Does Rounding Matter for Payment Efficiency?*

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Abstract

Theory predicts that dismissing the 1 and 2 euro cent coins from the denominational range of the euro leads to more payment efficiency. To examine whether this theory holds true in practice, we collected data for the Netherlands before and after September 1 2004, which marks the day that retail stores were allowed to round all amounts at 5 euro cents. The data consist of wallet contents for three cross sections of individuals. We propose a multivariate Poisson- log Normal model to analyze these data. We find that rounding leads to less 1 and 2 cent coins in wallets, but that still other coins are over or underrepresented, thereby suggesting that the euro range does not yet lead to fully efficient payment behavior.

Key words: Cash payment; Euro denomination; Multivariate Poisson–log Normal; maximum simulated likelihood;

JEL code: E51, C35, C15.

1 Introduction

January 1, 2002 marked the launch of the euro in 12 European countries. In the Netherlands the transition from the guilder to the euro involved a transition to a different denominational structure. The rather unique 1–$2\frac{1}{2}$–5 series for the guilder was replaced by the more common 1–2–5 series, which is generally accepted as the optimal denominational series of banknotes and coins. Indeed, Boeschoten and Fase (1989) already concluded that the Dutch guilder range was efficient to some extent, but that a transition to the more common 1–2–5 range would give benefits.

The euro coins have denominations 2, 1, 0.50, 0.20, 0.10, 0.05, 0.02 and 0.01,[1] and this range involves two more coins than the guilder range used to have, which were 5, 2.50, 1, 0.25, 0.10 and 0.05 guilders. Apart from the different denominational structure, the introduction of the euro coins also implied the introduction of 1 and 2 cent coins. Ever since the launch of the euro, the paying public faced inconveniences dealing with these smaller denominations. The Dutch Central Bank commissioned a research agency to see if the public would support the notion that retailers would to round at 5 euro cents,[2] and it seemed that they did. As a result, from September 1 2004 onwards, retailers in The Netherlands are allowed to round, although the 1 and 2 cent coins are still legal tender.

An interesting question is whether such rounding really helps to facilitate payment. As we will show in Section 2 below, there is a simple theoretical result that supports the positive effects of rounding. Of course, the issue is whether this theory holds in practice. To examine this, we collected Dutch data before and after September 1 2004 on wallet contents for three cross sections. In Section 3 we describe the data collection method and we provide a few basic statistics of the data obtained through this natural experiment. To analyze these data we resort to a multivariate Poisson - log Normal model, where we allow the intercept terms to vary across the cross sections. Details of this model are provided in Section 4, where we also compare it with other multivariate models for count data. The main reason to opt for our model is that the amounts of coins in individual wallets

[1] The euro banknotes have denominations 500, 200, 100, 50, 20, 10 and 5.

[2] Rounding implies that, say, €2.67 becomes €2.65 and that €2.68 becomes €2.70.

are likely to be correlated, whereas there is no reason to assume that this correlation should always be positive. In Section 5 we calibrate various models for our data, and we find that the log Normal gives the best fit. Based on the estimation results we conclude that rounding matters for the 1 and 2 cent coins, but that other coins are still over or underrepresented. Section 6 concludes and outlines a few topics for further research.

2 Theory

In this section we use a simple theoretical model to summarize predictions of what would happen if the 1 and 2 euro cent coins are dismissed from the denominational range. The theoretical model of individual payment behavior in Cramer (1983) is based on the 'principle of least effort'. If individuals would behave according to this principle, each amount would be paid such that the number of notes and coins exchanged is minimized. Such payment schemes are called efficient payment schemes.

Each amount has one or more efficient payment schemes, and some amounts have a very large number of ways to pay efficiently. An illustration is the amount of €11.30 that can be efficiently paid along three different ways, that is (*i*) $10 + 1 + 0.20 + 0.10$, (*ii*) $10 + 1 + 0.50$ and 0.20 returned and (*iii*) $10 + 2$ and $0.50 + 0.20$ returned. All other ways of paying this amount would lead to an exchange of 5 or more coins or notes.

The Cramer (1983) model provides an easy way to illustrate basic differences between denominational ranges, as it can be applied to any denominational range, and hence it can also be used to examine what happens if 1 and 2 euro cents coins are dismissed. Cramer (1983) also gives an algorithm to generate all efficient payment schemes for a given range of amounts. This algorithm is described in full detail in the appendix of Kippers et al. (2003).

As expected, a denominational range is viewed as more efficient than another if a smaller number of tokens will be exchanged on average, across all efficient payment schemes. Table 1 shows the results for rounding at 5 cent. We apply the Cramer algorithm to all amounts between €0.01 and €100, just like the example of €11.30 above, where the amounts are multiples of €0.01 in one case and are multiples of €0.05 in another, thus starting with the amount €0.05. The results in Table 1 are quite striking. The average number of tokens exchanged per payment scheme decreases from 5.83 to 4.93. Also, the maximum number of required tokens decreases from 8 to 7. This exercise tells us that payments can be done considerably more efficiently without the 1 euro cent and 2 euro cent coins. And, the number of coins needed to make efficient payments on average is smaller, hence wallet contents can be smaller.

3 Does the theory hold in practice?

To examine if the theory outlined in Section 2 holds in practice, we collected data on wallet contents before and after September 2004. Before we describe the data in more detail, we summarize a few assumptions that we need in order to be able to perform such an analysis.

The main assumption is that each paying individual, on average, intends to make an efficient payment, that is, he or she aims at exchanging the smallest possible amount of coins and notes when making a payment.

The next issue concerns the wallet content. This content is not random, as it depends on previous transactions. Hence, one may have made an efficient payment, but then for a next transaction it may occur that the wallet content does not allow for yet another efficient payment. To see if individuals make efficient payments, one therefore needs to know the content of the wallet at each transaction. To compare behaviour over time (before and after rounding), such data collection takes considerable effort, although potential simulation-based solutions are provided in Kippers et al. (2003). In this paper, we therefore assume that individuals intend to make efficient payments on average, and hence that their wallets contain coins that allow to do so, again on average.

How would such wallets look like? Again, we can rely on the algorithm of Cramer (1983), and we can see for all efficient payment schemes (where we limit the euro amounts to €100) how many coins are required on average. The results are displayed in the first column of Table 2. These numbers indicate that in order to make any efficient payment on average one needs 2 euro and 2 euro cent coins most, and 1 euro and 0.10 coins least.

In the second column of Table 2 we illustrate the fractions in for a wallet of size 15. For example, a wallet with 15 coins should contain 2.3 coins of 2 euro in order for the owner to make an efficient payment, on average.

Table 2 suggests an easy to implement method for practical data collection. Indeed, one only needs to collect wallet contents, and compare these contents with the theoretical fractions. Of course, this comparison amounts to a comparison of marginal distributions as it does not allow for correlation between the availability of coins in wallets. Indeed, individuals who carry too many 2 euro coins may also carry to many 1 euro coins. We postpone such an analysis to Section 5 below, as this requires more involved econometric models, those that we will discuss in Section 4.

In this section we now turn to a description of the data we have collected. In the spring of 2004 we became aware that the Dutch Central Bank was investigating the possibility of rounding at 5 euro cents. Therefore, we decided to collect wallet contents of a large number of individuals. We collected data at the Erasmus University, at a soccer club and in a waiting room of a physiotherapist. In the period February to June 2004 we collected 240 observations this way. The next cross section was taken in October 2004 and comprises of 211 observations. Finally, in January 2005 we collected yet another sample of 273 observations. We label the three samples as I, II and III. The main question we had for all cross sections was "Can I see your wallet content?" Interestingly, we did not encounter any problems whatsoever to get an answer to this question.

Some basic statistics on samples I, II and III are given in Tables 3, 4 and 5. The average wallet contents in these three samples were 13.5, 11.6 and 9.9 coins, respectively. Table 4 presents the observed mean, variance, minimum and maximum of all the coins in the wallets. Figure 1 to Figure 8 provide the histograms for each coin. It appears that in the second and third collection fewer coins of 1 cent and 2 cent are in the wallets. Table 5 shows that the number of 1 cent coins is highly correlated with the number of 2 cent coins in a wallet. Also other coins show strong correlation. When we put forward an econometric model to summarize these data, the model should therefore allow for such correlations (which are not necessarily positive).

4 The Multivariate Poisson–log Normal model

The standard and most simple model for counts, the type of data we have, is a Poisson model. It is easily extended to multivariate, but uncorrelated, counts. A Poisson model imposes equality of the mean and variance, which may be very restrictive. A mixed Poisson model with unobserved heterogeneity allows for overdispersion. A commonly applied mixed Poisson model is the Poisson Log-normal model, that assumes that normally distributed unobserved heterogeneity enters the Poisson regression. If the counts are uncorrelated we can estimate separate Poisson Log-normal models for the count of each coin.

4.1 Some models for multivariate counts

As can be observed from Table 5, the counts of the 8 euro coins are likely to have a non-trivial correlation structure across the outcomes of one individual. Modelling this correlation structure is important for valid inference. Winkelmann (2003) (Chapter 5) provides a comprehensive discussion of various multivariate count data models. He distinguishes five models for correlated counts, that is, three one-factor models and two multi-factor models. We provide a short review of these models.

Let $y_{ij}, i = 1, \ldots, n, j = 1, \ldots, 8$ (€2, €1, 50 cent, 20 cent, 10 cent, 5 cent, 2 cent, 1 cent) denote the count for individual i of coin j. Let $y_i = (y_{i1} \ldots y_{i8})'$ denote the vector of counts for individual i over the 8 different coins. In a one-factor model the correlation is generated through an individual–specific factor u_i that does not vary over the coins. This implies that the covariance structure is restricted to non-negative correlations.

The first (one-factor) model is the multivariate Poisson model (see Kocherlakota and Kocherlakota (1992) for an overview). Its derivation is based on the so-called trivariate reduction method. Suppose the count variables are defined as $y_{ij} = z_{ij} + u_i$, where $z_{ij}, j = 1, \ldots 8$ and u_i have independent Poisson distributions with $z_{ij} \sim \text{Poisson}(\lambda_{ij})$, and $u_i \sim \text{Poisson}(\gamma)$. The marginal distribution of y_{ij} is $\text{Poisson}(\lambda_{ij} + \gamma)$ and the correlation coefficient is given by

$$\mathrm{Corr}(y_{ij}, y_{ik}) = \frac{\gamma}{\sqrt{(\lambda_{ij}+\gamma)(\lambda_{ik}+\gamma)}} \qquad j \neq k \tag{1}$$

which is positive because $\gamma > 0$. As for the univariate Poisson count model, the conditional expectation and variance are equal. For our application this may be to restrictive.

Another one-factor multivariate count model is the multivariate negative binomial model (see Kocherlakota and Kocherlakota, 1992, p. 122). Although this model allows for overdispersion it also restricts the correlation between counts to be positive.

An alternative approach is a one-factor model in which the correlation is induced by an individual–specific multiplicative error term, that is, unobserved heterogeneity. If we assume that the unobserved heterogeneity u_i is zero-mean gamma distributed, as Hausman et al. (1984) do, and that $y_{ij} = \lambda_{ij} u_i$, the mixture multivariate density has a closed form solution. This multivariate Poisson-Gamma mixture model allows for overdispersion. However, it also restricts the covariances to be positive. In contrast to the two previous models this model does not have the equi-covariance property, but a covariance structure that is a product of both

$$\mathrm{Corr}(y_{ij}, y_{ik}) = \frac{\sigma^2 \lambda_{ij}\lambda_{ik}}{\sqrt{(\lambda_{ij}^2+\sigma^2\lambda_{ij})(\lambda_{ik}^2+\sigma^2\lambda_{ik})}} \qquad j \neq k \tag{2}$$

where σ^2 is the variance of u_i. A disadvantage of this model is that the covariances are not determined independently of the dispersion. Hence, finding a significant unobserved heterogeneity term can be an indicator of overdispersion, of correlation or of both.

In the multi-factor model based on a latent Poisson-normal model of van Ophem (1999), count data are interpreted as realizations of an underlying latent normally distributed variable. One problem with this model is that the support of the count data distributions is unbounded. And, this model yet concerns the bivariate case, and extensions to higher dimensional multivariate data seem cumbersome.

4.2 The preferred model

A flexible multi-factor model that allows for negative and positive correlations and can be derived for the bivariate and higher dimensions is the Multivariate Poisson–log Normal distribution, see Aitchison and Ho (1989). They assume that conditional on an (8×1) vector of individual and coin–specific random effects $\epsilon_i = (\epsilon_{i1} \dots \epsilon_{i8})'$, the distribution of y_i is independent Poisson

$$f(y_i|\epsilon_i) = \prod_{j=1}^{8} \frac{\exp(-\lambda_{ij} e^{\epsilon_{ij}}) \big(\lambda_{ij} e^{\epsilon_{ij}}\big)^{y_{ij}}}{y_{ij}!} \tag{3}$$

where ϵ_i is 8-variate normal distributed with covariance matrix Ω and mean $-\frac{1}{2}\text{diag}(\Omega)$. The non-zero mean specification facilitates the interpretation of the parameters and implies that the conditional expectation of the vector of counts is

$$\text{E}[y_i|\lambda_i, \Omega] = \lambda_i,$$

where $\lambda_i = (\lambda_{i1} \dots \lambda_{i8})'$ and the variance is

$$\text{Var}(y_i|\lambda_i, \Omega) = \Lambda_i + \Lambda_i \Big(\exp(\Omega) - ee'\Big) \Lambda_i,$$

where Λ_i is a diagonal matrix with λ_i on the diagonal and e is an (8×1) vector of ones. The correlations between the counts of one individual are

$$\text{Corr}(y_{ij}, y_{ik}) = \frac{\lambda_{ij}\big(\exp(\omega_{jk}) - 1\big)\lambda_{ik}}{\sqrt{\Big[\lambda_{ij} + \lambda_{ij}^2\big(\exp(\omega_{jj}) - 1\big)\Big]\Big[\lambda_{ik} + \lambda_{ik}^2\big(\exp(\omega_{kk}) - 1\big)\Big]}} \qquad j \neq k \tag{4}$$

which can be positive or negative depending on the sign of ω_{jk}, the (j,k) element of Ω.

A disadvantage of this model is that the likelihood function does not have an analytical solution for an arbitrary Ω. Computation of the likelihood requires the evaluation of an 8-variate integral with respect to the distribution of ϵ_i. The density for individual i is

$$f_i(y_i|\lambda_i, \Omega) = \int \prod_{j=1}^{8} \frac{\exp(-\lambda_{ij} e^{\epsilon_{ij}}) \big(\lambda_{ij} e^{\epsilon_{ij}}\big)^{y_{ij}}}{y_{ij}!} \phi_8\big(\epsilon_i| - \tfrac{1}{2}\text{diag}(\Omega), \Omega\big)\, d\epsilon_i, \tag{5}$$

where $\phi_8\big(\cdot| - \frac{1}{2}\text{diag}(\Omega), \Omega\big)$ is the density of an 8-variate normal distributed variable with mean vector $-\frac{1}{2}\text{diag}(\Omega)$ and covariance matrix Ω. In principle, this 8-variate integral could be approximated by a Gauss-Hermite quadrature. However, the number of evaluation points increases exponentially with the dimension of the integral. Maximum simulated likelihood (MSL) estimation provides a feasible and easy to implement alternative to the evaluation of these integrals.[3] In fact, to our knowledge the present paper is the first to analyze an 8-dimensional Poisson log–normal model.

Both for the Gauss-Hermite quadrature approach and for MSL, it is more efficient to transform the ϵ_i with a Cholesky decomposition. That is $v_i = L \cdot \epsilon_i + \frac{1}{2}\text{diag}(\Omega)$ where the lower triangular matrix L satisfies $LL^T = \Omega$ with rows L_j. The density in (5) then becomes

$$f_i(y_i|\lambda_i, \Omega) = \int \prod_{j=1}^{8} f_{ij}(y_{ij}|\lambda_{ij}, v_{ij}) \phi_8\big(v_i|0, I_8\big)\, dv_i, \tag{6}$$

with

$$f_{ij}(y_{ij}|\lambda_{ij}, v_{ij}) = \frac{\exp(-e^{x'_{ij}\beta_j + L_j \cdot v_i - 0.5\sigma_j^2})\big(e^{x'_{ij}\beta_j + L_j \cdot v_i - 0.5\sigma_j^2}\big)^{y_{ij}}}{y_{ij}!}$$

for $\lambda_{ij} = \exp(x'_{ij}\beta_j)$. To guarantee a positive definite covariance matrix we estimate the elements of the Choleksy decomposition.

4.3 Maximum Simulated Likelihood Estimation

Simulation can lead to an estimator with the same distribution of the maximum likelihood estimator (Hajivassiliou and Ruud (1994), Gouriéroux and Monfort (1996) and Stern (1997)). The increasing speed of computer power has made estimation by simulation more attractive. For a univariate Poisson-log normal model Hinde (1982) developed a simulation based method. Munkin and Trivedi (1999) discuss the estimation by maximum simulated likelihood for a bivariate Poisson-log normal model. We implement a maximum simulated likelihood (MSL) estimator along similar lines.

The density in (6) can be approximated by S simulated values $v_i^s, s = 1, \ldots, S$. A MSL estimator of $\theta = (\beta, \sigma, \rho)'$ is

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{n} \log\left[\frac{1}{S}\sum_{s=1}^{S}\prod_{j=1}^{8} f_{ij}(y_{ij}|\lambda_{ij}, v_{ij}^s)\right]$$

Gouriéroux and Monfort (1996) show that the MSL estimator is both consistent and asymptotically equivalent to the MLE if $\sqrt{n}/S \to 0$ as $S \to \infty$.

[3] Some authors prefer the term simulated maximum likelihood instead of maximum simulated likelihood.

For any fixed finite S the MSL estimator is not consistent. This bias arises because the log and the integral of the likelihood function do not commute. Therefore, the simulator of $\log(f_i)$ is biased even if the simulator of f_i is unbiased. Gouriéroux and Monfort (1991) give an expression for the bias of the MSL estimator. A way of reducing the inconsistency is to use a bias-adjusted log-likelihood, based on the second order Taylor expansion of $\log(\hat{f}_i)$ around $\log(f_i)$. The biased corrected MSL estimator then consists of maximizing the simulated likelihood function

$$\sum_{i=1}^{n} \log\left[\frac{1}{S}\sum_{s=1}^{S}\prod_{j=1}^{8} f_{ij}(y_{ij}|\lambda_{ij}, v_{ij}^{s})\right] + \frac{1}{2}\sum_{i=1}^{n} \frac{\sum_{s=1}^{S}\left\{\prod_{j=1}^{8} f_{ij}(y_{ij}|\lambda_{ij}, v_{ij}^{s}) - \frac{1}{S}\sum_{s=1}^{S}\prod_{j=1}^{8} f_{ij}(y_{ij}|\lambda_{ij}, v_{ij}^{s})\right\}^2}{\left\{\sum_{s=1}^{S}\prod_{j=1}^{8} f_{ij}(y_{ij}|\lambda_{ij}, v_{ij}^{s})\right\}^2} \tag{7}$$

The asymptotic covariance matrix of the MSL estimator can be estimated consistently and robustly with a sandwich estimator $H^{-1}UH^{-1}$. Here, H is the observed information matrix estimated by the Hessian of (7), and U is the expected information matrix estimated by the cross product of the gradients of (7).

5 Results

In this section we report on the estimation results when applying three different models for the data available. To test for overdispersion and correlation we consider (1) 8 Poisson models for the counts of each euro coin, (2) 8 Poisson-Log normal models for the counts of each euro coin. (this is a Multivariate Poisson-log normal model with a diagonal covariance structure), (3) 1 Multivariate Poisson-log normal model with an arbitrary covariance structure.

5.1 Estimation results

We compare the predictive performance of these models on the actual wallet data. From the estimated models we can also predict the fractions of coins in a wallet of given size and compare these results with the theoretical fractions of Section 3. For the first two models the fractions are independent of the size of the wallet. For the correlated counts model the fractions may change with the size of the wallet.

The only explanatory variable in the models concerns the moment of collection. We thus estimate for each coin three β's, that is, the intercept β_0, the additional intercept at the time of the second collection, β_{II} and, the additional intercept at the time of the third collection, β_{III}. For the Poisson Log-normal models the mean of the unobserved heterogeneity is equal to minus a half of the variance. This assures that for all three models the expected amount of a particular coin is e^{β_0} for the first collection and $e^{\beta_0+\beta_{II}}$ and $e^{\beta_0+\beta_{III}}$ for the second and third collection, respectively. The estimated β's can be found in Table 6, where the most obvious results concern the strong negative values of β_{II} and β_{III} for the 1 and 2 euro cent coins.[4]

For all three models the expected amount and variance of each coin for each collection is in Table 7. We observe that the availability of most coins is rather constant, except for the 1 and 2 euro coins. The correlation matrix among the amounts of each coin implied by the multivariate Poisson Log-normal model is shown in Table 8. Comparing this table with Table 5, we see that the model captures the empirical correlations rather good.

5.2 Forecasting

To assess the predictive power of the models we perform a validation test on the number of predicted coins in a wallet. For the Poisson model the estimated parameters directly imply the probabilities to observe a given number of a particular coin in a wallet. For the Poisson log-normal models these probabilities have to be simulated. We simulate from a normal 8-dimensional distribution using the estimated variance and, only for the correlated model, estimated correlation and then calculate, for each draw, the implied Poisson probabilities. Averaging over all simulation draws, here set at 10000, provides estimates of the marginal probabilities for the Poisson log-normal models. Multiplying

[4]The estimates of the variance for the Poisson Log normal model and the estimates of the elements of the Cholesky decomposition for the Multivariate Poisson Log normal model are not shown but are available upon request from the corresponding author.

these probabilities with the number of wallets we inspected gives the predicted number of wallets with a certain number of each coin.

The Poisson model and uncorrelated Poisson Log-normal model assume that the amount of each coin is independent from the amounts of other coins. The probability to observe some given number of coins in total in a wallet can be found from the marginal probabilities. The probability to observe, say, 5 coins in a wallet is the probability to observe all the possible combinations of the 8 euro coins that sum up to 5 coins. In the multivariate Poisson log-normal model the amount of counts for one coin depends on the number of all the other coins. However, conditional on the unobserved, log normal, heterogeneity vector, these counts are independent. Therefore, we can simulate the total number of coins in a wallet from the multivariate Poisson log normal model for each simulated heterogeneity vector, in the same way as for the two models that assume independence of the counts. Thus, averaging over all, again 10000, simulations and multiplying by the number of inspected wallets in each collection provides the predicted number of wallets with a certain number of coins.

A negligible amount of the wallets contain more than 16 coins of the same denomination or more than 30 coins in total. We therefore restrict the testing of the predictive power of the models whether they can predict up to a maximum of 16 coins of one particular coin or up to 30 coins in total well. Denote the predictive and observed number of wallets with k coins by P_k^p and P_k^o respectively. The value of the χ^2 test is equal to $\sum_k \frac{(P_k^p - P_t^o)^2}{P_k^p}$, which has a χ^2_{135} or χ^2_{31} distribution. A model gives a good prediction if the null hypothesis of predicting well cannot be rejected. A model performs best if it has the lowest value for the χ^2 test. Table 9 gives these Chi-squared tests. This table immediately shows that the Multivariate Poisson log normal is the only model that predicts the number of wallets with a certain number of a particular coin and with a certain number of coins in total well, as the null hypothesis is mostly not rejected.

5.3 Estimated fractions

In conjunction with the theory on efficient payment schemes, it is of interest to estimate the fraction of each coin in an arbitrary wallet. This fraction can be obtained from the model estimates. For the Poisson model and the uncorrelated Poisson log-normal model, which both assume that the number of each coin is independent of the number of other coins in a wallet, this fraction can easily derived from the expected amount of coins in a wallet, see Table 7. For example, the fraction of €2 coins in a wallet is equal to the expected amount of €2 coins, divided by the sum of the expected amounts for all coins. This has the familiar logit form. For the uncorrelated Poisson log-normal model we rely on the average of 10000 simulated 8-dimensional unobserved heterogeneity vectors.

For the correlated Multivariate Poisson log-normal model the amount of each coin in a wallet depends on the amount of all other coins in the wallet. It implies that the conditional probabilities are needed. This also implies that the fraction of a particular coin depends on the total number of coins in a wallet. For example, if we want the fraction of €2 coins in a wallet of size 5, we need the conditional probability of observing 0, 1, 2, 3 or 4 €2 coins given a total of 5 coins. If we denote these probabilities by $p(0|5), p(1|5), p(2|5), p(3|5), p(4|5)$ and $p(5|5)$ the fraction of €2 coins in a wallet of size 5 is then $\frac{1}{5} \cdot p(1|5) + \frac{2}{5} \cdot p(2|5) + \frac{3}{5} \cdot p(3|5) + \frac{4}{5} \cdot p(4|5) + p(5|5)$. Thus, in simulating these fractions we have to simulate all the possible conditional probabilities, given a range of possible total amounts of coins in a wallet and the joint probability of this amount of coins. After averaging over all (10000) simulation rounds we obtain estimated fractions.

These estimated fractions are reported in Table 10. For the correlated multivariate model the fractions for a wallet size of 5, 10, 15, 20 and 25 are reported. A number of interesting things are observed. We clearly see that the implied fractions for the €0.02 and €0.01 coins are much lower in the last two collections. In the correlated Poisson log-normal model the more coins a wallet contains the larger the fraction €0.02 and €0.01 coins and the lower the fraction €2 coins.

The fraction of €2 and €0.02 coins in a wallet are significantly below the efficient fraction and the fraction of €0.10 and €0.05 coins are above the efficient fraction. Before the rounding to 5 cent a wallet contained, on average, to few €0.50 and €0.20 coins. This inefficiency disappeared after the rounding was introduced. The fraction of €1 coins in a wallet matches with efficiency.

6 Conclusion

From our analysis we can conclude that rounding indeed matters. Rounding leads to less 1 and 2 euro cent coins in wallets, but some other coins are over- or underrepresented. This suggests that the euro range does not yet lead to fully efficient payment behavior. Apparently more experience with the euro range is needed, and hence future data collection is necessary to examine potential convergence towards an efficient use of available coins.

An interesting avenue for further research is to collect data on wallet contents in other European countries and see whether efficiency has been reached for those countries. The model presented in our paper can be used to verify or validate the hypothesis that rounding leads to more efficient payments.

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Table 1: Statistics on all efficient payment schemes for amounts between €0.01 and €100

Efficient payment schemes	all euro denominations[1)]	without 1 and 2 euro cent coins[2)]
number	36,591	5,957
average number of tokens exchanged	5.83	4.93
median	6	5
minimum	1	1
maximum	8	7

[1)] Amounts are multiples of €0.01. [2)] Amounts are multiples of €0.05.

Table 2: Theoretical fraction of coins in an average wallet

Coin	Average fraction	If wallet contains 15 coins
€2	0.152	2.303
€1	0.091	1.351
€0.50	0.116	1.723
€0.20	0.146	2.168
€0.10	0.091	1.351
€0.05	0.116	1.723
€0.02	0.156	2.317
€0.01	0.122	1.812

Table 3: Average fraction of coins in a wallet

	Cross section		
Coin	I	II	III
€2	0.083	0.122	0.110
€1	0.096	0.108	0.102
€0.50	0.104	0.119	0.117
€0.20	0.135	0.144	0.147
€0.10	0.127	0.169	0.149
€0.05	0.162	0.151	0.183
€0.02	0.143	0.100	0.099
€0.01	0.150	0.089	0.092
Average number of coins	13.5	11.6	9.9

Table 4: Sample statistics for each coin and each collection

		Collection		
Coin		I	II	III
€2	mean	0.971	1.232	0.934
	variance	(1.560)	(2.017)	(1.415)
	min #	0	0	0
	max #	9	8	8
€1	mean	1.158	1.190	0.883
	variance	(2.067)	(2.078)	(1.266)
	min #	0	0	0
	max #	8	8	5
€0.50	mean	1.117	1.166	1.037
	variance	(2.003)	(1.977)	(1.741)
	min #	0	0	0
	max #	10	12	10
€0.20	mean	1.825	1.602	1.421
	variance	(4.647)	(3.079)	(2.892)
	min #	0	0	0
	max #	13	10	10
€0.10	mean	1.788	1.820	1.498
	variance	(4.043)	(4.091)	(2.758)
	min #	0	0	0
	max #	11	12	8
€0.05	mean	2.133	1.872	1.919
	variance	(6.919)	(5.665)	(5.832)
	min #	0	0	0
	max #	20	18	19
€0.02	mean	2.221	1.384	1.051
	variance	(7.328)	(5.419)	(2.578)
	min #	0	0	0
	max #	18	17	10
€0.01	mean	2.271	1.299	1.121
	variance	(8.416)	(5.658)	(4.151)
	min #	0	0	0
	max #	21	15	15

Table 5: Observed sample correlation between the number of coins in a wallet

I

	€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
€2	1	0.24	0.25	0.15	0.18	0.04	0.22	0.00
€1	0.24	1	0.24	0.25	0.18	0.16	0.13	0.10
€0.50	0.25	0.24	1	0.27	0.23	0.05	0.11	0.13
€0.20	0.15	0.25	0.27	1	0.46	0.37	0.23	0.18
€0.10	0.18	0.18	0.23	0.46	1	0.36	0.34	0.21
€0.05	0.04	0.16	0.05	0.37	0.36	1	0.36	0.28
€0.02	0.22	0.13	0.11	0.23	0.34	0.36	1	0.49
€0.01	0.00	0.10	0.13	0.18	0.21	0.28	0.49	1

II

	€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
€2	1	0.41	0.17	0.27	0.24	0.19	0.19	0.19
€1	0.41	1	0.28	0.24	0.11	0.05	0.04	0.01
€0.50	0.17	0.28	1	0.12	0.31	0.11	0.08	0.22
€0.20	0.27	0.24	0.12	1	0.29	0.35	0.29	0.27
€0.10	0.24	0.11	0.31	0.29	1	0.33	0.35	0.38
€0.05	0.19	0.05	0.11	0.35	0.33	1	0.49	0.48
€0.02	0.19	0.04	0.08	0.29	0.35	0.49	1	0.71
€0.01	0.19	0.01	0.21	0.27	0.38	0.48	0.71	1

III

	€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
€2	1	0.31	0.12	0.08	0.11	0.04	-0.01	-0.04
€1	0.31	1	0.21	0.25	0.10	0.01	0.06	-0.01
€0.50	0.12	0.21	1	0.20	0.12	0.08	0.10	-0.03
€0.20	0.08	0.25	0.20	1	0.23	0.16	0.14	0.12
€0.10	0.11	0.10	0.12	0.23	1	0.40	0.10	0.12
€0.05	0.04	0.01	0.08	0.16	0.40	1	0.30	0.30
€0.02	-0.01	0.06	0.10	0.14	0.10	0.30	1	0.61
€0.01	-0.04	-0.01	-0.03	0.12	0.12	0.30	0.61	1

Table 6: Estimated parameters (with estimated standard errors in parentheses)

		Poisson		Poisson Log-Normal (uncorr.)		Multivariate Poisson Log-Normal (corr.)	
€ 2	β_0	-0.0296	(0.0657)	-0.2060	(0.0878)	0.0089	(0.0760)
	β_{II}	0.2384	(0.0904)	0.2386	(0.1070)	0.2340	(0.1051)
	β_{III}	-0.0386	(0.0907)	-0.0370	(0.1050)	-0.0676	(0.1052)
€ 1	β_0	0.1470	(0.0599)	-0.0613	(0.0869)	0.1981	(0.0763)
	β_{II}	0.0266	(0.0866)	0.0084	(0.1070)	0.0239	(0.1095)
	β_{III}	-0.2717	(0.0879)	-0.2886	(0.1063)	-0.3105	(0.1092)
€ 0.50	β_0	0.1103	(0.0612)	-0.0390	(0.0799)	0.1519	(0.0772)
	β_{II}	0.0431	(0.0886)	0.0235	(0.1040)	0.0536	(0.1109)
	β_{III}	-0.0744	(0.0854)	-0.0850	(0.1007)	-0.0971	(0.1069)
€ 0.20	β_0	0.6016	(0.0478)	0.3940	(0.0768)	0.6868	(0.0714)
	β_{II}	-0.1304	(0.0724)	-0.1826	(0.0968)	-0.1761	(0.1032)
	β_{III}	-0.2500	(0.0697)	-0.2739	(0.0933)	-0.3009	(0.0984)
€ 0.10	β_0	0.5808	(0.0483)	0.3946	(0.0735)	0.6900	(0.0685)
	β_{II}	0.0180	(0.0703)	0.0134	(0.0964)	-0.0687	(0.1009)
	β_{III}	-0.1766	(0.0691)	-0.1957	(0.0910)	-0.2657	(0.0953)
€ 0.05	β_0	0.7577	(0.0442)	0.5544	(0.0758)	0.8810	(0.0718)
	β_{II}	-0.1307	(0.0670)	-0.1374	(0.0964)	-0.2630	(0.1078)
	β_{III}	-0.1057	(0.0622)	-0.1261	(0.0914)	-0.2011	(0.0980)
€ 0.02	β_0	0.7979	(0.0433)	0.0839	(0.1543)	1.0254	(0.0763)
	β_{II}	-0.4730	(0.0728)	-0.2884	(0.1013)	-0.7150	(0.1228)
	β_{III}	-0.7479	(0.0732)	-0.6289	(0.0985)	-0.9705	(0.1139)
€ 0.01	β_0	0.8201	(0.0428)	0.2441	(0.1244)	1.0946	(0.0835)
	β_{II}	-0.5589	(0.0741)	-0.3681	(0.1138)	-0.8630	(0.1282)
	β_{III}	-0.7060	(0.0714)	-0.5421	(0.1080)	-0.9994	(0.1287)
Log-L		-10566.8		-9586.4		-9005.84	

Table 7: Expected mean and variance of the amount of coins in an average wallet

			Collection		
Coin	Estimation		I	II	III
€2	Poisson	mean	0.971	1.232	0.934
		variance	(0.971)	(1.232)	(0.934)
	Uncorrelated Poisson	mean	0.814	1.033	0.784
	Log-normal	variance	(0.959)	(1.267)	(0.919)
	Correlated Poisson	mean	1.009	1.279	0.943
	Log-normal	variance	(1.469)	(2.001)	(1.345)
€1	Poisson	mean	1.158	1.190	0.883
		variance	(1.158)	(1.190)	(0.883)
	Uncorrelated Poisson	mean	0.941	0.949	0.705
	Log-normal	variance	(1.162)	(1.174)	(0.829)
	Correlated Poisson	mean	1.219	1.249	0.894
	Log-normal	variance	(2.254)	(2.334)	(1.450)
€0.50	Poisson	mean	1.117	1.166	1.037
		variance	(1.117)	(1.166)	(1.037)
	Uncorrelated Poisson	mean	0.962	0.985	0.883
	Log-normal	variance	(1.160)	(1.192)	(1.051)
	Correlated Poisson	mean	1.164	1.228	1.056
	Log-normal	variance	(2.068)	(2.235)	(1.801)
€0.20	Poisson	mean	1.825	1.602	1.421
		variance	(1.825)	(1.602)	(1.421)
	Uncorrelated Poisson	mean	1.483	1.236	1.128
	Log-normal	variance	(2.077)	(1.648)	(1.471)
	Correlated Poisson	mean	1.987	1.666	1.471
	Log-normal	variance	(5.580)	(4.193)	(3.439)
€0.10	Poisson	mean	1.788	1.820	1.498
		variance	(1.788)	(1.820)	(1.498)
	Uncorrelated Poisson	mean	1.484	1.504	1.220
	Log-normal	variance	(2.030)	(2.065)	(1.589)
	Correlated Poisson	mean	1.994	1.861	1.528
	Log-normal	variance	(5.277)	(4.723)	(3.458)
€0.05	Poisson	mean	2.133	1.872	1.919
		variance	(2.133)	(1.872)	(1.919)
	Uncorrelated Poisson	mean	1.741	1.517	1.535
	Log-normal	variance	(2.613)	(2.180)	(2.212)
	Correlated Poisson	mean	2.413	1.855	1.974
	Log-normal	variance	(9.666)	(6.142)	(6.825)
€0.02	Poisson	mean	2.221	1.384	1.051
		variance	(2.221)	(1.384)	(1.051)
	Uncorrelated Poisson	mean	1.088	0.815	0.580
	Log-normal	variance	(1.648)	(1.130)	(0.739)
	Correlated Poisson	mean	2.788	1.364	1.057
	Log-normal	variance	(21.795)	(5.912)	(3.785)
€0.01	Poisson	mean	2.271	1.299	1.121
		variance	(2.271)	(1.299)	(1.121)
	Uncorrelated Poisson	mean	1.277	0.883	0.742
	Log-normal	variance	(2.017)	(1.238)	(0.993)
	Correlated Poisson	mean	2.988	1.261	1.100
	Log-normal	variance	(35.623)	(7.070)	(5.522)

Wallet sizes are given in Table 3.

Table 8: Correlation between the number of coins in a wallet implied by the estimated Multivariate Poisson-Log Normal model

				I				
	€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
€2	1	0.37	0.22	0.21	0.14	0.08	0.05	-0.01
€1	0.37	1	0.31	0.27	0.15	0.07	0.07	0.02
€0.50	0.22	0.31	1	0.20	0.19	0.09	0.04	0.05
€0.20	0.21	0.27	0.20	1	0.39	0.31	0.16	0.11
€0.10	0.14	0.15	0.19	0.39	1	0.41	0.23	0.15
€0.05	0.08	0.07	0.09	0.31	0.41	1	0.39	0.34
€0.02	0.05	0.07	0.04	0.16	0.23	0.39	1	0.74
€0.01	-0.01	0.02	0.05	0.11	0.15	0.34	0.74	1
				II				
	€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
€2	1	0.40	0.24	0.22	0.15	0.08	0.06	-0.01
€1	0.40	1	0.31	0.26	0.15	0.07	0.07	0.02
€0.50	0.24	0.31	1	0.20	0.19	0.09	0.04	0.04
€0.20	0.22	0.26	0.20	1	0.37	0.29	0.15	0.10
€0.10	0.15	0.15	0.19	0.37	1	0.39	0.22	0.14
€0.05	0.08	0.07	0.09	0.29	0.39	1	0.35	0.31
€0.02	0.06	0.07	0.04	0.15	0.22	0.35	1	0.66
€0.01	-0.01	0.02	0.04	0.10	0.14	0.31	0.66	1
				III				
	€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
€2	1	0.33	0.21	0.19	0.13	0.07	0.05	-0.01
€1	0.33	1	0.27	0.23	0.13	0.06	0.06	0.01
€0.50	0.21	0.27	1	0.19	0.18	0.08	0.04	0.04
€0.20	0.19	0.23	0.19	1	0.35	0.28	0.14	0.10
€0.10	0.13	0.13	0.18	0.35	1	0.38	0.20	0.13
€0.05	0.07	0.06	0.08	0.28	0.38	1	0.34	0.31
€0.02	0.05	0.06	0.04	0.14	0.20	0.34	1	0.63
€0.01	-0.01	0.01	0.04	0.10	0.13	0.31	0.63	1

Table 9: Chi-squared test of the 3 models on predictive power of the number of coins in a wallet

		Collection		
	Estimation	I	II	III
for each coin[1]	Poisson	536.7	357.3	435.0
	Uncorrelated Poisson Log-normal	437.8	218.8	443.1
	Correlated Poisson Log-normal	148.5	119.5	150.1
all coins[2]	Poisson	251.8	305.1	354.5
	Uncorrelated Poisson Log-normal	69.5	64.1	255.4
	Correlated Poisson Log-normal	65.8	44.3	62.0

[1] Based on predictions for 0 to 16 coins for each of the 8 coins. Critical 5% value for $\chi^2_{136} = 164.2$. [2] Based on predictions for 0 to 30 coins. Critical 5% value for $\chi^2_{31} = 44.99$.

Table 10: Estimated fraction per coin in a wallet based implied by the estimated models

		€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
Theoretical		0.152	0.091	0.116	0.146	0.091	0.116	0.156	0.122
						I			
		€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
Poisson		0.072(-)	0.086	0.083(-)	0.135	0.133(+)	0.158(+)	0.165	0.168(+)
Uncorrelated Poisson									
Log-normal		0.088(-)	0.100	0.103(-)	0.152	0.154(+)	0.175(+)	0.104(-)	0.123
Correlated	# coins								
Poisson	5	0.092(−)	0.121(+)	0.112	0.126(−)	0.142(+)	0.135(+)	0.119(−)	0.154(+)
Log-normal	10	0.078(−)	0.111(+)	0.099(−)	0.124(−)	0.139(+)	0.137(+)	0.136(−)	0.176(+)
	15	0.066(−)	0.098	0.087(−)	0.121(−)	0.135(+)	0.142(+)	0.151	0.199(+)
	20	0.057(−)	0.088	0.077(−)	0.118(−)	0.132(+)	0.145(+)	0.164	0.219(+)
	25	0.050(−)	0.080	0.070(−)	0.114(−)	0.127(+)	0.145(+)	0.176	0.238(+)
						II			
		€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
Poisson		0.107(-)	0.103	0.101	0.139	0.157(+)	0.162(+)	0.120(-)	0.112
Uncorrelated Poisson									
Log-normal		0.120(-)	0.109	0.114	0.139	0.168(+)	0.167(+)	0.087(-)	0.096(-)
Correlated	# coins								
Poisson	5	0.123(−)	0.133(+)	0.128	0.123	0.157(+)	0.134	0.092(−)	0.110
Log-normal	10	0.107(−)	0.126(+)	0.117	0.125	0.158(+)	0.140	0.103(−)	0.123
	15	0.094(−)	0.117(+)	0.106	0.127	0.158(+)	0.147(+)	0.114(−)	0.136
	20	0.084(−)	0.108	0.097	0.126	0.158(+)	0.151(+)	0.125(−)	0.150(+)
	25	0.076(−)	0.102	0.090(−)	0.124	0.155(+)	0.153(+)	0.136	0.165(+)
						III			
		€2	€1	€0.50	€0.20	€0.10	€0.05	€0.02	€0.01
Poisson		0.095(-)	0.089	0.105	0.144	0.152(+)	0.195(+)	0.107(-)	0.114
Uncorrelated Poisson									
Log-normal		0.107(-)	0.095	0.120	0.149	0.161(+)	0.197(+)	0.075(-)	0.095(-)
Correlated	# coins								
Poisson	5	0.107(−)	0.115(+)	0.130	0.129	0.152(+)	0.165(+)	0.087(−)	0.116
Log-normal	10	0.090(−)	0.105	0.116	0.131	0.153(+)	0.176(+)	0.098(−)	0.131
	15	0.077(−)	0.095	0.103	0.131	0.153(+)	0.186(+)	0.109(−)	0.147
	20	0.068(−)	0.087	0.093	0.129	0.150(+)	0.190(+)	0.120(−)	0.163(+)
	25	0.060(−)	0.078	0.084	0.126	0.146(+)	0.194(+)	0.131(−)	0.181(+)

A (-) indicates that the fraction is significantly (95%) below the theoretical fraction. A (+) indicates that the fraction is significantly above the theoretical fraction.

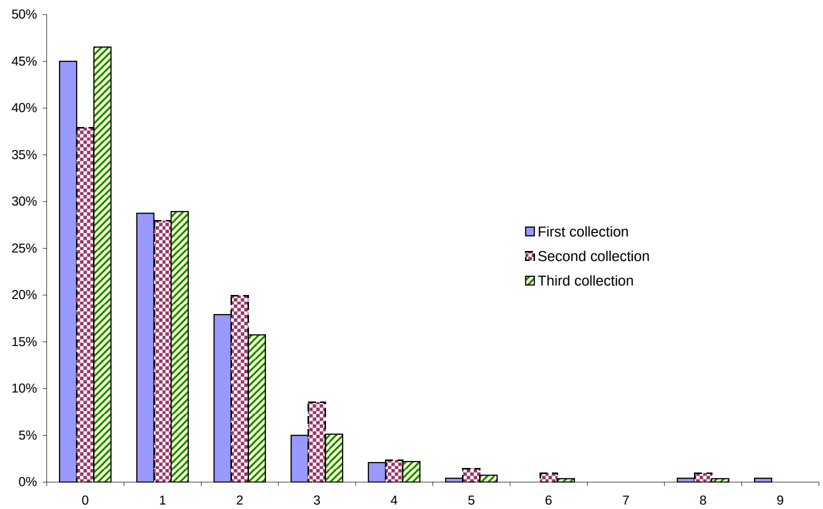

Figure 1: Frequency of €2 for each of the 3 collections

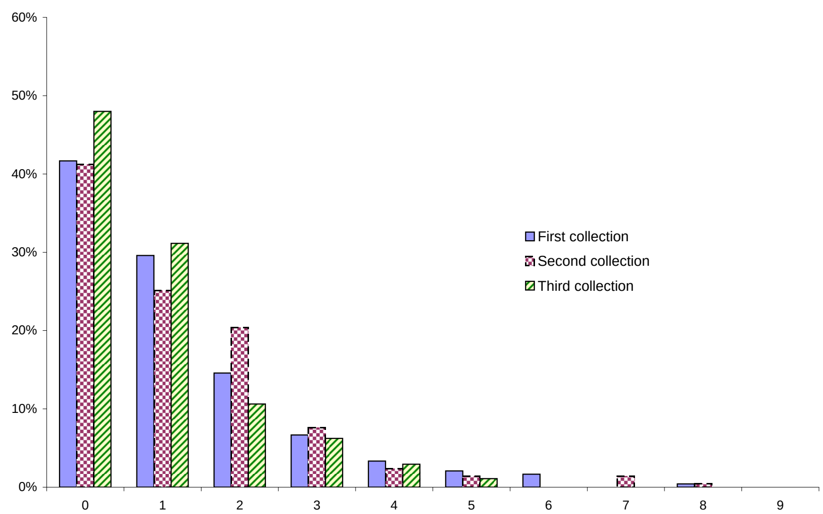

Figure 2: Frequency of €1 for each of the 3 collections

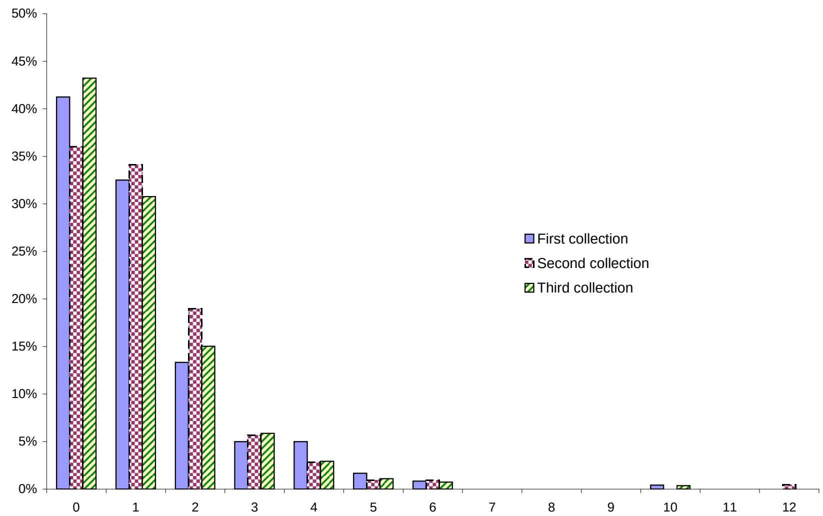

Figure 3: Frequency of €0.50 for each of the 3 collections

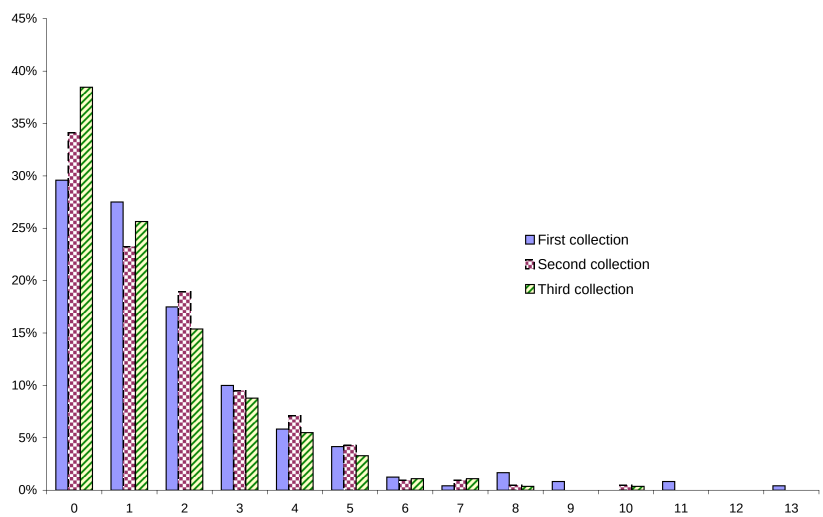

Figure 4: Frequency of €0.20 for each of the 3 collections

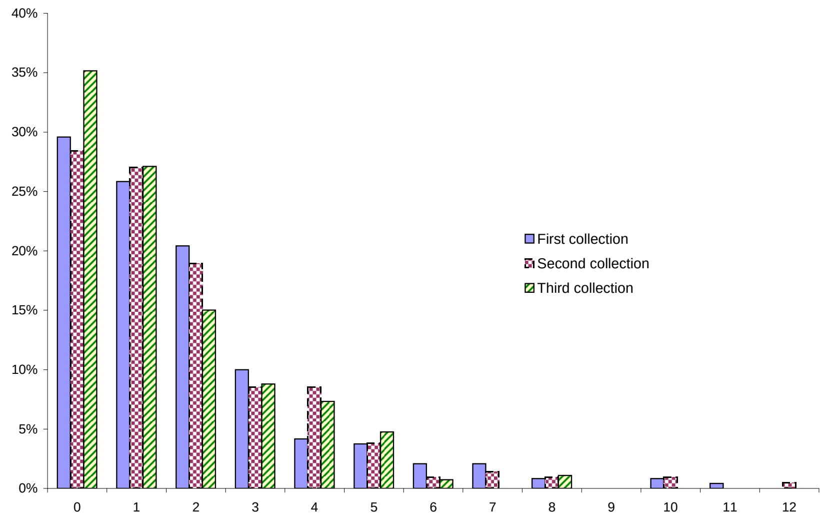

Figure 5: Frequency of €0.10 for each of the 3 collections

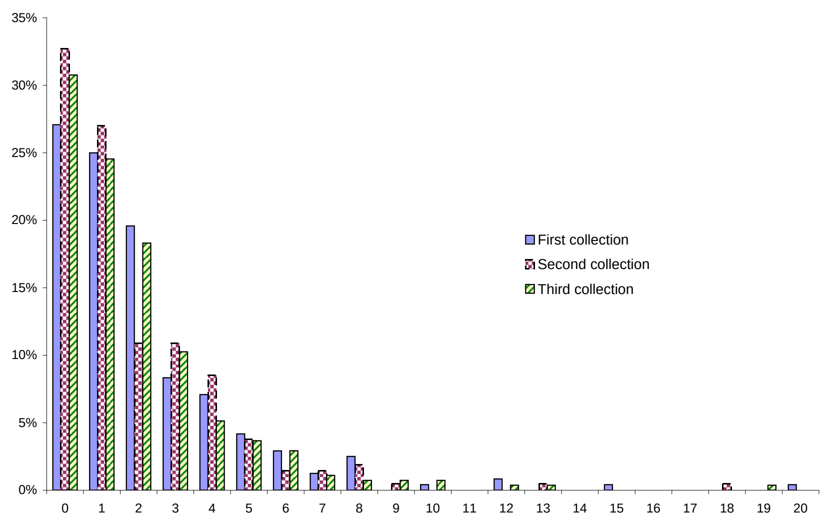

Figure 6: Frequency of €0.05 for each of the 3 collections

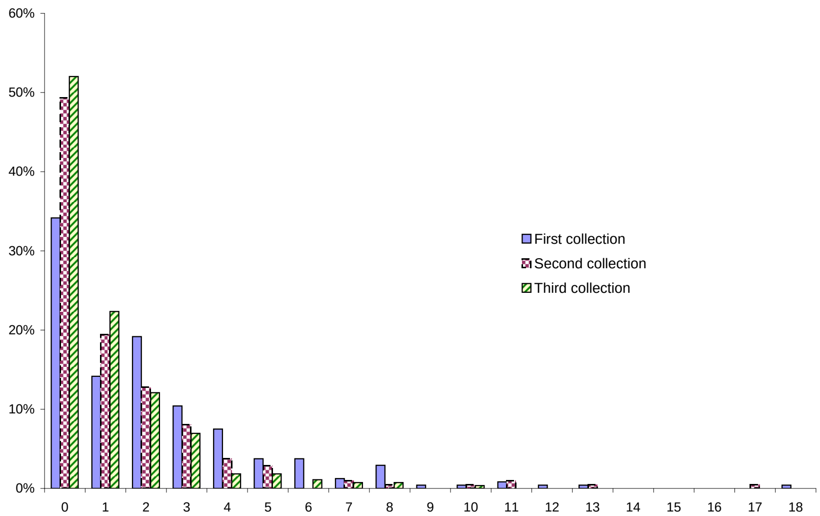

Figure 7: Frequency of €0.02 for each of the 3 collections

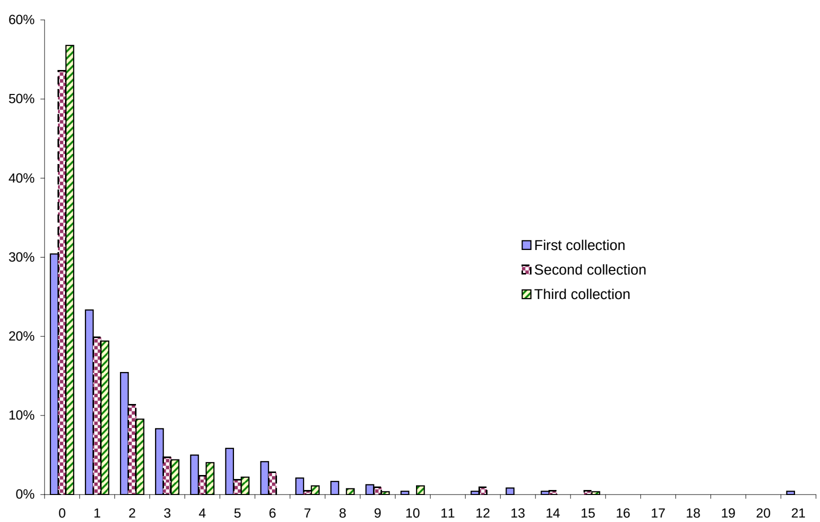

Figure 8: Frequency of €0.01 for each of the 3 collections