Uncertainty

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1. History and Introduction

In most economic decisions where agents face uncertainties, no probabilities are available. This point was first emphasized by Keynes (1921) and Knight (1921), and was reiterated by Greenspan (2004):

> … how … the economy might respond to a monetary policy initiative may need to be drawn from evidence about past behavior during a period only roughly comparable to the current situation. … In pursuing a risk-management approach to policy, we must confront the fact that only a limited number of risks can be quantified with any confidence." (p. 38)

Indeed, we often have no clear statistics available. Knight went so far as to call probabilities unmeasurable in such cases. Soon after Knight's suggestion, Ramsey (1931), de Finetti (1931), and Savage (1954) showed that probabilities can be defined in the absence of statistics after all, by relating them to observable choice.

For example, P(E) = 0.5 can be derived from an observed indifference between receiving a prize under event E and receiving it under not-E (the complement to E). Although widely understood today, the idea that something as intangible as a subjective degree of belief can be made observable through choice behavior, and can even be quantified precisely, was a major intellectual advancement.

Ramsey, de Finetti, and Savage assumed that the agent, after having determined the probabilities subjectively (as required by some imposed rationality axioms), proceeds as under expected utility for given objective probabilities. The Allais (1953) paradox (explained later) revealed a descriptive difficulty: for known probabilities, people often do not satisfy expected utility. Hence, we need to generalize expected utility. Another, more fundamental difficulty was revealed by the Ellsberg (1961) paradox (also explained later): for unknown probabilities, people behave in ways that cannot be reconciled with any assignment of subjective probabilities at all, so that further generalizations are needed.[1] Despite the importance and prevalence of unknown probabilities, understood since 1921, and the impossibility of modeling these through subjective probabilities, understood since Ellsberg (1961), decision theorists continued to confine their attention to decision under risk with given probabilities until the late 1980s. Wald's (1950) multiple priors model did account for unknown probabilities, but received little attention.

As a result of an idea of David Schmeidler (1989, first version 1982), the situation changed in the 1980s. Schmeidler introduced the first theoretically sound decision model for unknown probabilities without subjective probabilities, called rank-dependent utility or Choquet expected utility. At the same time, Wald's multiple priors model was revived when Gilboa & Schmeidler (1989) established its theoretical soundness. These discoveries provided the basis for nonexpected utility with unknown probabilities that had been sorely missing since 1921. Since the late 1980s, the table has turned in decision theory. Nowadays, most studies concern unknown probabilities. Gilboa (2004) contains recent papers and applications. This chapter will concentrate on conceptual issues of individual decisions in the absence of known probabilities.

Theoretical studies of nonexpected utility have usually assumed risk aversion for known probabilities (leading to concave utility and convex probability weighting), and ambiguity aversion for unknown probabilities (Camerer & Weber 1992, §2.3). These phenomena best fit with the existence of equilibria and can be handled using conventional tools of convex analysis (Mukerji & Tallon 2001). Empirically, however, a more complex fourfold pattern has been found. For gains with moderate and high likelihoods, and for losses with low likelihoods, risk aversion is prevalent indeed, but for gains with low likelihoods and losses with high likelihoods the opposite, risk seeking, is prevalent.

The fourfold pattern resolves the classical paradox of the coexistence of gambling and insurance, and leads for instance to new views on insurance. Whereas all classical studies of insurance explain the purchase of insurance through concave utility, empirical measurements of utility have suggested that utility is not very concave for losses, often exhibiting more convexity than concavity (surveyed by Köbberling, Schwieren, & Wakker 2006). This finding is diametrically opposite to what has been assumed throughout the insurance literature. According to modern decision theories, insurance is primarily driven by consumers' overweighting of small probabilities, rather than by marginal utility.

[1] The term "subjective probability" always refers to additive probabilities in this paper.

The fourfold pattern found for risk has similarly been found for unknown probabilities, and usually to a more pronounced degree. Central questions in uncertainty today concern how to analyze not only classical marginal utility but also new concepts such as probabilistic risk attitudes (how people process known probabilities), loss aversion and reference dependence (the framing of outcomes as gains and losses) and, further, states of belief and decision attitudes regarding unknown probabilities ("ambiguity attitudes").

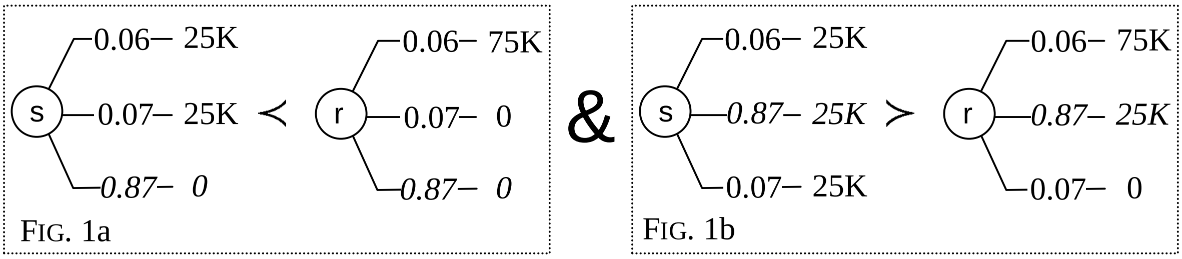
We end this introduction with some notation and definitions. Decision under uncertainty concerns choices between *prospects* such as $(E_1{:}x_1,\ldots,E_n{:}x_n)$, yielding *outcome* x_j if *event* E_j obtains, j = 1,…,n. Outcomes are monetary. The E_js are events of which an agent does not know for sure whether they will obtain, such as who of n candidates will win an election. The E_js are mutually exclusive and exhaustive. Because the agent is uncertain about which event obtains, he is uncertain about which outcome will result from the prospect, and has to make decisions under uncertainty.

2. Decision under Risk and Non-Expected Utility through Rank Dependence

Because risk is a special and simple subcase of uncertainty (as explained later), this chapter on uncertainty begins with a discussion of *decision under risk*, where the probability $p_j = P(E_j)$ is given for each event E_j. We can then write a prospect as $(p_1{:}x_1,\ldots,p_n{:}x_n)$, yielding x_j with probability p_j, j=1,…,n. Empirical violations of expected-value maximization because of risk aversion (prospects being less preferred than their expected value) led Bernoulli (1738) to propose expected utility, $\sum_{j=1}^{n} p_j U(x_j)$, to evaluate prospects, where U is the utility function. Then risk aversion is equivalent to concavity of U.

Several authors have argued that it is intuitively unsatisfactory that risk attitude be modeled through the utility of money (Lopes 1987, p. 283). It would be more satisfactory if risk attitude were also to be related to the way people feel about probabilities. Economists often react very negatively to such arguments, based as they are on introspection and having no clear link to revealed preference. Arguments against expected utility that are based on revealed preference were put forward by Allais (1953).

FIGURE 1. A version of the Allais paradox for risk



Circles indicate "chance nodes."

Figure 1 displays preferences commonly found, with K denoting $1000:

(0.06:25K, 0.07:25K, *0.87:0*) $\prec$ (0.06:75K, 0.07:0, *0.87:0*) and

(0.06:25K, *0.87:25K*, 0.07:25K) $\succ$ (0.06:75K, *0.87:25K*, 0.07:0) .

Preference symbols $\succcurlyeq$,$\succ$,$\prec$, and $\preccurlyeq$ are as usual. We denote the outcomes in a rank-ordered manner, from best to worst. In Fig. 1a, people usually prefer the "risky" (r) prospect because the high payment of 75K is attractive. In Fig. 1b, people usually prefer the "safe" (s) certainty of 25K for sure. These preferences violate expected utility because, after dropping the common (italicized) term 0.87U(0) from the expected-utility inequality for Fig. 1a and dropping the common term 0.87U(25K) from the expected-utility inequality for Fig. 1b, the two inequalities become the same. Hence, under expected utility either both preferences should be for the safe prospect or both preferences should be for the risky one, and they cannot switch as in Figure 1. The special preference for safety in the second choice (*the certainty effect*) cannot be captured in terms of utility. Hence, alternative, nonexpected utility models have been developed.

Based on the valuable intuition that risk attitude should have something to do with how people feel about probabilities, Quiggin (1982) introduced rank-dependent utility theory. The same theory was discovered independently for the broader and more subtle context of unknown probabilities by Schmeidler (1989, first version 1982), a contribution that will be discussed later. A *probability weighting function* $w : [0,1] \to [0,1]$ satisfies $w(0) = 0$, $w(1) = 1$, and is strictly increasing and continuous. It reflects the sensitivity of people towards probability. Assume that the outcomes of a prospect $(p_1{:}x_1,\ldots,p_n{:}x_n)$ are rank-ordered, $x_1 \geq \cdots \geq x_n$. Then its *rank-dependent utility* (*RDU*) is $\sum_{j=1}^{n}\pi_j U(x_j)$, where *utility* U is as before, and π_j, the *decision weight* of outcome x_j, is $w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1})$ (which is $w(p_1)$ for j=1).

Tversky & Kahneman (1992) adapted their widely used original prospect theory (Kahneman & Tversky 1979) by incorporating the rank dependence of Quiggin and Schmeidler. Prospect theory generalizes rank dependence by allowing different treatments of gains than of losses, which is desirable for empirical purposes. In this chapter on uncertainty, I will focus on gains, in which case prospect theory in its modern version, sometimes called cumulative prospect theory, coincides with RDU.

With rank dependence, we can capture psychological (mis)perceptions of unfavorable outcomes being more likely to arise, in agreement with Lopes' (1987) intuition. We can also capture decision attitudes of deliberately paying more attention to bad outcomes. An extreme example of the latter pessimism concerns worst-case analysis, where all weight is given to the most unfavorable outcome. Rank dependence can explain the Allais paradox because the weight of the 0.07 branch in Fig. 1b may exceed that in Fig. 1a:

$$w(1) - w(0.93) \;\geq\; w(0.13) - w(0.06). \qquad (2.1)$$

This inequality holds for w-functions that are steeper near 1 than in the middle region, a shape that is empirically prevailing indeed.

The following figures depict some probability weighting functions. Fig. 2a concerns expected utility, and Fig. 2b a *convex* w, which means that

$$w(p+r) - w(r) \qquad (2.2)$$

is increasing in r for all $p \geq 0$. This is equivalent to w' being increasing, or w'' being positive. Eq. 2.1 illustrates this property. Eq. 2.2 gives the decision weight of an outcome occurring with probability p if there is an r probability of better outcomes. Under convexity, outcomes receive more weight as they are ranked worse (i.e., r larger), reflecting *pessimism*. It implies low evaluations of prospects relative to sure outcomes, enhancing risk aversion.

Empirical studies have found that usually $w(p) > p$ for small p, contrary to what convexity would imply, and that $w(p) < p$ only for moderate and high probabilities p (inverse-S; Abdellaoui 2000; Bleichrodt & Pinto 2000; Gonzalez & Wu 1999; Tversky & Kahneman 1992), as in Figs. 3a and 3b. It leads to extremity-oriented behavior with both best and worst outcomes overweighted. The curves in Figs. 3a and 3b also satisfy Eq. 2.1 and also accommodate the Allais paradox. They predict risk seeking for prospects that with a small probability generate a high gain, such as in public lotteries. The inverse-S shape suggests a cognitive insensitivity to probability, generating insufficient response to intermediate variations of probability and then, as a consequence, overreactions to changes from impossible to possible and from possible to certain. These phenomena arise prior to any "motivational" (value-based) preference or dispreference for risk. Extreme cases of such behavior are in Figure 4 (where we relaxed the continuity requirement for w).

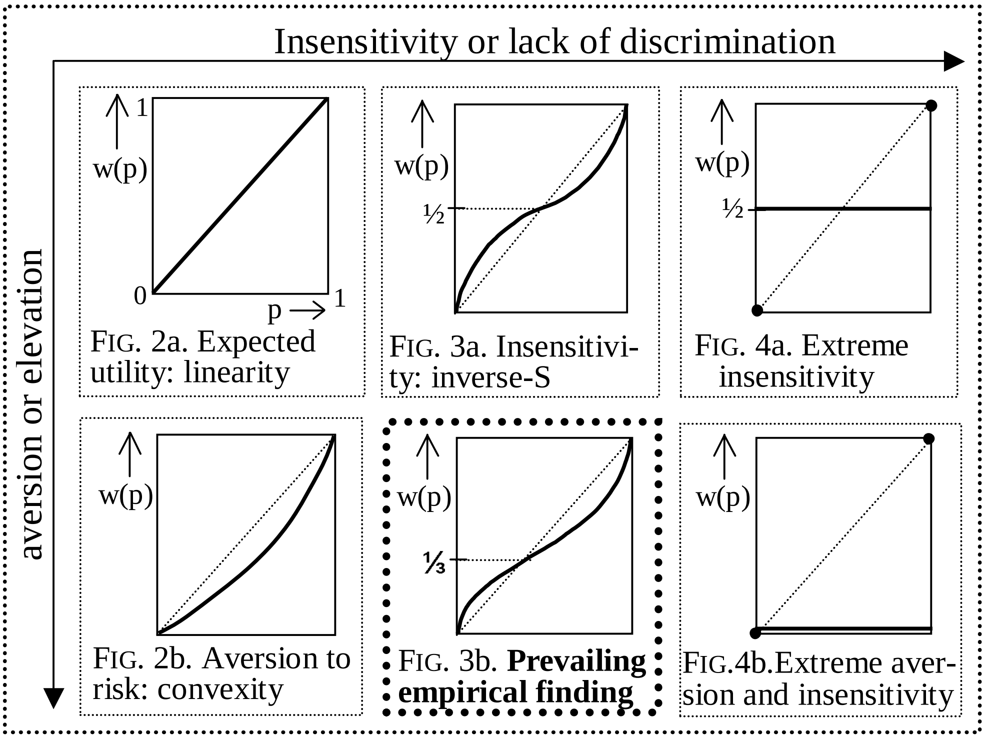


FIG. 2a. Expected utility: linearity

FIG. 3a. Insensitivity: inverse-S

FIG. 4a. Extreme insensitivity

FIG. 2b. Aversion to risk: convexity

FIG. 3b. **Prevailing empirical finding**

FIG.4b.Extreme aversion and insensitivity

Starmer (2000) surveyed nonexpected utility for risk. The main alternatives to the rank-dependent models are the betweenness models (Chew 1983; Dekel 1986), with Gul's (1991) disappointment aversion theory as an appealing special case. Betweenness models are less popular today than the rank-dependent models. An important reason, besides their worse empirical performance (Starmer 2000), is that models alternative to rank-dependence did not provide concepts as intuitive as the sensitivity to probability/information modeled through RDU's probability weighting w. For example, consider a popular special case of betweenness, called weighted utility. The value of a prospect is

$$\frac{\sum_{i=1}^{n} p_i f(x_i) U(x_i)}{\sum_{j=1}^{n} p_j f(x_j)} \tag{2.3}$$

for a function $f : \mathbb{R} \rightarrow \mathbb{R}^+$. This new parameter f can, similar to rank dependence, capture pessimistic attitudes of overweighting bad outcomes by assigning high values to bad outcomes. It, however, applies to outcomes and not to probabilities. Therefore, it captures less extra variance of the data in the presence of utility than w, because utility also applies to outcomes. For example, for fixed outcomes, Eq. 2.3 cannot capture the varying sensitivity to small, intermediate, and high probabilities found empirically. Both pessimism and marginal utility are entirely specified by the range of outcomes considered without regard to the probabilities involved. It seems more interesting if new concepts, besides marginal utility, concern the probabilities and the state of information of the decision maker rather than outcomes and their valuation. This may explain the success of rank-dependent theories and prospect theory.

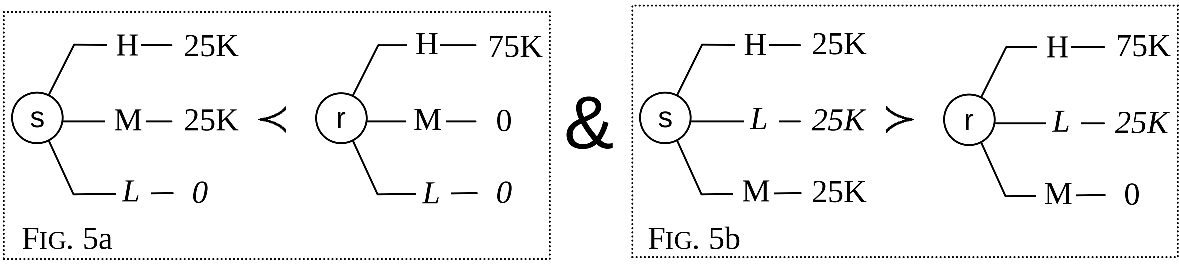
3. Phenomena under Uncertainty that Naturally Extend Phenomena under Risk

The first approach to deal with uncertainty was the Bayesian approach, based on de Finetti (1931), Ramsey (1931), and Savage (1954). It assumes that people assign, as well as possible, subjective probabilities $P(E_j)$ to uncertain events E_j. They then evaluate prospects $(E_1{:}x_1,\ldots,E_n{:}x_n)$ through their (subjective) expected utility $\sum_{j=1}^{n} P(E_j)U(x_j)$. This model was the basis of Bayesian statistics and of much of the economics of uncertainty (Greenspan 2004). The empirical measurement of subjective probabilities has been studied extensively (Fishburn 1986; Manski 2004; McClelland & Bolger 1994). We will confine our attention in what follows to models that were introduced in the last two decades, models that deviate from Bayesianism. To Bayesians (including this author) such models are of interest for descriptive purposes.

Machina & Schmeidler (1992) characterized *probabilistic sophistication*, where a decision maker assigns subjective probabilities to events with unknown probabilities and then proceeds as for known probabilities. The decision maker may, however, deviate from expected utility for known probabilities, contrary to the Bayesian approach, and Allais-type behavior can be accommodated.

The difference between objective, exogenous probabilities and subjective, endogenous probabilities is important. The former are stable, and readily available for analyses, empirical tests, and communication in group decisions. The latter can be volatile and can change at any time by mere further thinking of the agent. For descriptive studies, subjective probabilities may become observable only after complex measurement procedures. Hence, I prefer not to lump objective and subjective probabilities together into one category, as has been done in several economic works (Ellsberg 1961 p. 645; Epstein 1999). In this chapter, the term risk refers exclusively to exogenous objective probabilities. Such probabilities can be considered a limiting case of subjective probabilities, just as decision under risk can be considered a limiting case of decision under uncertainty (Greenspan 2004, pp. 36-37). Under a differentiability assumption for state spaces, Machina (2004) formalized this interpretation. Risk, while not occurring very frequently, is especially suited for applications of decision theory.

FIGURE 5. The certainty effect (Allais paradox) for uncertainty



The Allais paradox is as relevant to uncertainty as it is to risk (MacCrimmon & Larsson 1979, p. 364-365; Wu & Gonzalez 1999). Figure 5 presents a demonstration by Tversky & Kahneman (1992, Section 1.3). The analogy with Figure 1 should be apparent. The authors conducted the following within-subjects experiment. Let d denote the difference between the closing value of the Dow Jones on the day of the experiment and on the day after, where we consider the events H(igh): $d > 35$, M(iddle): $35 \geq d \geq 30$, L(ow): $30 > d$. The total Dow Jones value at the time of the experiment was about 3000. The right prospect in Fig. 5b is (H:75K, L:25K, M:0), and the other prospects are denoted similarly. Of 156 money managers during a workshop, 77% preferred the risky prospect r in Fig. 5a, but 68% preferred the safe prospect s in Fig. 5b. The majority preferences violate expected utility, just as they do under risk: After dropping the common terms P(L)U(0) and P(L)U(25K) (P denotes subjective probabilities), the same expected-utility inequality results for Fig. 5a as for Fig. 5b. Hence, either both preferences should be for the safe prospect, or both preferences should be for the risky one, and they cannot switch as in Figure 5. This reasoning holds irrespective of what the subjective probabilities P(H), P(M), and P(L) are.

Schmeidler's (1989) *rank-dependent utility* (*RDU*) can accommodate the Allais paradox for uncertainty. We consider a *weighting function* (or nonadditive probability or capacity) W that assigns value 0 to the vacuous (empty) event $\varnothing$, value 1 to the universal event, and satisfies monotonicity with respect to set-inclusion (if $A \supset B$ then $W(A) \geq W(B)$). Probabilities are special cases of weighting functions that satisfy *additivity*: $W(A \cup B) = W(A) + W(B)$ for disjoint events A and B. General weighting functions need not satisfy additivity. Assume that the outcomes of a prospect $(E_1{:}x_1,\ldots,E_n{:}x_n)$ are rank-ordered, $x_1 \geq \cdots \geq x_n$. Then the prospect's *rank-dependent utility* (*RDU*) is $\sum_{j=1}^{n} \pi_j U(x_j)$ where *utility* U is as before, and π_j, the *decision weight* of outcome x_j, is $W(E_1 \cup \cdots \cup E_j) - W(E_1 \cup \cdots \cup E_{j-1})$ ($\pi_1 = W(E_1)$). The decision weight of x_j is the marginal W contribution of E_j to the event of receiving a better outcome.

Quiggin's RDU for risk is the special case with probabilities $p_j = P(E_j)$ given for all events, and $W(E_j) = w(P(E_j))$ with w the probability weighting function. Tversky & Kahneman (1992) improved their 1979-prospect theory not only by avoiding violations of stochastic dominance, but also, and more importantly, by extending their theory from risk to uncertainty, by incorporating Schmeidler's RDU.

Figure 5 can, just as in the case of risk, be explained by a larger decision weight for the M branches in Fig. 5b than in Fig. 5a:

$$W(M \cup H \cup L) - W(H \cup L) \geq W(M \cup H) - W(H). \quad (3.1)$$

This inequality occurs for W-functions that are more sensitive to changes of events near the certain universal event $M \cup H \cup L$ than for events of moderate likelihood such as $M \cup H$. Although for uncertainty we cannot easily draw graphs of W functions, their properties are natural analogs of those depicted in Figures 2-4. W is *convex* if the marginal W contribution of an event E to a disjoint event R is increasing in R, i.e.

$$W(E \cup R) - W(R) \quad (3.2)$$

is increasing in R (with respect to set inclusion) for all E. This agrees with Eq. 3.1, where increasing R from H to $H \cup L$ leads to a larger decision weight for E = M. Our definition is equivalent to other definitions in the literature such as $W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$.[2]

[2] Take $E = A - B$, and compare $R = A \cap B$ with the larger $R = A$.

For probabilistic sophistication ($W(\cdot) = w(P(\cdot))$, convexity of W is equivalent to convexity of w under usual richness conditions, illustrating once more the close similarity between risk and uncertainty. Eq. 3.2 gives the decision weight of an outcome occurring under event E if better outcomes occur under event R. Under convexity, outcomes receive more weight as they are ranked worse (i.e., R larger), reflecting *pessimism*. Theoretical economic studies usually assume convex W's, implying low evaluations of prospects relative to sure outcomes.

Empirical studies have suggested that weighting functions W for uncertainty exhibit patterns similar to Fig. 3b, with unlikely events overweighted rather than, as convexity would have it, underweighted (Einhorn & Hogarth 1986; Tversky & Fox 1995; Wu & Gonzalez 1999). As under risk, we get extremity orientedness, with best and worst outcomes overweighted and lack of sensitivity towards intermediate outcomes (Chateauneuf, Eichberger, & Grant 2006; Tversky & Wakker 1995).

4. Phenomena for Uncertainty That Do not Show up for Risk: the Ellsberg Paradox

Empirical studies have suggested that phenomena found for risk hold for uncertainty as well, and do so to a more pronounced degree (Fellner 1961, p. 684; Kahneman and Tversky, 1979, p. 281), in particular regarding the empirically prevailing inverse-S shape and its extension to uncertainty (Abdellaoui, Vossmann, & Weber 2005; Kilka & Weber 2001; Weber 1994, pp. 237-238). It is plausible, for example, that the absence of known probabilities adds to the inability of people to sufficiently distinguish between various degrees of likelihood not very close to impossibility and certainty. In such cases, inverse-S shapes will be more pronounced for uncertainty than for risk. This observation entails a within-person comparison of attitudes for different sources of uncertainty, and such comparisons will be the main topic of this section.

For Ellsberg's paradox, imagine an urn K with a known composition of 50 red balls and 50 black balls, and an ambiguous urn A with 100 red and black balls in unknown proportion. A ball is drawn at random from each urn, with R_k denoting the event of a red ball from the known urn, and the events B_k, R_a, and B_a defined similarly. People prefer to bet on the known urn than on the ambiguous urn, and common preferences are:

$(B_k{:}100,R_k{:}0) \succ (B_a{:}100,R_a{:}0)$ and $(B_k{:}0,R_k{:}100) \succ (B_a{:}0,R_a{:}100)$.

Such preferences are also found if people can themselves choose the color to bet on so that there is no reason for suspecting an unfavorable composition of the unknown urn. Under probabilistic sophistication with probability measure P, the two preferences would imply $P(B_k) > P(B_a)$ and $P(R_k) > P(R_a)$. However, $P(B_k) + P(R_k) = 1 = P(B_a) + P(R_a)$ yields a contradiction, because two big numbers cannot give the same sum as two small numbers. Ellsberg's paradox consequently violates probabilistic sophistication and, a fortiori, expected utility. Keynes (1921, p. 75) discussed the difference between the above two urns before Ellsberg did, but did not put forward the choice paradox and deviation from probabilistic sophistication as Ellsberg did. We will now analyze the example assuming RDU.

In many studies of uncertainty, such as Schmeidler (1989), expected utility is assumed for risk, primarily for the sake of simplicity. Then, $W(B_k) = W(R_k) = 0.5$ in the above example, with these W values reflecting objective probabilities. Under RDU, the above preferences imply $W(B_a) = W(R_a) < 0.5$, in agreement with convex (or eventwise dominance, or inverse-S; for simplicity of presentation, I will confine my attention to convexity hereafter) weighting functions W. This finding led to the widespread misunderstanding that it is primarily the Ellsberg paradox that implies convex weighting functions for unknown probabilities, a condition that was sometimes called ambiguity aversion. I have argued above that it is the Allais paradox, and not the Ellsberg paradox, that implies these conclusions, and will propose another interpretation of the Ellsberg paradox hereafter, following works by Amos Tversky in the early 1990s.

First, it is more realistic not to commit to expected utility under risk when studying uncertainty. Assume, therefore, that $W(B_k) = W(R_k) = w(P(B_k)) = w(P(R_k)) = w(0.5)$ for a nonlinear probability weighting function. It follows from the Ellsberg paradox that $W(B_a) = W(R_a) < w(0.5)$. This suggests:

HYPOTHESIS. *In the Ellsberg paradox, the weighting function is more convex for the unknown urn than for the known urn.* □

Thus, the Ellsberg paradox itself does not speak to convexity in an absolute sense, and does not claim convexity for known or for unknown probabilities. It speaks to convexity in a relative (within-person) sense, suggesting *more* convexity for unknown probabilities than for known probabilities. It is, for instance, possible that the weighting function is concave, and not convex, for both known and unknown probabilities, but is less concave (and thus more convex) for the unknown probabilities (Wakker 2001, Section 6; cf. Epstein 1999 pp. 589-590 or Ghirardato & Marinacci 2002, Example 25).

With only information about observed behavior, and without additional information about the compositions of the urns or the agent's knowledge thereof, we cannot conclude which of the urns is ambiguous and which is not. It would then be conceivable that urn K were ambiguous and urn A unambiguous, and that the agent satisfied expected utility for A and was optimistic or ambiguity seeking (concave weighting function, Eq. 3.2 decreasing in R) for K, in full agreement with the Ellsberg preferences. Which of the urns is ambiguous and which is not is based on extraneous information, being our knowledge about the composition of the urns and about the agent's knowledge thereof. This point suggests that no endogenous definition of (un)ambiguity is possible.

The Ellsberg paradox entails a comparison of attitudes of one agent with respect to different sources of uncertainty. It constitutes a *within-agent* comparison. Whereas the Allais paradox concerns violations of expected utility in an absolute sense, the Ellsberg paradox concerns a relative aspect of such violations, finding more convexity (or eventwise dominance, or inverse-S) for the unknown urn than for the known urn. Such a phenomenon cannot show up if we study only risk, because risk is essentially only one source of uncertainty. Apart from some volatile psychological effects (Kirkpatrick & Epstein 1992; Piaget & Inhelder 1975), it seems plausible that people do not distinguish between different ways of generating objective known probabilities.

Uncertain events of particular kinds can be grouped together into sources of uncertainty. Formally, let *sources* be particular algebras of events, which means that sources are closed under complementation and union, and contain the vacuous and universal events. For example, source $\mathcal{A}$ may concern the performance of the Dow Jones stock index tomorrow, and source $\mathcal{B}$ the performance of the Nikkei stock index tomorrow. Assume that A from source $\mathcal{A}$ designates the event that the Dow Jones index goes up tomorrow, and B from source $\mathcal{B}$ the event that the Nikkei index goes up tomorrow. If we prefer (A:100, not-A,0) to (B:100, not-B:0), then this may be caused by a special source preference for $\mathcal{A}$ over $\mathcal{B}$, say if $\mathcal{A}$ comprises less ambiguity for us than $\mathcal{B}$. However, it may also occur simply because we think that event A is more likely to occur than event B. To examine ambiguity attitudes we have to find a way to "correct" for differences in beliefs.

One way to detect *source preference* for $\mathcal{A}$ over $\mathcal{B}$ is to find an $\mathcal{A}$-partition $(A_1,\ldots,A_n)$ and a $\mathcal{B}$-partition $(B_1,\ldots,B_n)$ of the universal event such that for each j, $(A_j{:}100, \text{not-}A_j{,}0) \succ$

(B_j:100, not-B_j:0) (Nehring 2001, Definition 4; Tversky & Fox 1995; Tversky & Wakker 1995). Because both partitions span the whole universal event, we cannot have stronger belief in every A_j than B_j (under some plausible assumptions about beliefs), and hence there must be a preference for dealing with $\mathcal{A}$ events beyond belief. The Ellsberg paradox is a special case of this procedure.

Under the above approach to source preference, there is a special role for probabilistic sophistication. For a source of ambiguity $\mathcal{A}$ for which not some of its events are more ambiguous than others, it is plausible that $\mathcal{A}$ exhibits source indifference with respect to itself. This condition can be seen to amount to the additivity axiom of qualitative probability (if A_1 is as likely as A_3, and A_2 is as likely as A_4, then $A_1 \cup A_2$ is as likely as $A_3 \cup A_4$ whenever $A_1 \cap A_2 = A_3 \cap A_4 = \varnothing$), which, under sufficient richness, implies probabilistic sophistication under RDU, and does so in general (without RDU assumed) under an extra dominance condition (Fishburn 1986; Sarin & Wakker 2000). Probabilistic sophistication, then, entails a *uniform degree* of ambiguity (with respect to source preference) of a source.

In theoretical economic studies it has usually been assumed that people are averse to ambiguity, corresponding with convex weighting functions. Empirical studies, mostly by psychologists, have suggested a more varied pattern, where different sources of ambiguity can arouse all kinds of emotions. For example, Tversky & Fox (1995) found that basketball fans exhibit source preference for ambiguous uncertain events related to basketball over events with known probabilities, which entails ambiguity seeking. This finding is not surprising in an empirical sense, but its conceptual implication is important: attitudes towards ambiguity depend on many ad hoc emotional aspects, such as a general aversion to deliberate secrecy about compositions of urns, or a general liking of basketball. Uncertainty is a large domain, and fewer regularities can be expected to hold universally for uncertainty than for risk, in the same way as fewer regularities will hold universally for the utility of nonmonetary outcomes (hours of listening to music, amounts of milk to be drunk, life duration, etc.) than for the utility of monetary outcomes. It means that there is much yet to be discovered about uncertainty.

5. Models for Uncertainty Other than Rank-Dependence

5.1. Multiple Priors

An interesting model of ambiguity by Jaffray (1989), with a separation of ambiguity beliefs and ambiguity attitudes, received, unfortunately, little attention. A surprising case of unknown probabilities can arise when the expected utility model perfectly well describes behavior, but utility is state-dependent. The (im)possibility of defining probability in such cases has been widely discussed (Drèze 1987; Grant & Karni, 2005; Nau 2006).

The most popular alternative to Schmeidler's RDU is the multiple priors model introduced by Wald (1950). It assumes a set of probability measures $\mathcal{P}$ plus a utility function U, and evaluates each prospect through its minimal expected utility with respect to the probability distributions contained in $\mathcal{P}$. The model has an overlap with RDU: If W is convex, then RDU is the minimal expected utility over $\mathcal{P}$ where $\mathcal{P}$ is the CORE of W, i.e. the set of probability measures that eventwise dominate W. Drèze (1961, 1987) independently developed a remarkable analog of the multiple priors model, where the maximal expected utility is taken over $\mathcal{P}$, and $\mathcal{P}$ reflects moral hazard instead of ambiguity. Drèze also provided a preference foundation. Similar functionals appear in studies of robustness against model misspecification in macroeconomics (Hanssen & Sargent, 2001).

Variations of multiple priors, combining pessimism and optimism, employ convex combinations of the expected utility minimized over $\mathcal{P}$ and the expected utility maximized over $\mathcal{P}$ (Ghirardato, Maccheroni, & Marinacci 2004, Proposition 19). Such models can account for extremity orientedness, as with inverse-S weighting functions and RDU. Arrow & Hurwicz (1972) proposed a similar model where a prospect is evaluated through a convex combination of the minimal and maximal utility of its outcomes (corresponding with $\mathcal{P}$ being the set of all probability measures). This includes maximin and maximax as special cases. Their approach entails a level of ambiguity so extreme that no levels of belief other than "sure-to-happen," "sure-not-to-happen," and "don't know" play a role, similar to Figure 4, and suggesting a three-valued logic. Other non-belief-based approaches, including minimax regret, are in Manski (2000) and Savage (1954), with a survey in Barberá, Bossert, & Pattanaik (2004).

Other authors proposed models where for each single event a separate interval of probability values is specified (Budescu & Wallsten 1987; Kyburg 1983; Manski 2004). Such interval-probability models are mathematically different from multiple priors because there is no unique relation between sets of probability measures over the whole event space and intervals of probabilities separately for each event. The latter models are more tractable than multiple priors because probability intervals for some relevant event are easier to specify than probability measures over the whole space, but these models did not receive a preference foundation and never became popular in economics. Similar models of imprecise probabilities received attention in the statistics field (Walley 1991).

Wald's multiple priors model did receive a preference axiomatization (Gilboa & Schmeidler 1989), and consequently became the most popular alternative to RDU for unknown probabilities. The evaluating formula is easier to understand at first than RDU. The flexibility of not having to specify precisely what "the" probability measure is, while usually perceived as an advantage at first acquaintance, can turn into a disadvantage when applying the model. We then have to specify exactly what "the" set of probability distributions is, which is more complex than specifying only one probability measure exactly (cf. Lindley 1996).

The simple distinction between probability measures that are either possible (contained in $\mathcal{P}$) or impossible (not contained in $\mathcal{P}$), on the one hand adds to the tractability of the model, but on the other hand cannot capture cognitive states where different probability measures are plausible to different degrees. To the best of my knowledge, the multiple priors model cannot yet be used in quantitative empirical measurements today, and there are no empirical assessments of sets of priors available in the literature to date. Multiple priors are, however, well suited for general theoretical analyses where only general properties of the model are needed. Such analyses are considered in many theoretical economic studies, where the multiple priors model is very useful.

The multiple priors model does not allow deviations from expected utility under risk, and a desirable extension would obviously be to combine the model with nonexpected utility for risk. Promising directions for resolving the difficulties of the multiple priors model are being explored today (Maccheroni, Marinacci, & Rustichini 2005).

5.2. Model-Free Approaches to Ambiguity

Dekel, Lipman, & Rustichini (2001) considered models where outcomes of prospects are observed but the state space has not been completely specified, as relevant for incomplete contracts. Similar approaches with ambiguity about the underlying states and events appeared in psychology in repeated-choice experiments by Herwig et al. (2003), and in support theory (Tversky & Koehler 1994). This section discusses two advanced attempts to define ambiguity model-free that have received much attention in the economic literature.

In a deep paper, Epstein (1999) initiated one such approach, continued in Epstein & Zhang (2001). Epstein sought to avoid any use of known probabilities and tried to endogenize (non)ambiguity and the use of probabilities.[3] For example, he did not define risk neutrality with respect to known probabilities as we did above, but with respect to subjective probabilities derived from preferences as in probabilistic sophistication (Epstein 1999, Eq. 2.2). He qualified probabilistic sophistication as ambiguity neutrality (not uniformity as done above). Ghirardato & Marinacci (2002) used another approach that is similar to Epstein's. They identified absence of ambiguity not with probabilistic sophistication, as did Epstein, but, more restrictively, with expected utility.

The above authors defined an agent as ambiguity averse if *there exists* another, hypothetical, agent who behaves the same way for unambiguous events, but who is ambiguity neutral for ambiguous events, and such that the real agent has a stronger preference for sure outcomes[4] versus ambiguous prospects than the hypothetical agent. This definition concerns traditional between-agent within-source comparisons (Yaari 1969). The stronger preferences for certainty are, under rank-dependent models, equivalent to eventwise dominance of weighting functions, leading to nonemptiness of the CORE (Epstein 1999, Lemma 3.4; Ghirardato & Marinacci 2002, Corollary 13). These definitions of ambiguity aversion are not very tractable because of the "there exists" clause. It is difficult to establish which ambiguity neutral agent to take for the comparisons. To mitigate this problem, Epstein (1999, Section 4) proposed eventwise derivatives as models of local probabilistic sophistication. Such derivatives only exist for continua of events with a linear structure, and are difficult to elicit. They serve their purpose only under restrictive circumstances (ambiguity aversion throughout plus constancy of the local derivative, called coherence; see Epstein's Theorem 4.3).

In both above approaches, ambiguity and ambiguity aversion are inextricably linked, making it hard to model attitudes towards ambiguity other than aversion, or to distinguish between ambiguity-neutrality or -absence (Epstein 1999, p. 584 1st para; Epstein & Zhang, p. 283; Ghirardato & Marinacci 2002, p. 256, 2nd para). Both approaches have difficulties

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[3] He often used the term uncertainty as equivalent to ambiguity.

[4] Or unambiguous prospects, but these can be replaced by their certainty equivalents.

distinguishing between the two Ellsberg urns. Each urn in isolation can be taken as probabilistically sophisticated with, in our interpretation, a uniform degree of ambiguity, and Epstein's definition cannot distinguish which of these is ambiguity neutral (cf. Ghirardato & Marinacci 2002, middle of p. 281). Ghirardato & Marinacci's definition can, but only because it selects expected utility (and the urn generating such preferences) as the only ambiguity-neutral version of probabilistic sophistication. Any other form of probabilistic sophistication, i.e. any nonexpected utility behavior under risk, is then either mismodeled as ambiguity attitude (Ghirardato & Marinacci 2002, pp. 256-257), or must be assumed not to exist.

We next discuss in more detail a definition of (un)ambiguity by Epstein & Zhang (2001), whose aim was to make (un)ambiguity observable by expressing it directly in terms of a preference condition. They called an event E *unambiguous* if

$$(E{:}c,\ E_2{:}\gamma, E_3{:}\beta, E_4{:}x_4,\ldots,E_n{:}x_n) \succcurlyeq (E{:}c,\ E_2{:}\beta, E_3{:}\gamma, E_4{:}x_4,\ldots,E_n{:}x_n) \text{ implies}$$
$$(E{:}c', E_2{:}\gamma, E_3{:}\beta, E_4{:}x_4,\ldots,E_n{:}x_n) \succcurlyeq (E{:}c', E_2{:}\beta, E_3{:}\gamma, E_4{:}x_4,\ldots,E_n{:}x_n) \qquad (5.1)$$

for all partitions $E_2,\ldots,E_n$ of not-E, and all outcomes c, c´, $\gamma \succcurlyeq \beta$, $x_4,\ldots,x_n$, with a similar condition imposed on not-E. In words, changing a common outcome c into another common outcome c´ under E does not affect preference, but this is imposed only if the preference concerns nothing other than to which event (E_2 or E_3) a good outcome γ is to be allocated instead of a worse outcome β. Together with, mainly, their Axiom 4, which the authors interpret as richness, Eq. 5.1 implies that probabilistic sophistication holds on the set of events satisfying this condition, which in the interpretation of the authors designates absence of ambiguity (rather than uniformity). As we will see next, it is not clear why Eq. 5.1 would capture the absence of ambiguity.

EXAMPLE. Assume that events are subsets of [0,1), E = [0, 0.5), not-E = [0.5, 1), and E has unknown probability π. Every subset A of E has probability $2\pi\lambda(A)$ (λ is the usual Lebesgue measure, i.e. the uniform distribution over [0,1)) and every subset B of not-E has probability $2(1-\pi)\lambda(B)$. Then it seems plausible that event E and its complement not-E are ambiguous, but conditional on these events ("within them") we have probabilistic sophistication with respect to the conditional Lebesgue measure and without any ambiguity. Yet, according to Eq. 5.1, events E and not-E themselves are unambiguous, both preferences in Eq. 5.1 being determined by whether $\lambda(E_2)$ exceeds $\lambda(E_3)$. □

In the example, the definition in Eq. 5.1 erroneously ascribes the unambiguity that holds for events conditional on E, so “within E,” to E as a whole. Similar examples can be devised where E and not-E themselves are unambiguous, there is "nonuniform" ambiguity conditional on E, this ambiguity is influenced by outcomes conditional on not-E through nonseparable interactions typical of nonexpected utility, and Eq. 5.1 erroneously ascribes the ambiguity that holds within E to E as a whole.

A further difficulty with Eq. 5.1 is that it is not violated in the Ellsberg example with urns A and K as above (nor if the uncertainty regarding each urn is extended to a "uniform" continuum as in Example 5.8ii of Abdellaoui & Wakker 2005), and cannot detect which of the urns is ambiguous. The probabilistic sophistication that is obtained in Epstein & Zhang (2001, Theorem 5.2) for events satisfying Eq. 5.1, and that rules out the two-urn Ellsberg paradox and its continuous extension of Abdellaoui & Wakker (2005), is mostly driven by their Axioms 4 and 5 and the necessity to consider also intersections of different-urn events for $E_2, \ldots, E_n$ in Eq. 5.1 (see their Appendix E). This imposes, in my terminology, a uniformity of ambiguity over the events satisfying Eq. 5.1 which, rather than Eq. 5.1 itself, rules out the above counterexamples.

5.3. Multi-Stage Approaches to Ambiguity

Several authors have considered two-stage approaches with intersections of first-stage events A_i, $i = 1,\ldots,\ell$ and second-stage events K_j, $j = 1,..,k$, so that $n = \ell k$ events A_iK_j result, and prospects $(A_iK_j:x_{ij})_{i=1}^{\ell}{}_{j=1}^{k}$ are considered. It can be imagined that in a first stage it is determined which event A_i obtains, and then in a second stage, conditional on A_i, which event K_j obtains. Many authors considered such two-stage models with probabilities given for the events in both stages, the probabilities of the first stage interpreted as ambiguity about the probabilities of the second stage, and non-Bayesian evaluations used (Levi 1980; Segal 1990; Yates & Zukowski 1976).

Other authors considered representations

$$\sum_{i=1}^{\ell} Q(A_i)\varphi\left(\sum_{j=1}^{k} P(K_j)U(x_{ij})\right) \qquad (5.2)$$

for probability measures P and Q, a utility function U, and an increasing tranformation φ. For φ the identity or, equivalently, φ linear, traditional expected utility with backwards induction results. Nonlinear φ’s give new models. Kreps & Porteus (1979) considered Eq.

5.2 for intertemporal choice, interpreting nonlinear φ's as nonneutrality towards the timing of the resolution of uncertainty. Ergin & Gul (2004) and Nau (2006) reinterpreted the formula, where now the second-stage events are from a source of different ambiguity than the first-stage events. A concave φ, for instance, suggests stronger preference for certainty, and more ambiguity aversion, for the first-stage uncertainty than for the second.

Klibanoff, Marinacci, & Mukerji (2005) considered cases where the decomposition into A- and K-events is endogenous rather than exogenous. This approach greatly enlarges the scope of application, but their second-order acts, i.e. prospects with outcomes contingent on aspects of preferences, are hard to implement or observe if those aspects cannot be related to exogenous observables.

Eq. 5.2 has a drawback similar to Eq. 2.3. All extra mileage is to come from the outcomes, to which also utility applies, so that there will not be a great improvement in descriptive performance or new concepts to be developed.

6. Conclusion

The Allais paradox reveals violations of expected utility in an absolute sense, leading to convex or inverse-S weighting functions for risk and, more generally, for uncertainty. The Ellsberg paradox reveals deviations from expected utility in a relative sense, showing that an agent can deviate more from expected utility for one source of uncertainty (say one with unknown probabilities) than for another (say, one with known probabilities). It demonstrates the importance of within-subject between-source comparisons.

The most popular models for analyzing uncertainty today are based on rank-dependence, with multiple priors a popular alternative in theoretical studies. The most frequently studied phenomenon is ambiguity aversion. Uncertainty is, however, a rich empirical domain, with a wide variety of phenomena, and with ambiguity aversion and ambiguity insensitivity (inverse-S) as prevailing but not universal patterns. The possibility of relating the properties of weighting functions for uncertainty to cognitive interpretations such as insensitivity to likelihood-information makes RDU and prospect theory well suited for links with other fields such as psychology, artificial intelligence (Shafer 1976), and brain imaging studies (Camerer, Loewenstein, & Prelec 2004).

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