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OPTIMUM SAVINGS AND UTILITY MAXIMIZATION OVER TIME

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The problem dealt with in this article is whether we can indicate, with the help of measured economic concepts, the rate of savings—as a function of time—which maximizes utility over time. The author believes that his attempt has been unsuccessful, but hopes that the nature of the difficulties encountered may be of some help in future attempts to solve this problem—a problem regarding the most important decision to be taken for any development program.

1. IN THIS ESSAY I propose to report on an unsuccessful attempt to find a simple solution to the problem of optimum savings. By the problem of optimum savings I refer to the question of what constitutes the best savings programme over time, starting with a given income. This problem is one which a nation has to solve when making its development plans. Should the nation save 12% of its national income, or 25%, or some other figure? For long periods the large western countries have saved about 12%, averaging over business cycles. The communist countries save something like 25% or more. Is it possible, by an econometric device, to find the optimum rate?

2. Of course, the problem has to be simplified. Let us therefore assume that no physical distinction needs to be made between consumer goods and investment goods and speak only of the national product. Part of each year's product is transformed into capital equipment and then contributes to the national product of all following years. Let us assume that the law of production is simply that of a constant capital-output ratio, which we indicate by $\varkappa$. To illustrate our argument we shall choose $\varkappa = 3$ years, which is a realistic figure as long as we assume that it remains constant. At certain moments, however, we will drop this assumption. It is well then to remember that, for the Cobb-Douglas production function, the marginal capital-output ratio is about four times as high as the average capital-output ratio.

Initial income is considered given and indicated by y_0. This implies that, in the case of a constant capital-output ratio, the nation's capital stock equals $\varkappa y_0$.

3. Our attempt to rationalize the optimum savings problem is the following. It is assumed that the savings programme should be such as to maximize over time the utility drawn from consumption in each successive year. Utility $U(t)$ in any one year t is assumed to depend only on consumption

c_t during that year. We assume it to be measurable and that the expression we wish to maximize is:

$$\int_0^\infty U(t')dt' .$$

It can easily be shown that the same result is obtained even if we do not assume measurability of utility itself, but only the measurability of the marginal rate of substitution between consumption in any one year t_1 and in another year t_2. For simplicity of expression we shall assume measurability of utility.

In order to obtain numerical results we must specify the utility function more precisely. This specification will be expressed in terms of the marginal utility u_t for year t. We assume it to obey a two-parameter relationship

$$u_t = \frac{1}{(c_t - \bar{c})^{v}} \tag{3.1}$$

where $\bar{c}$ means the subsistence minimum[1] and $-v$ may be called the elasticity of marginal utility with respect to "surplus consumption," $c_t - \bar{c}$.

4. Rough numerical values for $\bar{c}$ and v can be derived from measurements Ragnar Frisch carried out as early as 1931.[2] Frisch gives two figures for what he calls the flexibility of marginal utility with respect to consumption,

$$\chi = \frac{\partial u}{\partial c} \frac{c}{u} \tag{4.1}$$

which, on our specification, equals $-vc/(c-\bar{c})$. The estimate for American workers is -1, and that for French workers is -3.5. Assuming that for both countries the subsistence consumption level, $\bar{c}$, and v are the same and assuming, in addition, that the absolute level of consumption in the United States was, at the time of measuring, double that in France, we have:

$$\frac{2vc_F}{2c_F - \bar{c}} = 1 , \quad \frac{vc_F}{c_F - \bar{c}} = 3.5 . \tag{4.2}$$

This yields:

$$\frac{\bar{c}}{c_F} = \frac{5}{6} , \quad v = 0.6 .$$

[1] The suggestion to introduce this variable was made to me by professor P. Masani of Bombay University.

[2] Ragnar Frisch, *New Methods of Measuring Marginal Utility,* Tübingen, 1931. These findings have been confirmed by those published later in R. Frisch, *Confluence Analysis,* Oslo, 1934, and subsequent new attempts made by L. Johansen in "Et sett av etterspørsel koeffisienter for en multi-sektor-analyse med variable priser" (mimeographed), Sosialøkon. Inst., Oslo, 24 March 1958.

It is of course possible to make alternative assumptions with regard to the subsistence levels in the two countries; but the American result will always require that $v < 1$, which is a strategic point in our further analysis.

5. We are now able to give a precise form to the simplest version of our problem. It can be expressed by the use of only two variables, income y and consumption c, both varying with time. At any point of time, consumption should be such that its marginal utility equals the marginal utility of savings; the former representing the sacrifice to be made in order to save an additional unit and the latter the advantages to be reaped. These advantages evidently are represented by the satisfaction, i.e., the utility to be obtained from the additional future product to which the savings will give rise. This additional product will be available for all future years at a constant quantity of $1/\varkappa$ for each unit saved now. Hence at any given time:

$$u_t = \frac{1}{\varkappa} \int_t^{\infty} u_{t'} dt' \,. \tag{5.1}$$

With the frontier between consumption and savings determined in this way the maximum of satisfaction over time will be obtained.

For our further analysis it is more useful to replace this equation by another, by first differentiating both sides:

$$\frac{du_t}{dt} = -\frac{1}{\varkappa} u_t \tag{5.2}$$

and then substituting the expression (3.1) for u_t. This leads to:

$$\frac{\dot{c}}{c - \bar{c}} = \frac{1}{\varkappa v} \,. \tag{5.3}$$

Furthermore, our variables must, at any time, satisfy the balance equation telling us that savings and consumption add up to income. Since savings are equal to the rate of increase of capital, which itself equals $\varkappa y$, the balance equation reads:

$$c + \varkappa \dot{y} = y \tag{5.4}$$

6. The solution of this system of two equations may be undertaken by first integrating (5.3), which leads to

$$c - \bar{c} = C e^{t/\varkappa v} \quad (C \text{ arbitrary}). \tag{6.1}$$

Substituting (6.1) into (5.4) yields:

$$y - \varkappa \dot{y} = C e^{t/\varkappa v} + \bar{c} \,. \tag{6.2}$$

This being a nonhomogeneous differential equation, its solution consists of the solution of the corresponding homogeneous equation, plus a particular solution. The solution of the homogeneous equation runs:

$$y = Y_1 e^{t/\varkappa}. \tag{6.3}$$

A particular solution of equation (6.2) may be attempted by putting

$$y = Y_2 e^{t/\varkappa v} + \bar{c} \tag{6.4}$$

which imposes on Y_2 the following condition:

$$Y_2\left(1 - \frac{1}{v}\right) = C. \tag{6.5}$$

The complete solution of the system now runs:

$$y = Y_1 e^{t/\varkappa} + Y_2 e^{t/\varkappa v} + \bar{c}, \tag{6.6}$$

$$c = \left(1 - \frac{1}{v}\right) Y_2 e^{t/\varkappa v} + \bar{c}. \tag{6.7}$$

The appearance of two arbitrary constants seems to be consistent with the fact that a second-order equation in y would have been obtained if the system of equations (5.3) and (5.4) had been written in only one variable, by solving c from (5.4) and substituting it into (5.3). Economically, however, only one initial position must be given, namely y_0, which represents only one condition for both Y_1 and Y_2:

$$y_0 = Y_1 + Y_2 + \bar{c}. \tag{6.8}$$

We are then left with a one-dimensional family of "solutions"; but for $v < 1$ it appears on closer inspection that only one of these is economically possible. This is due to two boundary conditions, namely $y \geq 0$, $c \geq \bar{c}$, which, it appears, must be added. Since for $v < 1$ the second term in (6.6) grows more rapidly than the first, we must have, in order that $y \geq 0$, $Y_2 \geq 0$. At the same time, in order that $c \geq \bar{c}$, we must have $Y_2 \leq 0$, and hence the only possibility is $Y_2 = 0$. The corresponding solution is:

$$y = (y_0 - \bar{c})e^{t/\varkappa} + \bar{c}; \quad c = \bar{c}. \tag{6.9}$$

7. Before discussing its implications, we may, for the sake of curiosity, mention the solution for $v > 1$, which looks much more "normal." In this case, since the first term in (6.6) now grows more rapidly than the second, in order that $y \geq 0$ we must have:

$$Y_1 \geq 0. \tag{7.1}$$

In order that $c \geq \bar{c}$ we must have

$$Y_2 \geq 0. \tag{7.2}$$

These two conditions leave us with a one-dimensional family of solutions, but among these there is one which clearly gives the maximum utility over time, namely the one with the maximum admissible value of Y_2. Since this clearly is the one for which $Y_1 = 0$, we now have:

$$y = (y_0 - \bar{c})e^{t/\varkappa v} + \bar{c}\,, \tag{7.3}$$

$$c = \left(1 - \frac{1}{v}\right)(y_0 - \bar{c})e^{t/\varkappa v} + \bar{c}\,. \tag{7.4}$$

8. The solution (6.9) evidently means that consumption must be kept at the subsistence level forever and that consequently savings will be equal to income minus subsistence consumption. Income grows very rapidly as a consequence of the high level of savings.

This seems so far from a maximum of utility over time that we must ask whether an apparent solution has crept into our mathematics. In order to answer this question it may be wise to introduce a finite horizon T, to consider a case with discrete time units, and even to start with a further simplification. Since the strategic factor seems to be the low level of v, to begin with, let us assume it to be zero, i.e., let us assume that marginal utility is constant for all levels of consumption, as soon as $c > \bar{c}$. Consumption cannot fall below $\bar{c}$, however; at this level its marginal utility suddenly jumps to ∞.

The problem then reduces to maximizing consumption itself, since utility is a linear function of consumption. In order to simplify our mathematics to the utmost, we change the time unit and make it equal to $\varkappa$ years, meaning that the capital-output ratio equals one, and income at any time equals capital k_t, the amount at the beginning of the unit time period t. We assume capital at the beginning of period 1, or k_1, to be given again. Now, at any time the decision regarding consuming or saving places before the individual (or the nation) the choice of either consuming a unit now or having one unit more to consume for all future time. The latter alternative will always add more to consumption than the former. Since there is the horizon T, however, consumption after period T does not interest the individual. Up to period $T - 2$ his choice will be to save everything he can, i.e., to live at the subsistence level. In period $T - 1$ he is indifferent whether to save or to consume, since for every unit not consumed then he will have one unit to consume in the next time unit T. In period T he will consume his whole income (and maybe even his capital, but this is not relevant to our argument). In order to find the total quantity of consumption available, let us first assume that the individual chooses to save also in time period $T - 1$. The following table will illustrate what happens:

TABLE I

Period	Capital = Production	Consumption	Savings
1	k_1	$\bar{c}$	$v = k_1 - \bar{c}$
2	$k_1 + v$	$\bar{c}$	$k_1 + v - \bar{c} = 2v$
3	$k_1 + v + 2v$	$\bar{c}$	$k_1 + v + 2v - \bar{c} = 4v$
4	$k_1 + v + 2v + 4v$	$\bar{c}$	$8v$, etc.

Evidently, $k_t = k_1 + v + 2v + 4v + \ldots 2^{t-1}v = k_1 + (2^t - 1)v$.
Hence,

(8.1) $$k_{T-1} = k_1 + (2^{T-1} - 1)v \quad \text{and} \quad k_T = k_1 + (2^T - 1)v .$$

Total consumption amounts to

(8.2) $$\sum_1^T c_t = (T - 1)\,\bar{c} + y_T = (T - 1)\,\bar{c} + k_1 + (2^T - 1)v = T\bar{c} + 2^T v .$$

Assuming now that in period $T - 1$ the choice is made in favour of consum tion, we find:

Consumption in period $T - 1 = c_{T-1} = y_{T-1} = k_{T-1} = k_1 + (2^{T-1} - 1)$

During period T the same consumption is now possible as during T — leading to $\Sigma_1^T\, c_t = (T - 2)\bar{c} + 2k_1 + (2^{T-1} - 1)2v = T\bar{c} + 2^T v$. . it should be, this is identical with (8.2).

From these results it is also clear that this consumption is larger than t minimum conceivable consumption which would simply be equal to 1 In this case of a finite horizon, therefore, the pattern of maximum a minimum satisfaction from consumption do not coincide, except for t first $T - 2$ time units. Such coincidence over a finite period, however, w extend over an ever increasing period if we let T grow beyond limit.

9. There is a second way in which we may elucidate the nature of o problem. Let us return to our first, continuous model, but introduce finite horizon T, as in the second model. This evidently means that in equ tion (5.1) the upper limit of the integral becomes T instead of ∞; intere ingly enough, however, this does not change the subsequent relatio (5.2), (5.3), or our solutions (6.6) and (6.7). It does introduce, howev one element which can now be treated in a more precise way, name a second boundary condition. With a finite horizon we can be sure th it makes no sense to plan any savings for the last time unit contained the period $0 \leq t \leq T$. We may therefore add the boundary condition:

(9.1) $$y_T = c_T$$

leading to

$$Y_1 e^{T/\varkappa} + Y_2 e^{T/\varkappa v} = \left(1 - \frac{1}{v}\right) Y_2 e^{T/\varkappa v}$$

or

$$\frac{Y_2}{Y_1} = -v \frac{e^{T/\varkappa}}{e^{T\varkappa/v}} = -ve^{-T(1/v-1)/\varkappa} . \tag{9.2}$$

Accordingly, the solution becomes:

$$\begin{aligned} y &= Y_1(e^{T/\varkappa} - ve^{-T(1/v-1)/\varkappa} e^{t/\varkappa v}) + \\ c &= Y_1 (1 - v) e^{-T(1/v-1)/\varkappa} e^{t/\varkappa v} + \bar{c} . \end{aligned} \tag{9.3}$$

Clearly, Y_2 need not be zero now, but can and even must be negative; up to $t = T$ this will not violate the boundary condition previously introduced ($y \geq 0$), and what happens after $t = T$ does not interest the planning individual or nation. In this case c will not be pressed down to $\bar{c}$; yet the difference will become very small for a very large T. For T growing beyond any limit and for t finite we shall again find $c = \bar{c}$; but not for $t = T$. In fact, for $t = T$ we get

$$c_T = Y_1(1 - v)e^{T/\varkappa} + \bar{c} ,$$

and hence

$$\lim_{T=\infty} c_T = \infty .$$

10. Our two examples show that the solution (6.9) is correct for finite values of t and suggest the following "explanation": for the low values of v assumed here ($v \leq 1$) and found by Frisch, it looks promising, at any time, to save "as much as possible"—i.e., everything beyond the subsistence level of consumption, because a higher contribution to "satisfaction" (utility) will be obtained "later." But this "later" is continually postponed and in fact never occurs.

It is interesting to note that our solution, for $v \leq 1$, shows one other characteristic, namely, that of being independent of the value of $\varkappa$. This implies that a change in $\varkappa$ during the process of choice between saving and consuming does not affect the result of the choice; i.e., that the same result would also be obtained when $\varkappa$ is not constant, but dependent on the level of savings.[3]

11. Before drawing more general conclusions from our analysis, we introduce one additional feature. So far we have assumed that any increase

[3] Cf. Branko Horvat, "The Optimum Rate of Saving: a Note," *Economic Journal*, LXVIII (1958), p. 157; A. K. Sen, "A Note on Tinbergen on the Optimum Rate of Saving," *Economic Journal*, LXVII (1957), p. 745.

in consumption, whatever the level already attained, leads to an increase in utility. Conceivably this might be the cause of our strange results. We therefore assume now that there is a limit to increasing one's utility and that, therefore, there exists a saturation level of consumption, to be written as $c_m + \bar{c}$; c_m representing the difference between the saturation and the subsistence levels. The assumption may be specified by the hypothesis that:

$$u = \left(\frac{c_m}{c - \bar{c}} - 1\right)^v . \tag{11.1}$$

It can be shown that this changes the consumption path over time (6.1) into

$$c - \bar{c} = \frac{c_m}{1 + Ce^{-t/\varkappa v}} \quad (C \text{ arbitrary}) , \tag{11.2}$$

which is a logistic curve with an asymptote $c = \bar{c} + c_m$. The corresponding solution for y can only be found explicitly for $1/v$ an integer. As one could expect, it shows an asymptote $y = c + \bar{c}_m$. The details will be discussed elsewhere.

From a wider sociological point of view this solution looks more satisfactory than the previous ones, with their unlimited rise in consumption and production. In the present approach an end is seen to the "economic phase" in the life of mankind; an end reached when the saturation level has been attained. This must also have been on the minds of some Keynesian authors when they introduced the concept of "bliss" into their theories of savings.[4] Yet for realistic values for c_m—i.e., several times as large as $\bar{c}$—the values found here for the optimum level of savings are, like our previous results, very high and tend to postpone consumption, if not to infinity, to a remote future.

12. The conclusion to be drawn from the preceding sections is that no clue for the establishment of an "optimum savings programme" can be derived from the simple maximization over time of instantaneous utility. The answers thus obtained do not enable us to choose between the practice followed in communist and noncommunist countries; in a way they are "plus communistes que les communistes." Evidently, some essential element has been missed in the theories tried. There seem to be two ways of specifying this element. One, following the conventional approach, is to introduce a time discount factor. This method has been avoided on purpose, since, in the opinion of many economists, such a time discount should not govern

4 F. P. Ramsay, "A Mathematical Theory of Saving," *Economic Journal*, XXXVIII (1928), p. 543.

a nation's decisions—in contrast with an individual's decisions, where it should. Even if it were accepted as an element in a nation's planning, we must answer the difficult question of what its numerical value should be.

The other way of specifying the missing element in our theory is to refer to the "distribution over generations" or to "the need for continuity". In fact, more than one element is involved. With reference to generations, we note the fact that in each consecutive time unit a different group of people or generation is living and that postponing consumption also means allocating it to another generation. The missing element is some sort of balance between the consumption levels of consecutive generations. If the desired balance is complete equity, no saving should be undertaken as long as the size of the population remains unchanged. If, in a practical programme, a compromise between maximum utility and equity were sought, the optimum would depend on the relative weights given to both principles.

With respect to "continuity" we have reference to something which applies not only to the relation between generations but also during an individual's life. An individual seeks not only a maximum over his prospective lifetime, but also some degree of continuity in his consumption level. The precise meaning of this is that large differences in consumption between various time units adversely affect total utility. This amounts to saying that the feeling of utility at any moment depends not only on the consumption of that moment but also on consumption in other time periods and that our formula (3.1) should be generalized accordingly. It is characteristic of the underdeveloped state of our science that virtually nothing empirical is known about this generalization and that, as a consequence, nothing sensible can be said about the optimum level of savings at present.

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