WEIGHTED CONSTRAINTS IN FUZZY OPTIMIZATION

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Abstract	Many practical optimization problems are characterized by some flexibility in the problem constraints, where this flexibility can be exploited for additional trade-off between improving the objective function and satisfying the constraints. Especially in decision making, this type of flexibility could lead to workable solutions, where the goals and the constraints specified by different parties involved in the decision making are traded off against one another and satisfied to various degrees. Fuzzy sets have proven to be a suitable representation for modeling this type of soft constraints. Conventionally, the fuzzy optimization problem in such a setting is defined as the simultaneous satisfaction of the constraints and the goals. No additional distinction is assumed to exist amongst the constraints and the goals. This report proposes an extension of this model for satisfying the problem constraints and the goals, where preference for different constraints and goals can be specified by the decision-maker. The difference in the preference for the constraints is represented by a set of associated weight factors, which influence the nature of trade-off between improving the optimization objectives and satisfying various constraints. Simultaneous weighted satisfaction of various criteria is modeled by using the recently proposed weighted extensions of Archimedean) fuzzy t-norms. The weighted satisfaction of the problem constraints and goals are demonstrated by using a simple general, and it can also be applied to fuzzy mathematical programming problems and multi-objective fuzzy optimization.	
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Weighted Constraints in Fuzzy Optimization

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Abstract

Many practical optimization problems are characterized by some flexibility in the problem constraints, where this flexibility can be exploited for additional trade-off between improving the objective function and satisfying the constraints. Especially in decision making, this type of flexibility could lead to workable solutions, where the goals and the constraints specified by different parties involved in the decision making are traded off against one another and satisfied to various degrees. Fuzzy sets have proven to be a suitable representation for modeling this type of soft constraints. Conventionally, the fuzzy optimization problem in such a setting is defined as the simultaneous satisfaction of the constraints and the goals. No additional distinction is assumed to exist amongst the constraints and the goals. This report proposes an extension of this model for satisfying the problem constraints and the goals, where preference for different constraints and goals can be specified by the decision-maker. The difference in the preference for the constraints is represented by a set of associated weight factors, which influence the nature of trade-off between improving the optimization objectives and satisfying various constraints. Simultaneous weighted satisfaction of various criteria is modeled by using the recently proposed weighted extensions of (Archimedean) fuzzy t-norms. The weighted satisfaction of the problem constraints

and goals are demonstrated by using a simple fuzzy linear programming problem. The framework, however, is more general, and it can also be applied to fuzzy mathematical programming problems and multi-objective fuzzy optimization.

Keywords

Fuzzy constraints, weighted aggregation, weighted t-norms, fuzzy optimization, soft decision making.

1 Introduction

Optimization is an important activity in many fields of science and engineering. A lot of modeling, design, control and decision making problems can be formulated in terms of mathematical optimization. The classical framework for the optimization is the minimization (or maximization) of the objectives, given the constraints for the problem to be solved. Many design problems, however, are characterized by multiple objectives, where a trade-off amongst various objectives must be made, leading to under or over-achievement of different objectives. Moreover, some flexibility may be present for specifying the constraints of the problem. Furthermore, some of the objectives in decision making may be known only approximately. In management decisions, for instance, many of the objectives can be expressed approximately in linguistic terms, but a precise mathematical formula is not available. Also, the decision constraints may be relaxed in some situations, as long as the decision objectives can be improved. These types of problems require an extension of the classical optimization and constraint framework in order to deal with the flexibility of the constraints and with the approximate specification of the objectives.

Fuzzy set theory provides ways of representing and dealing with the flexible or soft constraints, in which the flexibility in the constraints can be exploited to obtain additional trade-off between improving the objectives and satisfying the constraints. Various fuzzy optimization methods have been proposed in the literature in order to deal with different aspects of soft constraints. In one formulation of fuzzy optimization due to Zimmermann [21], concepts from Bellman and Zadeh model of fuzzy decision making [1] are used for formulating the fuzzy optimization problem. In this formulation, fuzzy sets represent both the (aspired) problem goals

and the flexible (soft) constraints. The optimal trade-off amongst the problem goals and the constraints is determined by the maximizing fuzzy decision, in which the optimal decision is found by maximizing the simultaneous satisfaction of the optimization objectives and the constraints. The asymmetry between the problem goals and the problem constraints disappears in this formulation, and the fuzzy goals and the constraints are aggregated to a single function that is maximized. It should be noted that this framework is general enough to handle crisp constraints as well as fuzzy constraints.

In the fuzzy optimization model of Zimmermann, simultaneous satisfaction of the decision goals and the constraints is sought. No further distinction is made amongst the constraints and the goals. When there is a possibility to make a trade-off between improving the objective and satisfying the constraints, however, the user of the optimization algorithm (i.e the designer, the decision maker, the controller, etc.) can choose to trade a particular constraint or goal preferentially with respect to the other ones. Within the classical framework, constraints of different importance are distinguished by ordering them hierarchically according to their importance and to admit them into the optimization problem one by one, often by first starting with the most constraining set and then gradually removing the constraints one at a time. In addition to the more conventional hierarchical ordering approach, fuzzy optimization admits another model for dealing with the preference structure imposed on a constraint set by introducing weight factors that represent the importance of the constraints for the optimization problem. Since there is no distinction between the fuzzy goals and the fuzzy constraints in Zimmermann's formulation of fuzzy optimization, the weight factors can also be applied to the optimization objectives. This report extends Zimmermann's fuzzy optimization framework with weighted aggregation of the fuzzy objectives and the fuzzy constraints. Within the extended framework, the trade-off amongst the objectives and various constraints can be influenced by changing the associated weight factors. Recently proposed weighted extensions of fuzzy t-norm operators are used for the aggregation [7].

The proposed framework is rather general, and it can be applied to various fuzzy non-linear programming problems with multiple objectives and constraints. In this article, the application of the framework to fuzzy linear programming is considered. The main concepts are illustrated by using a small optimization problem as an example. It is assumed that a general optimization algorithm is available and has been implemented for performing the final (crisp) optimization

in order to obtain the optimal solution to the fuzzy optimization problem. Various well-known algorithms with different complexity can be used for this purpose. Examples are interior-point methods, sequential quadratic programming, exhaustive search or even heuristic search. For the working example in this report, an optimization algorithm based on Nelder and Mead's simplex minimizer [14] is used.

The outline of the report is as follows. Section 2 describes the general fuzzy optimization framework used in the remainder of this report. Section 3 discusses the application of the fuzzy optimization framework of Section 2 to fuzzy linear programming problems. Zimmermann's solution to fuzzy linear programming is presented. Weighted aggregation of fuzzy sets is introduced in Section 4. Popular methods for weighted fuzzy aggregation are considered as well as proposals based on recent developments in the field of fuzzy aggregation. The weighted aggregation methods considered are used in Section 5 for formulating the weighted combination of constraints within the fuzzy linear programming framework. The proposed formulation is illustrated in Section 6 by presenting a small example. Finally, conclusions are given in Section 7.

2 Fuzzy Optimization

Fuzzy optimization is the name given to the collection of techniques that formulate optimization problems with flexible, approximate or uncertain constraints and goals by using fuzzy sets. In general, fuzzy sets are used in two different ways in fuzzy optimization.

1. To represent uncertainty in the constraints and the goals (objective functions).

2. To represent flexibility in the constraints and the goals.

In the first case, fuzzy sets represent generalized formulations of intervals that are manipulated according to rules which are extensions of the interval calculus by using the α-cuts of fuzzy sets. In the second case, fuzzy sets represent the degree of satisfaction of the constraints or of the aspiration levels of the goals, given the flexibility in the formulation. Hence, the constraints (and the goals) that are essentially crisp are assumed to have some flexibility that can be exploited for improving the optimization objective. This framework is suitable for the representation of interaction and possible trade-off amongst the constraints and the objectives of the optimization, as discussed in this report. Consequently, the remainder of the report considers the latter case,

where the fuzzy sets represent the flexibility in the constraints and the goals. Further, the term *fuzzy optimization* also refers to a formulation in terms of the flexibility of the constraints.

The general formulation for *fuzzy optimization* in the presence of flexible goals and constraints is given by

$$\begin{aligned} &\text{fuzzy maximize}_{\mathbf{x}\in X} \ [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})] \\ &\text{subject to} \qquad\quad g_i(\mathbf{x}) \tilde{\leq} 0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{1}$$

In (1), the tilde sign denotes a fuzzy satisfaction of the constraints. The sign $\tilde{\leq}$ thus denotes that $g_i(\mathbf{x}) \leq 0$ can be satisfied to a degree smaller than 1. The fuzzy maximization corresponds to achieving the highest possible aspiration level for the goals $f_1(\mathbf{x})$ to $f_n(\mathbf{x})$, given the fuzzy constraints to the problem. This optimization problem can be solved by using the approach of Bellman and Zadeh to fuzzy decision making [1].

Consider a decision making problem where the decision alternatives are $\mathbf{x} \in X$. A fuzzy goal F_j, $j = 1, 2, \ldots, n$ is a fuzzy subset of X. Its membership function $F_j(\mathbf{x})$, $\mathbf{x} \in X$, with $F_j : X \to [0, 1]$ indicates the degree of satisfaction of the decision goal by the decision alternative $\mathbf{x} \in X$. Similarly, a number of fuzzy constraints G_i, $i = 1, 2, \ldots, m$ can be defined as fuzzy subsets of X. Their membership functions $G_i(\mathbf{x})$, $\mathbf{x} \in X$ denote the degree of satisfaction of the fuzzy constraint G_i by the decision alternative $\mathbf{x} \in X$. According to Bellman and Zadeh's fuzzy decision making model, the fuzzy decision D is defined as the confluence of the fuzzy goals and constraints, i.e.

$$D(\mathbf{x}) = F_1(\mathbf{x}) \circ F_2(\mathbf{x}) \circ \cdots \circ F_n(\mathbf{x}) \circ G_1(\mathbf{x}) \circ G_2(\mathbf{x}) \circ \cdots \circ G_m(\mathbf{x}), \tag{2}$$

where $\circ$ denotes an aggregation operator for fuzzy sets. Since the goals and the constraints must be satisfied simultaneously, Bellman and Zadeh proposed to use an intersection operator, i.e. a fuzzy t-norm for the aggregation. The optimal decision alternative $\mathbf{x}^*$ is then the argument that maximizes the fuzzy decision, i.e.

$$\mathbf{x}^* = \arg\max_{\mathbf{x}\in X} D(\mathbf{x}). \tag{3}$$

The optimization problem is then defined by

$$\max_{\mathbf{x}\in X} F_1(\mathbf{x}) \wedge \cdots \wedge F_n(\mathbf{x}) \wedge G_1(\mathbf{x}) \wedge \cdots \wedge G_m(\mathbf{x}). \tag{4}$$

Note that both the goals and the constraints are aggregated. Hence, the goals and the constraints are treated equivalently, which is why the model is said to be symmetric.

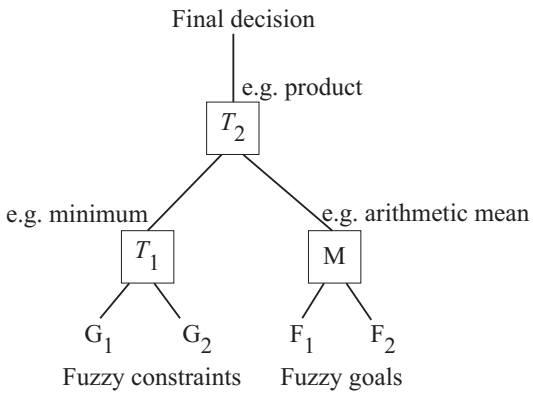


Figure 1: Hierarchical aggregation of goals and constraints.

The symmetric model is not always appropriate, however, since the aggregation of the goals and the constraints may have different requirements. Often, for example, some trade off amongst the goals is allowed or may even be desirable, which may be modeled by an averaging operation. The aspiration level for some goals may then be unreachable, but this is not a problem if compensation is allowed amongst the goals. The constraints, however, should not be violated, i.e. their aggregation must be conjunctive. In that case, the goals and the constraints can not be combined uniformly by using a single aggregation operator. In the simplest case, the goals must be combined by using one operator and the constraints must be combined by using another operator. The aggregated results must then be combined at a higher level by using a third aggregation operator, which has to be conjunctive (i.e. both the aggregated goals and the aggregated constraints should be satisfied). This leads to the hierarchical aggregation scheme like the one shown in Fig. 1. An example of an application of this scheme can be found in [15].

Clearly, the above formulation of fuzzy optimization is closely related to the penalty function methods known from classical optimization theory. The aggregated goals correspond to an overall objective function, which is maximized. The constraints extend this objective function by using fuzzy t-norms. This approach is similar to the addition of a penalty function to an optimization objective function in classical optimization. After combining the objectives and the constraints, the resulting optimization is unconstrained, but possibly non-convex. Furthermore, gradient descent methods may not be suitable for the maximization due to possible and likely discontinuity in the first derivative of the final aggregated function. Derivative-free search and optimization algorithms such as genetic algorithms [5], simulated annealing [12], branch-

and-bound [2] or Nelder and Mead's simplex algorithm [14] can be used to solve this type of optimization problems.

3 Fuzzy Linear Programming

Fuzzy linear programming can be viewed as a special case of the general multiple objective multiple constraint fuzzy optimization. Let there be n decision variables. The general *fuzzy linear programming* (FLP) problem is then formulated as

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{fuzzy maximize}} && \mathbf{c}^T\mathbf{x} \\ &\text{subject to} && \mathbf{A}\mathbf{x} \tilde{\leq} \mathbf{b} \\ & && \mathbf{x} \geq \mathbf{0} \end{aligned} \quad . \tag{5}$$

Zimmermann [21] has considered the fuzzy linear programming problem formulated as a symmetric problem in terms of (4). In this formulation, the vectors $\mathbf{c}$ and $\mathbf{b}$ as well as the matrix $\mathbf{A}$ have crisp elements. The fuzziness arises because of the definition of fuzzy maximization and the approximate inequality $\tilde{\leq}$. These are defined by the fuzzy goal and the fuzzy constraints whose membership functions represent the degree to which $\mathbf{x} \in \mathbb{R}^n$ satisfies the fuzzy goal or the fuzzy constraints. The membership function of the fuzzy goal is given by $F(\mathbf{c}^T\mathbf{x})$, while the membership functions of the fuzzy constraints are given by $G_i(\mathbf{a}_i^T\mathbf{x})$, $i = 1, 2, \ldots, m$, where $\mathbf{a}_i^T$ represents row i of the matrix $\mathbf{A}$. The optimal vector $\mathbf{x}^*$ is found by

$$D(\mathbf{x}^*) = \sup_{\mathbf{x}\in\mathbb{R}^n} F(\mathbf{c}^T\mathbf{x}) \wedge G_1(\mathbf{a}_1^T\mathbf{x}) \wedge \cdots \wedge G_m(\mathbf{a}_m^T\mathbf{x}). \tag{6}$$

Usually, shouldered trapezoidal fuzzy sets are used as membership functions. The fuzzy constraints G_i are then defined by their membership function

$$G_i(\mathbf{a}_i^T\mathbf{x}) = \begin{cases} 1 & \mathbf{a}_i^T\mathbf{x} < b_i \\ \dfrac{p_i + b_i - \mathbf{a}_i^T\mathbf{x}}{p_i} & b_i \leq \mathbf{a}_i^T\mathbf{x} \leq b_i + p_i \\ 0 & b_i + p_i < \mathbf{a}_i^T\mathbf{x} \end{cases} \quad . \tag{7}$$

Next, a fuzzy set representing the satisfaction of the aspiration level for the objective is specified as

$$F(\mathbf{c}^T\mathbf{x}) = \begin{cases} 1 & z_u < \mathbf{c}^T\mathbf{x} \\ \dfrac{\mathbf{c}^T\mathbf{x} - z_l}{z_u - z_l} & z_l \leq \mathbf{c}^T\mathbf{x} \leq z_u \\ 0 & \mathbf{c}^T\mathbf{x} < z_l \end{cases} \quad . \tag{8}$$

The coefficients z_l and z_u are obtained by solving the conventional linear programming problems

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b} \ , \\ &\mathbf{x} \geq \mathbf{0} \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b}+\mathbf{p} \ , \\ &\mathbf{x} \geq \mathbf{0} \end{aligned} \tag{10}$$

respectively, where $\mathbf{p}$ is a vector of relaxation coefficients p_i, $i = 1, 2, \ldots, m$.

Let now

$$\mathbf{a}_0^T = \mathbf{c}^T,$$

and

$$G_0(\mathbf{a}_0^T\mathbf{x}) = \begin{cases} 1 & z_0 < \mathbf{a}_0^T\mathbf{x} \\ \dfrac{\mathbf{a}_0^T\mathbf{x} + p_0 - z_0}{p_0} & z_0 - p_0 \leq \mathbf{a}_0^T\mathbf{x} \leq z_0 \\ 0 & \mathbf{a}_0^T\mathbf{x} < z_0 - p_0 \end{cases},$$

with $z_u = z_0$ and p_0 the relaxation of the aspiration level of the goal $\mathbf{c}^T\mathbf{x}$. We have then $G_0(\mathbf{a}_0^T\mathbf{x}) = F(\mathbf{c}^T\mathbf{x})$. The solution to the fuzzy linear programming is now given by the conjunction of all fuzzy sets as

$$D(\mathbf{x}) = \bigwedge_{i=0}^{m} G_i(\mathbf{a}_i^T\mathbf{x}), \tag{11}$$

where $\bigwedge$ is the minimum operator. The solution is found by seeking an optimal $\mathbf{x}^* \in \mathbb{R}^n$ such that

$$D(\mathbf{x}^*) = \sup_{\mathbf{x}\in\mathbb{R}^n} D(\mathbf{x}). \tag{12}$$

After introducing an additional variable λ, the solution to the optimization (12) can be found by solving the conventional linear programming problem [21]

$$\begin{aligned} \text{maximize} \quad & \lambda \\ \text{subject to} \quad & \mathbf{c}^T\mathbf{x} \geq z_0 - (1-\lambda)p_0 \\ & \mathbf{a}_i^T\mathbf{x} \leq b_i + (1-\lambda)p_i, \quad i = 1, 2, \ldots, m \\ & \mathbf{x} \geq \mathbf{0} \\ & \lambda \geq 0 \\ & \lambda \leq 1 \end{aligned} \ . \tag{13}$$

The optimization (12) can not be reduced to the problem (13) when the membership functions are not trapezoidal, or when t-norms other than the minimum are used for the aggregation. In that case, the optimization must be solved by a more general optimization algorithm that can deal with nonlinear programming problems such as sequential quadratic programming [4].

4 Weighted Fuzzy Aggregation

Weighted aggregation has been used quite extensively especially in fuzzy decision making, where the weights are used to represent the relative importance that the decision maker attaches to different decision criteria. Almost always an averaging operator has been used for the weighted aggregation, such as the generalized means [3, 16], fuzzy integrals [6] or the ordered weighted average (OWA) operators [19]. Consequently, the weighted aggregation of fuzzy sets has been studied with averaging type of operators. The generalized means extend naturally to weighted equivalents. The weighted generalized mean operator has been used in many fields, and it has been studied in the context of fuzzy set aggregation in [3, 9, 10]. The OWA operators and the fuzzy integrals are inherently weighted operators, which do not need a separate extension to the weighted case. Applications of these operators have also been reported in the literature (see e.g. [6, 20]).

The averaging operators are suitable for modeling compensatory aggregation. They are not suitable, however, for modeling simultaneous satisfaction of aggregated criteria. Since the goal in fuzzy optimization is the simultaneous satisfaction of the optimization objectives and the constraints, t-norms must be used to model the conjunctive aggregation. In order to use the weighted aggregation in fuzzy optimization, weighted aggregation using t-norms must thus be considered.

The axiomatic definition of t-norms does not allow for weighted aggregation. In order to obtain a weighted extension of t-norms, some of the axiomatic requirements must be dropped. Especially the commutativity and the associativity properties must be dropped, since weighted operators are by definition not commutative. The commutativity and the associativity requirements must be relaxed to hold only in case of equal weight factors, which is a special case of weighted aggregation.

Weighted aggregation of fuzzy sets by using t-norms has been considered first by Yager

in [17]. He proposed to modify the membership functions with the associated weight factors before the fuzzy aggregation. The weighted aggregation is then the aggregation of the modified membership functions. A generalized form of this idea leads to the weighted aggregation function [18]

$$D(\mathbf{x}, \mathbf{w}) = T[I(G_1(\mathbf{x}), w_1), I(G_2(\mathbf{x}), w_2), \ldots, I(G_m(\mathbf{x}), w_m)], \tag{14}$$

where $\mathbf{w}$ is a vector of weight factors $w_i \in [0, 1]$, $i = 1, 2, \ldots, m$ associated with the aggregated membership functions $G_i(\mathbf{x})$, T is a t-norm and I is a function of two variables that transforms the membership functions. Usually, the power-raising method is used for the transformation and the minimum operator for the t-norm, so that the aggregation function becomes

$$D(\mathbf{x}, \mathbf{w}) = \bigwedge_{i=1}^{m} [G_i(\mathbf{x})]^{w_i} \tag{15}$$

The weighted t-norm aggregation (14) and its special cases like (15) have long been motivated on intuitive grounds [17], and an axiomatic framework for the extension of t-norms with weight factors has been unavailable. Nevertheless, the aggregation function (15) is quoted often (see e.g. [13]) in various publications regarding fuzzy weighted aggregation, especially in multicriteria decision making, without regard to mathematical analysis for the requirements regarding the implication functions I and the conditions under which they can be applied. This is possibly one reason why weighted aggregation of fuzzy sets has not been considered in fuzzy optimization previously.

Recently, weighted aggregation of fuzzy sets has been investigated in more detail in a generalized framework [7, 10, 11], where weighted counterparts of fuzzy t-norms have also been proposed based on a sensitivity analysis of weighted fuzzy aggregation. The analysis provides a general mechanism for introducing weight factors into Archimedean t-norms and t-conorms by considering several requirements that can be imposed on a weighted aggregation operator. The results are extensions of fuzzy aggregation operators such as the t-norms and the t-conorms to their weighted counterparts. The analysis indicates, for example, that the power raising method, apart from the idempotent case, can only be used with strict Archimedean t-norms, but not with the nilpotent t-norms. An application of the weighted counterparts of Archimedean t-norms can be found in [8].

Weighted counterparts of several Archimedean t-norms as studied in [7] are used in this report. The specific operators considered are the weighted extension of the product t-norm

given by

$$D(\mathbf{x}, \mathbf{w}) = \prod_{i=1}^{m} [G_i(\mathbf{x})]^{w_i}, \tag{16}$$

the extension of the Hamacher t-norm given by

$$D(\mathbf{x}, \mathbf{w}) = \begin{cases} \dfrac{1}{1 + \sum_{i=1}^{m} w_i \dfrac{1 - G_i(\mathbf{x})}{G_i(\mathbf{x})}} & \text{if } \forall i, G_i(\mathbf{x}) > 0 \\ 0 & \text{if } \exists i, G_i(\mathbf{x}) = 0 \end{cases} \tag{17}$$

and the extension of the Yager t-norm given by

$$D(\mathbf{x}, \mathbf{w}) = \max\left(0, 1 - \sqrt[s]{\sum_{i=1}^{m} w_i (1 - G_i(\mathbf{x}))^s}\right), \quad s > 0. \tag{18}$$

Note that the extension (16) of the product t-norm according to the sensitivity based analysis is the same as the application of (14) with the product operator as T and the power raising as I. However, the extensions (17) and (18) can not be obtained from (14).

5 Fuzzy Linear Programming with Weighted Aggregation

In fuzzy optimization, the importance of the constraints can be used to indicate which constraints should be satisfied preferentially, in a similar fashion to the aggregation of decision criteria in fuzzy decision making. Since it is possible to satisfy a constraint partially in fuzzy optimization, the weight factors indicate to what degree various constraints can be interchanged. The extension of fuzzy linear programming with weighted aggregation follows naturally from the formulation in (6). The fuzzy linear programming problem is formulated in the usual way by specifying the constraints, the flexibility in the constraints (fuzzy sets representing the allowed relaxation) and the coefficients for the objective function from which the aspiration level for the optimization goal is computed. Additionally, the user must now specify a set of weight factors that indicate the importance of the corresponding constraints or the objective in the fuzzy aggregation used for fuzzy optimization.

Following the notation in (11), note that there are $m + 1$ fuzzy sets to be aggregated in the problem, where m is the number of constraints. Hence, $m + 1$ weight factors must be specified. Furthermore, the membership functions are functions of $\mathbf{a}_i^T\mathbf{x}$, i.e. $G_i(\mathbf{a}_i^T\mathbf{x})$, $i = 0, 1, \ldots, m$

with $\mathbf{a}_i \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. The solution to the fuzzy linear programming in the presence of weight factors is then given by the weighted conjunction of all fuzzy sets according to

$$D(\mathbf{x}, \mathbf{w}) = T\left(\mathbf{w}, G_0(\mathbf{a}_0^T\mathbf{x}), G_1(\mathbf{a}_1^T\mathbf{x}), \ldots, G_m(\mathbf{a}_m^T\mathbf{x})\right). \tag{19}$$

The solution is then found by seeking an optimal $\mathbf{x}^* \in \mathbb{R}^n$ such that

$$D(\mathbf{x}^*, \mathbf{w}) = \sup_{\mathbf{x}\in\mathbb{R}^n} D(\mathbf{x}, \mathbf{w}). \tag{20}$$

In case of using Yager's approach to weighted fuzzy aggregation, (19) becomes

$$D(\mathbf{x}, \mathbf{w}) = \bigwedge_{i=0}^{m} \left[G_i(\mathbf{a}_i^T\mathbf{x})\right]^{w_i}. \tag{21}$$

When using the extension of the product t-norm, one obtains

$$D(\mathbf{x}, \mathbf{w}) = \prod_{i=0}^{m} \left[G_i(\mathbf{a}_i^T\mathbf{x})\right]^{w_i}. \tag{22}$$

When using the extension of the Hamacher t-norm, one obtains

$$D(\mathbf{x}, \mathbf{w}) = \frac{1}{1 + \sum_{i=0}^{m} w_i \frac{(1 - G_i(\mathbf{a}_i^T\mathbf{x}))}{G_i(\mathbf{a}_i^T\mathbf{x})}}, \tag{23}$$

and when using the extension of the Yager t-norm, one obtains

$$D(\mathbf{x}, \mathbf{w}) = \max\left(0, 1 - \sqrt{\sum_{i=0}^{m} w_i[1 - G_i(\mathbf{a}_i^T\mathbf{x})]^2}\right). \tag{24}$$

Note that a value of $s = 2$ is used in (24). This value is also used in the example in Section 6.

Just like in fuzzy decision making, one of the questions to answer is how to select which t-norm to use for the aggregation. This question is more general to fuzzy optimization. The problem of selecting an aggregation operator is not studied explicitly in this report. We suffice by applying several t-norms and by observing their influence on the optimization results.

The weight factors represent the relative importance of various constraints and the objective with respect to one another. The general assumption is that the higher the weight of a particular constraint, the larger its importance on the aggregation result. Hence, final optimization result will be closer to the more important constraints. If the objective is more important, the constraints will be relaxed to a larger degree in order to increase the objective function. The

user can specify preferences regarding the outcome of the optimization by changing the weight factors.

Note that a similar effect can also be achieved by modifying the membership functions that represent the flexible constraints and the objective. If a constraint should have more influence on the result, it can be set tighter by reducing the fuzzy spread of the membership function, i.e. by reducing the value of p_i. However, the analysis of the optimization problem is simplified if the specification of the constraints can be separated from the specification of the aggregation to determine the solution. The membership functions then represent the actual constraints imposed on the problem. The weight factors represent the preferences of the user regarding the optimal solution, and they indicate the influence of the constraints on the total aggregation. By separating the specification of the constraints from the specification of the aggregation, a more transparent problem specification can be obtained. The membership functions can be modified to study the influence of the constraint flexibility on the optimization results. The weight factors can be modified to study the sensitivity of the optimization results to the preference information articulated by the user for satisfying the different constraints.

One of the issues regarding the use of weight factors in any problem specification is to find a common scale for the weight factors corresponding to the constraints and the objective. The problem is to determine a suitable normalization for the weight factors so that the influence of the weight factors on different constraints can be compared. Often, the normalization is done so that the sum of the weight factors equals to 1, i.e.,

$$\sum_{i=0}^{m} w_i = 1. \tag{25}$$

This normalization is suitable for weighted aggregation by using averaging operators. Essentially, the increase in the importance of one of the constraints is coupled to a decrease in the importance of the remaining constraints. Since the sum of the weight factors is 1, only an averaging result can be obtained. The conjunctive aggregation with t-norms is not compatible with averaging aggregation, and hence a different normalization is needed. We propose to use

$$\bigvee_{i=0}^{m} w_i = 1 \tag{26}$$

as the normalization, where $\vee$ denotes the maximum operator. Equation (26) has the desired property that the influence of a constraint diminishes to zero as the corresponding weight factor

decreases to zero, while the influence of a constraint can not increase arbitrarily due to the maximal value.

6 Example

In this section, weighted fuzzy linear programming is illustrated by using a small fuzzy linear programming problem based on an example from [13]. The solution of the fuzzy linear programming problem is studied by using the t-norms from Section 5 for aggregation. The influence of different weighting is studied for three scenario's.

1. With equal weigh factors.

2. With unequal weight factors with preference for one of the constraints.

3. With unequal weight factors with preference for increasing the objective function.

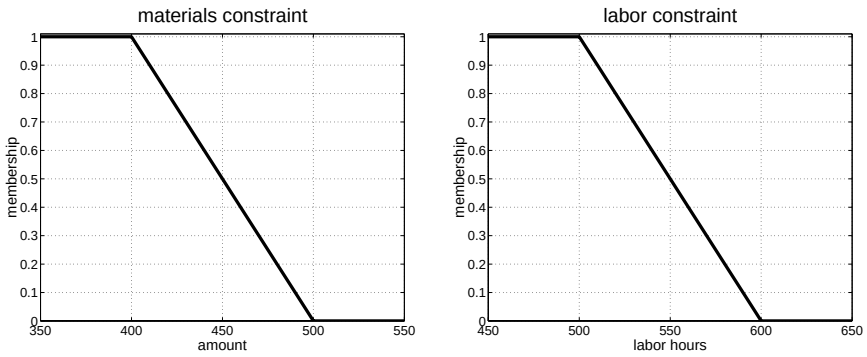
Consider a company that makes two products, P_1 and P_2. Product P_1 has a k$0.40 profit per unit, while product P_2 has a k$0.30 profit per unit. Product P_1 takes twice as long to produce than product P_2. The total labor time per day is 500 hours, and it may be extended to 600 hours per day due to special arrangements for overtime work and hiring of external labor resources. The supply of material is sufficient for at least 400 units of both products, but it may possibly be extended to 500 units per day. The problem is to determine the number of units to produce per day of each product P_1 and P_2 in order to maximize the total profit.

Let x_1 and x_2 represent the number of units of the products P_1 and P_2, respectively. Then the fuzzy optimization problem is formulated as

$$\begin{array}{lll} \underset{\mathbf{x}\in\mathbb{R}^2}{\text{fuzzy maximize}} & z = 0.4x_1 + 0.3x_2 & \\ \text{subject to} & x_1 + x_2 \mathrel{\tilde{\leq}} 400 & \text{material} \\ & 2x_1 + x_2 \mathrel{\tilde{\leq}} 500 & \text{labor hours} \\ & \mathbf{x} \geq \mathbf{0} & \end{array} . \tag{27}$$

Note that

$$\begin{aligned} \mathbf{a}_0^T = \mathbf{c}^T &= [0.4 \quad 0.3], \\ \mathbf{a}_1^T &= [1.0 \quad 1.0], \\ \mathbf{a}_2^T &= [2.0 \quad 1.0]. \end{aligned}$$



(a) (b)

Figure 2: Membership functions that represent the flexibility in (a) the material amount, (b) the labor hours.

Assume that the membership functions for the constraints are defined as piecewise linear membership functions

$$G_1(x_1+x_2) = \begin{cases} 1 & x_1+x_2 < 400 \\ \dfrac{500-(x_1+x_2)}{100} & 400 \leq x_1+x_2 \leq 500 \\ 0 & 500 < x_1+x_2 \end{cases}, \tag{28}$$

and

$$G_2(2x_1+x_2) = \begin{cases} 1 & 2x_1+x_2 < 500 \\ \dfrac{600-(2x_1+x_2)}{100} & 500 \leq 2x_1+x_2 \leq 600 \\ 0 & 600 < 2x_1+x_2 \end{cases}. \tag{29}$$

Figure 2 shows the membership functions for the constraints.

We now determine the membership function for the aspiration level of the objective (profit). The coefficients z_l and z_u for the membership function for the objective are determined by solving (9) and (10), respectively. The solution obtained is $z_l = 130$ and $z_u = 160$, which leads to the membership function

$$G_0(\mathbf{c}^T\mathbf{x}) = \begin{cases} 1 & 160 < 0.4x_1+0.3x_2 \\ \dfrac{\mathbf{c}^T\mathbf{x}-130}{30} & 130 \leq 0.4x_1+0.3x_2 \leq 160 \\ 0 & 0.4x_1+0.3x_2 < 130 \end{cases}. \tag{30}$$

The relaxation coefficients can be read directly from (28), (29) and (30) as $p_1 = p_2 = 100$, and $p_0 = 30$.

By using (13), the optimal solution of the fuzzy linear programming is obtained for $\lambda^* = 0.5$ as $x_1^* = 100$ and $x_2^* = 350$. The optimal profit z^* is found to be k\$145. Note that this is an

intermediate result between z_l and z_u. The number of labor hours needed per day is 550, and 450 products are produced. Hence, by relaxing the constraints within the allowed flexibility bounds, the profits have increased from k$130 to k$145. Note that this solution corresponds to a non-weighted aggregation using the minimum operator according to (11).

Similarly, other aggregation operators can be used to determine the optimal solution. We now report the result of optimization using the operators (22), (23) and (24) when the weight factors are equal, i.e $w_i = 1$, $i = 0, 1, 2$. In all these cases, the optimization is performed by the simplex algorithm of Nelder and Mead [14], using the optimal solution due to (13) as the initial estimate. The optimization results are rounded to the nearest integer, since the products are assumed to be fully completed on each day. When the product operator (22) is used for the aggregation, the optimal solution is found to be $x_1^* = 50$ and $x_2^* = 400$, which corresponds to a profit of k$140. The aggregation using the Hamacher operator (23) leads to $x_1^* = 79$, and $x_2^* = 370$ corresponding to $z^* = 142.6$. When the Yager operator (24) is used, $x_1^* = 79$ and $x_2^* = 364$ with $z^* = 140.8$. Observe that all four operators considered make different trade-offs amongst the constraints and the objective. The minimum operator leads to the largest profit, but of course it requires also the largest relaxation of the constraints. The minimum operator relaxes the two constraints to an equal degree (0.5 membership), while the product operator relaxes only the constraint on the materials. The other two operators are positioned between the minimum and the product. The optimal solutions with each operator are also listed in Table 1. Figure 3 shows the regions within which the trade-off amongst the constraints and the objective function occurs. Also the contour lines (lines of equal function value) for the aggregated decision function that is maximized are shown.

Suppose, now, that there is preference for satisfying one of the constraints to a larger degree. The employees, for example, may find the constraint on the materials less important than the constraint on labor hours. After all, making long overtime may not always be desirable, and it may be difficult to arrange for outside labor resources. Accordingly, let the corresponding weight factors be $w_0 = 1$ (profit), $w_1 = 0.5$ (material) and $w_2 = 1$ (labor hours). Table 1 shows the results of optimization with the modified weight factors. Note that the requirement for labor time has decreased for all the solutions. This corresponds to the fact that the satisfaction of the labor time constraint is more important to the user. Hence, the trade-off is achieved by relaxing the materials constraint more. This is indeed what one would expect from decreasing the weight

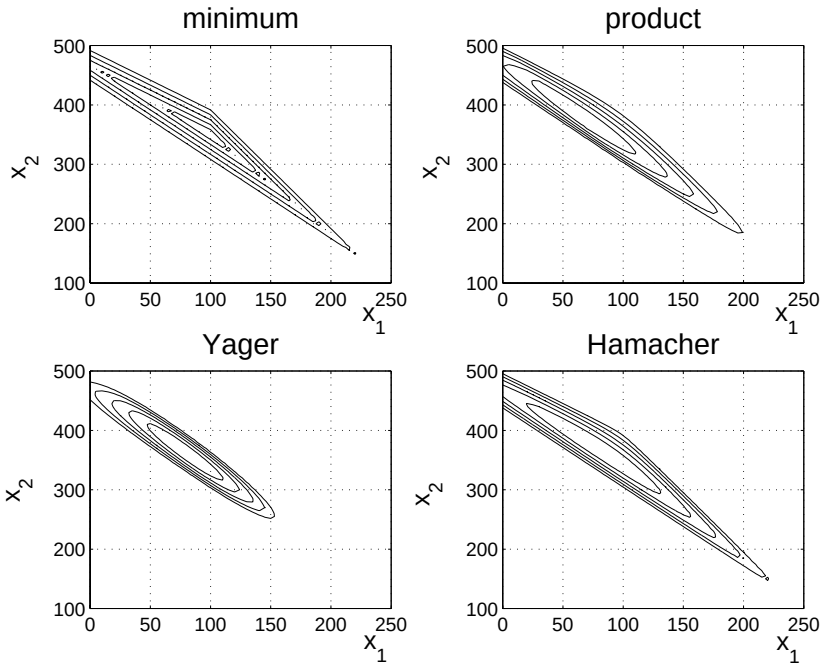


Figure 3: Regions within which the optimal solution is sought for different fuzzy aggregation operators.

or importance of the materials constraint. Note that the weight of the materials constraint has also decreased compared to the weight of the profits. Hence, the influence of the profit on the overall aggregation has also increased. This is visible in the increased profits compared to an aggregation with equal weight factors. Observe that the Yager operator leads to a comparable profit as with the minimum operator in the non-weighted case. The relaxation of the labor constraint is much less, however, which may be a more preferable solution in this case.

The management of the company may want to put more emphasis on the daily profits. Let this goal be represented by the weight factors $w_0 = 1$, $w_1 = 0.25$ and $w_2 = 0.5$. The optimal results obtained from different aggregation operators are given in Table 1. As expected, the profits increase further by relaxing the constraints more. The profit according to the minimum operator is very close to the profit according to the Yager operator. However, the constraint on the labor hours is relaxed to a smaller degree with the Yager operator, which may be a more acceptable solution, given the context. The merits of one trade-off against the other must be decided upon by the decision maker, but this example shows that the range of fuzzy set operators together with different weight combinations can be used to explore various trade-off possibilities. As an illustration, Fig. 4 shows the region within which the Yager operator leads

Aggregation	w_0	w_1	w_2	x_1	x_2	Material	Labor	Profit
Minimum	1.0	1.0	1.0	100	350	450	550	145.0
Product	1.0	1.0	1.0	50	400	450	500	140.0
Yager	1.0	1.0	1.0	79	364	443	522	140.8
Hamacher	1.0	1.0	1.0	79	370	449	528	142.6
Minimum	1.0	0.5	1.0	76	391	467	543	147.7
Product	1.0	0.5	1.0	33	433	466	499	143.1
Yager	1.0	0.5	1.0	50	417	467	517	145.1
Hamacher	1.0	0.5	1.0	60	400	460	520	144.0
Minimum	1.0	0.25	0.5	75	402	477	552	150.6
Product	1.0	0.25	0.5	36	443	479	515	147.3
Yager	1.0	0.25	0.5	33	456	489	522	150.0
Hamacher	1.0	0.25	0.5	66	400	466	532	146.4

Table 1: Optimal solutions (rounded to the nearest integer) obtained for various aggregation and weight factor combinations.

to the optimal solution, and the solutions corresponding to different weight combinations. Note how the optimal solution moves towards to edge of the initial feasible region as the weight of the labor constraint and the profit increases.

7 Conclusions

An optimization model based on weighted fuzzy aggregation has been proposed for satisfying the constraints and the objectives in fuzzy optimization. The model is an extension of the solution framework proposed by Zimmermann [21]. In the proposed model, the user of the optimization scheme is able to convey preference information for satisfying various goals and constraints. For example, certain constraints can be more important to satisfy, and hence the solution should take into account this preference information provided by the user. The difference in the importance is represented as a set of weight factors. Weighted extensions of t-norms are used for the aggregation. These operators can model simultaneous satisfaction of the constraints and the goals, while taking the difference in the importance into account.

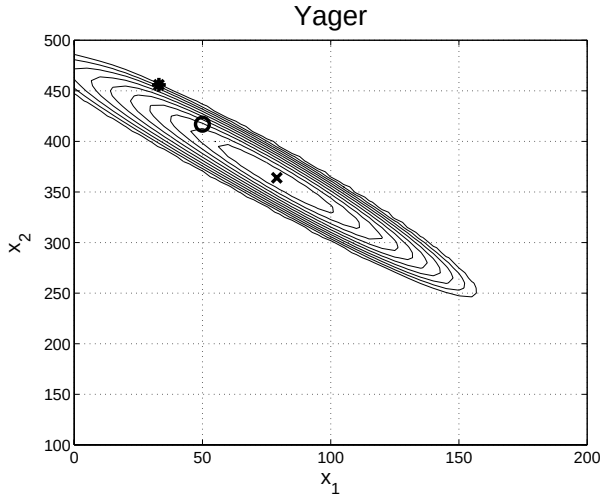


Figure 4: Solutions when using the Yager operator for different combinations of weight factors: $\times$ - non-weighted solution, $\circ$ - more weight on labor constraint, $*$ - more weight on profit.

A main advantage of the proposed method is that it allows the user to concentrate on the actual limitations in a problem during the specification of the flexible constraints. The difference in preference information is then incorporated during the specification of the weight factors. In this way, a separation can be achieved between the requirements of the problem and the preferences of the user regarding the preferred solution. In classical optimization, the preference is specified as a hierarchical ordering of problem constraints that can be removed one by one if needed. The fuzzy optimization setting provides the user with an alternative method to the hierarchical ordering of constraints.

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