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THE APPRAISAL OF ROAD CONSTRUCTION: TWO CALCULATION SCHEMES

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1. Apart from the general difficulties involved in the appraisal of investment projects, there are some particular ones in the appraisal of road building projects. First of all, the "product" of a road is not usually sold by its owner, but made freely available. There are no prices therefore which might help to orient the investor. The effects have to be estimated in an indirect way. They consist of cost reductions for existing production and of the creation of new production possibilities. The second difficulty is that the estimation of these effects requires insight into the influence exerted on the general economic situation of a country or a region by transportation facilities. This insight is not readily available since economic science as a rule disregards the complications of transportation costs and of geographical price disparities which are the heart of the problem. Practical methods of estimation have been developed mostly by engineers and, although of considerable help, usually neglect the question of the indirect and secondary effects. The present note tries to develop a method which, on the one hand, fulfills certain requirements of an economic nature and, on the other hand, remains manageable as far as numerical calculation is concerned.

2. It should be stated beforehand that some important aspects of project appraisal in general are left out of consideration. The problem will be treated in a static way, that is, the economic situation is assumed to be stationary before as well as after the construction of the road, and an attempt is made to estimate only the rise in national product that will have been obtained by that construction. The question of how to discount future product is not considered; neither is the question whether in fact product is the correct measuring rod of welfare.

3. In order to offer a solution to the problems characteristic of the appraisal of a road project, the schemes to be discussed must, as a minimum, show the following features. First, they must cover, in principle, the production of all commodities, implying that demand for these commodities is not considered given but itself derives from the incomes earned in production. Second, the schemes have to distinguish between geographically separated "centers" and to represent their mutual economic relationships. Movements of goods from one center to another will give rise to transportation costs and these will be among the data, alongside the data on supply and demand functions for each of the goods considered in each of the centers. The problem to be considered will then be, of course, to vary one or more of these data on transportation costs and to study the effect on total production. The central problem that will come up for consideration is the one of competition between suppliers of the same good situated at different centers. Two approaches to this phenomenon will be undertaken, each of them giving rise to one scheme. In scheme I the elasticity of substituting one supplier for another will be assumed, for all buyers, to be infinite; that is, as soon as supplier k is charging a lower price than supplier k', at the location of buyer i, the latter will buy all he wants of this commodity from supplier k. In scheme II the elasticity of substitution will be taken to be finite; if, because of a transportation improvement, supplier k becomes relatively cheaper to buyer i than supplier k', there will be a certain shift from k' to k, but not a complete shift. Its size will

depend on the change in relative prices. Here we are applying Triffin's method of dealing with monopolistic competition,[1] taking it for granted that there are always numerous reasons why a shift of this type is only partial. It should be added that the shift will, nevertheless, be larger, the longer the period we are interested in and that the choice of the elasticity of substitution should be dependent on the length of that period.

4. Before presenting our schemes in the form of numerical examples, we will summarize their theoretical contents with the aid of general formulas. In these formulas the following symbols will be used.

The numbers of centers will be indicated by subscripts i and k, each running from 1 to n, the total number of centers; i will be, as a rule, the center buying a certain commodity, k the center selling it. When placed in succession, two subscripts will indicate a movement of the commodity considered from the first to the second subscript. Movements of payments will, as a consequence, be opposite. In scheme I there will, in addition, be a superscript h, indicating the commodity considered. In scheme II it will be assumed that each center only produces one commodity, and it is not necessary therefore to add a superscript.

The economic variables considered are:

p^h_k the price charged for good h at center k;

v^h_{ki} the quantity of commodity h supplied by center k to center i;

V^h_{ki} the money value of v^h_{ki}, at the price of delivery at center i. This price is obtained by multiplying its price at the center of production, p^h_k, by a transportation coefficient T_{ki} (see below).

The data considered are:

T_{ki} the transportation coefficient for transportation between k and i. It will be assumed that this coefficient is independent of the type of good transported, but this is by no means necessary. The removal of this assumption does not introduce any difficulty.

δ^h_i and δ_i are coefficients appearing in the demand equations; δ^h_i is i's propensity to spend income on commodity h (in scheme II the commodity will be indicated by the sector of origin k); and δ_i, occurring in scheme II only, is the influence exerted by a unit change in price $p_k T_{ki}$ (of commodity k in center i) on the value V_{ki} of commodity k bought by center i. Since it is only relative price which is assumed to be relevant in this respect, the same coefficient, but with inverted sign, also occurs before a term $\frac{1}{n-1}\sum_{h'\neq k} p_{h'} T_{h'i}$ representing the average of all other prices quoted in center i.

$\bar{\sigma}^h_i$ and σ^h_i are coefficients appearing in the supply equations for commodity h in center i (in scheme II this is necessarily commodity i only).

5. The equations used in the two schemes are:

Scheme I (infinite elasticity of substitution between competitors)

Definitions:

$$V^h_{ki} = v^h_{ki} P^h_k T_{ki} \qquad \text{(I 1)}$$

Demand equations (for center i):

$$V^h_{ki} = \bar{\delta}^h_i \sum_{h'k'}\sum V^{h'}_{ik'} \qquad \text{for } k = S^h(i) \text{ and } i = S^{h'}(k') \qquad \text{(I 2a)}$$

$$V^h_{ki} = 0 \qquad \text{for } k \neq S^h(i). \qquad \text{(I 2b)}$$

Demand equations (I 2a) are valid only for demand exerted by i at the cheapest center k (taking account of transportation from k to i) for the commodity h; demand equations (1 2b) apply to the demand exerted at the other centers.[2]

It will be clear that in equations (I 2a) the expression behind δ^h_i represents center i's income, when k' is, for each h', the center supplied by i.

Supply equations (for center i):

$$p^h_i \sum_k v^h_{ik} = \bar{\sigma}^k_i p^h_i - \sigma^h_i \sum_{h'\neq h} p^{h'}_{k'} T_{k'i} \qquad \text{where } i = S^h(k) \text{ and } k' = S^{h'}(i). \qquad \text{(I 3)}$$

It will be understood that $\sum_k v^h_{ik}$ is the total quantity of product h supplied to other centers by center i; only such k's have to be included which center i is actually supplying. The supply relation assumed is a hyperbolic function of the relative price $p^h_i / \sum p^{h'}_{k'} T_{k'i}$:

[1] R. Triffin, *Monopolistic Competition and General Equilibrium Theory* (Cambridge, 1940).

[2] This is expressed by the symbol S, standing for "supply center to."

$$\sum v_{ik}^h = \bar{\sigma}_i^h - \sigma_i^h \frac{\sum p_{k'}^{h'} T_{k'}}{p_i^h}$$

expressing that $\bar{\sigma}_i^h$ is the "capacity" limit; σ_i^h can be related to the elasticity of supply, as will be shown in our numerical examples.

Scheme II (finite elasticity of substitution between competitors)

Definitions:

$$V_{ki} = v_{ki}\, p_k\, T_{ki} \qquad \text{(II 1)}$$

Demand equations (for center i):

$$V_{ki} = \bar{\delta}_i^k \sum_{k'} V_{ik'} - \delta_i\, p_k\, T_{ki} + \frac{\delta_i}{n-1} \sum_{k' \neq k} p_{k'}\, T_{k'i} \qquad \text{(II 2)}$$

Since each center k is now producing one commodity only, no superscripts to indicate commodities are necessary; $\sum_{k'} V_{ik'}$ represents center i's income, and $p_k T_{ki}$ and $p_{k'} T_{k'i}$ the prices paid by center i; all these are assumed to influence the value of its demand V_{ki} for good k, and these price terms represent the finite elasticity of substitution between producers. They replace the set of alternative demand equations used in scheme I.

Supply equations (for center i):

$$p_i \sum_k v_{ik} = \bar{\sigma}_i p_i - \sigma_i \sum_{k \neq i} p_k\, T_{ki}\,. \qquad \text{(II 3)}$$

This equation is not essentially different from the supply equation in scheme I.

6. The numerical examples to be presented are characterized by three centers in each case, and the following specific assumptions.

(i) The propensities to spend are the same for the three centers[3];

(ii) The transportation coefficients are not different between commodities and are independent of the sense of transportation.

These restrictions are not essential to the applicability of the schemes but only simplify them. Since the two schemes are different with regard to the other data, they will now be discussed one by one.

It is assumed, as appears from Table 1, that there are four commodities, No. 1 is produced by center 1 only, No. 2 by center 2, No. 3 by center 3 (which are supposed to have important comparative advantages in each of these goods, respectively); commodity 4, however, is produced by both center 2 and center 3; center 2 supplies this commodity to both itself and center 1.

Scheme I. *Initial situation.*

(i) TABLE 1. — MATRIX OF MONEY FLOWS, V_{ki}^h

		Consuming centers (buying centers): $i = 1$				$i = 2$				$i = 3$				
Commodities	$h \rightarrow$	1	2	3	4	1	2	3	4	1	2	3	4	Total receipts
Producing centers	$k = 1$	48				32				40				120
(selling centers)	2		24		12		16		8		20			80
	3			36				24				30	10	100
Total expenditure			120				80				100			300

(ii) TABLE 2. — TRANSPORTATION MATRIX

The transportation coefficients T_{ki} are supposed to be:

	$i = 1$	2	3
$k = 1$	1	1.1	1.3
2	1.1	1	1.2
3	1.3	1.2	1

(iii) TABLE 3. — PRODUCER PRICES

$p^h{}_k$	$h = 1$	2	3	4
$k = 1$	1	(2)	(2)	(2)
2	(2)	1	(2)	1.1
3	(2)	(2)	1	1

In order to keep the example simple, it has been assumed that the supply prices of goods 1, 2, 3 and 4, if produced by one of the countries not actually supplying them in the initial situation, would be so much higher that they are not able to compete, even after the change in transportation costs to be considered. Figures 2 have been assumed everywhere in those cases for p_k^h; since these figures will not play an active role in the scheme, they have

[3] This assumption was also made by F. D. Graham in his *Theory of International Values.*

(iv) TABLE 4. — CONSUMER PRICES

q^h_{ki}	$i = 1$				$i = 2$				$i = 3$			
h	1	2	3	4	1	2	3	4	1	2	3	4
$k = 1$	*1*	(2)	(2)	(2)	*1.1*	(2.2)	(2.2)	(2.2)	*1.3*	(2.6)	(2.6)	(2.6)
2	(2.2)	*1.1*	(2.2)	*1.21*	(2)	*1*	2	*1.1*	(2.4)	*1.2*	(2.4)	1.32
3	(2.6)	(2.6)	*1.3*	1.3	(2.4)	(2.4)	*1.2*	1.2	(2)	(2)	*1*	*1*

been given in parentheses; so have the derived figures for $q^h_{ki} = p^h_k T_{ki}$ indicating the supply prices at the consuming centers. If the potential supply prices were nearer those of the centers actually supplying these commodities in the initial stage, their role might have been an active one. This will only be demonstrated in this example with regard to commodity 4 as supplied by center 3. In the initial situation center 3 cannot compete with center 2 in supplying center 1 with good 4: its price is 1.3, as against 1.21 charged by center 2. It cannot compete either in supplying center 2, where its price is 1.2, as against 1.1 charged by center 2 itself. On the other hand, center 2 cannot compete with center 3 in the latter's own market: here center 2 would have to charge 1.32, whereas center 3 itself supplies the commodity at a price 1.

The figures on prices are such that in each column of the table for q the cheapest supplier can be easily traced; the minimum prices are italicized, and the money flows in the V^h_{ki} table (Table 1) correspond to these choices.

7. The *problem* to be solved by numerical computation is to find the new equilibrium situation corresponding to a change in the transportation matrix given by

$$T'_{ik} = \begin{bmatrix} 1 & 1.1 & 1.1 \\ 1.1 & 1 & 1.2 \\ 1.1 & 1.2 & 1 \end{bmatrix}$$

meaning that transportation costs between centers 1 and 3 have been reduced to one-third of their previous value.

The additional data needed, according to the general formulas for scheme I, are:

(i) the marginal propensities to spend income on the four commodities; it is assumed that these are equal to the average propensities, meaning that:

$$\delta_1 = 0.4; \quad \delta_2 = 0.2; \quad \delta_3 = 0.3; \quad \delta_4 = 0.1.$$

(ii) all relevant supply equations. As a consequence of our simplifications (i.e. the high supply prices shown in brackets in the table for q^h_{ki}) only the following supply equaing supply equations are relevant:

$$\left.\begin{aligned}
&\text{Comm. 1, center 1:} \\
&\quad p^1_1 \sum_k v^1_{1k} = 216\, p^1_1 - 29 \sum p^{h'}_{k'} T_{k'1} \\
&\text{Comm. 2, center 2:} \\
&\quad p^2_2 \sum_k v^2_{2k} = 165\, p^2_2 - 32 \sum p^{h'}_{k'} T_{k'2} \\
&\text{Comm. 3, center 3:} \\
&\quad p^3_3 \sum_k v^3_{3k} = 156\, p^3_3 - 22 \sum p^{h'}_{k'} T_{k'3} \\
&\text{Comm. 4, center 2:} \\
&\quad p^4_2 \sum_k v^4_{2k} = 363\, p^4_2 - 116 \sum p^{h'}_{k'} T_{k'2} \\
&\text{Comm. 4, center 3:} \\
&\quad p^4_3 \sum_k v^4_{3k} = 213\, p^4_3 - 58 \sum p^{h'}_{k'} T_{k'3}
\end{aligned}\right\} \quad (\text{I } 3')$$

The particular nature of each of the sums has been explained before, when equations (I 3), of which the five equations just presented are examples, were discussed. The figures in the supply equations have to be chosen in such a way that (a) the initial values of the variables satisfy them and (b) the elasticities coincide with the evidence available.

8. The *solution* will have to be sought in the following way. According to what was observed about the nature of the scheme, it can only be by trial and error that the answer can be ascertained. On the basis of the changes in transportation costs a provisional new price matrix is constructed, assuming that only the changes in transportation costs are relevant. This will at most hold as a first approximation and exact calculations of the price changes will have to be made later. The provisional price matrix is given in Table 5.

TABLE 5. — PROVISIONAL CONSUMER PRICES

q^h_{ki}	$i = 1$				$i = 2$				$i = 3$			
h	1	2	3	4	1	2	3	4	1	2	3	4
$k = 1$	1				1.1				1.1			
2		1.1		1.21		1		*1.1*		1.2		1.32
3			1.1	*1.1*			1.2	1.2			*1.1*	

From this new matrix the cheapest suppliers in each column can again be determined; because of the extreme simplicity of our example, the only change taking place is that center 1 will now buy good 4 from center 3 instead of from center 2. Upon this assumption we now recalculate the money flows.

Indicating by V_{ki} the total money flow paid for commodities moving from k to i, we will have the following equations:

$$V_{11} = 0.4(V_{11} + V_{12} + V_{13})$$
$$V_{21} = 0.2(V_{11} + V_{12} + V_{13})$$
$$V_{31} = (0.3 + 0.1)(V_{11} + V_{12} + V_{13})$$

$$V_{12} = 0.4(V_{21} + V_{22} + V_{23})$$
$$V_{22} = (0.2 + 0.1)(V_{21} + V_{22} + V_{23})$$
$$V_{23} = 0.3(V_{21} + V_{22} + V_{23})$$
$$V_{13} = 0.4(V_{31} + V_{32} + V_{33})$$
$$V_{23} = 0.2(V_{31} + V_{32} + V_{33})$$
$$V_{33} = (0.3 + 0.1)(V_{31} + V_{32} + V_{33})$$

The last factor in each equation represents the income of the buying center. It is preceded by one numerical coefficient if from the supplying center considered only one commodity is bought. The equation for V_{21}, for example, refers to commodity 2 only, with a propensity of 0.2, since now sector 1 buys commodity 2 only from sector 2. The equation for V_{31}, on the contrary, refers to commodities 3 and 4 with propensities of 0.3 and 0.1. Upon this choice the solution of the equations yields:

$$V_{11} = 48 \qquad V_{21} = 24 \qquad V_{31} = 48$$
$$V_{12} = 26.7 \qquad V_{22} = 20 \qquad V_{32} = 20$$
$$V_{13} = 45.3 \qquad V_{23} = 22.7 \qquad V_{33} = 45.3$$

After splitting up again V_{31}, V_{22} and V_{33} we obtain the provisional new money flows matrix.

The equations are homogeneous, as a consequence of the static character of the scheme. One money-flow variable may accordingly be chosen freely, and this choice only affects the absolute level of prices but not quantities and relative prices. We assume that total income remains 300.

9. The next step consists of the exact calculation of the prices from the money flows. This can be done with the help of the supply equations (I 3), already partly specified in (I 3′) and still further to be specified as follows. The left-hand side of each equation (I 3′) can be expressed in terms of the V^h_{ki} and T_{ki} because

$$p^h_i \sum_k v^h_{ik} = \sum_k \frac{V^h_{ik}}{T_{ik}}.$$

On both sides the sums have to be carefully specified; they should refer, on the left-hand side, to all actual supply flows and, on the right-hand side of (I 3′), to all prices actually paid for all other goods in the center concerned. Both specifications depend on the choices made by the consuming centers with regard to their suppliers. In our particular case it can be easily found that the resulting equations are:

$$\frac{48}{1} + \frac{26.7}{1.1} + \frac{45.3}{1.1} = 216\, p^1_1 - 29\,(1.1\, p^2_2 + 1.1\, p^3_3 + 11 p^4_3)$$

$$\frac{24}{11} + \frac{13.3}{1} + \frac{22.7}{1.2} = 165\, p^2_2 - 32\,(1.1\, p^1_1 + 1.2\, p^3_3 + p^4_2)$$

$$\frac{36}{1.1} + \frac{20}{12} + \frac{34}{1} = 156\, p^3_3 - 22\,(1.1\, p^1_1 + 1.2\, p^2_2 + p^4_3)$$

TABLE 6. — PROVISIONAL ESTIMATE OF MONEY FLOWS

V^h_{ki}	$i = 1$				$i = 2$				$i = 3$				
h	1	2	3	4	1	2	3	4	1	2	3	4	Total
$k = 1$	48				26.7				45.3				120
2		24				13.3		6.7		22.7			66.7
3			36	12			20				34	11.3	113.3

$$\frac{6.7}{1} \qquad = \qquad 363\, p_2^4 - 116\,(1.1\, p_1^1 + \quad p_2^2 + 1.2 p_3^3)$$

$$\frac{12}{1.1} \qquad + \frac{11.3}{1} = \qquad 213\, p_3^4 - \quad 58\,(1.1\, p_1^1 + 1.2\, p_2^2 + \quad p_3^3)$$

The solutions appear to be:

$$p_1^1 = 0.95; \quad p_2^2 = 0.95; \quad p_3^3 = 0.98;$$

$$p_2^4 = 1.04; \quad p_3^4 = 0.97$$

10. We now have to test whether these prices justify our provisional choice with regard to the supplying centers. They actually do; in particular, center 3 will now be the supplier to center 1 of good 4, since the price at which it is able to supply, $p_3^4\, T_{31} = 0.97 \times 1.1 = 1.07$ is lower than the price $p_2^4\, T_{21} = 1.04 \times 1.1 = 1.14$ at which center 2 can supply center 1. All other choices appear to have been correct too. This means that the values found for V_{ki}^h and the p's are the new equilibrium values indeed and can be used to estimate the changes in the volumes of production and hence in real national product. We will proceed to these estimates below. Before doing so, we should discuss, however, what would have been the course to follow had it been found that the provisional choices about the supplying centers were not correct. Evidently other choices should then have been made, until the test of the p values would have confirmed the choice. Since the number of possibilities is sometimes large, this process of trial and error may require considerable time. It may be devised along the lines of linear programming methods. Under certain conditions of convergency of the dynamics of the system considered we may hope, in addition, that the second choice of suppliers can be based simply on the prices found in the first round. It would lead us too far in this note on practical methods to enter into these questions.

11. In our numerical examples where this difficulty does not occur we are therefore now able to proceed to the final step: the estimation of the change in real national product. From the figures so far obtained, the following table is easily calculated:

The total increase in production or real income therefore amounts to 22. It is interesting to compare this figure with the reduction in transportation costs obtained for the initial pattern of transportation, a figure often calculated as a measure for the advantages to be obtained. In the initial situation the quantities transported between centers 1 and 3 are (see Table 1) 40/1.3 + 36/1.3 = 58. Transportation costs diminish from 0.3 per unit to 0.1 per unit or by 0.2 per unit. The total decrease therefore is 11.6, a figure considerably lower than the increase in production calculated in Table 7. A "multiplier" of 1.9 should have been applied to the cost reduction figure in order to obtain the increase in national product.

TABLE 7. — INCREASE IN NATIONAL PRODUCT AT INITIAL PRICES AS A CONSEQUENCE OF REDUCTION IN TRANSPORTATION COSTS

Supply Center	Commod.	Volume of Production			Initial price	Increase at initial price
		Initial	Final	Increase		
1	1	108	113/0.95 = 119	+11	1	11
2	2	55	54/0.95 = 57	+ 2	1	2
3	3	78	84/0.98 = 86	+ 8	1	8
2	4	17.3	6.7/1.04 = 6.4	−11	1.1	−12
3	4	10	22.2/0.97 = 22.9	+13	1	13
					Total	22

12. We will now consider a numerical example of scheme II. Since each center is here producing one commodity only, both the money flows and the prices can be represented by a simple 3-by-3 matrix. Since, in addition, this scheme can be described, for the initial as well as the final situation, by the same system of equations, we will follow a slightly different course in presenting this example. For the theorist, the data of the problem are the de-

mand and supply coefficients, the two transportation matrices (for initial and final state), and the arbitrarily chosen general price level (which in a static set-up is of no importance for the "real" variables). These we choose as follows:

(i) Propensities to spend income on commodities 1, 2, and 3, respectively:

$$\bar{\delta}^1 = 0.5; \qquad \bar{\delta}^2 = 0.3; \qquad \bar{\delta}^3 = 0.2.$$

(ii) Substitution coefficients:

$$\delta_1 = 0.25, \qquad \delta_2 = 0.15, \qquad \delta_3 = 0.1.$$

(iii) Supply coefficients:

$$\bar{\sigma}_1 = 1.2 \qquad \bar{\sigma}_2 = 0.7 \qquad \bar{\sigma}_3 = 0.5$$
$$\sigma_1 = 0.25 \qquad \sigma_2 = 0.15 \qquad \sigma_3 = 0.1$$

(iv) Transportation coefficients:

Initial / Final

$$T = \begin{Bmatrix} 1 & 1.2 & 1.6 \\ 1.2 & 1 & 1.4 \\ 1.6 & 1.4 & 1 \end{Bmatrix} \qquad T' = \begin{Bmatrix} 1 & 1.2 & 1.2 \\ 1.2 & 1 & 1.4 \\ 1.2 & 1.4 & 1 \end{Bmatrix}$$

(v) Total money flow $V_{00} = \sum_k \sum_i V_{ki} = 300.$

From these data the theorist can calculate all variables in the initial as well as the final state and make the comparison needed to appraise the "road project" implied in the reduction of transportation costs between centers 1 and 3. For the practical application of scheme II we will have to assume that all the initial values of the variables are given and that the coefficients of the demand and supply relations will have to be in line with these values. They will, of course, also depend on other evidence. In practice they will often have to be estimated and these estimates can only be rough ones.

13. The following arrangement of the calculations seems to be the simplest and may be of help in the theoretical as well as in the practical set-up. Write V_{10}, V_{20}, and V_{30} for the money incomes of the three centers, where $V_{10} = V_{11} + V_{12} + V_{13}$ etc., i.e. the zero stands for summation over all values of an index; consequently we may indicate by V_{00} the total of all incomes or all flows in our scheme.

Equations (II 2) may now be written:

$$V_{11} = 0.5\, V_{10} - 0.25\,(1 p_1 - \tfrac{1}{2} \times 1.2\, p_2 - \tfrac{1}{2} \times 1.6\, p_3)$$

$$V_{12} = 0.5\, V_{10} - 0.15\,(1.2 p_1 - \tfrac{1}{2}\, p_2 - \tfrac{1}{2} \times 1.4\, p_2) \text{ etc.}$$

Working out the numerical coefficients we obtain the following set:

$$V_{11} = 0.5\, V_{10} - 0.25\, p_1 + 0.15\, p_2 + 0.20\, p_3 \qquad (13.1)$$
$$V_{12} = 0.5\, V_{20} - 0.18\, p_1 + 0.08\, p_2 + 0.11\, p_3 \qquad (13.2)$$
$$V_{13} = 0.5\, V_{30} - 0.16\, p_1 + 0.07\, p_2 + 0.05\, p_3 \qquad (13.3)$$
$$V_{14} = 0.3\, V_{10} + 0.13\, p_1 - 0.30\, p_2 + 0.20\, p_3 \qquad (13.4)$$
$$V_{22} = 0.3\, V_{20} + 0.09\, p_1 - 0.15\, p_2 + 0.11\, p_3 \qquad (13.5)$$
$$V_{23} = 0.3\, V_{30} + 0.08\, p_1 - 0.14\, p_2 + 0.05\, p_3 \qquad (13.6)$$
$$V_{31} = 0.2\, V_{10} + 0.13\, p_1 + 0.13\, p_2 - 0.40\, p_3 \qquad (13.7)$$
$$V_{32} = 0.2\, V_{20} + 0.09\, p_1 + 0.08\, p_2 - 0.21\, p_3 \qquad (13.8)$$
$$V_{33} = 0.2\, V_{30} + 0.02\, p_1 + 0.07\, p_2 - 0.10\, p_3 \qquad (13.9)$$

These equations, upon addition three by three yield:

$$V_{10} = 0.5\, V_{00} - 0.59\, p_1 + 0.30\, p_2 + 0.36\, p_3 \qquad (13.10)$$
$$V_{20} = 0.3\, V_{00} + 0.30\, p_1 - 0.59\, p_2 + 0.36\, p_3 \qquad (13.11)$$
$$V_{30} = 0.2\, V_{00} + 0.30\, p_1 + 0.30\, p_2 - 0.71\, p_3 \qquad (13.12)$$

As two tests we may calculate the total expenditure flow of each center which has to add up identically to its income:

$$V_{11} + V_{21} + V_{31} \equiv V_{10}, \text{ etc. and}$$
$$V_{10} + V_{20} + V_{30} \equiv V_{00}$$

which, apart from rounding errors, appears to be the case. Substituting (13.10), (13.11), and (13.12) into the preceding equations, we obtain all flows V_{11} to V_{33} in terms of V_{00} and the three p's. Since we are free in our choice of V_{00}, we choose it 300, as in the previous example.

From the expressions for the V_{ik} we may now derive the total producers' value of each commodity, to be denoted by $p_1 v_{10}$, $p_2 v_{20}$ and $p_3 v_{30}$ respectively:

$$p_1\, v_{10} = \frac{V_{11}}{T_{11}} + \frac{V_{12}}{T_{12}} + \frac{V_{13}}{T_{13}} \qquad (13.13)$$

$$p_2 v_{20} = \frac{V_{21}}{T_{21}} + \frac{V_{22}}{T_{22}} + \frac{V_{23}}{V_{23}} \quad (13.14)$$

$$p_3 v_{30} = \frac{V_{31}}{T_{31}} + \frac{V_{32}}{T_{32}} + \frac{V_{33}}{T_{33}}. \quad (13.15)$$

Equating these expressions to what they should be according to the supply functions, we obtain:

$$132 - 0.58\,p_1 + 0.26\,p_2 + 0.43\,p_3 = 1.20\,p_1 - 0.30\,p_2 - 0.40\,p_3 \quad (13.16)$$

$$78 + 0.26\,p_1 - 0.54\,p_2 + 0.35\,p_3 = -0.18\,p_1 + 0.70\,p_2 - 0.21\,p_3 \quad (13.17)$$

$$45 + 0.25\,p_1 + 0.23\,p_2 - 0.55\,p_3 = -0.16\,p_1 - 0.14\,p_2 + 0.50\,p_3. \quad (13.18)$$

Upon solution these equations yield the following values for the prices:

$$p_1 = 276 \qquad p_2 = 273 \qquad p_3 = 246$$

from which all other variables can be computed. Of particular importance for our problem are the expressions (13.13) to (13.15), for which we obtain:

$$p_1 v_{10} = 151; \quad p_2 v_{20} = 89; \quad p_3 v_{30} = 41; \quad \text{total: } 281. \quad (13.19)$$

14. In order to determine the consequences of the improvement in transportation represented by matrix T' we have to repeat the calculation for the new values of the T_{ki}, where only T_{13} and T_{31} are affected. As will be seen from our equations (II 1) to (II 3), they do interfere both in the demand and the supply equations, however. It appears that the new values of the prices and the producers' values are:

$$p'_1 = 243 \qquad p'_2 = 244 \qquad p'_3 = 235$$

$$p'_1 v'_{10} = 147 \qquad p'_2 v'_{20} = 77 \qquad p'_3 v'_{30} = 55.$$

In order to make the appropriate comparison with the initial state we have, however, to value the new production values at the old prices, yielding:

$$p_1 v'_{10} = 167; \quad p_2 v'_{20} = 86; \quad p_3 v'_{30} = 58; \quad \text{total: } 311. \quad (14.1)$$

Comparing (13.19) and (14.1) we find an increase in national product of 30, or 10.7 per cent. It is interesting again to compare this result with the decrease in transportation costs on the flows actually in existence under the initial conditions. Since transportation costs were $(T_{13} - 1)\ (V_{13} + V_{31})$ and at the new transportation coefficients $(T'_{13} - 1)\ (V_{13} + V_{31})$, the decrease is $(T_{13} - T'_{13})\ (V_{13} + V_{31})$, which appears to be 8. The "multiplier" to be applied to this figure in order to arrive at the exact consequences of the transportation improvement therefore amounts, in this example, to $30/8 = 3.8$.

For the reader's convenience, the figures obtained in both the initial and the final state for the money flows are shown in Table 8. They have been calculated on the slide rule and their last digit is not accurate.

15. This note may be concluded with a few remarks on possible generalizations of the schemes shown. With a larger number of centers the transportation coefficients will, or at least may, show interdependence. There will be a certain road net and there may be alternative ways of transportation between any two centers. In particular, a road improvement project will influence the choice and the same road improvement may influence T_{ki} between several pairs of centers. All this must of course be mapped out carefully.

Scheme II may also be used for application to problems where one real (geographical) center produces more than one commodity: two theoretical (computational) centers may have the same location.

A complication requiring more adaptation of

TABLE 8. — APPROXIMATE VALUES OF MONEY FLOWS IN EXAMPLE II, INITIAL AND FINAL SITUATION

Initial				Final			
V_{ki}	$i = 1$	2	3	V'_{ki}	$i = 1$	2	3
$k = 1$	99	48	11	$k = 1$	92	42	28
2	50	39	9	2	40	37	9
3	9	10	27	3	29	7	21

the schemes derives from vertical disaggregation between industries. As the schemes are, all phases in the production of a final good have been assumed to be vertically integrated. Disaggregation would necessitate the introduction of demand functions for raw materials — to take the simplest case — in addition to final goods. While this can be done without too much difficulty, we have not attempted to bring this feature into the schemes of this note.