THE OPTIMUM RATE OF SAVING

1. THE most important choice before the government of a developing country is the choice concerning the rate of savings, *i.e.*, the proportion of national income to be devoted to investment, and hence not available for consumption. So far this choice has been made, it seems, on rather arbitrary grounds. In the free-enterprise economies of the nineteenth century it was left to the citizens individually, and the national average probably reflected the income distribution rather than any deliberate policy. In Russia, the first country to make a deliberate choice, it seems that no theoretical concepts have been at the basis of the choice. In the western democracies after the Second World War the situation was not very different in this respect: although saving was controlled to a large extent by the governments, the basis of the figures chosen was just as arbitrary.

Since the choice is of such a far-reaching impact on the economy, the question seems in place whether a more rational basis can be given to such a choice.

2. There need not be, in principle, any difference between the choice an individual makes and the choice to be made for a nation as a whole. For the sake of convenience we will use the terminology applying to an individual's choice. Two types of factors seem to enter into the problem: (i) those referring to the objective consequences of the choice, and (ii) those pertaining to the valuation of these consequences. The consequences of an extra unit of savings are: (*a*) a reduction of present consumption, and (*b*) an increase in future production and hence consumption possibilities. The valuation of these consequences will be based on the significance to the individual of present and future consumption, being the ultimate aim of economic activity. A description of both groups of factors will have to be based on certain simplifying assumptions. Since the results to be obtained will to some extent depend on the nature of these assumptions, it seems appropriate to vary them somewhat so as, if possible, to cover the most relevant situations that may arise.

3. Concerning the *objective consequences* of a variation in savings rate we make the assumption of the existence of a fixed "capital coefficient," *i.e.*, ratio between the additional investments and the resulting increase in future income. This coefficient we will indicate by K; it is well known that for long periods values of about 3 have been observed, whereas recently lower values down to $1\frac{1}{2}$ have been found. When using these figures we should not overlook, however, that they imply the assumption that all increases in incomes are due to increases in capital. This may be a considerable overestimate of what can be obtained by saving only. This state of affairs is illustrated also by the fact that the average saver gets much less increase in income as a consequence of his saving. A capital coefficient of 3 would

imply a yield of 33%, whereas the ordinary saver usually gets only a few %. Even the entrepreneur-saver will often have returns of no more than 10 or 20%. There may therefore be scope to apply considerably higher capital coefficients in our interpretations, up to figures of, say, 20.

Our formulæ will permit us to calculate alternative results, depending on the value of K. Indicating investment by J and income by Y, and assuming to start with that there is a negligible time lag between investment and the subsequent rate of increase $\dot{Y}$, we will have:

$$J = K\dot{Y} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In order not to make our problem too complicated, we will further assume that a decision on the savings rate will be taken once for all, meaning that the rate chosen will continue to be applied during all other time units. Indicating by c the proportion of income consumed, and by C the absolute quantity of consumption, we will therefore have:

$$C = cY \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since, in addition,

$$Y = C + J \quad . \quad . \quad . \quad . \quad (3)$$

the future development of consumption will be determined by these two coefficients:[1]

$$Y(1 - c) = K\dot{Y} \quad . \quad . \quad . \quad . \quad (4)$$

$$Y_t = Y_0 \, exp\left(\frac{1-c}{K}t\right) = Y_0 e^{\frac{1-c}{K}t} \quad . \quad . \quad . \quad (5)$$

$$C_t = cY_0 \, exp\left(\frac{1-c}{K}t\right) \quad . \quad . \quad . \quad . \quad (6)$$

Here the subscripts t and 0 refer to time periods.

An alternative assumption with regard to the objective consequences of savings is that there be a lag of one time unit (for which a year was taken when we gave numerical values for K), leading to slightly different equations:

$$J_t = K(Y_{t+1} - Y_t) \quad . \quad . \quad . \quad . \quad (1')$$

$$Y_{t+1} = \frac{1 - c + K}{K} Y_t \quad . \quad . \quad . \quad . \quad (4')$$

and

$$C_t = cY_0\left(\frac{1 - c + K}{K}\right)^t \quad . \quad . \quad . \quad . \quad (6')$$

4. As to the *valuation* by the individual (or the nation) of the results to be obtained from whatever decision on savings, we assume that the criterion will be the "utility" to be derived from all consumption, present and future. This utility we assume to be measurable, and to have the following characteristics:

(i) it is the sum total of the utilities to be derived from the consumption in each of the infinite series of time units;

(ii) it diminishes, for the same volume of consumption per time

[1] The reader will observe the similarity of this treatment to that given by R. F. Harrod, "An Essay in Dynamic Theory," ECONOMIC JOURNAL, XLIX (1939), p. 14.

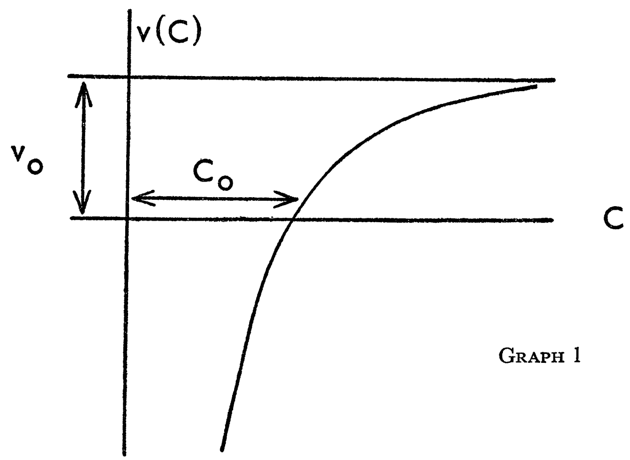
unit, if the time of consumption becomes more remote, proportional to a " discount factor " n^{-t}, where $n - 1$ may be called the " psychological discount rate," comparable to an interest rate;

(iii) it increases, for a given time of consumption, with increasing volume C of consumption, but proportional to a function $v(C)$ showing decreasing marginal rates of increase; *i.e.*, $v'(C) > 0$, $v''(C) < 0$. More specifically two functions $v(C)$ have been used alternatively, namely:

(*a*) $$v(C) = \log C \qquad (7)$$

(*b*) $$v(C) = v_0\left(\frac{1 - C_0}{C^E}\right) \qquad (7')$$

The former of these two functions needs no further explanation; evidently the marginal utility $v'(C) = \frac{1}{C}$ diminishes with increasing C, but never becomes zero, whereas $v(C)$ has no upper boundary. The latter of the functions, illustrated by graph 1, is a hyperbola with a horizontal asymptote



GRAPH 1

at $v(C) = v_0$, *i.e.*, this $v(C)$ has an upper boundary. Both have in common that for no finite value of C, $v'(C) = 0$, *i.e.*, saturation is not supposed to be reached for any finite value of C. In the second function an arbitrary constant E has been included, enabling us to produce alternative shapes of curves. For large values of E the saturation level will be approached more quickly than for small values. For $E = 1$ the hyperbola is equilateral. The value of E will influence the marginal utility of consumption, on which statistical estimates have been made by Frisch.[1] Indicating the flexibility of the marginal utility of consumption by f, we have

$$f = \frac{C}{v'}\frac{\partial v'}{\partial C} = -(E + 1)$$

According to Frisch's findings, this flexibility was about —3·5 for French

[1] R. Frisch, *New Methods of Measuring Marginal Utility* (Tübingen, 1931): *Statistical Confluence Analysis by Means of Complete Regression Systems* (Universitetets Økonomiske Institutt, Publikasjon nr 5, Oslo, 1934).

workers and about -1 for American workers. The corresponding values for E are $E = 2{\cdot}5$ and $E = 0$ respectively.

Reference should be made finally to the well-known article about our subject published already in 1928 by Ramsey.[1] Ramsey, in his most imaginative pioneering approach, laid more stress on the existence of some finite income for which a state of complete satisfaction (" bliss ") would be attained. His approach is somewhat less easy for numerical application, however, since it is difficult to make numerical assumptions on this income; but for high values of E the value v_0 will be " almost " reached for " low " values of C already.

Indicating the utility of all consumption by U, we will have, according to our characteristics, in case (a):

$$U_a = \int_0^\infty \frac{\log C_t}{n^t} dt \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and in case (b):

$$U_b = v_0 \int_0^\infty \left(1 - \frac{C_0}{C_t^E}\right) \frac{dt}{n^t} \quad . \quad . \quad . \quad (8')$$

5. We will now investigate the question for what values of c, with given values of n and K, these expressions for U are a maximum. In order to give the answer we evidently have to substitute either (6) or (6′) into (8) or (8′), to carry out the integration process, and subsequently to determine the maximum as a function of c. We will consider only three out of the four possible cases, since these will supply us with a sufficient insight into the answer. These cases will be numbered I (using (6) and (8)), II ((6) and (8′)) and III ((6′) and (8′)).

Case I. Here we find:

$$U_a = \int_0^\infty \frac{\log Y_0}{n^t} dt + \int_0^\infty \frac{\log c}{n^t} dt + \int_0^\infty \frac{(1-c)t}{n^{tK}} dt \quad . \quad (9)$$

We will use two auxiliary formulæ for two integrals:

$$\int_0^\infty \frac{dt}{n^t} = \frac{1}{\log n} \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\int_0^\infty \frac{tdt}{n^t} = \frac{1}{\log^2 n} \quad . \quad . \quad . \quad . \quad (11)$$

With their aid we find:

$$U_a = \frac{\log Y_0 + \log c}{m} + \frac{1-c}{K}\frac{1}{m^2} \quad . \quad . \quad . \quad (12)$$

where m has been written for $\log n$; it so happens that approximately m equals the interest rate $n - 1$, since for small m, $\log(1 + m) = m$. Even for a value of $m = 0{\cdot}3$, $\log(1 + m) = 0{\cdot}26$, meaning that *qua* order of magnitude the approximation even holds for such high values as $0{\cdot}3$.

[1] F. P. Ramsey, " A Mathematical Theory of Saving," Economic Journal, XXXVIII (1928), p. 543.

In order to find the optimum value of c we have to determine the maximum of U_a with respect to c; this optimum satisfies the relation

$$\frac{dU_a}{dc} = 0$$

which leads to

$$\frac{m}{c} = \frac{1}{K} \text{ or } c = mK \quad . \quad . \quad . \quad . \quad . \quad (13)$$

With $m = 0{\cdot}05$ and $0{\cdot}10$ and $K = 2$, 3 and 10 respectively, we find:[1]

	m	=	0·05	0·10	0·30
$K=$					
2	c	=	0·10	0·20	0·60
3		=	0·15	0·30	0·90
10		=	0·50	1·00	.
20		=	1·00	.	.

For the lower values of K and m these figures are amazingly low: they correspond to savings rates between 0·9 and 0·7. In order to obtain the value most commonly observed, *i.e.*, $c = 0{\cdot}9$, we would have to assume either values for K of 9 and higher, or, if $K = 3$, a psychological discount rate of $\frac{0{\cdot}9}{K}$, *i.e.*, 0·3.

6. *Case II.* Here,

$$U_b = v_0 \int_0^\infty \frac{dt}{n^t} - \frac{v_0 C_0}{c Y_0} \int_0^\infty \frac{dt}{\left(e^{\frac{1-c}{K}E} n\right)^t} \quad . \quad . \quad . \quad (9')$$

which, with the aid of (10) and (11), reduces to:

$$U_b = \frac{v_0}{\log n} - \frac{v_0 C_0}{c^E Y_0^E} \cdot \frac{1}{\log n + E \dfrac{1-c}{K}} \quad . \quad . \quad (12')$$

Writing m again for log n, we will have to look for the value of c which renders

$$c^E \left(m + \frac{1-c}{K} E \right)$$

a maximum. We find

$$c = \frac{Km + E}{E + 1} \quad . \quad . \quad . \quad . \quad . \quad (13')$$

For $E = 0$ this formula coincides with (13). For some higher values for E and the same values of K and m as considered above we find:

	$E = 1$				$E = 2{\cdot}5$		
$K =$	$m =$	0·05	0·10	0·30	0·05	0·10	0·30
2	$c =$	0·55	0·60	0·80	0·74	0·77	0·89
3		0·58	0·65	0·95	0·76	0·80	0·97
10		0·75	1·00	.	0·86	1·00	.
20		1·00	.	.	1·00	.	.

[1] Dots will be shown where values for $c > 1$ are found; they will have to be replaced by the boundary condition $c = 1$.

Again for the lower values of K and m and $E = 1$, we find consumption quota much lower than observation yields. For $E = 2{\cdot}5$ more realistic figures are found. Evidently we can obtain figures as close to 1 as we want if only E is taken high enough, and then irrespective of what m and K are. Figures near 1 may also be obtained for the values of K approaching 10 or for values of m approaching 0·30.

Case III. Here finally

$$U_b = v_0\int_0^{\infty}\frac{dt}{n^t} - \frac{v_0C_0}{c\,Y_0}\int_0^{\infty}\frac{dt}{\left(n\,\dfrac{1-c+K}{K}\right)^t} \qquad (9'')$$

or

$$U_b = \frac{v_0}{m} - \frac{v_0C_0}{cY_0}\,\frac{1}{m + \log\dfrac{1-c+K}{K}} \qquad (12'')$$

Now the expression $cm + c \log (1 - c + K) - c \log K$ should be a maximum. This leads to a transcendental equation for c which for $m = 0{\cdot}1$ and $K = 3$ yields an optimum value of $c = 0{\cdot}7$. Again we find in order to obtain a value for c equal to almost 1 we have to assume a value of m of some 0·3 or more.

7. Some *tentative conclusions* may be drawn. They will have to be tentative, since evidently quite some influence appears to be exerted by: (i) the form of the utility function assumed; (ii) the values given to K; (iii) the values given to m; and (iv) those given to E.

(i) As regards the *utility function*, it would seem that our second function already somewhat overemphasises the phenomenon of diminishing marginal utility, since it assumes a ceiling for utility which according to many economists does not exist. The existence of such a ceiling will reduce the attractiveness to save, since additional units of future consumption will not add very much to U any more. The saving rates found with U_b therefore tend already, in our opinion, to be under-estimated.

(ii) The values to be given to the *capital coefficient* K, we saw, have to depend on whether we assume that, together with increased savings, there will be forthcoming the increased technological knowledge, organisation, etc., that, in the history of developed countries, accompanied the growth of their capital assets. In other words, whether together with savings the growth of knowledge and organisation can also be accelerated. If this is assumed to be the case, we should give to K the values of about 3 or even a little less; if not, K may be considerably higher, up to 20.

(iii) The values to be given to the *psychological discount rate* m may for individuals be quite high, as some well-known facts about money-lending rates show. This evidently particularly applies to individuals living near starvation. For a nation as a whole some would even maintain that it should be taken equal to zero.

(iv) The values of E evidently also depend on the level of well-being, and may therefore not be, at least for individuals, independent of the values for m.

Our first tentative conclusion from the tables shown is that optimum savings rates may well be considerably higher than observed saving rates, especially for developed countries (where E and m are low).

This conclusion also would apply for all countries if it is maintained that m should be taken equal to almost zero. The conclusion does not, however, apply if, with moderate values of m, K is considered to be above 10 anyhow.

A second tentative conclusion is that in order to explain observed personal rates of saving prevailing in most countries we must assume that either the capital coefficient is supposed to be much higher than the values usually now presented or the psychological discount rate is considerably higher than normal interest rates.

A third and last tentative conclusion is that the preceding conclusions are not influenced very much by a possible longer lag between investment and the resulting increase in income. This is shown, on the one hand, by a comparison of Cases II and III; on the other hand, it follows from a change in time units, which we might undertake. Suppose we take two years instead of one year as our time unit. This would reduce by 50% the value of K and raise by about 100% the value of m. Their product would not change appreciably, and hence the optimum values found for c would not either.

8. Some *final remarks* have to be added. The problem discussed is a well-known problem with some well-known difficulties. If the two characteristics (ii) and (iii) of the utility function introduced in Section 4 are neglected, the problem becomes intractable; if total consumption over an infinite period is to be maximised, the maximum will be found to correspond to the highest savings rate possible; *i.e.*, 1, which evidently is an unrealistic proposition. Even if the discount factor is introduced, this difficulty does not vanish. For values of m below the rate of increase in income again the optimum rate of savings will be 1. For values of m above that rate it will suddenly fall down to zero. This evidently even qualitatively is not a realistic picture of the choice problem before us. These experiences should be kept in mind when the idea of an optimum rate of savings is loosely being discussed in terms of a " maximum income or maximum consumption over a certain period." Only the introduction of both the characteristics seems to make the problem a tractable one.

Admittedly the approach chosen here is in many respects oversimplified still. It has been assumed that only one choice is made with regard to c and that this is a choice for all times. Evidently a more flexible approach is conceivable, where c would change as time goes on. Also the utility function might express two features not introduced explicitly—although present implicitly in our models: the feature of a minimum subsistence level and the one of a possible ceiling to consumption. As we already stated, we do not think that our conclusions will be invalidated by such refinements; but we have to admit that further research might reveal such limitations. The purpose of this paper is no other than to stimulate this further research.

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