optimal solution has non-unique contracts. Notice that there are no examples with only two retailer types. In the next section, we will prove in Theorem 4.1 that the optimal menu for two types always has unique contracts.

Optimal Menu			F	H	f	h_1	h_2	h_3	Objective	x_1	z_1	x_2	z_2	x_3	z_3
1x	2LeftRight	3UpLeft	1	1	1	5	9	10	6.179 510 077	0.816 497	0.154 678	0.544 331	0.086 637	0.544 331	0.086 637
1Up	2LeftRight	3UpLeft	1	1	1	4	9	10	6.026 932 378	0.894 427	0.078 461	0.547 903	0.092 520	0.547 903	0.092 520
1UpRight	2LeftRight	3x	1	5	6	1	2	6	16.017 368 801	2	0.535 898	2	0.535 898	1.128 152	0.238 786
1UpRight	2LeftRight	3Up	1	4	9	1	2	6	15.694 403 017	2.459 866	0.646 029	2.459 866	0.646 029	1.414 214	0.214 297

(a) Examples for $K = 3$ with unconstrained holding costs, unit rates ($d = p = 1$), and unit weights ($\omega_1 = \omega_2 = \omega_3 = 1$).

Optimal Menu			F	H	f	h_1	h_2	h_3	Objective	x_1	z_1	x_2	z_2	x_3	z_3
1x	2LeftRight	3UpLeft	1	1	1	3	4	5	35.106 395 133	1	0.053 004	0.715 282	0.023 977	0.715 282	0.023 977
1Up	2LeftRight	3UpLeft	1	1	1	3	5	7	36.825 868 698	1	0.050 510	0.646 084	0.067 423	0.646 084	0.067 423
1UpRight	2LeftRight	3x	1	3	7	1	2	3	84.397 091 826	2.708 013	0.197 270	2.708 013	0.197 270	1.632 993	0.286 427
1UpRight	2LeftRight	3Up	1	3	1	1	2	3	52.964 992 389	1.035 276	0.069 350	1.035 276	0.069 350	0.816 497	0

(b) Examples for $K = 3$ with equidistant holding costs, unit rates ($d = p = 1$), and weights $\omega_1 = 10$, $\omega_2 = 1$, $\omega_3 = 10$.

Table 2: Examples of non-unique contracts in the optimal solution for three retailer types.

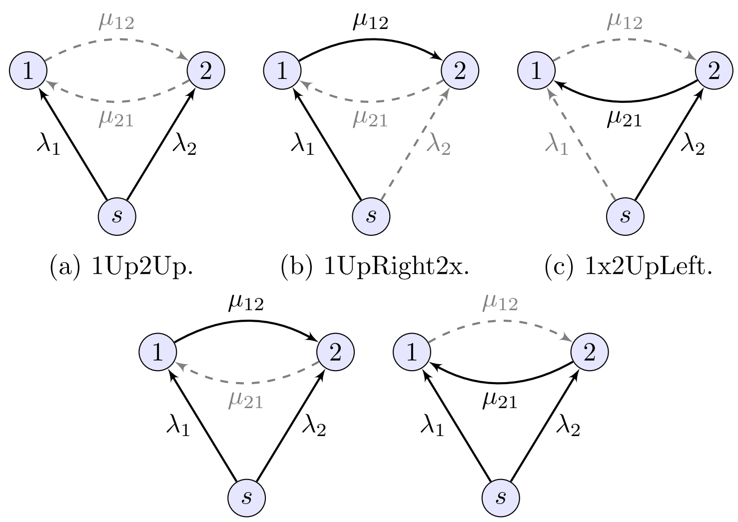
4 Optimal menus of contracts

The KKT conditions lead to a list of candidate solutions, one of which is the optimal solution. The analysis of the KKT conditions excludes certain KKT menus from consideration, which allows us to focus on the remaining cases. When determining formulas for these KKT menus, we can often reuse parts of the solution of subproblems or symmetric cases. We have determined the formulas for all KKT menus for two and three retailer types. These lists of KKT menus are minimal, i.e., if we omit any menu there are instances for which we would fail to determine the optimum.

In the next sections, we provide and discuss example instances and their optimal KKT menu for two and three retailer types. These instances have been solved by determining the best KKT menu, which are derived in Appendices B.2 and B.3. As a verification step, all instances have also been solved using a cutting-plane procedure (see Section 2). Furthermore, to check the minimality of the lists of KKT menus, we have verified that exactly one KKT menu is optimal.

4.1 Two retailer types

From our analysis we can reduce the number of KKT menus significantly if there are only two retailer types ($K = 2$). From the $2^4 = 16$ cases, there remain 5 cases that can occur, see Figure 5. The details of the derivation of the corresponding KKT menus are given in Appendix B.2. All 5 KKT menus can be optimal, see Table 3 for example instances and their optimal solution. We conclude that our analysis is tight for $K = 2$, i.e., we cannot exclude any of these KKT menus.



(a) 1Up2Up. (b) 1UpRight2x. (c) 1x2UpLeft.

(d) 1UpRight2Up. (e) 1Up2UpLeft.

Figure 5: All KKT graphs for two retailer types.

Optimal Menu		F	H	f	h_1	h_2	**Objective**	x_1	z_1	x_2	z_2
1x	2UpLeft	2	1	1	1	2	4.363 081 101	1.732 051	0.055 748	1.224 745	0.041 241
1Up	2Up	1	1	1	1	2	2.878 315 178	1.414 214	0	1.154 701	0.020 726
1Up	2UpLeft	1	1	1	2	4	3.120 955 865	1.154 701	0.020 726	0.828 427	0.035 534
1UpRight	2x	1	2	4	1	2	5.139 837 026	2.236 068	0.078 461	1.581 139	0.164 500
1UpRight	2Up	1	1	4	1	2	3.125 827 677	2.343 146	0.050 253	1.825 742	0.016 632

Table 3: Example instances for two retailer types with unit rates and weights.

Although the KKT approach is viable for two retailer types, there is a faster and easier solution approach. Theorem 4.1 provides closed-form formulas for the optimal menu of contracts in this case. The theorem shows that the contracts for the types are unique (i.e., not the same) and that each order quantity lies between the default and joint order quantity for that type. These properties do not hold in general for three or more types, which will be discussed in Section 4.3.

Theorem 4.1. *For two retailer types ($K = 2$) the unique optimal menu of contracts is given by:*

$$x_1 = \min\left\{\sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}}, \max\left\{\frac{2\sqrt{2df}}{\sqrt{h_1}+\sqrt{h_2}}, \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}}\right\}\right\},$$

$$x_2 = \max\left\{\sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_1}{\omega_2}(h_2 - h_1)}}, \min\left\{\frac{2\sqrt{2df}}{\sqrt{h_1}+\sqrt{h_2}}, \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}}\right\}\right\},$$

$$z_1 = df\frac{1}{x_1} + \frac{1}{2}h_1 x_1 - \sqrt{2dfh_1} + \max\left\{0, \sqrt{2df}(\sqrt{h_1} - \sqrt{h_2}) + \frac{1}{2}(h_2 - h_1)x_2\right\},$$

$$z_2 = df\frac{1}{x_2} + \frac{1}{2}h_2 x_2 - \sqrt{2dfh_2} + \max\left\{0, \sqrt{2df}(\sqrt{h_2} - \sqrt{h_1}) + \frac{1}{2}(h_1 - h_2)x_1\right\},$$

where we ignore any imaginary values if $h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2) < 0$.

Furthermore, we have that $x_1 > x_2$ and that each order quantity x_k lies in the closed interval with endpoints x_R^{k*} and x_J^{k*}. Finally, $x_k = x_R^{k*}$ if and only if $x_S^* = x_R^{k*}$, which implies $x_J^{k*} = x_R^{k*}$.

Proof. The proof is given in Appendix A. □

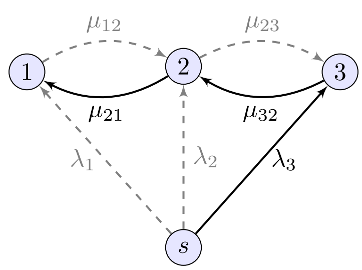
The closed-form formulas and properties in Theorem 4.1 for the optimal menu of contracts can be determined relatively easily using only calculus for differentiable convex functions. So there is no need to evaluate multiple menus or even use KKT conditions to determine the optimal solution.

4.2 Three retailer types

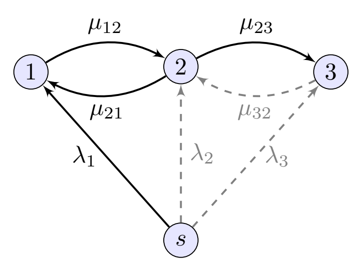
For three retailer types ($K = 3$) our results reduce the number of KKT menus from $2^7 = 128$ to only 23. For the details we refer to Appendix B.3. For each of the 23 KKT menus, we have found instances where that menu is optimal. We conclude that our analysis is tight for $K = 3$. See Table 4 for example instances and their optimal solution. Observe that such examples can already be found in a small integer range for the cost parameters.

The analysis and these examples show certain structures in the optimal solution that do not occur in related literature. For example, Voigt and Inderfurth 2011 study multiple retailer types, but due to their cost functions only the KKT graph in Figure 6a occurs. For our model, many more possible optimal structures exist, which complicates the analysis.

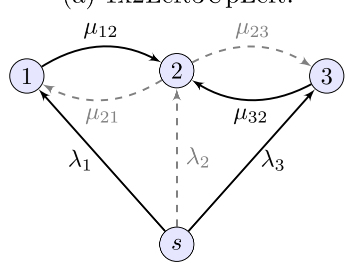
Furthermore, in certain optima the same contract is offered to multiple types: see menus with cycles in their KKT graph, such as in Figure 6b. We also draw additional attention to the solution corresponding to Figure 6c. Here, type 2 wants to lie having either lower holding cost h_1 or higher cost h_3. As a final note, there is no monotonicity or general ordering in the side payments.



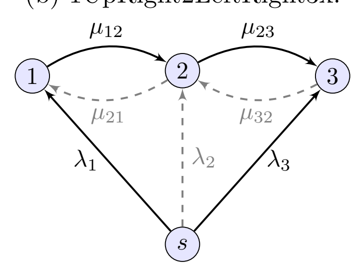
(a) 1x2Left3UpLeft.



(b) 1UpRight2LeftRight3x.



(c) 1UpRight2x3UpLeft.



(d) 1UpRight2Right3Up.

Figure 6: Example KKT graphs for three retailer types that occur in optimal solutions.

Optimal Menu			F	H	f	h_1	h_2	h_3	Objective	x_1	z_1	x_2	z_2	x_3	z_3
1x	2UpLeft	3Up	1	1	1	3	4	20	6.082 721 306	1	0.079 821	0.816 497	0.029 311	0.436 436	0.331 090
1x	2UpLeft	3UpLeft	1	1	1	3	4	6	5.149 148 707	1	0.079 821	0.816 497	0.029 311	0.635 674	0.016 054
1x	2Left	3UpLeft	1	1	1	3	4	5	5.069 000 755	1	0.099 524	0.816 497	0.049 014	0.707 107	0.019 703
1x	2LeftRight	3UpLeft	1	1	1	5	9	10	6.179 510 077	0.816 497	0.154 678	0.544 331	0.086 637	0.544 331	0.086 637
1Up	2x	3UpLeft	1	1	1	1	3	4	4.656 338 798	1.414 214	0	1	0.079 821	0.816 497	0.029 311
1Up	2Up	3Up	1	1	1	1	2	9	4.960 229 811	1.414 214	0	1.154 701	0.020 726	0.632 456	0.184 548
1Up	2Up	3UpLeft	1	1	1	1	2	3	4.451 529 277	1.414 214	0	1.154 701	0.020 726	0.898 979	0.011 352
1Up	2UpLeft	3Up	1	1	2	3	4	8	4.605 022 879	1.224 745	0.006 009	1.071 797	0.009 619	0.816 497	0.058 622
1Up	2UpLeft	3UpLeft	1	1	1	2	3	5	4.819 285 508	1.154 701	0.020 726	0.898 979	0.011 352	0.712 788	0.022 634
1Up	2UpRight	3x	1	3	6	1	8	9	9.216 936 179	1.870 829	0.678 448	1.183 216	0.005 830	1.080 123	0.025 909
1Up	2UpRight	3Up	1	1	7	1	3	4	4.895 439 736	2.828 427	0.147 430	2.005 148	0.017 995	1.788 854	0.007 513
1Up	2Left	3UpLeft	1	1	1	2	3	4	4.742 532 117	1.154 701	0.020 726	0.898 482	0.011 470	0.758 372	0.006 931
1Up	2LeftRight	3UpLeft	1	1	1	4	9	10	6.026 932 378	0.894 427	0.078 461	0.547 903	0.092 520	0.547 903	0.092 520
1UpRight	2x	3Up	1	2	3	1	2	5	6.763 963 962	2	0.050 510	1.414 214	0.086 044	1.069 045	0.001 630
1UpRight	2x	3UpLeft	1	6	1	1	7	8	10.836 135 802	0.775 694	0.262 802	0.554 700	0.002 932	0.517 411	0.002 344
1UpRight	2Up	3Up	1	1	3	1	2	3	4.400 022 554	2.029 224	0.043 520	1.632 993	0.006 009	1.414 214	0
1UpRight	2Up	3UpLeft	1	5	1	1	6	7	9.845 849 029	0.819 955	0.215 343	0.603 023	0.003 279	0.555 112	0.002 673
1UpRight	2UpRight	3x	1	2	5	1	2	3	7.859 316 091	2.619 717	0.056 184	2	0.027 864	1.549 193	0.079 140
1UpRight	2UpRight	3Up	1	1	5	1	2	3	4.726 682 033	2.619 717	0.056 184	2.010 179	0.025 384	1.732 051	0.007 602
1UpRight	2Right	3x	1	3	3	1	2	3	9.236 588 251	2	0.050 510	1.414 214	0.086 044	1.154 701	0.173 530
1UpRight	2Right	3Up	1	2	4	1	2	5	7.181 899 382	2.239 120	0.077 549	1.584 379	0.161 041	1.195 229	0.010 156
1UpRight	2LeftRight	3x	1	5	6	1	2	6	16.017 368 801	2	0.535 898	2	0.535 898	1.128 152	0.238 786
1UpRight	2LeftRight	3Up	1	4	9	1	2	6	15.694 403 017	2.459 866	0.646 029	2.459 866	0.646 029	1.414 214	0.214 297

Table 4: Example instances for three retailer types with unit rates and weights.

4.3 Differences between two or more types

In this section, we discuss observed differences in the optimal menus of contracts for two and three retailer types. First of all, the bounds on the order quantities in Theorem 4.1 are a unique property for the case with only two retailer types. To be more specific, an optimal order quantity x_k is not bounded by x_R^{k*} or x_J^{k*} when there are more than two retailer types. For an example, see Table 5 where type 2 is not bounded by its default or joint order quantity. In fact, this can even occur if the retailer type and the supplier both desire the same order quantity, i.e., $x_R^{k*} = x_S^* = x_J^{k*}$, as the example in Table 5 shows for type $k = 2$.

Furthermore, for three or more types it can be optimal to have duplicate contracts in the menu, as we have shown in Sections 3.6 and 4.2. Recall that by Theorem 3.15 we need additional assumptions to guarantee to have unique contracts. In contrast, for two types we know from Theorem 4.1 that the optimal menu never contains duplicate contracts.

Unfortunately, our analysis suggests that for each number of retailer types K we need to solve some cases from scratch. That is, we were unable to reuse old results for $K-1$ types in a scalable way to solve the problem for K types. For example, menus such as those corresponding to Figures 6c and 6d are troublesome. Therefore, the analytical KKT approach does not seem to be a generalisable solution approach. For a general number of retailer types, the scalable technique described in Section 2 is preferred.

To conclude, there are significant differences in the qualitative properties of the optimal menu of contracts for two types compared to more than two types.

F	H	f	h_1	h_2	h_3	Optimal Menu			Objective
3	15	1	1	5	7	1UpRight	2Right	3Up	29.149 616

Type k	x_k	z_k	x_R^{k*}	x_J^{k*}	$\phi_S(x_k) + z_k$	$\phi_S(x_R^{k*})$
1	0.844 369	0.192 288	1.414 214	0.707 107	10.078 003	12.727 922
2	0.638 707	0.059 480	0.632 456	0.632 456	9.546 772	9.486 833
3	0.603 023	0.027 234	0.534 522	0.603 023	9.524 842	9.621 405

Table 5: Example instance for $K = 3$ with unit rates and weights.

5 Discussion and conclusion

Before we conclude with the main insights of our research, we discuss consequences of two model assumptions. First, using the expected costs as objective function is common in the literature. However, this can lead to the peculiar situation where the supplier's action to offer a menu of contracts results in an increase in the supplier's costs compared to taking no such action. See Section 5.1 for more details. Second, we discuss the screening capability of the contracting model in Section 5.2. Finally, we conclude our findings in Section 5.3.

5.1 Unfavourable realisations

The objective of the contracting model is to minimise the expected costs of the supplier based on the K scenarios, where each scenario corresponds to a retailer type. Therefore, it could be that for a certain realisation of the scenarios the supplier is worse off using the menu of contracts instead of accepting the retailer's default option. The example in Table 5 shows that this can indeed happen: for a realisation of the scenario of retailer type 2 the supplier would be better off with the default option. We note that examples for two retailer types also exist.

If this phenomenon is not allowed, then we have to add the following constraints to the model to prevent it:

$$\phi_S(x_k) + z_k \leq \phi_S(x_R^{k*}), \qquad \forall\, k \in \mathcal{K}. \tag{5.1}$$

We can interpret the new constraints (5.1) as the individual rationality constraints for the supplier. Notice that the menu with all default order quantities and zero side payments is still feasible. Furthermore, adding these constraints to the default contracting model will lead to an optimal objective value at least that of the default model.

If we consider the alternative convex model of Section 2.2, the equivalent constraints of (5.1) are convex. Therefore, we can still efficiently solve the alternative model after adding the IR constraints for the supplier. Of course, the theoretical analysis has to be redone after adding these constraints.

5.2 Screening capability

The contracting model is often called a screening model, i.e., it allows the supplier to correctly identify the retailer's type. The idea is as follows. We have shown that there exists an optimal solution where the retailer types do not lie about their type (also known as the revelation principle). Therefore, by observing the contract chosen by the retailer, we can determine its true type. Unfortunately, this idea has some issues, which we discuss in this section.

First, we have the issue of the retailer's indifference between contracts. Consider a type $k \in \mathcal{K}$. If the IC constraint where type k compares to the contract for type $l \in \mathcal{K}$ is tight, then type k is indifferent between contracts (x_k, z_k) and (x_l, z_l). We have assumed that in this case the supplier can convince the retailer to choose contract (x_k, z_k) without any additional cost. Without this assumption, we need to model the IC constraints as strict inequalities or add a secondary objective for the retailer to determine its choice. For example, the retailer could be inequity averse, a topic analysed in Voigt 2015.

Second, it may be optimal to assign the same contract to multiple retailer types. We have showed in Section 4 that this phenomenon can indeed occur if and only if there are more than two types. For such an optimal solution, we cannot distinguish those types by the retailer's choice. Of course, we can modify the contract to be unique at a small cost in objective value, by the hidden convexity of the feasible region. Another possibility is to make assumptions on the retailer types to guarantee unique contracts as seen in Section 3.6.

To conclude, the screening capability of the contracting model should be treated with care, especially if there are more than two retailer types.

5.3 Main insights

Our model extends the current literature by having a general number of retailer types with type-dependent default options. The inclusion of type-dependent default options and both ordering/setup costs and holding costs for the retailer and supplier increases the structural complexity of optimal menus of contracts.

Our analysis shows that an optimal menu of contracts for three or more retailer types has different structural properties than an optimal menu for two types. For two retailer types the contracts in the optimal menu are unique (see Theorem 4.1). Whereas for three or more retailer types, it may be optimal to present the same contract to multiple types. This insight affects the screening capability of the contracting model, as discussed above. However, if the distribution of the retailer types is uniform and equidistant, then we are guaranteed to have different contracts for each type (see Theorem 3.15).

Furthermore, for two retailer types the order quantities in the optimal menu lie between their default and joint order quantities (see Theorem 4.1). This does not hold for more than two types, as we have counterexamples.

Besides the monotonicity in order quantities, there are also other common properties. In any optimal menu of contracts the order quantity for at least one type is its joint order quantity (see Lemma 3.14). If the retailer's true type is that specific type, then perfect supply chain coordination takes place. Similarly, in any optimal menu the resulting costs for at least one type is the same as its default costs.

By considering more than two types we observe additional properties of the retailer's lying behaviour. For example, consider three adjacent types with different contracts, say types 1, 2, and 3 with holding costs $h_1 < h_2 < h_3$ respectively. The following situation cannot occur: types 1 and 3 simultaneously want to lie having (higher, respectively lower) holding cost h_2. However, it can happen that type 2 simultaneously wants to lie having lower (h_1) or higher (h_3) holding cost. To our knowledge, such behaviour is uncommon in the literature.

To conclude, we can efficiently solve our model for a general number of retailer types by a change of decision variables (see Theorem 2.1). Changing the perspective from side payments to information rents reveals the hidden convexity of our model. Remarkably, the current literature seems to focus on formulations using side payments. As our case illustrates, the change of perspective can lead to valuable insights or solution approaches. Therefore, for other contracting models we would like to promote the use or investigation of an alternative formulation using information rents.

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A Proof of Theorem 4.1

In this appendix, we give the proof of Theorem 4.1. The proof only requires some basic calculus for differentiable convex functions and the results from Section 3.2. In particular, we do not use the KKT conditions in any way. First, we give two lemmas that relate certain values that appear in optimal contracts and are needed to prove the theorem.

Lemma A.1. *For $k \in \mathcal{K}$, x_J^{k*} lies in the closed interval with endpoints x_R^{k*} and x_S^*, and is equal to either endpoint if and only if $x_R^{k*} = x_S^*$.*

Proof. This follows from simple algebraic manipulation. Let $\sim$ denote the ordering relation between two numbers, i.e., $\sim \in \{=, \geq, >, \leq, <\}$. We have:

$$\begin{aligned}
x_J^{k*} \sim x_R^{k*} \quad &\iff \quad \frac{2d(f+F)}{h_k + H\frac{d}{p}} \sim \frac{2df}{h_k} \quad \iff \quad h_k(f+F) \sim f(h_k + H\tfrac{d}{p}) \quad \iff \quad h_k F \sim fH\tfrac{d}{p} \\
&\iff \quad \frac{2dF}{H\frac{d}{p}} \sim \frac{2df}{h_k} \quad \iff \quad x_S^* \sim x_R^{k*},
\end{aligned}$$

and a similar equivalence for the supplier: $x_J^{k*} \sim x_S^* \iff x_R^{k*} \sim x_S^*$. □

Lemma A.2. *For $k, l \in \mathcal{K}$ with $k < l$, we have:*

$$x_R^{k*} > \frac{2(\phi_R^{l*} - \phi_R^{k*})}{h_l - h_k} > x_R^{l*}.$$

Proof. The square root function is concave, so we can relate its gradient as follows, using $h_k < h_l$:

$$\frac{1}{2\sqrt{h_k}} > \frac{\sqrt{h_l} - \sqrt{h_k}}{h_l - h_k} > \frac{1}{2\sqrt{h_l}}.$$

Therefore, we have

$$\begin{aligned}
x_R^{k*} &= \sqrt{\frac{2df}{h_k}} > 2\sqrt{2df}\left(\frac{\sqrt{h_l} - \sqrt{h_k}}{h_l - h_k}\right) = \frac{2(\phi_R^{l*} - \phi_R^{k*})}{h_l - h_k}, \\
x_R^{l*} &= \sqrt{\frac{2df}{h_l}} < 2\sqrt{2df}\left(\frac{\sqrt{h_l} - \sqrt{h_k}}{h_l - h_k}\right) = \frac{2(\phi_R^{l*} - \phi_R^{k*})}{h_l - h_k}.
\end{aligned}$$

This proves the lemma. □

We continue with the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemma 3.3 any feasible solution must satisfy $x_1 \geq x_2$. Assuming $x_1 \geq x_2$ the optimal side payments are determined by the shortest paths in the corresponding IRIC graph (see also Corollary 3.5). There are only two paths possible: directly from node s or via the other retailer node. Thus, the side payments are:

$$\begin{aligned}
z_1 &= -\min\left\{\phi_R^{1*} - \phi_R^1(x_1), \left(\phi_R^{2*} - \phi_R^2(x_2)\right) + \left(\phi_R^1(x_2) - \phi_R^1(x_1)\right)\right\} \\
&= \max\left\{\phi_R^1(x_1) - \phi_R^{1*}, \phi_R^1(x_1) - \phi_R^{2*} + \frac{1}{2}(h_2 - h_1)x_2\right\} \\
&= \phi_R^1(x_1) - \phi_R^{1*} + \max\left\{0, \phi_R^{1*} - \phi_R^{2*} + \frac{1}{2}(h_2 - h_1)x_2\right\},
\end{aligned}$$

and likewise

$$z_2 = \phi_R^2(x_2) - \phi_R^{2*} + \max\left\{0, \phi_R^{2*} - \phi_R^{1*} + \frac{1}{2}(h_1 - h_2)x_1\right\}.$$

Therefore, the contribution of x_1 to the objective function is

$$\omega_1\left(\phi_S(x_1) + \phi_R^1(x_1)\right) + \omega_2 \max\left\{0, \phi_R^{2*} - \phi_R^{1*} + \frac{1}{2}(h_1 - h_2)x_1\right\}. \tag{A.1}$$

The expression in (A.1) is a continuous convex function in x_1 with one non-differentiable point. Consequently, its minimiser is the value of x_1 such that the derivative is zero or changes from negative to positive. The derivative of (A.1) is given by

$$\begin{cases} \omega_1\left(-d(f+F)\frac{1}{x_1^2} + \frac{1}{2}\left(h_1 + H\frac{d}{p}\right)\right) + \frac{1}{2}\omega_2(h_1 - h_2) & \text{if } x_1 < \frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1} \\ \omega_1\left(-d(f+F)\frac{1}{x_1^2} + \frac{1}{2}\left(h_1 + H\frac{d}{p}\right)\right) & \text{otherwise} \end{cases}.$$

This implies that there are three critical values for x_1:

$$x_1 = x_J^{1*}, \qquad x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}}, \qquad x_1 = \frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1}.$$

Notice that we have the following relation indicated by $\sim \in \{=, \geq, >, \leq, <\}$:

$$\begin{aligned} & \omega_1\left(-d(f+F)\frac{1}{x_1^2} + \frac{1}{2}\left(h_1 + H\frac{d}{p}\right)\right) + \frac{1}{2}\omega_2(h_1 - h_2) \sim 0 \\ \iff \quad & \left(h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)\right) x_1^2 \sim 2d(f+F). \end{aligned}$$

So if $h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2) \leq 0$, the gradient is strictly negative for all $x_1 < 2(\phi_R^{2*} - \phi_R^{1*})/(h_2 - h_1)$. In this case, the minimiser is given by

$$x_1 = \max\left\{\frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1}, x_J^{1*}\right\}.$$

Otherwise, $h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2) > 0$ and all critical values are well-defined. Using the fact that

$$x_J^{1*} = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} < \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}},$$

we end up with three cases (orderings of the critical values) for x_1, see Figure 7. Determining the minimiser for these cases is straightforward and leads to the optimal value of x_1:

$$x_1 = \min\left\{\sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}}, \max\left\{\frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1}, x_J^{1*}\right\}\right\}.$$

The proof for x_2 is similar. Its contribution to the objective value is

$$\omega_2\left(\phi_S(x_2) + \phi_R^2(x_2)\right) + \omega_1 \max\left\{0, \phi_R^{1*} - \phi_R^{2*} + \frac{1}{2}(h_2 - h_1)x_2\right\}.$$

The corresponding derivative is

$$\begin{cases} \omega_2\left(-d(f+F)\frac{1}{x_2^2}+\frac{1}{2}\left(h_2+H\frac{d}{p}\right)\right)+\frac{1}{2}\omega_1(h_2-h_1) & \text{if } x_2 > \frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1} \\ \omega_2\left(-d(f+F)\frac{1}{x_2^2}+\frac{1}{2}\left(h_2+H\frac{d}{p}\right)\right) & \text{otherwise} \end{cases},$$

with three critical values for x_2:

$$x_2 = x_J^{2*}, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}}, \qquad x_2 = \frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}.$$

In contrast to the case for x_1, these critical values are always well-defined. See also Figure 7 for the minimisers, given by the formula

$$x_2 = \max\left\{\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}}, \min\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}, x_J^{2*}\right\}\right\}.$$

It remains to verify the final claims on these optimal values for x_1 and x_2. The fact that $x_1 > x_2$ follows from the formulas for the optimal order quantities and

$$\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}} < \sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}}} < \sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}}} < \sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}}.$$

Thus, the formulas do indeed give feasible order quantities (as $x_1 \geq x_2$ must hold by Lemma 3.3).

Finally, the statement that the optimal order quantities lie between the default and joint order quantities follows from the formulas and

$$x_R^{1*} > \frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1} > x_R^{2*},$$

which has been proved in Lemma A.2. The details are as follows. We have:

$$x_R^{1*} > x_J^{1*} \quad\Longrightarrow\quad \begin{cases} x_1 & \leq \max\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}, x_J^{1*}\right\} < x_R^{1*} \\ x_1 & \geq \min\left\{\sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}}, x_J^{1*}\right\} = x_J^{1*} \end{cases},$$

$$x_R^{1*} \leq x_J^{1*} \quad\Longrightarrow\quad \begin{cases} x_1 & = \min\left\{\sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}}, x_J^{1*}\right\} = x_J^{1*} \end{cases},$$

$$x_R^{2*} \geq x_J^{2*} \quad\Longrightarrow\quad \begin{cases} x_2 & = \max\left\{\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}}, x_J^{2*}\right\} = x_J^{2*} \end{cases},$$

$$x_R^{2*} < x_J^{2*} \quad\Longrightarrow\quad \begin{cases} x_2 & \geq \min\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}, x_J^{2*}\right\} > x_R^{2*} \\ x_2 & \leq \max\left\{\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}}, x_J^{2*}\right\} = x_J^{2*} \end{cases}.$$

Moreover, $x_k = x_R^{k*}$ if and only if it equals the joint order quantity x_J^{k*} and thus corresponds with the supplier's own optimal order quantity x_S^* (see also Lemma A.1). □

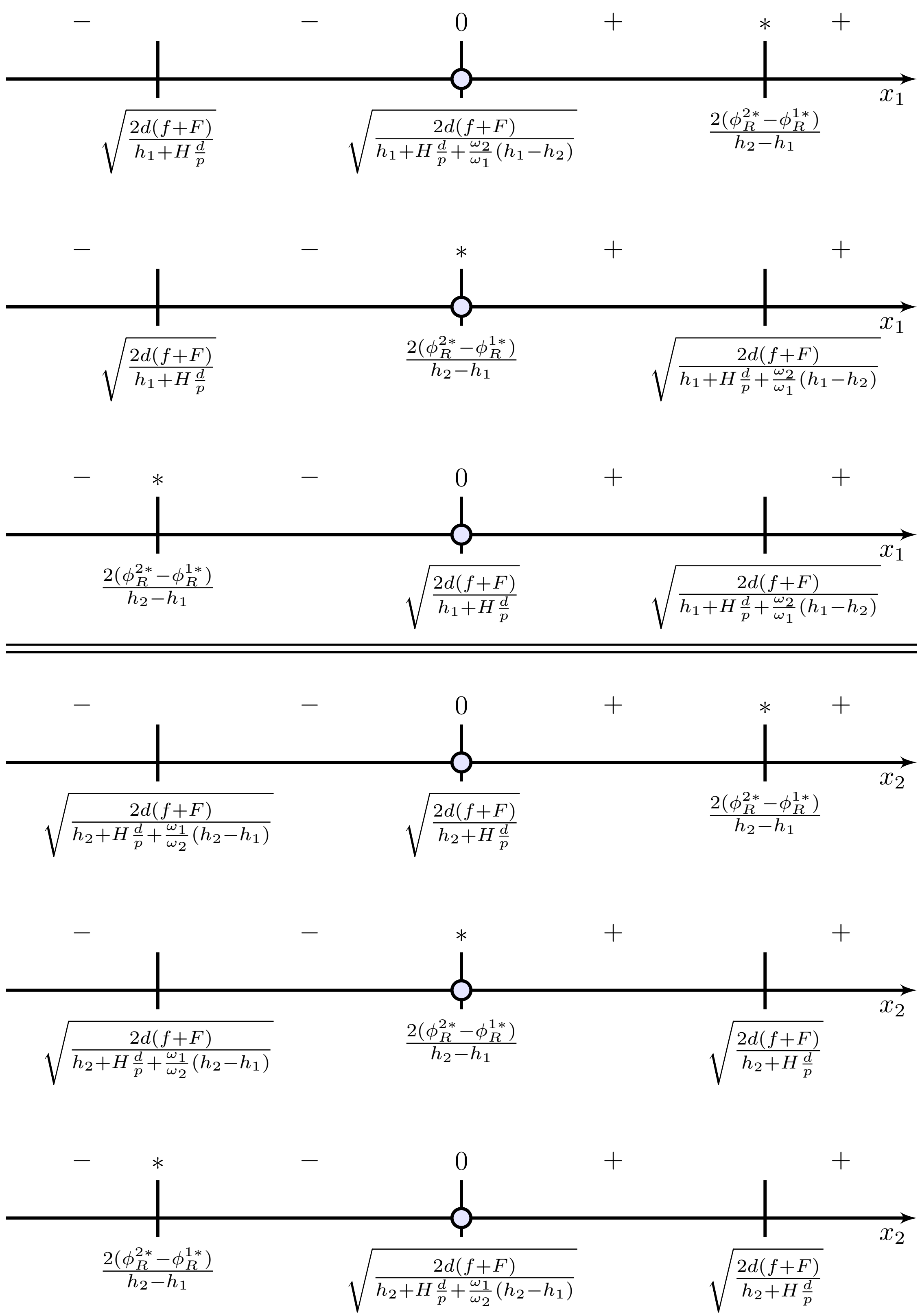


Figure 7: The sign of the derivative of the contribution of x_1 and x_2 to the objective value. An asterisk denotes that the point is non-differentiable. The circle indicates the minimiser.

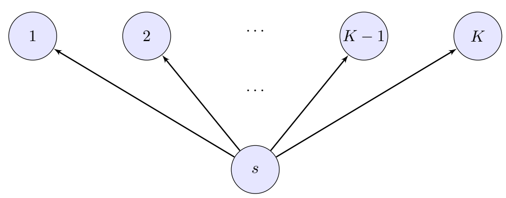
B Derivation of KKT menus

In this appendix, we derive the menus of contracts that follow from the KKT conditions. We only consider the cases with two or three retailer types. First, in Section B.1 we give the KKT menus for certain generalisable patterns that are used for both two and three types. In Section B.2 we derive all KKT menus for two types. For three types, we only show the analysis for certain cases from which all results can be reproduced, see Section B.3.

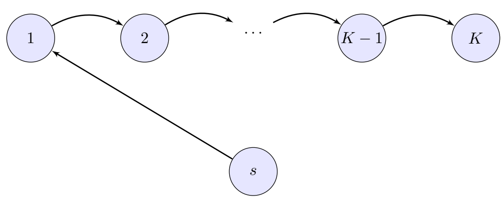
B.1 Simple KKT menus

We can distinguish two types of KKT menus based on the corresponding KKT graph. If the KKT graph is a spanning tree, we call the corresponding menu a *simple* KKT menu. The other cases give so-called *complex* KKT menus. Recall that by Lemma 3.7 the KKT graph is either a spanning tree or a strict superset of a spanning tree.

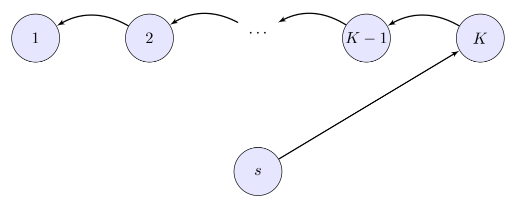
There are three fundamental patterns for simple KKT menus: the Up-tree, Right-tree, and Left-tree, see Figure 8. That is, for a simple menu each connected component is one of these patterns. Note that by Lemma 3.8 we have no 'T-pattern'. In the next sections, we derive the corresponding KKT menus for each of these simple patterns.



(a) Up-tree pattern.



(b) Right-tree pattern.



(c) Left-tree pattern.

Figure 8: The fundamental patterns for simple KKT menus.

B.1.1 Up-tree pattern

For the Up-tree pattern no retailer node has outgoing arcs. Therefore, we can apply Lemma 3.13 to determine the order quantities, i.e., we have $x_k = x_J^{k*}$ for all $k \in \mathcal{K}$. The side payments follow from complementary slackness: since $\lambda_k > 0$ for all $k \in \mathcal{K}$, it must hold that $\phi_R^k(x_k) - z_k = \phi_R^{k*}$. This leads to the KKT contract for all $k \in \mathcal{K}$:

$$x_k = \sqrt{\frac{2d(f+F)}{h_k + H\frac{d}{p}}}, \qquad z_k = \phi_R^k(x_k) - \phi_R^{k*}.$$

B.1.2 Right-tree pattern

Based on the spanning tree in the KKT graph, we can determine formulas for the side payments. Complementary slackness with respect to $\lambda_1 > 0$ implies that $z_1 = \phi_R^1(x_1) - \phi_R^{1*}$. Likewise,

$\phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1$ must hold, since $\mu_{1,2} > 0$. After substituting the known value for z_1, we obtain: $z_2 = \phi_R^2(x_2) + \phi_R^1(x_1) - \phi_R^2(x_1) - \phi_R^{1*}$. In general, for retailer type $k \in \mathcal{K}$ we have:

$$z_k = \phi_R^k(x_k) + \sum_{i=1}^{k-1} \left(\phi_R^i(x_i) - \phi_R^{i+1}(x_i)\right) - \phi_R^{1*}.$$

For the order quantities, first notice that adding up all KKT stationarity conditions (3.3) leads to $\sum_{k\in\mathcal{K}} \lambda_k = \sum_{k\in\mathcal{K}} \omega_k$. For the Right-tree pattern, we have $\lambda_1 = \sum_{k\in\mathcal{K}} \omega_k$. Consequently, the conditions (3.3) imply:

$$\begin{aligned}
\omega_1 - \lambda_1 + \mu_{1,2} = 0 \qquad &\Longrightarrow \qquad \mu_{1,2} = \lambda_1 - \omega_1 = \sum_{i=2}^{K} \omega_i, \\
\omega_k - \mu_{k-1,k} + \mu_{k,k+1} = 0 \qquad &\Longrightarrow \qquad \mu_{k,k+1} = \mu_{k-1,k} - \omega_k = \sum_{i=k+1}^{K} \omega_i, \\
\omega_K - \mu_{K-1,K} = 0 \qquad &\Longrightarrow \qquad \mu_{K-1,K} = \omega_K.
\end{aligned}$$

Thus, KKT stationarity conditions state

$$\omega_k \left(-\frac{2d(f+F)}{x_k^2} + h_k + H\frac{d}{p}\right) + (h_k - h_{k+1}) \sum_{i=k+1}^{K} \omega_i = 0,$$

which for $k \in \mathcal{K}$ implies the order quantity:

$$x_k = \sqrt{\frac{2d(f+F)}{h_k + H\frac{d}{p} + (h_k - h_{k+1})\frac{1}{\omega_k}\sum_{i=k+1}^{K}\omega_i}}.$$

Note that the order quantities be imaginary (infeasible).

B.1.3 Left-tree pattern

For the Left-tree pattern, the analysis is by symmetry similar to the Right-tree pattern. We have

$$\begin{aligned}
x_k &= \sqrt{\frac{2d(f+F)}{h_k + H\frac{d}{p} + (h_k - h_{k-1})\frac{1}{\omega_k}\sum_{i=1}^{k-1}\omega_i}} > 0, \\
z_k &= \phi_R^k(x_k) + \sum_{i=k+1}^{K} \left(\phi_R^i(x_i) - \phi_R^{i-1}(x_i)\right) - \phi_R^{K*}.
\end{aligned}$$

Here, the order quantities are always well-defined.

B.2 KKT menus for two types

For two retailer types ($K = 2$) we can reduce the number of KKT menus to consider from $2^4 = 16$ to 5 cases. Table 6 and Figure 5 provide the details of these 5 cases. In the following sections we derive formulas for the KKT menus. As expected, we see the same possible optimal order quantities as derived in Theorem 4.1. Furthermore, there is a bijection between the optimal order quantity x and the KKT graph. See Section 4.1 for numerical examples.

Case	Menu		λ_1	λ_2	μ_{12}	μ_{21}
1	1Up	2Up	>0	>0	$=0$	$=0$
2	1UpRight	2x	>0	$=0$	>0	$=0$
3	1x	2UpLeft	$=0$	>0	$=0$	>0
4	1UpRight	2Up	>0	>0	>0	$=0$
5	1Up	2UpLeft	>0	>0	$=0$	>0

Table 6: All cases of Lagrange multipliers for two retailer types.

B.2.1 Case 1Up2Up

Since this is a simple KKT menu, the derivation is already given in Section B.1.1. The menu of contracts is as follows:

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} > 0, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}} > 0,$$

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

B.2.2 Case 1UpRight2x

This menu is a Right-tree pattern, see Section B.1.2. The KKT menu is given by:

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}}, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}} > 0,$$

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}.$$

Note that x_1 might be infeasible if h_2 is large. Likewise, z_2 could be negative (infeasible) for certain cost parameters.

B.2.3 Case 1x2UpLeft

By symmetry, this case is similar to the 1UpRight2x contract, with the roles of types 1 and 2 interchanged:

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} > 0, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_1}{\omega_2}(h_2 - h_1)}} > 0,$$

$$z_1 = \phi_R^1(x_1) - \phi_R^1(x_2) + \phi_R^2(x_2) - \phi_R^{2*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

Note that z_1 could be negative (infeasible) for certain cost parameters.

B.2.4 Case 1UpRight2Up

By complementary slackness and $\lambda_1, \lambda_2 > 0$, we can directly derive the side payments:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

Furthermore, since node 2 has no outgoing arcs, x_2 is the joint order quantity:

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}} > 0.$$

Finally, by complementary slackness and $\mu_{12} > 0$, we have $\phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1$. Substituting the formulas for the side payments results in:

$$\phi_R^{2*} = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*} \qquad \Longleftrightarrow \qquad \frac{1}{2}(h_2 - h_1)x_1 = \phi_R^{2*} - \phi_R^{1*}.$$

Hence, type 1 has order quantity:

$$x_1 = \frac{2\left(\phi_R^{2*} - \phi_R^{1*}\right)}{h_2 - h_1} > 0.$$

B.2.5 Case 1Up2UpLeft

Again by symmetry, we can reuse the analysis of the 1UpRight2Up contract to obtain:

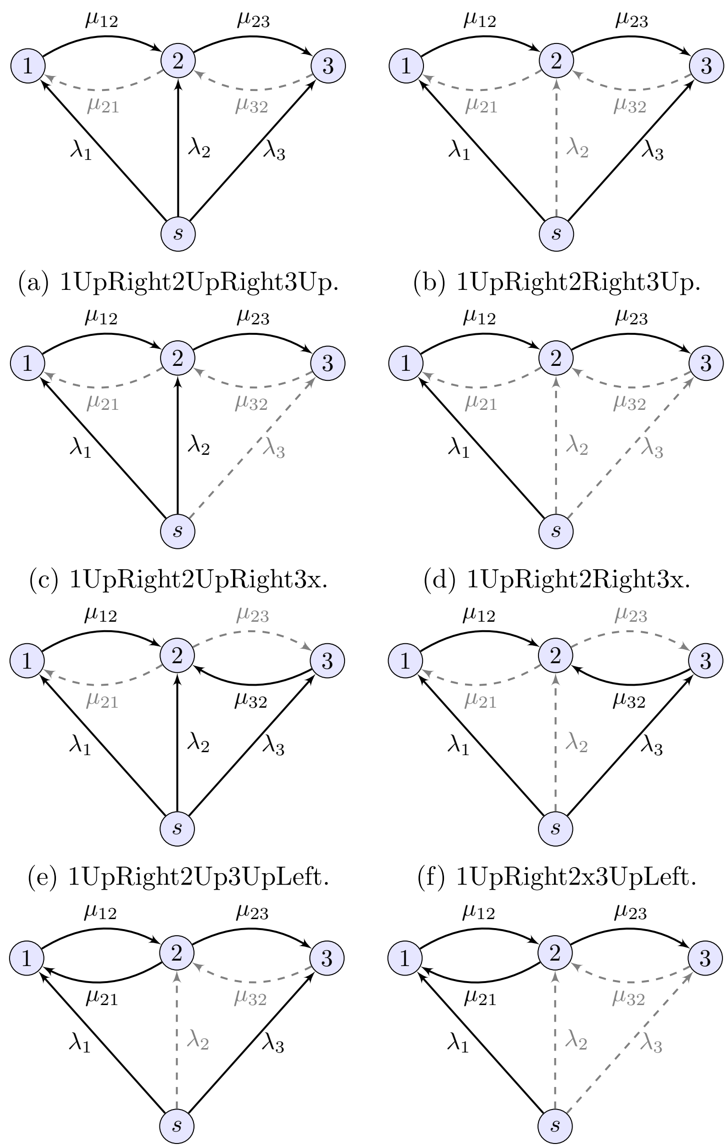
$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} > 0, \qquad x_2 = \frac{2\left(\phi_R^{2*} - \phi_R^{1*}\right)}{h_2 - h_1} > 0,$$
$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

B.3 KKT menus for three types

In this section we derive the menus in case of three retailer types ($K = 3$), see also Section 4.2. Table 7 provides an overview of all possible cases. The cases indicated as reducible can be solved by reusing KKT contracts for $K = 2$ and by using Lemma 3.13. That is, one retailer type gets offered the joint order quantity according to Lemma 3.13. For the other two types we can use the contracts for $K = 2$ derived earlier. This leaves 14 new cases to solve, but due to symmetry in the cases only 8 important cases remain. These are shown in Figure 9 and solved below. Compare this to the $2^7 = 128$ cases that would need to be analysed without using our results.

Case	**Menu**			**Reducible**	$\boldsymbol{\lambda_1}$	$\boldsymbol{\lambda_2}$	$\boldsymbol{\lambda_3}$	$\boldsymbol{\mu_{12}}$	$\boldsymbol{\mu_{21}}$	$\boldsymbol{\mu_{23}}$	$\boldsymbol{\mu_{32}}$
1	1Up	2Up	3Up	Yes	>0	>0	>0	$=0$	$=0$	$=0$	$=0$
2	1UpRight	2Up	3Up	Yes	>0	>0	>0	>0	$=0$	$=0$	$=0$
3	1Up	2UpLeft	3Up	Yes	>0	>0	>0	$=0$	>0	$=0$	$=0$
4	1Up	2UpRight	3Up	Yes	>0	>0	>0	$=0$	$=0$	>0	$=0$
5	1Up	2Up	3UpLeft	Yes	>0	>0	>0	$=0$	$=0$	$=0$	>0
6	1UpRight	2UpRight	3Up	No	>0	>0	>0	>0	$=0$	>0	$=0$
7	1UpRight	2Up	3UpLeft	No	>0	>0	>0	>0	$=0$	$=0$	>0
8	1Up	2UpLeft	3UpLeft	No	>0	>0	>0	$=0$	>0	$=0$	>0
9	1Up	2UpRight	3x	Yes	>0	>0	$=0$	$=0$	$=0$	>0	$=0$
10	1UpRight	2UpRight	3x	No	>0	>0	$=0$	>0	$=0$	>0	$=0$
11	1UpRight	2x	3Up	Yes	>0	$=0$	>0	>0	$=0$	$=0$	$=0$
12	1Up	2x	3UpLeft	Yes	>0	$=0$	>0	$=0$	$=0$	$=0$	>0
13	1UpRight	2Right	3Up	No	>0	$=0$	>0	>0	$=0$	>0	$=0$
14	1Up	2Left	3UpLeft	No	>0	$=0$	>0	$=0$	>0	$=0$	>0
15	1UpRight	2x	3UpLeft	No	>0	$=0$	>0	>0	$=0$	$=0$	>0
16	1UpRight	2LeftRight	3Up	No	>0	$=0$	>0	>0	>0	>0	$=0$
17	1Up	2LeftRight	3UpLeft	No	>0	$=0$	>0	$=0$	>0	>0	>0
18	1x	2UpLeft	3Up	Yes	$=0$	>0	>0	$=0$	>0	$=0$	$=0$
19	1x	2UpLeft	3UpLeft	No	$=0$	>0	>0	$=0$	>0	$=0$	>0
20	1UpRight	2Right	3x	No	>0	$=0$	$=0$	>0	$=0$	>0	$=0$
21	1UpRight	2LeftRight	3x	No	>0	$=0$	$=0$	>0	>0	>0	$=0$
22	1x	2Left	3UpLeft	No	$=0$	$=0$	>0	$=0$	>0	$=0$	>0
23	1x	2LeftRight	3UpLeft	No	$=0$	$=0$	>0	$=0$	>0	>0	>0

Table 7: All cases of Lagrange multipliers for three retailer types.



(a) 1UpRight2UpRight3Up. (b) 1UpRight2Right3Up.

(c) 1UpRight2UpRight3x. (d) 1UpRight2Right3x.

(e) 1UpRight2Up3UpLeft. (f) 1UpRight2x3UpLeft.

(g) 1UpRight2LeftRight3Up. (h) 1UpRight2LeftRight3x.

Figure 9: Relevant KKT graphs for three retailer types. Not showing reducible or symmetric cases.

B.3.1 Case 1UpRight2UpRight3Up

As usual, the side payments follow directly from the complementary slackness conditions:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.$$

Since node 3 has no outgoing arcs, we can apply Lemma 3.13 to obtain the order quantity:

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

Finally, we use the complementary slackness conditions of the remaining active arcs. First, consider the following:

$$\phi_R^{2*} = \phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1 = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*} = \frac{1}{2}(h_2 - h_1)x_1 + \phi_R^{1*}.$$

Likewise,

$$\phi_R^{3*} = \phi_R^3(x_3) - z_3 = \phi_R^3(x_2) - z_2 = \phi_R^3(x_2) - \phi_R^2(x_2) + \phi_R^{2*} = \frac{1}{2}(h_3 - h_2)x_2 + \phi_R^{2*}.$$

Solving both equalities leads to:

$$x_1 = \frac{2(\phi_R^{2*} - \phi_R^{1*})}{(h_2 - h_1)}, \qquad x_2 = \frac{2(\phi_R^{3*} - \phi_R^{2*})}{(h_3 - h_2)}.$$

B.3.2 Case 1UpRight2Right3Up

This case is more difficult to solve. The side payments are straightforward to determine:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.$$

Lemma 3.13 specifies the order quantity for retailer type 3:

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

Furthermore, we have

$$\begin{aligned}\phi_R^{3*} &= \phi_R^3(x_3) - z_3 = \phi_R^3(x_2) - z_2 = \phi_R^3(x_2) - \phi_R^2(x_2) + \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*} \\ &= \frac{1}{2}(h_3 - h_2)x_2 + \frac{1}{2}(h_2 - h_1)x_1 + \phi_R^{1*},\end{aligned}$$

that is,

$$(h_2 - h_1)x_1 + (h_3 - h_2)x_2 = 2(\phi_R^{3*} - \phi_R^{1*}).$$

We also need to rewrite two KKT stationarity conditions, resulting in:

$$\begin{aligned}\frac{\omega_1}{(h_1 - h_2)}\left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p}\right) + \mu_{12} = 0,\\ \frac{\omega_2}{(h_2 - h_3)}\left(-2d(f+F)\frac{1}{x_2^2} + h_2 + H\frac{d}{p}\right) + \mu_{23} = 0.\end{aligned}$$

Combining these two equations and using $\omega_2 - \mu_{12} + \mu_{23} = 0$ leads to:

$$\frac{\omega_1}{(h_1 - h_2)}\left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p}\right) - \frac{\omega_2}{(h_2 - h_3)}\left(-2d(f+F)\frac{1}{x_2^2} + h_2 + H\frac{d}{p}\right) + \omega_2 = 0,$$

which is equivalent to

$$\frac{-2d(f+F)\omega_1}{(h_1 - h_2)}\frac{1}{x_1^2} - \frac{-2d(f+F)\omega_2}{(h_2 - h_3)}\frac{1}{x_2^2} = \frac{\omega_2}{(h_2 - h_3)}\left(h_2 + H\frac{d}{p}\right) - \frac{\omega_1}{(h_1 - h_2)}\left(h_1 + H\frac{d}{p}\right) - \omega_2.$$

To conclude, we have to solve a pair of equations for which we can use Lemma B.1 in Section B.3.9. As proved in the lemma, there exists a unique strictly positive and real solution. Both exact closed-form formulas and an efficient numerical solution method exist to solve these equations. Unfortunately, the formulas for x_1 and x_2 are too verbose to state here and will be omitted.

B.3.3 Case 1UpRight2UpRight3x

The side payments are:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^3(x_2) + \phi_R^2(x_2) - \phi_R^{2*}.$$

As seen before, the order quantity of type 3 follows from Lemma 3.13:

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

Complementary slackness states that

$$\phi_R^{2*} = \phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1 = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*},$$

that is,

$$x_1 = \frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1}.$$

Finally, from the KKT stationarity conditions we have $\mu_{23} = \omega_3$ and

$$\omega_2\left(-2d(f+F)\frac{1}{x_2^2} + h_2 + H\frac{d}{p}\right) + \omega_3(h_2 - h_3) = 0,$$

leading to

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_3}{\omega_2}(h_2 - h_3)}}.$$

B.3.4 Case 1UpRight2Right3x

This is a simple KKT menu, namely the Right-tree pattern (see Section B.1.2). The menu of contracts is given by the order quantities

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2+\omega_3}{\omega_1}(h_1 - h_2)}}, \quad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_3}{\omega_2}(h_2 - h_3)}}, \quad x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}},$$

and side payments

$$\begin{aligned}
z_1 &= \phi_R^1(x_1) - \phi_R^{1*}, \\
z_2 &= \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}, \\
z_3 &= \phi_R^3(x_3) - \phi_R^3(x_2) + \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}.
\end{aligned}$$

B.3.5 Case 1UpRight2Up3UpLeft

We apply the general solution technique to find:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*},$$

and

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}}.$$

Now use complementary slackness:

$$\phi_R^{2*} = \phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1 = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*}.$$

We obtain a similar equation for x_3. Hence, we have

$$x_1 = \frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1}, \qquad x_3 = \frac{2(\phi_R^{3*} - \phi_R^{2*})}{h_3 - h_2}.$$

B.3.6 Case 1UpRight2x3UpLeft

This case is one of the more difficult cases. The side payments are:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.$$

The order quantity for type 2 is straightforward:

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}}.$$

We use complementary slackness to find the following equation:

$$\begin{aligned}
& \phi_R^2(x_1) - z_1 = \phi_R^2(x_2) - z_2 = \phi_R^2(x_3) - z_3 \\
\iff \quad & \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*} = \phi_R^2(x_3) - \phi_R^3(x_3) + \phi_R^{3*} \\
\iff \quad & (h_2 - h_1)x_1 + (h_3 - h_2)x_3 = 2(\phi_R^{3*} - \phi_R^{1*}).
\end{aligned}$$

The following derivation has been used before. We rewrite two KKT stationarity conditions:

$$\begin{aligned}
\frac{\omega_1}{(h_1 - h_2)}\left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p}\right) + \mu_{12} = 0, \\
\frac{\omega_3}{(h_3 - h_2)}\left(-2d(f+F)\frac{1}{x_3^2} + h_3 + H\frac{d}{p}\right) + \mu_{32} = 0.
\end{aligned}$$

Next, combine both equations and use $\omega_2 - \mu_{12} - \mu_{32} = 0$:

$$\frac{\omega_1}{(h_1 - h_2)}\left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p}\right) + \frac{\omega_3}{(h_3 - h_2)}\left(-2d(f+F)\frac{1}{x_3^2} + h_3 + H\frac{d}{p}\right) + \omega_2 = 0,$$

which is equivalent to

$$\frac{-2d(f+F)\omega_1}{(h_1 - h_2)}\frac{1}{x_1^2} - \frac{-2d(f+F)\omega_3}{(h_2 - h_3)}\frac{1}{x_3^2} = \frac{\omega_3}{(h_2 - h_3)}\left(h_3 + H\frac{d}{p}\right) - \frac{\omega_1}{(h_1 - h_2)}\left(h_1 + H\frac{d}{p}\right) - \omega_2.$$

Thus, we solve the pair of equations using Lemma B.1. As stated in the lemma, a unique strictly positive and real solution exists. The formulas for x_1 and x_3 are too verbose and will be omitted.

B.3.7 Case 1UpRight2LeftRight3Up

From Corollary 3.11 we know that $x_1 = x_2$ and $z_1 = z_2$. Hence, we have side payments

$$z_1 = z_2 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.$$

As before, the order quantity of type 3 is:

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

Using complementary slackness results in:

$$\phi_R^{3*} = \phi_R^3(x_3) - z_3 = \phi_R^3(x_2) - z_2 = \phi_R^3(x_1) - z_1 = \phi_R^3(x_1) - \phi_R^1(x_1) + \phi_R^{1*}.$$

Solving for x_1 gives:

$$x_1 = x_2 = \frac{2(\phi_R^{3*} - \phi_R^{1*})}{h_3 - h_1}.$$

B.3.8 Case 1UpRight2LeftRight3x

The 2-cycle implies that $x_1 = x_2$ and $z_1 = z_2$. First, we give the side payments:

$$z_1 = z_2 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^3(x_2) + \phi_R^1(x_1) - \phi_R^{1*},$$

where by Lemma 3.13

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

The KKT stationarity conditions state that $\mu_{23} = \omega_3$, hence $\mu_{12} - \mu_{21} = \omega_2 + \omega_3$. Adding up the KKT conditions

$$\begin{aligned}\omega_1\left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p}\right) + \mu_{12}(h_1 - h_2) = 0,\\ \omega_2\left(-2d(f+F)\frac{1}{x_1^2} + h_2 + H\frac{d}{p}\right) + \mu_{21}(h_2 - h_1) + \omega_3(h_2 - h_3) = 0,\end{aligned}$$

leads to:

$$(\omega_1 + \omega_2)\left(-2d(f+F)\frac{1}{x_1^2} + H\frac{d}{p}\right) + \omega_1 h_1 + \omega_2 h_2 + (\omega_2 + \omega_3)(h_1 - h_2) + \omega_3(h_2 - h_3) = 0.$$

To conclude, the order quantity for types 1 and 2 is equal to:

$$x_1 = x_2 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_3}{\omega_1 + \omega_2}(h_1 - h_3)}}.$$

B.3.9 Special system of equations

In this section we discuss a special system of equations that needs to be solved for certain KKT contracts. See Lemma B.1 for the details.

Lemma B.1. *Consider the pair of equations of the following form:*

$$\alpha_1 x_1 + \alpha_2 x_2 = \gamma_1, \qquad \beta_1 \frac{1}{x_1^2} - \beta_2 \frac{1}{x_2^2} = \gamma_2,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 \in \mathbb{R}_{>0}$ and $\gamma_2 \in \mathbb{R}$ are given parameters. These equations always have a unique strictly positive real solution, i.e., satisfying $x_1, x_2 \in \mathbb{R}_{>0}$.

Proof. First, suppose $\gamma_2 = 0$. We have

$$\beta_1 \frac{1}{x_1^2} = \beta_2 \frac{1}{x_2^2} \qquad \Longleftrightarrow \qquad x_1 = \sqrt{\tfrac{\beta_1}{\beta_2}} x_2.$$

Thus, the other equation implies that

$$\alpha_1 \sqrt{\tfrac{\beta_1}{\beta_2}} x_2 + \alpha_2 x_2 = \gamma_1 \qquad \Longleftrightarrow \qquad x_2 = \frac{\gamma_1}{\alpha_1 \sqrt{\frac{\beta_1}{\beta_2}} + \alpha_2} > 0.$$

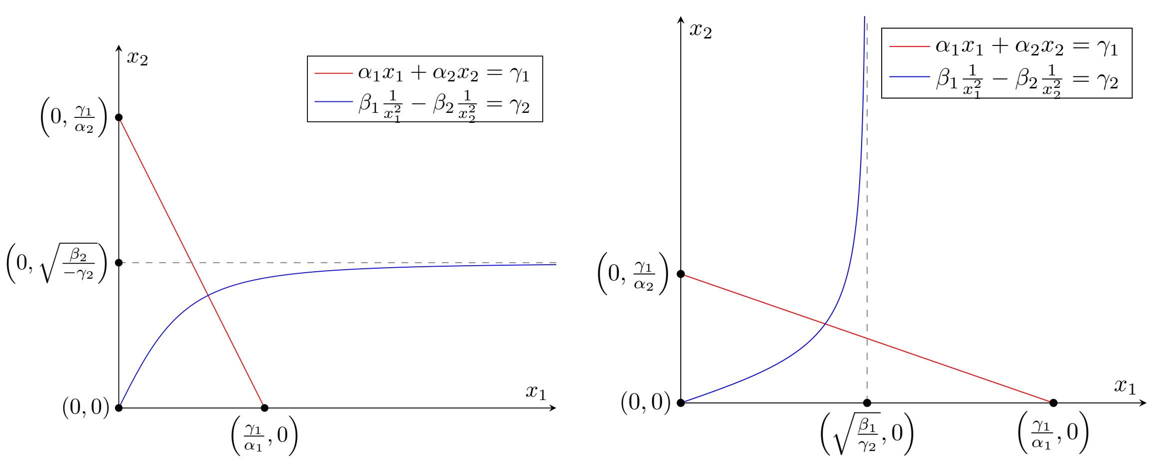
This proves the claim for $\gamma_2 = 0$.

Next, consider the case that $\gamma_2 < 0$. Notice that we can solve this case by finding the roots of a forth degree polynomial, for which an exact closed-form formula exists. This polynomial follows from substitution of one equation in the other. What remains is to show that exactly one of these four roots is strictly positive and real. To do so, solve the non-linear equation for x_2:

$$x_2 = \sqrt{\frac{\beta_2}{\beta_1 \frac{1}{x_1^2} - \gamma_2}} = \frac{\sqrt{\beta_2} x_1}{\sqrt{\beta_1 - \gamma_2 x_1^2}}.$$

This is a well-defined strictly positive solution for $x_1 > 0$. Furthermore, we have $x_2 = \gamma_1/\alpha_2 - (\alpha_1/\alpha_2)x_1$. Figure 10a shows the corresponding curves in the positive quadrant. Since the limits for $x_1 \to 0$ and $x_1 \to \infty$ are well-defined, this proves that a unique strictly positive real solution exists.

The case that $\gamma_2 > 0$ is similar, see Figure 10b, and will not be shown. □



(a) Curve if $\gamma_2 < 0$.

(b) Curve if $\gamma_2 > 0$.

Figure 10: Solution curve in the positive quadrant.

For completeness sake, we show how to numerically find the solution efficiently. Again, we assume $\gamma_2 < 0$ ($\gamma_2 > 0$ is similar). Consider the function θ for $x \in \mathbb{R}_{\geq 0}$:

$$\theta(x) = \frac{\sqrt{\beta_2} x}{\sqrt{\beta_1 - \gamma_2 x^2}} + \frac{\alpha_1}{\alpha_2} x - \frac{\gamma_1}{\alpha_2} \in \left[-\frac{\gamma_1}{\alpha_2}, \infty \right),$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}(x) = \frac{\beta_1 \sqrt{\beta_2}}{(\beta_1 - \gamma_2 x^2)^{3/2}} + \frac{\alpha_1}{\alpha_2} \in \left(\frac{\alpha_1}{\alpha_2}, \sqrt{\frac{\beta_2}{\beta_1}} + \frac{\alpha_1}{\alpha_2} \right].$$

Solving $\theta(x) = 0$ is equivalent to finding the value x_1. We only need to search in the bounded domain $x \in (0, \gamma_1/\alpha_1)$, since

$$\alpha_1 x_1 = \gamma_1 - \alpha_2 x_2 < \gamma_1 \qquad \Longrightarrow \qquad x_1 < \gamma_1/\alpha_1,$$

and $x_1, x_2 > 0$. As $\alpha_1/\alpha_2 > 0$, the derivative of θ is never zero. For example, if $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, then the derivative lies between 1 and 2. This suggests that methods such as Newton-Raphson should work very well and numerical results confirm fast and accurate convergence in typically less than 10 iterations.