ANNUAL SURVEY: SUGGESTIONS ON QUANTITATIVE BUSINESS CYCLE THEORY

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CONTENTS

SECTION I. INTRODUCTION

1. The aim of business cycle theory is to explain certain movements of economic variables. Therefore, the basic question to be answered is in what ways movements of variables may be generated. In answering this question it is useful to distinguish between *exogen* and *endogen* movements, the former being movements during which certain data vary, while, in the latter, the data are supposed to be constant. Since in a *static* theory (e.g., the Walrasian system) the values of all the variables are determined by the data at the same moment, this theory does not admit endogen movements. A dynamic theory is therefore necessary, a theory being called "dynamic" when variables relating to different moments appear in one equation. Of course, exogen movements are possible in a dynamic theory as well as in a static one. Business cycle research shows that a dynamic theory is necessary to explain facts and so business cycle theory is not possible within the field of static theory. We have to pay attention, therefore, to a number of fundamental dynamic notions. The first thing is to distinguish between the "mechanism" and "exterior influences." The mechanism is the system of relations existing between the variables; at least one of

these relations must be dynamic.[1] This system of relations defines the structure of the economic community to be considered in our theory. Such a mechanism may perform certain kinds of swinging movements that are characteristic of the system as such ("Eigenschwingungen"), and, under certain circumstances, it may show other fluctuations whose character is dependent, to a smaller or larger extent, on the nature of the exterior influences, the "impulses."[2] Frequently, the impulses present themselves as given initial conditions of the variables—comparable with the shock generating a movement of a pendulum—or as given changes of the data entering the equations. Examples of such shocks will be given later.

It is important to distinguish between the *mathematical form* of the mechanism, i.e., of the equations defining it, and the *economic sense* of these equations. The mathematical form determines the nature of the possible movements, the economic sense being of no importance here. Thus, two different economic systems obeying, however, the same types of equations may show exactly the same movements. But, it is evident that for all other questions the economic significance of the equations is of first importance and no theory can be accepted whose economic significance is not clear.

From the foregoing, it is evident that all business cycle theories may be looked upon as generalizations of static theory in the sense that one or more "spots" in the system of static equations describing the economic community are "dynamized." In the next section I shall give a survey of the most important dynamic relations existing in reality, paying special attention to those which have been more or less verified by statistical investigations. I shall then treat certain of the recent attempts which give—with the help of rigorous simplifications—a closed system of equations, a closed system being one that contains just as many variables as equations. As a rule, the analytical form of the equations is simplified as much as possible, otherwise no explicit solution would ever be possible. Such a system admits of one or more solutions and leads to a definite movement, initial conditions being given. This deserves the name of a business cycle theory, although it may be a very much simplified one, but "open" systems cannot represent theories, since they do not give a complete system of hypotheses sufficient to determine the movement of the variables. Many literary

[1] In the next chapter a large number of examples of dynamic relations between economic variables will be given. The above definition of statics and dynamics falls into line with the one given by Frisch, *Nationaløkonomisk Tidsskrift*, 1929.

[2] The term "impulse" is here taken to cover both discontinuous and continuous changes (cf. 25).

theories seem to fall in this category, in that they do not clearly state all relations which are necessary or are tacitly included. Examples will be given. I have attempted also to discuss some of the generally known recent theories from this point of view. The use of mathematics is of peculiar value in this field.

SECTION II. THE FACTS

2. In this section a survey will be given of the most important dynamic relations existing in real economic life which may, or must, be chosen as starting points for an adequate business cycle theory. It is evident that in principle there is a very large number of dynamizations. In almost every part of the economic system reality shows divergencies from static theory. It is virtually impossible to give them all their places in any given "closed" theory. Some simplifications must, therefore, be made and, as in all applied sciences, the question is how to find the happy medium between the complexity of the real world and the simplicity of an amenable theory. Suppose that we confront a complete static system with its demand and supply equations, budget equations, monetary equations, and technical and other institutional equations. We must ask how to group the subjects, the markets, the institutions, etc. We must inquire which dynamizations must be chosen as the more important ones.

The procedure in this section will be as follows: I shall give a certain highly simplified grouping of the elements of economic life. The principal remaining relations will be discussed and special attention will be paid to their dynamic features. In particular, those dynamic features which have been verified statistically, or that seem to possess a high degree of evidence, will be treated. But the number of relations enunciated is still too large to make an explicit solution possible. In the next sections, therefore, further simplified systems are given and partially solved. It will be possible, in the end and to a certain degree, to combine such solvable cases. The solutions of the simple cases to be discussed will, however, themselves exhibit remarkable features.

The grouping of the elements, which has its statistical counterpart in the calculation of index numbers of all sorts, is characteristic of what Frisch calls *macrodynamic* economics in contrast with *microdynamic* economics; of course, this subdivision also fits the static type. In the macrodynamic field, I propose the following grouping of subjects, objects, and markets:

Subjects: 1. Producers:
Entrepreneurs: A. Bankers
B. Others

Wage earners
Capitalists
Government
2. Consumers } being largely the same
3. Savers } subjects as before

The usual reservations must be made; certain individuals may be both extrepreneurs and capitalists, both wage earners and capitalists, both consumers and savers.

Objects: 1. Original factors of production:
Nature
Labor
Capital Disposal
Short term
Long term
2. Products: Raw Materials—for Consumables
—for Means of Production
Finished Products—Consumables
—Means of Production
3. Property rights: Stocks and shares

Markets: Generally speaking, a market may be defined for each of these objects. Depending on the viewpoint of the investigator, the market for the production factor, "nature," may be the market for hiring land or, when agriculture is placed outside the "cycle generating community," the agricultural raw materials' market.

We have now to sum up which groups of subjects contribute to the supply or the demand in each of the groups of markets and also how these supply and demand contributions behave in dependence on the variables adopted. Within certain limits, this choice of variables is free. We can state *a priori,* however, that categories remaining constant, or nearly constant, throughout a business cycle should not be taken as variables. It is, on the contrary, a natural simplification of our problem that we may consider such categories as constants.[3] Variables may be, e.g., the general price level, or subdivisions of it, wage rates, profit margins, the quantity of investment goods produced or commanded, money rates, etc. Further, it is preferable *to measure all quantities from their long term equilibrium value as a base,* i.e., in deviations from this equilibrium value.

3. In consequence of the very important rôle which the decisions of entrepreneurs play in different markets, it is useful to begin with an analysis of the activities of these subjects. They form the basis for the

[3] This procedure is analogous to Ehrenfest's method of "adiabatic variations" in theoretical physics.

supply of products as well as for the demand for production factors. The following subdivision of these activities seems to be reasonable. There are activities relating to: (A) current production, (B) replacing of worn-out means of production, (C) new investment, and (D) speculation.

Current production.—Current production is carried on with the help of existing equipment, which may or may not be used fully. The capacity of equipment itself is a result of past increases and decreases, which, in general, depend on earlier values of variables such as profit margins, etc. The total capacity of existing equipment will, therefore, have the character of an integral over time of these margins, the integral being possibly taken with a certain weight function.

The influence of total *existing* capacity on production does not seem to be important in the few cases where information is available, except in the case of shipping. One of the reasons might be the small deviations of capacity from its trend, which may be considered as its long term equilibrium value.[4]

The second factor determining output, viz., the percentage of capacity in use, shows much wider variations. This percentage will depend, roughly speaking, on the profit expectancies at the moment in which the decision to produce is taken. This means that there is a time lag between these profit expectancies and the quantity delivered.

Profit expectancies do not necessarily coincide with profits earned at the moment of decision nor with those really earned on the production planned at that moment. There is, however, a tendency to correspondence between profits earned at the moment of decision and profit expectancies, as the former are an important factor in the evaluation of the latter.

Discrepancies arise, e.g., out of psychological influences such as general confidence or lack of confidence and out of facts indicating an impending change. Such facts may be of an exogen character, as inventions, crops, etc., or they may be of an endogen character, as the value of price velocities dp/dt, p being the price level and t time.

Profits earned depend on the method of calculation and the special values of the entities included in that method of calculation. As to the method, it may be remembered that costs of materials may be taken as relating to the moment of delivery or to the moment of purchase of the individual materials used. In the latter case another lag is introduced.

The number and character of entities included in the calculation of profits are influenced in the first place by the "horizon" of the entrepreneur, i.e., the period of time over which he is calculating his profits.

[4] Cf., however, 12.

This may sound theoretical. Each case of "distress selling," however, to which an important place in business cycle theory is given by Fisher and others, represents an example of a certain shortening of the entrepreneur's (in this case the dealer's) horizon. This case is so complicated that a separate treatment seems necessary; I can only mention it here. In certain cases it may be formulated in Goudriaan's way: below a certain price level the supply curve is negatively sloped. I doubt, however, whether, in a first approximation to the business cycle problem, these negatively sloped supply curves should be introduced. The frequency of this phenomenon seems not to be such that for production as a whole any increase of production during the heavy price fall of 1929–1932 can be observed. A simpler example of the influence of horizon is given below (8, raw material markets).

The most important categories included in the profit calculation are prices and quantities of products and production factors.

As to prices, the most variable ones included are, in general, the prices of products and raw materials,[5] which may be combined in the price margin. In mineral and agricultural production particularly, the behavior of price margins is important for the course of profits and, therefore, for that of production.[6] For industry this relation has not been investigated sufficiently. It is possible, and for the post war period even probable, that the rigid way in which prices of industrial products are formed has weakened this relation, where it existed before.[7] On the other hand, Mitchell attaches great importance to the course of prices as a cause of the movement of production. The great difficulty in statistical research in this field is the scarcity of finished products prices for the pre-war period. Thus far, our knowledge of the influence of prices on production is meagre.

Other prices appearing in the profit calculation are those of the factors of production. Among those the first is wages. My impression is that the influence of wages on the cyclical movements of profit margins has been exaggerated and that this is partly due to political controversies. Nobody will deny, of course, that the price of labor has some influence. There are two things, however, that tend to reduce this

[5] Raw materials are factors of production, when we consider industry (and finance, trade, etc.) apart from agriculture. Raw materials are, then, so to say, the contributions of nature to the system. The choice of this more restricted system rests on good grounds, as cyclical movements are particularly industrial and financial phenomena; agriculture only shows them irregularly.

[6] Hanau, "Die Prognose der Schweinepreise," *Vierteljahrshefte zur Konjunkturforschung*, Sonderheft 18. Berlin, 193; Bean, "The Farmer's Response to Price," *Journal of Farm Economics* (1929), p. 368; Tinbergen, "De Wisselwerking tuschen loon en werkgelegenheid," *De Ned. Conjunctuur* (September 1933), p. 10.

[7] Wagemann, *Konjunkturlehre* (Berlin, 1928), p. 174.

influence on *cyclical* movements of production below what is generally assumed. First, the variability of wages is much smaller than that of prices. In accounting for changes in profit margins, prices are, therefore, far more important. Second, at least for enterprises already existing, only in variable costs are the wages responsible for changes in production, as constant costs do not influence it. Of course, constant costs may be very important for the level around which profits are oscillating and, therefore, for the welfare of entrepreneurs. For the cyclical variations of production volume they are, however, only of secondary interest.[8]

Perhaps these two circumstances explain the statistical fact that the influence of wage rates on production cannot be shown clearly.[9]

The prices of the second principal factor of production, *i.e.*, money rates, also seem to have only a small influence on profit margins for current production. Speaking of money rates, we should distinguish between short term rates and long term rates. As far as can be shown statistically, both seem to have only a very small influence on the variations of production occurring at the same time. The chief reasons for this fact may perhaps be expressed as follows: the total amount of short term interest included in costs of production is small, whereas the variations in the amount of long term interest are small. This is rarely contested as to current production in its limited sense. It is, however, contested as to trade and building activity, to both of which the above mentioned facts do not apply. In accordance with this, prewar building activity in Germany[10] shows cycles fairly contrary to general business cycles. For England and the United States, however, building cycles are concurrent with general cycles. As to trading, the influence of short money rates has not yet been shown statistically. Thus, the influence of money rates on current production and trading volume should also be investigated more exactly.[10a]

[8] They may be of some interest in so far as they influence the moment at which a failure may occur and this failure may lead to reorganization. Wages also influence the demand side of the finished product's market; this will be mentioned later on.

[9] Cf. my paper mentioned in note 6, parts of which are reproduced in Methorst and Tinbergen, "Les recherches relatives à la conjuncture au Bureau Central de Statistique des Pays-Bas," *Revue de l'Institut Intern. de Stat.* I (1934), p. 37. As is stated there, in my opinion the figures given by Rueff on this problem, and discussed at the Tokio (1930) Session of the Institute, indicate that the chief influence on the price-wage relation originates from prices. (Cf. *Bull. de l'Inst. Int. de Stat.*, Tome XXV, 3e livr., p. 765.)

[10] Cf. Hunscha, "Die Dynamik des Baumarkts," *Vierteljahrshefte zur Konjunkturforschung*, Sonderheft 17, Berlin 1930.

[10a] After I had finished this survey, C. F. Roos' admirable paper on building activity in St. Louis was published in his *Dynamic Economics*, Bloomington, Ind.,

In so far as long term money rates influence current production volume, this influence may have a considerable lag; in the profit calculation money rates of long ago may appear, owing to long term loans.

Apart from prices, different "*quantities*" have influence upon profits. The most important example in our present line of argument (cf. p. 249) is the quantity of production factors required for a unit of product, the productivity. It is evident, for example, that an increase in productivity, other things being equal, raises profits. In some cases the influence of productivity on quantity produced has been proved statistically to be important.[11] Changes in productivity themselves may be of a different nature. Distinction must be made between sporadic changes caused by important inventions and systematic changes caused by systematic improvement of methods used. Of course the limits between those two cases are not clearly defined but the distinction seems practical. The first category is especially emphasized by Schumpeter (his "new combinations"). The second category may be distinguished again in two sub-categories, automatic and non-automatic changes. By automatic changes, I mean changes caused by the variations of quantities produced, leading automatically to variations in overhead labor per unit of product. By non-automatic are meant all other systematic changes. An example of a non-automatic systematic change may be indicated. An examination of the statistical material available shows that, in different branches, a deviation of total production of minus one per cent from its trend is followed after about half a year by an increase of productivity per man of one per cent per annum above its trend value.[12]

These variations in productivity are related to a phenomenon emphasized by Hayek, i.e., *variations in roundabout way*. The Böhmian notion of a roundabout way may be considered as a means of measuring the method of production used. Assuming that technical knowledge remains unchanged, there exists a one-dimensional range of possible methods of production. Each of these methods is, in general, characterized by (a) a certain length of roundabout way, (b) a certain degree of mechanization, and (c) a certain productivity. Supposing, on the other hand, each special (a) or each (b) or each (c) represented by only one possible method, it would be indifferent whether we used (a) or (b) or (c) as a measure for that method, and there might be the

1934; here the very distinct and large influence of the credit position (the number of foreclosures) was proved to exist.

[11] Cf. the papers mentioned in note 7.

[12] Cf. Tinbergen, "De invloed van de conjunctuur op de arbeidsproductiviteit," *De Nederlandsche Conjunctuur* (June 1934), p. 13.

possibility that Hayek's phenomenon and ours would be one and the same. There is, however, doubt as to the validity of both hypotheses just mentioned. There is a large possibility that the length of roundabout way varies only very slowly and continuously, and then it has no significance for business cycle theory. It is very difficult, if not impossible, to get statistical evidence on these questions, and for that reason it is doubtful whether this phenomenon should be mentioned at all in this section, dealing as it does with factual evidence.

Up to this point we have assumed, in accordance with Walrasian theory, that the entrepreneur fixes his output in response to the price situation, independent of the quantity demanded. It is even characteristic of the Walrasian line of argument that, when there is inequality of demand and supply, prices will change first of all, and it is only these changed prices which, in turn, induce the producer to change his supply. We may formulate this by saying that the initiative in producing is taken by the producer, who bases his activity on prices, etc. For large sections of the economic community, however, another form of "market organization" exists, that is, one in which the initiative is taken by the buyers, and where the determining factor for production is the order, or in which the initiative is still taken by the producer, who bases his acts on expected orders rather than on prices. Modern budget methods, in fact, work in that manner.[13] It may be observed that, in these circumstances, the producer is no longer a real "entrepreneur," but only a representative of his buyers. I make no objection to this observation; the principal thing is that our equations will be different. This phenomenon of "passive entrepreneurs" presents itself in a certain degree when production is carried out "on order" and not "for stock." It must be recognized, however, that the investing activity of entrepreneurs will be "passive" only in a very small number of cases.

I have now terminated the description of the more important factors which in an "observable" way influence profits and, therefore, the amount of current production. A further question concerns the quantitative relation between profits and the quantity produced. In general,

[13] The fact that, especially in recent business cycles, prices show a lag behind production does not of necessity prove that, on the average, prices are not the determining factors. They may be, for other factors may be present, showing a lead before production, such that the proper combination of prices and these factors shows a small lead also. As examples of these factors the following may be mentioned: (a) the price derivative with respect to time (changing velocity of prices) and (b) labor costs of production, moving *inversely* parallel to the integral of price-deviations from trend over time. A more serious indication against what is called here the "Walrasian" version is the poor correlation existing between profit margins and production for separate *industries.*

we can assume that the larger the profits, the larger the quantity produced. It seems, however, more exact to formulate the relation as follows: the amount produced depends on that part of the total capacity for which profitable exploitation is possible or, still more exactly, for which exploitation is more profitable than idleness. The precise form of this relation depends on the distribution of the different classes of profitability. This distribution is, in many cases, close to a Gaussian one.[14] But, for small variations, a linear approximation may be used. In the following sections, many examples of this approximation will be discussed.

I have already stated that, in general, a certain period of time will elapse between the decision to produce and the emergence of the product. This is especially true of investment goods. For ships this time lag amounts to about a year, for most machinery a quarter to half a year, for large buildings and roads even longer. In the production of consumables a lag of some months also exists, its length depending on the stages included.[15] It is an important question for statistical research whether this lag should perhaps include the growing of the agricultural raw materials, representing on the average perhaps half a year, or the planning of investment works. The answer depends on the exact form of the theory considered. But, in several cases there is good reason for not including them. This is especially true when there are surplus stocks of raw materials sufficient for the larger volume of production required. As soon as the volume required exceeds these possibilities, a larger lag becomes necessary, and thus the lag must, at least for this reason, increase during the upward movement of business. It is interesting, in this respect, to remark that, during the downward movement, the shortest lag possible will be prevalent, since raw material stocks will be sufficient all the time.

It should be remarked, in addition, that for cases in which only the lag for the industrial process, and not that for the agricultural process, is taken into account, the corresponding profit margin, too, should be taken for industry only (cf. 13, 15).

Apart from the total length of time production required, the shape of the activity distribution curve of the process is important. By this

[14] Cf. various investigations of the Bureau of Labor Statistics (e.g., nos. 411, 412, 441, 474, 475) which, although they do not give exactly what is needed here, make it seem probable that the distribution under discussion is a Gaussian one.

[15] For some data computed from Dutch production statistics, see *De Ned Conj.* (August, 1934), p. 32. These data relate chiefly to metal industries; data on residential building are added. Only one branch, ship building, shows production periods of over one year. Iron-construction and machinery show figures of 3 to 8 months.

I mean the function indicating how many units of factors of production are to be used per time unit and per unit of product at each moment between the beginning and the end of the process.[16] This function is of a certain importance—at least in principle—for the amount and the "timing" of the buying power of the factors used.

4. Replacing of Worn-Out Capacity.—What is said in the following relates essentially to such means of production as have a "long" duration of life. Although the limit between "long" and "short" is not sharp, the distinction has great importance. It should be chosen in the neighborhood of the lag of production. For some purposes the "very long living" means of production may be considered separately. Among these we reckon all means lasting more than, say, 15 years.

Generally speaking, the history of an individual machine or other long living means of production will be characterized by decreasing productivity. There will come a moment, therefore, at which it no longer pays to use that machine, depending not only on the velocity with which productivity decreases but also on the economic situation in general. When the moment has come, one of two things may occur. The machine may be replaced by a new one or it may not. Probably a large part of worn-out machinery will be replaced. Besides, other new machines will be installed. This activity is not discussed here but under "New Investment."

To begin with, we consider a very much simplified case in which the following conditions are fulfilled:[17]

(a) all worn-out machines are replaced by new ones,

(b) the moment of replacement is independent of the economic situation,

(c) there is only one sort of machine, and

(d) there are no individual or random differences.

The consequence of (d) is that there is, for each machine, a fixed duration of "life," that of (c) and (d) that this life time is the same for all machinery. In consequence of (a), the number of machines replaced will be equal to the number worn out, which will be the number "born" at the moment just a life time before.

It is evident that the four premises mentioned are not fulfilled in reality. By replacing them by more complicated hypotheses, we gradually approximate real phenomena. Assumption (a) is perhaps the least unrealistic. It might be changed slightly by assuming that there

[16] Cf. Frisch, "Propagation Problems and Impulse Problems in Dynamic Economics," *Economic Essays in Honour of Gustav Cassel*, London, 1933, p. 181, 182 ("advancement function").

[17] This theoretical set-up has been considered by the Norwegian economist Dr. Kr. Schönheyder, *Statsokonomisk Tidsskrift*.

is a certain proportion between machines worn out and machines replaced.[18] It is evident that, by so assuming, we loosen the dependence existing between replacement and total demand for machinery; the smaller our proportion, the larger the part of total demand consisting of new investment—assuming total increase in production capacity is given. This is only possible when there is a certain surplus capacity, for then immediate replacement is not necessary.

A similar less stringent dependence between demand for new machinery and its construction one life time before presents itself when assumption (b) is generalized and the moment of replacement is supposed to depend on the economic situation. In these two cases, the generation of a new boom will depend in a less marked degree on the life time of the machinery and in a larger degree on other cyclical tendencies in the system, or on external (exogenous) changes in productivity.

Another generalization is introduced by changing assumption (c) and assuming that there are different sorts of machines, each sort having its own life time. Then there will be—supposing now, for simplicity's sake, that (a) and (b) again hold—a category of machine being replaced every three years, another every four years, and so on.

A somewhat different scheme develops when, instead of assumption (c), we generalize assumption (d) by considering the possibility of random differences in life time. The result is, then, that at the same "spot" in the economic system a series of different "life periods" is observed.

Statistical information on these questions is scarce. Only a few "life time distributions" of certain machines and transportation equipment are available[19] and they all show a wide spread. Apart from that, the divergences in average life time between different sorts of means of production are large.[20] Moreover, recent investigations on the volume of replacement in comparison with that of new investment show that—at least for Germany since 1924—new investment oscillates very much more widely than replacement.

New Investment.—I now pass to the activity that, at present, is looked on as the most important one in connection with business cycles, new investment activity.

In general, it may be said that decisions to invest will be based on considerations very similar to those leading to current production.

[18] The "coefficient of upkeep," Frisch, mimeographed lectures.

[19] Cf. data for automobiles in *Quarterly Journal of Economics*, Feb. 1933, and for railway rolling stock, "De levensduur van spoorwegmaterieel," *De Ned Conjunctuur*, March, 1934, p. 22.

[20] A list is given by Vos, *De Socialistische Gids* (1932), p. 861.

When profit expectancies are large, more will be produced and invested; when they are smaller, less will be invested and produced. It is not quite the same profit calculations, however, which induce more or less investment and lead to more or less product. Some of the cost elements, for example, which are fixed for an already existing enterprise are variable for planning.[21] Something analogous may be said of the financing of new investment especially. Although, in general, capital will not be taken at fixed interest, and so in the strict sense long term interest is not an important cost element, at least for big enterprises the condition of the capital market is generally considered as of importance to the success of financing. For current production, however, the condition of the capital market is not of great importance.

The parallelism just mentioned leads to a parallelism in the production of consumers' goods and capital goods.[22] In fact, however, it is only a tendency that may be approximated all the better when there is a surplus capacity. When capacity is fully utilized, there may be a technical impossibility of such parallelism. It is then necessary that the increase in current production be preceded by an increase in capacity. Writing $x(t)$ for current production (current demand), and supposing that at every moment capacity is fully used, while all elements of capacity are equally productive, capacity must be $cx(t)$, where c is a constant. It follows that new investment is given[23] by $c\dot{x}(t)$, which forms another sort of tendency.[24] If the movement is cyclical, new investment should *lead* current production (current demand) by about a quarter period.

These formulae are related to the widespread opinion that "production of investment goods shows the business cycle earlier than production of consumers' goods." The evidence available is not, however, uniformly affirmative. An inspection of the chief sources shows the following situation.

[21] Cf. Schneider, *Theorie der Produktion*, Vienna, 1934, pp. 27–29.

[22] Although it does not exactly relate to entrepreneurial activity, a factor may be mentioned here that tends to strengthen this parallelism, viz., the spending of wages by workers, both in consumers' goods and capital goods industries, chiefly on consumers' goods. So, when capital goods industries are very active, this will be a factor for an almost unlagged corresponding activity of consumers' goods industries and conversely when activity is low.

[23] Cf. the discussion between Clark and Frisch in the *Journal of Political Economy*, 1931–32.

[24] As long as cyclical movements occur, the two relations mentioned are incompatible in their rigid form. It is only possible to combine them, therefore, when the one or the other is "generalized" by the introduction of more variables, as prices of consumers' goods, idle capacity, etc. (cf. 13).

Capital issues, on which Mitchell chiefly rests, show, indeed, for the period 1890–1913, a (not very clear) lead of about one year relative to general business activity (employment, prices, etc.). For the post-war period, this seems not to be the case. The correlation is not high and it is improved by adding some forms of short capital supplied. The improved series show, however, a lead of only about 3 months. On the other hand, J. M. Clark's data on car loadings and construction of new cars (*Overhead Costs* p. 395) did show a connection between new investment and $x(t)$ which, for the $3\frac{1}{2}$ year cycle, corresponds to new investment, leading about one year.

Indexes of production of investment goods and of consumers' goods, (for Germany[25] and for England[26] before the war and many countries after the war) do not always show a very good correlation in pre-war periods. So far as correlation exists (and for post-war periods it is even high) no definite lag in either direction is to be found. There are pre-war business cycles for which production of investment goods leads and others of which the opposite is true. All this is also valid for Swedish figures concerning employment in the two groups over a long pre-war period (see Figure 1).

Statistics relating to individual industries, such as railways,[27] shipping, cotton spinning, and blast furnaces, show divergent results.

In this connection, a knowledge of production periods in these industries is useful. Some figures are available for Holland[28] showing average lags of no more than 6 months for iron construction, machinery and other metal branches, except ship building, which shows one year and more.

In a recent correspondence between the present author and Professor Frisch on these seemingly diverging data, the latter suggested the explanation that the Mitchell-Clark relation is a relation between consumption, x, and "capital-starting," y, while the data of Figure 1 show the connection between consumption, x, and the "carry-on-activity," z. It is of course, quite conceivable that y moves approximately synchronously with x (the Mitchell-Clark relation) and, at the same time, z moves approximately synchronously with x. According to Frisch, further experimental calculations on erratic-shock-maintained swings of the systems discussed in his paper "Propagation Problems and Impulse Problems . . . " actually give movements where both the above kinds of synchronisms are present.

[25] Cf. "Die Industriewirtschaft," *Vierteljahrshefte zur Konjunkturforschung*, Sonderheft 31, 1933.

[26] Cf. W. Hoffmann, "Ein Index der industriellen Produktion für Grossbritannien seit dem 18 Jahrhundert," *Weltw. Archiv.*, Sept. 1934, p. 383.

[27] Cf. Clark, *The Economics of Overhead Cost.*

[28] Cf. *De Ned. Conjunctuur*, August 1934, p. 32.

Scarcely any statistical analyses as to the determining factors of new investments have been made. The relative influence of economic categories (prices, wages, money rates, "general confidence" etc.,) and of technical ones (exogenous factors such as new inventions, etc.) is not, therefore, known with any exactitude. An inspection of English capital issues by branches of industry suggests that the influence of general factors is rather small. There is need of further analysis along these lines.

5. *The Labor Market.*—As has already been stated, the demand for labor is largely determined by the decisions of entrepreneurs to invest

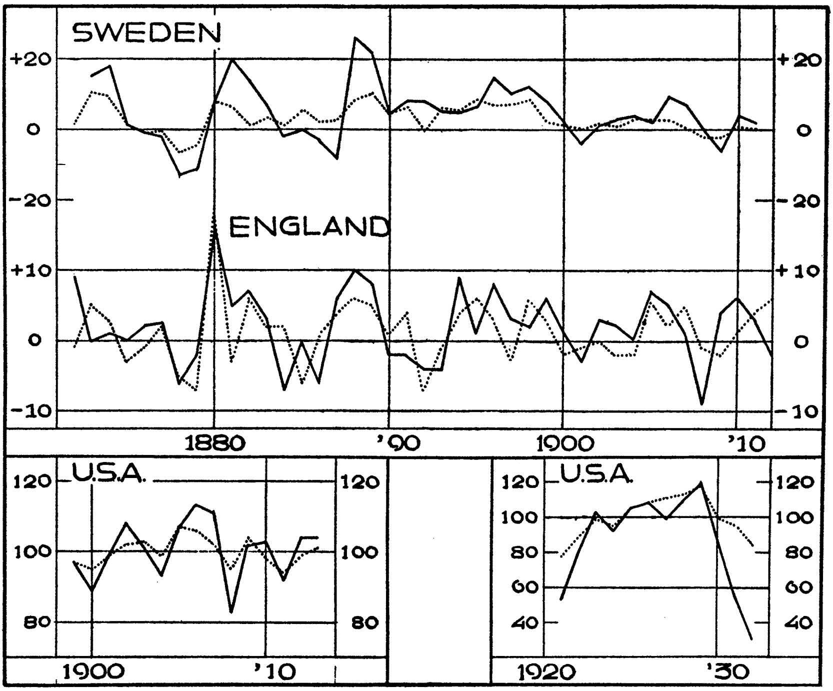

FIGURE 1.—Activity in consumers' goods (------) and capital goods industries (——). Sweden: Percentage increase of workers occupied. England: Percentage increase of production (Hoffmann, *Weltw. Archiv,* 40 (1934), 383). U.S.A., pre-war: Trend percentages of production. U.S.A. post-war: production indexes.

or to produce non-investment goods and, therefore, as a first approximation, by profit calculations. These may manifest themselves after certain lags, depending on the period of production and other characteristics of the production process. One of the determining factors is the wage level, which, in turn, is determined in the market now under discussion.

Granting free competition, the question of how employment is determined would be rather simple. Supply would be given by the number of workers present and would show no cyclical variations.

The assumption of free competition, however, has no value as an explanation of the movements of the labor market, since under its aegis no unemployment is possible. This does not mean, however, that the problem of interpreting wage movements is solved by stating that wage formation is monopolistic. For even an ideal monopolistic fixation of wages would also mean an immediate adaptation to varying circumstances, i.e., a specific statical reaction. The reaction is, however, a dynamical one that may be formulated, as a first approximamation, as a simple lag of, say, one year.

There is also another possible way to interpret wage movements. Wage changes may be assumed to be inversely proportional to the excess unemployment, i.e., to the excess of unemployment over a certain normal level. The strength of underbidding will grow with such an excess; the power to *increase* wages will grow as unemployment is smaller. Wages themselves will, then, move inversely proportional to the integral curve of excess unemployment, i.e., almost proportional to the integral curve of excess profits. For sine curves, this will simply mean a lagging behind profits but with a lag of two years (one-fourth of the period of the eight year cycle). For non-sine curves, the lag may be smaller, especially for real business cycles; apart from that there may be assumed a second influence on wages, viz., prices. Combining these two influences, one finds one year lags also possible, as prices move about simultaneously with profits. Statistical investigation undertaken by the present author shows for different countries and periods that this second explanation agrees slightly better with the facts; the difference in correlation coefficients is, however, very small.

A third interpretation is given by Mitnitzky,[29] a synthesis of a static and a dynamic relation. Basing his argument on carefully examined statistical material, he concludes that during the upward movement there is no lag between employment and wages whereas during the downward movement a lag exists. Since Mitnitzky does not choose any exact mathematical relation to represent his theory, a comparison with other interpretations is not easy; moreover, his relation is essentially of a character that cannot be expressed simply in mathematical terms. For this last reason, it is not suitable for use in a simple mathematical scheme of cycles, but for more detailed analysis his theoretical set-up seems to correspond well with the facts.[30]

[29] "Lohn und Konjunktur vor dem Kriege," *Archiv. f. Sozialw. u. Sozialpol.* LXVIII (1932), p. 318.

[30] There is, however, a disagreement with a fact pointed out by Keynes,

Apart from the regularities just mentioned, the influence of large strikes or lockouts, such as in England in 1926, should be mentioned. These must be considered as exogenous influences.

The amplitude of the cyclical wage movements is rather small, as shown by available data. As a first approximation to a more complicated theoretical scheme, therefore, wages may, without violation of the facts, be supposed to be constant (i.e., to have only a trend movement).

6. *The Credit Markets.*—Quantitative dynamic investigations into the formation of prices in the credit markets are not numerous.[31] In the face of the great importance attached by many theorists to credit problems in the understanding of business cycles, this is remarkable. Moreover, it is regrettable that static theories on interest seldom pay much attention to a fact which in reality is of great significance, viz., the existence of different markets for long and short credits. These markets behave so differently that only in an extremely rough theoretical consideration may they be treated together.

First a few general remarks.

a. We consider the market for new credits only. For short credits this does not differ much from all credits outstanding; for long term credits, of course, it does.

b. New capital issues belong to this subject only in so far as they concern fixed interest securities. Stock issues, at least formally, are not credit transactions, and, besides, a market cannot be considered if no price is contracted.

c. A third remark relates to a very general quantitative aspect of credit problems. A chief function of credit is to provide for the differences between expenses and income. The amount of new credits necessary at a given rate of interest is given by this difference. This is essentially the fundamental idea introduced by Knut Wicksell and later utilized by Keynes and others. The movements of this difference and, therefore, of credit demand at a given rate of interest may be quite different, depending on the tendencies that govern the movements of expenses and incomes of individuals and, above all, of enterprises. Figure 2 illustrates this schematically. Assuming expenses and incomes to move synchronously, but expenses to show a larger amplitude, new

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Treatise on Money, I, 303, where it is stated that: "after a large proportion of the unemployed factors have been absorbed into employment, the entrepreneurs . . . will begin to offer higher rates of remuneration."

[31] Cf., however, "Money Rates and Loan-Deposit Ratio" ("demand-supply ratio"), Persons, *Review of Econ. Stat.*, 1924, p. 266. Interrelations between "deposits" and "loans and discounts," and their ratio, U. S. National Banks outside New York, adjusted figures, Persons, *loc. cit.*, 1924, p. 281.

credits will move synchronously also (case I). Assuming that expenses and incomes show the same amplitude, but a phase difference (case II), new credits show a still larger phase difference. It is clear that there is a wide variety of other possibilities.

d. Credit markets are especially susceptible to variations of "confidence." Certain events may suddenly shock confidence in state fi-

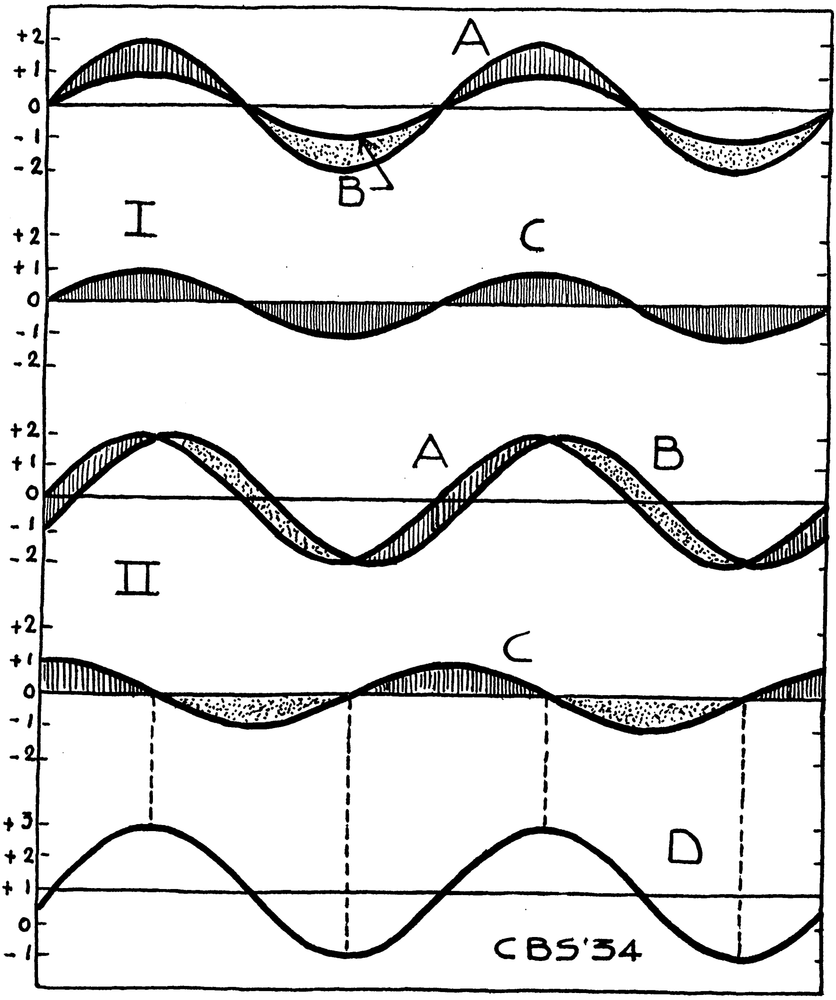

FIGURE 2.—Schematic representation of credit demand (at fixed interest rate). C, as difference between expenses A and income B. Case I: synchronous movements of A and B, A's amplitude surpassing B's. Case II: equal amplitudes of A and B; B lagging behind A. D: total credit amount demanded since beginning of operations (method of Bolza).

nances or in the economic future in general, and, in consequence, interest rates will rise (e.g., the autumn of 1931).

e. Interrelations between short and long term credit markets are to be sought particularly on the supply side. Saving may, in many cases, as readily take the form of supplying short as long term credit. Apart from prices in both markets, general aspects of the economic future (the "degree of confidence") may influence decisions in one direction or the other. A systematic investigation into these interrelations by Lorenz[32] showed that they would be best described in the following dynamic way: month to month changes of long term rates correlate highly with the margin between short and long term rates. The results of this interesting investigation have not yet been sufficiently utilized in further analyses; it seems that especially the causal nexus: short interest rate—long rate—confidence—investment—general activity, might be investigated more extensively.

The total supply appearing in credit markets (and to be distributed over their divisions in the way just indicated) originates from several sources: (1) Income of private persons not spent or hoarded or invested directly; (2) Temporarily idle funds of enterprises (these will, of course, supply the short credit markets particularly); and (3) Credits created by banks.

Different sorts of dynamic relations are introduced by the fact that the amounts mentioned under (1) and (2) often originate from earlier periods. Exact investigations at these points are difficult, as many statistical data are lacking. The elasticity of the monetary system resulting from the existence of the third source has been discussed so fully by competent authors that I need not go into detail here. It may be remarked again, however, that statistical information is far from ideal on this point. It should be remembered that a larger or smaller *elasticity* of the credit supply does not form a dynamic element in our sense but is an important static feature of the system.

Turning now to discussion of the short term credit market (the *money market*), we first notice that this market shows a highly variable price. In a first approximation this must be ascribed to varying demand.[33] Demand may be separated into three categories: (1) Demand for stock speculation; (2) Demand for business; (3) Demand by governments, etc. Of these categories, the first and third one seem to show

[32] Eine Differentialgleichung der Wirtschaftsforschung und ihr Integral. *Blätter für Vers.- Mathematik und verwandte Gebiete,* Beilage zur Zeitschrift für die gesamte Vers.-Wissenschaft, Bd. 29, Heft 3, p. 212.

[33] Cf. *De Ned. Conj.* Aug. 1934, pp. 78 ff. where it is stated that, except in the case of New York before the war, the influence of supply could not be traced.

only a very slight sensitiveness to price, i.e., to money rates.[34] The second category must be divided into categories according to liquidity. "Advances" (England) and "all other loans" (U.S.A.) do not show a distinct sensitiveness to rates, but bills discounted (except treasury bills) do (cf. Figure 3). They form, however, only a very small portion of total credits outstanding and, therefore, it seems logical to say that not much "regulation of demand" by money rates takes place.

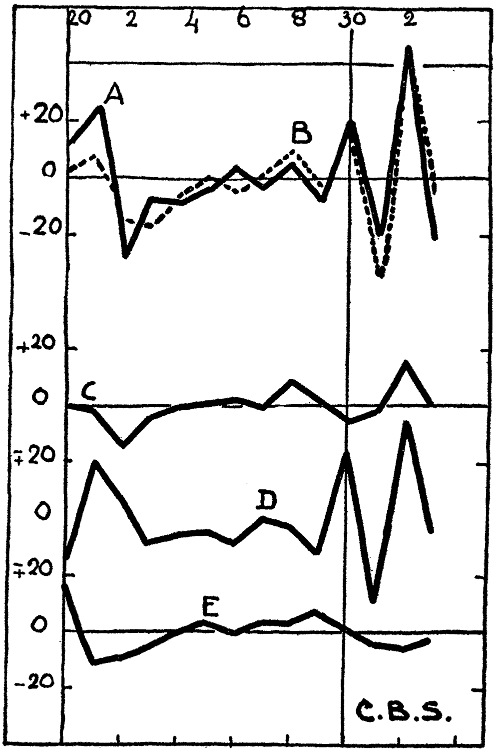

FIGURE 3.—Annual percentage change in bills discounted by English joint stock banks, actual (A) and calculated (B) from:
C. influence of annual percentage change in treasury bill circulation;
D. influence of change in money rates; and
E. influence of activity, measured by total value of foreign trade (*De Ned. Conj.*, Aug. 1934, p. 24).

The conclusions given here depend, however, in some degree on what is a good index of the demand for credit at a given rate, for only when such an index is known can the relative influence of money rates and other factors on demand be estimated by correlation analysis.

[34] Cf. Owens and Hardy, *Interest Rates and Stock Speculation* (Brookings Institution); *De Ned. Conj.* Aug. 1934, p. 18.

A few words may be added on the factors that govern demand, apart from money rates. Demand for stock speculation shows, of course, high correspondence with stock sales at the exchange and with stock prices, demand by governments with deficits on their services. Advances to business show the well-known and interesting feature of maxima early in each business cycle; they are replaced, in later phases of the cycle, by capital issues. Neither increase in advances nor issues shows a high correlation with business activity, but their sum does.

Still less can be said of the long term credit markets (capital markets). The most striking feature is the large constancy of long term money rates throughout long periods. Quantitative researches into demand and supply of new capital (bonds) have not been made.

7. *The bond and stock markets.*—Closely connected with the market for new bonds is that for "old" ones; in general, it may be said that they move parallel and, theoretically speaking, may be considered to form one market. Looking at this total market, other supply and demand sources join those mentioned above. Supply of ("old") bonds (or demand for the corresponding credit, to maintain our previous terminology) is yielded by speculators or by those who must liquidate some reserves; demand comes from savers or speculators, both terms being used in the widest sense. Quantitative analyses from this point of view (e.g., considering the factors that induce an enterprise to liquidate reserves or to build up reserves) have not yet been made.

In contrast with new stock issues, dealing in old stocks can be said to have a market, although not in the sense valid for bonds, for, whereas bond yields are to a large extent known beforehand, stock yields are not.

Price formation of stocks has been investigated by Donner.[35] His investigation shows that prices are governed, as might be expected, chiefly by profit figures. As a smaller investigation made by the author suggested,[36] movements of stock prices are not proportional to profits, but show a smaller amplitude. In years of high profits the possibility of reactions, and in years of low profits that of improvement, is discounted.

Prices on the stock exchange seem to be of importance for the course of business in two ways. First, they seem more or less to regulate the amount of new capital issues, and, second, they form a basis for credits to, and consumption by, speculators, and so influence demand for commodities. Exact investigations are not yet at our disposal but it may be mentioned that the second influence, if strong enough, would

[35] O. Donner, "Unternehmungsertrag und Zins als Faktoren des Aktienmarkts," *Weltw. Archiv*, XL (1934), 116.

[36] *De Ned. Conj.* Dec. 1932, p. 12. ECONOMETRICA, July 1933, p. 247.

be important also for the fact that the consumption of speculators would depend on their profits, i.e., on the velocity of the rise of stock prices (or on the velocity of decline, in the case of a bear speculator).

8. *Markets for consumables.*—The principal aspects of the supply side are discussed above under the heading "entrepreneurial activity." It may be emphasized that the exogeneous influence of crops is large, especially for consumables.

As to the *demand* side, the most important point seems to be that the great bulk of consumers save only a very small and stable part of their incomes. The remaining part is spent independent of the height of the price level. Of course, the distribution over different articles is not independent of the prices of these articles and of the general price level. But, with a high degree of approximation, one may say that the volume of goods and services bought by these consumers is determined simply by the amount of their incomes and the general price level, i.e., the cost of living.[37] So the amount of income is the most important determining factor. This depends, in turn, to a large degree, on employment and, therefore, on profit calculations of entrepreneurs. As already mentioned, dynamic elements may enter here. Employment shows a "distributed lag" after profit expectancies, and wages per hour probably depend on a time integral over excess profits (cf. 5), etc. There may be another lag between income payment and income spending, depending on the so-called income period.[38] I doubt, however, whether this lag has much influence since it is small compared with the other lags entering in the problem.

Apart from this great bulk of consumers, there are, of course, others for whom the distribution of their income between saving and spending seems to be more variable. These are small in number but their incomes are high and, especially for more elastic consumptions, their demand will be of importance. As a first approximation, the part saved will largely depend on profit expectancies.

Perhaps pure speculators' income is most interesting in this respect. Since, for the speculator, the income will depend on the rate of increase of stock prices, it will reach a maximum before stock prices themselves. Fluctuations in these incomes will be very large and, even though they form only a moderate part of total income, the fluctuations will be of importance for total income fluctuations.[39] As far as

[37] Robert M. Walsh, "Empirical Tests for Price Theories, Fisher, Foster and Catchings, Keynes," *Qu. J. of Econ.* XLVIII (1934), 546.

[38] This income period is given an important role by Koopmans, "Zum Problem des 'Neutralen' Geldes"; *Beiträge zur Geldtheorie, herausgegeben von F. A. Hayek* (Vienna, 1933).

[39] This point is stressed by Limperg especially.

income statistics allow a judgment, this statement seems to hold for the United States in the post-war period.

Raw Material Markets.—A consideration of raw materials markets apart from finished products markets has several purposes. If we consider agriculture as a field outside the real stage of business cycles—a defensible view-point—raw material markets are to be considered as the exponents of some of the "irregular shocks" working on the system, i.e., the shocks caused by varying crops. Whether or not we adhere to this view-point, the distinction of raw material from finished product markets is, of course, an expression for the existence of a vertical division of labor. Although vertical division of labor does not add principally new elements in our search for dynamic relations, it has, as will be shown further on, quantitative influence. In this connection, it should also be noticed that raw material markets and markets for finished goods differ essentially on the point of stock holding, this being a dynamic feature par excellence.

Stocks of finished products offer the less interesting features of the two. As far as statistical experience goes, they are, roughly speaking, proportional to production one year before and are small compared with production.[40] Raw material stocks seem much more important and seem, too, to be theoretically interesting. In an earlier issue of this journal,[41] I tried to sketch the essence of the mechanism by which these stocks generate. To recapitulate briefly—and, therefore, perhaps not exactly—, just after a crop, producers are in the possession of a supply consisting of this crop plus the carry-over from the previous year. They expect a certain average crop and demand for other years, depending on price. They distribute their supply over these different years in such a way as to have a maximum of (partly expected) profit. The demand expectancies may depend not only on general business conditions at the moment, but also on their velocity of movement.

9. *Markets for instrumental goods.*—The supply side of this market may be considered a special case of entrepreneurial activity as to current production. The special features of this market are (a) a large part of production is production on order; (b) the period of production is rather long and reaches, in ship building, for example, 1–2 years, and (c) a large proportion of cost is overhead cost.

The demand side is governed by the replacement and new-investment activities of entrepreneurs. An important difference between this market and that for consumables is the fact that purchases are not limited to the income or receipts of the buyers. To effectuate the purchase of instrumental goods, credit may be attracted and, therefore,

[40] *De Ned. Conj.* March 1933, p. 77.

[41] ECONOMETRICA I (1933), 247.

profit expectancies and probably technical circumstances such as wear-and-tear are predominant.

For the market as a whole, demand may be considered predominant, an assumption in accordance with the known positive correlation between cyclical changes in prices and production of pig iron and other metals.

The Circulation of Money.—Some authors draw special attention to the amount of money in circulation and the velocity of circulation. These factors do not, however, constitute new elements apart from production, prices, and new credits. Whether a higher sales volume is completed with more money or with money circulating more rapidly seems of only secondary interest to the business cycle theorist. Experience shows that both things happen more or less in proportion, and thus velocity of circulation is a practical symptom of business activity.[42] Whether it is a necessary link in the casual chain of events, or whether the introduction of the notion more readily explains the business cycle, seems open to doubt.

SECTION III. RECENT THEORIES

10. *The non- and semi-mathematical theories.*—In section II, I have summed up a number of dynamic relations entering a system of equations when one tries to approximate reality closer than by static equilibrium equations. In the present section I propose to discuss some of the recent theoretical schemes. I start with a discussion of the general structure of these schemes and the quantitative assumptions on which they are based. I shall discuss the mathematical equations of the schemes, whether or not these equations are explicitly given. In a further section, the solutions of these equations and the conclusions based on them will be treated; in those sections the common features of most schemes are placed in the centre. In the present section the individual features will demand our chief attention. I will begin with two rather well known systems, Hayek's and Keynes'.

Both are examples of "open" systems. The number of variables referred to in the discussion is larger than the relations that are precisely and explicitly stated. An attempt at a mathematical "translation" leads the reader to many unsolved questions. This is not to deny that many original and valuable suggestions are made and, by stating the various assumptions and relations with mathematical precision, one may contribute to a better understanding of the unsolved problems.

As to Hayek's theory, I see as its central thought the statement that changes in the price system not only lead to changes in quantities

[42] Snyder, *Review of Econ. Stat.*, 1925, p. 256.

produced but also in the method of production, such as the degree of mechanization. The question is legitimate whether these changes are really important. I have the feeling—and I admit it is a feeling—that technique changes continuously only, and not cyclically. We need figures.

Suppose, however, that Hayek is right. Then, I venture to doubt that the Böhmian notion of "roundabout way" is a suitable measure of "method used" in these problems (cf. 4). It should not be forgotten that Böhm uses this notion especially for static equilibrium problems, or perhaps for slowly changing equilibria.[43] The notion of "roundabout way" is less useful for the analysis of rapidly changing situations. For the simple case used as an example by Hayek which, in principle, consists of a production without long lasting means of production, no great difficulty arises. In that case "roundabout way" is identical with "period of production" or "number of stages passed" (when each stage demands that a constant period be run through). The latter is, indeed, a useful notion for dynamic problems also. But, in reality, roundabout way depends in a high degree on the life time of machines used; the average time distance between productive act and consumption of the result, i.e., roundabout way, is greatly influenced by that life time. But, the specific notions, (a) life time, (b) production period for machinery, and (c) period necessary for the production of finished articles with the help of the machinery, are more apt for the description of dynamic processes than the single and general notion of roundabout way. The important questions of how production reacts on price and wage situations and how replacement activity behaves cannot be answered through the notion of roundabout way.

Two examples from Hayek's own line of argument may be given. He asks[44] what is the influence of an increase in consumers' income on the receipts of the producers of highest order means of production and replies that these producers will have increasing receipts only after the length of the roundabout way. This is unrealistic. As soon as the producers of consumers' goods have higher receipts, they invest more; and whether they replace worn-out machinery by new depends, in addition, on whether there is worn-out machinery at that moment. An element such as "confidence" in monetary policies might play a rôle, but I doubt that the length of roundabout way has much to do with it.

Again, Hayek seems to conclude from the lengthening of roundabout way in boom times that this entails a relative shortage of consumers'

[43] See the interesting remarks made by Hill, *Economic Journal* XLIII (1933), 599, on this question.

[44] *Preise und Produktion*, p. 58–68.

goods.[45] He seems to hold that the period of production is also lengthened and that, therefore, in the time of "transition" from the shorter to the longer period of production, there is a certain shortage. This is, indeed, true in the simple case where no long living means of production are used, but it is by no means necessary when such means are used. It is easily shown that roundabout way may increase, whereas period of production, i.e., time distance between beginning of raw material production and end of finished goods production, decreases. It is only necessary that the percentage of production cost relating to amortization rise. For a number of industries the period of production does, indeed, decrease during the boom period.[46]

We return to the unsolved questions rising from the fact that certain relations are not expressly stated but are, nevertheless, necessary to reach a closed system. This applies expecially to the way in which money rates are formed. As to the demand and supply functions of credit, Hayek is vague. This becomes a fundamental difficulty because he attaches so much weight to interest rates ("long" or "short" rates?). It is not clear whether he supposes them to depend on prices, or price derivatives, etc., or on any or what combination of these variables. It is not clear whether or not certain lags are supposed to exist here.

Thus, when one tries to find out what dynamic relations play the the principal rôle in Hayek's system, and whether his system in all circumstances leads to cycles—and, therefore, to an automatic return of prosperity—there remain many difficulties, which is, after all, Hayek's own view.[47]

11. Keynes probably does not pretend to give a closed theory. In his *Treatise on Money* there are three parts which give very pertinent remarks on the business cycle problem. First, there is the part in which the general exchange equations are discussed. So much has been written on these equations[48] that I confine myself to one very general remark. In reading this part of Keynes' work, one realizes clearly that business cycle problems have not so much to do with the movements of *prices* and *quantities* considered as separate variables but rather with the fluctuation of *values* (prices multiplied by quantities).

The second part in Keynes which deserves special attention is Chapter 27, dealing with the notion of "working capital." This seems very important to business cycle theory and admits of mathematical treatment. It is intimately related to the notions (used by Frisch, Kalecki,

[45] *Preise und Produktion*, p. 84.

[46] Cf. "De duur van eenige productieprocessen," *De Ned. Conj.*, August 1934, p. 32.

[47] Cf. his contribution to "Der stand und die nächste Zukunft der Konjunktur forschung," *Festschrift fur Arthur Spiethoff* (München, 1933).

[48] Cf. Hansen and Tout, ECONOMETRICA I (1933), p. 119.

and myself) of the time shape of production. Writing $f(x)$ for the time shape of production, i.e., for the amount of productive activity applied per unit of product and per unit of time at x time units after the starting of production (the "advancement function" in Frisch's terminology), then the accumulated amount of activity invested in the production of one product unit at the moment t, production having begun at time τ, will be

$$\int_{\tau}^{t} dx\, f(x).$$

When $a(\tau)$ units per time unit at time τ have been started, the amount invested in *these* units will be

$$a(\tau) \int_{\tau}^{t} f(x)dx.$$

Production processes at many different moments τ will have been started and all will contribute to the general total of activity invested, provided the products have not yet been finished. Assuming a fixed lag between the initiating and the finishing of the process, this general total amounts to

$$\int_{t-\theta}^{t} a(\tau)dt \int_{\tau}^{t} f(x)dx.$$

When $f(x)$ is expressed in value units instead of physical units, we have working capital. Assuming $f(x)$ to depend only on τ, for example, the double integral reduces to a simple one, and so on. It is easily seen that, as soon as working capital enters our equations, these will be dynamic.

The function of working capital in Keynes' system is not, however, quite clear. Probably it has influence on the price formation of money rates, but, like Hayek, Keynes does not work this out rigorously. The only thesis stated clearly in the chapters on working capital is that its increase is bound to a certain maximum depending on the nation's savings. This is a type of quantitative reasoning of a remarkable sort which enters many arguments on business cycles. It is used sometimes and, in a certain respect also by Keynes, to calculate the length of periods. For that purpose, it is not only assumed that its velocity is bound to a maximum but also that the uniform growing process, corresponding with that maximum velocity, is ended when a certain maximum possible amplitude is reached.[49] Further, it is supposed, in addition, that the growing process is immediately followed by its reverse, a declining process. And, in a similar way, the length of the period is calculated. It has the advantage—aside from the question

[49] Cf. what is said about these questions in 22.

whether it forms a suitable approximation to reality—that the mathematics used are primitive. But, when one tries to write the equations on which the system rests, these are less simple as they contain discontinuous terms.

The third part in Keynes' *Treatise* relating to business cycle mechanism is found in Chapter 20, "An Exercise in the Pure Theory of the Credit Cycle." This chapter gives a very exactly described "standard case" of a moving system of a very simple nature. It is assumed that the quantity of consumption goods, the production of which is started, jumps, at a given moment t, from one constant value to another. It is shown that, corresponding to it, wage income and working capital increase steadily until the production lag has elapsed and they remain constant at the height corresponding to the new production. At that very moment, t_2, the supply of finished consumption goods makes the jump from the old to the new constant value. The price of these goods rises steadily during the period t_1t_2 and falls suddenly to the old level at the moment t_2.

The system described in this standard case is not closed. We are not told what factors regulate the quantity produced. On the other hand, more attention is given to the financing of the production extension.

The standard case is generalized in different ways; influence of other methods of wage setting and financing and the influence of some form of speculation is considered; the length of the price cycle and the fact that the system is not closed are, however, not changed. Thus, we must look on this chapter—as its author does—as really an exercise. Some of its features will be met in the schemes to be discussed later.

12. *The Mathematical Theories.*—I now turn to a discussion of some business cycle theories that form macrodynamic closed systems and are formulated mathematically. The first publication of this sort was a paper in Polish by Kalecki in *Proba Teorij Konjunktury,* Warszawa, 1933.[50] A few months later appeared Frisch's "Propagation Problems and Impulse Problems." Both theories were presented at the Leyden meeting of the Econometric Society in 1933. Kalecki draws our attention chiefly to the following concepts.

Let $I(t)$ be the physical volume of investment orders at moment t. Then, assuming *a constant period of production* for investment goods, deliveries of newly produced investment goods, $L(t)$, will be equal to:

$$L(t) = I(t - \theta). \qquad (12.1)$$

Assuming, further, a *constant volume of replacement* U, the *increase* K' of the total volume of capital goods present (the volume itself being K) will be:

[50] For an English version of Kalecki's theory, see "A Macrodynamic Theory of Business Cycles," in this issue of ECONOMETRICA, pages 327–344.

$$K'(t) = L(t) - U. \tag{12.2}$$

On the other hand, we get, starting again from $I(t)$, an expression for the total volume of unfilled orders for investment goods (supposing each order is immediately put in hand) or of the total work in execution at t:

$$W(t) = \int_{t-\theta}^{t} I(\tau)d\tau. \tag{12.3}$$

The productive activity in investment goods industries will be proportional to this expression: $A = W/\theta$.

At this point, the most important assumption of Kalecki's system comes in. It is supposed that the volume of investment orders, $I(t)$, is determined by an equation of the type

$$I(t) = C' + mA(t) - nK(t), \tag{12.4}$$

where C', m, and n, are constants; C' is here written with a prime because it is not exactly the same as Kalecki's C. The justification of equation (12.4) is given by Kalecki by introducing it in the form:

$$\frac{I(t)}{K(t)} = m' \frac{C_0' + W(t)}{K(t)} - n, \tag{12.5}$$

indicating that the relative investment (relative to capital already present) is governed by the expression on the right side representing a linear function of relative profits.

It is easily seen that equations (12.1) to (12.4) lead to a mixed difference and differential equation. Kalecki shows that one of the components which this system may show—and the one which economically is the most important—is a periodic movement with a period which, assuming realistic values for the constants, turns out to be 10 years.[51]

The reader will observe that there are two dynamic features in Kalecki's system, viz., the production lag and the appearance of K—which has an integral character—in equation (12.4). These features have been mentioned in 4; they are essential to this theory insofar as the omission of one of them would lead to a non-oscillatory system. My impression is that the supposed influence of K is rather questionable.[52] A happy feature of Kalecki's system is the fact that he places capital goods production in the center. A very remarkable feature is that the

[51] A fuller discussion of these numerical results is given by Kalecki, *loc. cit.*, and by Frisch and Holme, ECONOMETRICA III (1935), 225–239.

[52] This is illustrated by the way in which revival generates in this scheme. Although profits are still decreasing, it is by the decrease in capital that percentage profits increase and in Kalecki's argument investment is stimulated again. Is this sufficiently realistic?

very small number of variables included is sufficient to get a "closed system." The fact that other very much discussed concepts do not appear explicitly in it does not mean that they would not show any business cycle. For money rates, Kalecki states *expressis verbis* that they are assumed to be parallel to $W(t)$. The same thing might be assumed about prices which, remarkably enough, do not appear at all in his theory. Although general economic theory is accustomed to give the central place to prices and although I myself, in an attempt to give business cycle schemes, did so too, I do not consider this feature of Kalecki's system a great disadvantage. In many cases, statistical investigation leads to the suggestion that the rôle of prices is not so important as we have become accustomed to think. It is, in fact, quite possible that investment activity is predominant, a fact recognized and emphasized by many modern authors.

Like many other mathematical schemes, Kalecki's system will hold good even when quite different assumptions are taken as a base. Thus, the term with K in equation (12.4) might be interpreted by pointing to the influence of productivity on profits (cf. 4).

A striking difference from one of the schemes to be discussed further on, in 18, consists in the assumption of constancy of replacement. The schemes later to be mentioned assume just the opposite, i.e., that replacement demand after the life time of equipment has elapsed is the principal factor in the new revival. It is interesting to note that these schemes do not need the two dynamic features that are essential in Kalecki's system.

The Kalecki solution is, in one sense, of restricted value; he chooses such a solution as has zero dampening. As Frisch points out,[53] the solutions of the Kalecki equations, particularly their dampening characteristics, are very sensitive to variations in the data; it is an interesting question whether this is in accordance with facts. On the other hand, the period is not so sensitive, and this is an advantage in comparison with my own schemes.

To conclude, the structure of Kalecki's system entails certain difficulties of great practical importance. It is not clear, for example, how a price stabilization should be represented by it. What constants will be changed by price stabilization, or what additional terms will appear in the equations? How should other disturbing factors, such as changing crops, react on them?

The reader will have remarked, nevertheless, that there are important features in the theory and that the exact form in which it is presented creates the possibility of a clear and fruitful discussion.

13. A second case of a really closed system of equations representing

[53] *Op. cit.* (note 51).

business cycle mechanism was given by Frisch.[54] Its important feature is that it makes a distinction between the mechanism and the external shocks operating on that mechanism and shows, by a very interesting example, what shapes of cycles appear when such shocks are assumed to occur. For that reason, the special mechanism assumed by Frisch may have been intended to have merely an illustrative character. Its economic foundation is not clear in every point. Three variables are distinguished:

$x(t)$, volume of consumption,

$y(t)$, production of capital goods started, the same variable as Kalecki's $I(t)$, and

$z(t)$, called by Frisch "carry-on-activity", equivalent to Kalecki's variable $A(t)$.

The following equations are assumed to exist between x, y, and z:

$$z(t) = \frac{1}{\epsilon}\int_{t-\epsilon}^{t} y(\tau)d\tau \qquad (13.1)$$

(ϵ being the construction period of capital goods),

$$y(t) = mx(t) + \mu\dot{x}(t), \text{ and} \qquad (13.2)$$

$$\dot{x}(t) = c - \lambda(rx + sz). \qquad (13.3)$$

The significance of the first of these equations is clear. That of the second is based on the hypotheses that replacement of capital goods is proportional to consumption $x(t)$ and new investment is proportional to $\dot{x}(t)$. I discussed the relative validity of these hypotheses in 4. Sufficient statistical evidence on (13.2) has not yet been given.[55] The third equation is based on the notion of "encaisse désirée," introduced by Walras, and it sets forth that as the need for that encaisse désirée $\lambda(rx+sz)$, depending on consumption and production, increases, the rate of growth of consumption $\dot{x}(t)$ diminishes. This may be true. But it is desirable that these relations be analyzed economically and investigated statistically before they are made a basis for a theory, especially when the number of competing possibilities is as large as here. It might be that the "brake" included in the mechanism of equation (13.3) works in a different way.

Again, Frisch chooses his constant ϵ, the construction period of capital goods, equal to 3 years. I think this is too long, even though one considers—as Frisch does in this connection—the planning and erection of new plants and new enterprises. No doubt the construction

[54] "Propagation Problems and Impulse Problems in Dynamic Economics," *Economic Essays in Honour of Gustav Cassel* (1933), p. 171.

[55] Cf. what has been said in 4 C.

of some parts of equipment needs several years, but there are numerous cases where a certain extension of consumption can be obtained by much shorter investment processes. The discussion of this point is difficult, however, because the relation is not easily verified. One of the direct consequences of the assumption of this lag is, for example, that production starting, $y(t)$, and carry-on-activity, $z(t)$, move almost in opposite directions. I think this does not correspond with reality. Moreover, the length of the periods found is highly dependent on the value of ϵ.

Nevertheless, Frisch's method is important and so is his result. The mechanism appears to lead to movements with periods of $3\frac{1}{2}$ and $8\frac{1}{2}$ years and the influence of m, μ, c, λ, r, and s, on these periods is but small.

14. A remarkable scheme has been presented by Roos,[56] who considers rather in detail the entrepreneur's fixing of prices and production. Assuming a demand function

$$y = \gamma(ap(t) + b + h\dot{p}(t)) = \gamma u, \qquad (14.1)$$

and a cost function

$$Q = Au^2 + Bu + C + D\dot{u}^2 + F\dot{p}^2 + G\dot{p} + H\dot{u}, \qquad (14.2)$$

he calculates profits over a given time interval

$$= \int_{t_1}^{t_2} (\gamma pu - Au^2 - Bu - C - D\dot{u}^2 - F\dot{p}^2 - G\dot{p} - H\dot{u})E(t_1, t)dt \qquad (14.3)$$

which is maximized under the condition that (14.1) is satisfied; u being quantity produced, p price, y quantity sold, $E(t_1, t)$ a "discount function," reducing money amounts at time t to time t_1 and being equal to $1-\int_{t_1}^{t}\delta(r)dr$, where $\delta(r)$ is the interest rate.

All other symbols represent, for the moment, constants.

The problem thus posed is a variable end point variation problem. Its solutions must satisfy a fourth order differential equation. They may have different forms, as exponential and periodical components can be combined in different ways.

Aside from many details and some questions of principle treated in a very elegant way by Roos, his view of the problem of economic oscillations is that every change in constants causes jumps in other individual solutions. In that way oscillatory and exponential, damped and anti-damped movements follow one another.

It cannot be denied that there are some very suggestive elements in

[56] "A Mathematical Theory of Price and Production Fluctuations and Economic Crises," *Journal of Political Economy*, Oct. 1930, pp. 501–502.

Roos' theory. Above all, the idea of the successive pursuance of curves of different shape is of importance and its analogy with Frisch's "shock" theory is evident.

As to the mechanism itself, 1 have several objections. It seems a little unbalanced in its economic construction. The questions as to how the changes in production affect incomes and, therefore, demand, and how they influence money rates, are not considered.[57] On the other hand, the assumed cost function is very complicated, much more complicated than is necessary, as a first approximation, to describe facts. This applies particularly to the constants D and F. They play an important rôle in Roos' equations; when $D=F=0$ they generate to much simpler ones. When, in addition, also $A=0$, no oscillations would occur at all. Furthermore, in Roos' theory no estimation of the length of the periods is undertaken. This would meet with several problems, of course, as a statistical determination of the constants introduced would be very difficult.

In some respects, Vinci has also presented a mathematical theory of business cycles, in the sense that he has given a system of eight equations.[58] Vinci has not yet published anything on the *solutions* but, nevertheless, I should like to make a few remarks on his attempt. The interesting feature in it, which is, no doubt, an advantage in comparison with the simpler schemes, is the larger number of equations. To the assumptions on which his special equations are based, I have serious objections. He supposes, for example, that the physical volume of production is determined by the velocity with which profits change and the velocity with which income spent changes (Eq. I). There is very little evidence that these velocities are the chief factors. The same might be said of his equations II, V, VI, VII, and VIII. It should be added that Vinci himself sees the necessity of empirical investigations in this respect.

15. *Other mathematical schemes. Short lag schemes.*—The schemes of Kalecki, Frisch, Vinci, and myself,[59] are examples of the wide variety of possibilities of explaining economic oscillations. The various schemes put different emphases on the various factors. In my own schemes, prices occupy a central place. It is convenient to divide these schemes into *short lag schemes* and *long lag schemes.* "Short lags" are short in comparison to the period of the cycles, whereas the "longer" ones are of the same order of magnitude. The first sub-group admits of a special

[57] At this time Dr. Tinbergen had not seen Roos' *Dynamic Economics*, Bloomington, 1934. ASSISTANT EDITOR.

[58] "Significant Developments in Business Cycle Theory," ECONOMETRICA II (1934), pp, 136–139.

[59] *Zeitschrift für Nationalökonomie* V (1934), 289; Wagemann-Festschrift.

method of treatment that will presently be discussed. For the reason already mentioned, all schemes are very much simplified. They are macrodynamic. No difference is made between separate categories of goods (except between such categories as investment goods and consumption goods, or, in some cases, between raw materials and finished goods), nor between separate categories of factors of production (in many respects labor and capital behave quite similarly). In later sections of this survey some of the resulting limitations will be removed. It may be emphasized, however, as is pointed out also in the papers quoted, that the formulas obtained often have a much wider validity than the simplified scheme for which they are deduced.

The mathematical method of analysis is simplified by the introduction of the notion of "equilibrium values" for each variable. By equilibrium values, the constant values (contrary to the variable values or functions of time) that satisfy the equations are meant. Such equilibrium values will be indicated by capital characters and the variable values will be measured as deviations from these equilibrium values. These deviations are indicated by the corresponding small characters.

Pure lag schemes.—The simplest cases considered, which are useful also as a methodological starting point, are those in which the only dynamic feature included is a production lag. This entails two consequences: the number of finished products shows this lag as compared with the number of products started and the number of production factor units used (chiefly labor) shows, as a first approximation, half this lag. We introduce the following symbols:

Price of finished consumers' goods, $P+p(t)$,

Number of products started (consumers' goods), $Z+z(t)$,

Number of products sold (consumers' goods), $Y+y(t)$,

Income spent by consumers, $X+x(t)$,

Increase of stocks of products, $V+v(t)$.

It will easily be seen that $Z=Y$, whereas $V=0$.

Furthermore, suppose the following relations to exist between these variables:[60]

$$x(t) = \frac{2k}{a} z(t-1), \tag{15.1}$$

$$\epsilon v(t) = z(t-2) - y(t), \qquad (\epsilon = 1 \text{ or } 0) \tag{15.2}$$

$$\epsilon(z(t-2\eta) - ap(t)) = 0, \qquad (\epsilon = 1 \text{ or } 0) \tag{15.3}$$

$$z(t) = \epsilon' ap(t) + (1-\epsilon')y(t), \qquad (0 \leqq \epsilon' \leqq 1) \tag{15.4}$$

$$(Y + y(t))(P + p(t)) = X + x(t). \tag{15.5}$$

[60] These relations are all assumed to be linear; a more general conception is given by Haldane, *Review of Economic Studies* (1934), p. 186.

The economic meaning of these equations is the following. Eq. (15.1) expresses a hypothesis regarding the oscillations of income spent, viz., that they are proportional[61] to the oscillations of productive activity. This holds exactly, for example, when all factors of production have constant earning rates and when entrepreneurs are also supposed to spend in proportion to activity. This may seem an unrealistic hypothesis. More detailed considerations will show, however, that numerous cases of realistic value can be brought under this formulation (cf. 8). As an example, it may be remarked that our formulas hold when the amounts spent by entrepreneurs (e.g., farmers) show other movements, provided that these are synchronous with the value of products sold. In that case, equal terms are to be added to the left and right sides of equation (15.5) and these terms neutralize each other (cf. what is said below on equation (15.5)).

Equation (15.2) expresses, for $\epsilon=1$, a purely technical relation between production finished, $z(t-2)$, sales, $y(t)$, and stock increase, $v(t)$. For $\epsilon=0$, it indicates that no stock variation occurs.

Equation (15.3) may be called the price setting equation. It expresses the fact, for $\epsilon=1$, that, when production is high, prices are fixed at a high level, and when production is low, prices are also low. This relation may be considered as an expression for the result of price calculations by the producer. According to different choices of η, price setting shows more or less of a lag behind planning. For $\epsilon=0$, this equation is abolished, as price setting by the producer is impossible in the case that all products have to be sold. Market relations then determine prices. The introduction of ϵ, therefore, simply means the introduction of alternative forms of selling policy.

Equation (15.4) may be called a production planning equation. It gives the relation between market condition, represented by price $p(t)$, and sales, $y(t)$, on one side, and new production started, $z(t)$, on the other, which is an expression for the considerations leading to the fixing of $z(t)$. It is assumed that both "past performance" and price situation may act on $z(t)$, but, by varying ϵ', the one or the other factor may be given more stress. As limiting cases, we have $\epsilon'=1$ where production is entirely determined by prices, and $\epsilon'=0$ where it is entirely determined by "past performance."

Equation (15.5) may be called the market equation. It expresses the fact that the amount spent by consumers equals the product of prices and quantities of consumers' goods sold. As the reader will already have seen, equations (15.1) to (15.5) represent different cases, depending on the special choice made for ϵ and ϵ' in the first place, and

[61] The proportionality factor is written $2k/a$ to accord with the paper mentioned.

for the other constants. Values for ϵ and ϵ' are not quite independent, as $\epsilon'=1$ must necessarily entail $\epsilon=0$ or $\eta=0$. Indeed, $\epsilon'=1$ means that $z(t)=ap(t)$ and this would be contrary to (15.3) unless $\epsilon=0$ or $\eta=0$. A certain choice for ϵ does not, however, restrict possibilities for ϵ'.

In terms of the "literary" business cycle theories, these schemes may be characterized as follows. In the first place, they show the case of a general over-production and over-investment. Second, in these schemes the variable p is proportional to the deviation of the natural rate of interest (in the Wicksell-Mises-Hayek sense) from its own long run equilibrium value and also from the money rate of interest. Third, in one of the more complicated versions (viz., after the introduction of integral terms below) the notion of scarcity of capital (Spiethoff) is introduced, although in a very simple way. In this respect, there are many opportunities of improvement of the schemes. This question of the introduction of the influence of capital scarcity is a suitable example of the vagueness of many "literary" theorists as to the description of relations introduced. Many mathematical forms may compete here: By what variables is the demand and the supply of credit determined? Are the changing velocities of these variables an important factor to be included? Nothing explicit is to be found in the literature; nevertheless, it makes a big difference which one of these demand and supply functions is used as basis. Some, for example, yield no cyclical movements at all. For others, quite different phase differences between some of the variables will occur, and so forth.

We proceed to the treatment of two special cases of the scheme given by (15.1) to (15.5).

A. $\epsilon=0,\ \epsilon'\neq 1$.

Equations (15.1) to (15.4) then become: $x(t)=2k/a\ z(t-1)$; $z(t-2)=y(t)$; $z(t)=\epsilon' ap(t)+(1-\epsilon')y(t)$, and, substituting this in Eq. (15.5), we get for small movements:

$$z(t-2)+\frac{z(t)}{\epsilon' a}-\frac{1-\epsilon'}{\epsilon' a}z(t-2)=\frac{2k}{a}z(t-1),\ \text{or}$$

$$z(t)-2k\epsilon' z(t-1)+(\epsilon' a+\epsilon'-1)z(t-2)=0. \quad (15.6)$$

The solution is found[62] by putting $z(t)=C\xi^t$, which leads to the following equation for ξ:

$$\xi^2-2k\epsilon'\xi+(\epsilon' a+\epsilon'-1)=0,$$

$$\xi=k\epsilon'\pm\sqrt{k^2\epsilon'^2-\epsilon' a-\epsilon'+1}.$$

[62] Cf. Frisch and Holme, "The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics," ECONOMETRICA, *loc. cit.*, for this and the following problems.

It follows that the movements are periodical when

$$k^2\epsilon'^2 + 1 < \epsilon'(a + 1),$$

and exponential when

$$k^2\xi'^2 + 1 \geqq \epsilon'(a + 1).$$

In addition, we have $|\xi|^2 = \epsilon'(a+1) - 1$.

The special case $\epsilon' = 1$ is identical with that treated in my *Zeitschrift* paper. A fuller discussion of the solutions will be given, together with that of the other cases, in 16.

B. $\epsilon = 1, \epsilon' \neq 1$.

Equations (15.1) to (15.4) now become:

$$x(t) = \frac{2k}{a} z(t - 1); \; v(t) = z(t - 2) - y(t); \; z(t - 2\eta) = ap(t);$$

$$z(t) = \epsilon' ap(t) + (1 - \epsilon')y(t).$$

Of these, the second is to be considered as the definition of $v(t)$ only. Substitution in (15.5) gives, after some rearrangement,

$$z(t) - \frac{2k}{a} z(t - 1) + \left(\frac{1}{a} - \frac{\epsilon'}{1 - \epsilon'}\right) z(t - 2\eta) = 0. \qquad (15.7)$$

For $\eta = \frac{1}{2}$, no oscillatory movements (except, in some cases, with period 2) are possible. For $\eta = 1$, we get for ξ, introduced by $z(t) = C\xi^t$,

$$\xi^2 - \frac{2k}{a}\xi + \frac{1}{a} - \frac{\epsilon'}{1 - \epsilon'} = 0,$$

$$\xi = \frac{k}{a} \pm \sqrt{\frac{k^2}{a^2} - \frac{1}{a} + \frac{\epsilon'}{1 - \epsilon'}}.$$

The "wave condition" now becomes

$$\frac{k^2}{a^2} - \frac{1}{a} + \frac{\epsilon'}{1 - \epsilon'} < 0, \quad \text{whereas,}$$

$$|\xi|^2 = \frac{1}{a} - \frac{\epsilon'}{1 - \epsilon'}.$$

Introduction of integral terms.—The schemes just given are admittedly much simplified. We shall now show different possibilities of making closer approximations to reality. A first possibility is no longer to suppose remunerations of the factors of production to be constant. As is shown in more detail in the paper cited above, a good approximation to reality consists in supposing that wage rates show year-to-year changes proportional to employment deviations from trend (cf. also section II, 5 of this survey). Using profit margin as the central variable, this second approximation means an introduction of integral terms $\int_0^t d\tau p(\tau)$, where p is the profit margin.

The same occurs when productivity changes are taken into account or, as in Kalecki's case, total capital accumulated since some initial date.

Needless to say, when an integral term is added, oscillations are also possible in cases where, for example, only two terms of the type of equation (15.6) or (15.7) are present.

Introduction of derivatives.—Another type of approximation leads to the introduction of derivatives. Examples of great importance are different sorts of speculative influences. Apart from prices or profit margins, the change in prices or margins may have influence on productive activity— as is assumed also, though not always on exactly the same grounds, by Amoroso,[63] Fisher,[64] Roos,[65] Evans,[66] and Vinci[67] (cf. 16). The interpretation of such relations is given in 4.

We shall treat, also, a set of examples of these cases. Amending Case A (for $\epsilon'=1$, i.e., the "pure lag case" treated in my *Zeitschrift* article) we get Case

C. $z(t)=ap(t)+ab\ \dot{p}(t)$; $y(t)=z(t-2)$; $x(t)=2kp(t-1)+bk\ p(t)-bk\,p(t-2)$, leading to the following final equation for $p(t)$, which we take now as the central variable.

$$(1-bk)p(t)-2kp(t-1)+(a+bk)p(t-2)+ab\dot{p}(t-2)=0. \quad (15.8)$$

In 16 a further discussion of this case will be given.

Case D. We may generalize Case C by supposing the supply of consumables to be, in the units already introduced,

$$1+ap(t+n)+a'\dot{p}(t+n), \quad (15.9)$$

and the amount spent for consumables,

$$1+2kp(t+m)+2k'\dot{p}(t+m). \quad (15.10)$$

The meaning of these formulas is clear, when n is negative, $m=\frac{1}{2}n$, and $k'/K=a'/a$, as they then turn into those of Case C (with a high

[63] "Ciò che è chiaro e ciò che è oscuro nelle fluttuazioni dei prezzi," *Atti dell' Istituto Nazionale delle Assicurazioni* IV (1932).

[64] *Theory of Interest*, 1930.

[65] Though on somewhat other grounds; cf. "A Mathematical Theory of Competition," *Amer. J. of Math.*, XLVII (1925), and many other publications. See also note 4.

[66] Cf. *A Math. Introduction to Economics.*

[67] ECONOMETRICA II (1934), p. 136.

degree of approximation). When these conditions are not fulfilled, however, it is not difficult to give them a suitable interpretation.

For example, n may also be positive. This means that production no longer is fixed by the price situation but, conversely, as we already discussed, that prices are determined by production, i.e., (15.9) is then analogous to (15.3). The term $a'\dot{p}(t+n)$ may then be attributed to the circumstance that price setting is influenced by historical cost prices.

When, in this case, $m>n$ and $k'/k=a'/a$, our previous expression for the amount spent still holds good. This amount is then determined by the productive activity corresponding with the supply of consumables, activity in capital goods industries being assumed parallel to that activity.

It is, however, also possible that $m \leqq n$. This may be the case when there are retarded elements in the amount spent, such as the lagging of salaries behind activity,[68] or even some forms of saving.

It is equally possible that $k/k'=a/a'$ no longer holds. This is possible when, in the amount spent, certain elements appear that are not parallel to activity. When these elements are parallel to prices (as wage rates may be), k will be larger than would correspond with the proportionality just mentioned. When they are parallel to price changes (as speculation gains may be), k' will be larger.

The possibilities mentioned show that, until all questions raised are answered by detailed factual research, there is a very large number of possible schemes. This number may be further enlargened by introducing the peculiarities corresponding with $\epsilon' \neq 1$ in Case A. Our considerations are, therefore, only to be looked upon as *examples*.

Expressions (15.9) and (15.10) lead us to the equation (neglecting second order terms),

$$ap(t+n)+a'\dot{p}(t+n)+p(t)-2kp(t+m)-2k'\dot{p}(t+m)=0. \qquad (15.11)$$

Mixed form of equations

By introducing integral, derivative, and lag terms, still more general types of equations, or systems of equations, are obtained. Those will be reached when, as wages and productivity change, speculative influences and the fact of different lags for different industries or enterprises are considered. As far as one equation with only one variable finally results from elimination, this equation will have the form:

$$\sum_1^n a_i p(t-t_i) + \sum_1^n b_i \dot{p}(t-t_i) + \sum_1^n c_i \int_0^{t_i} p(\tau)d\tau = 0. \qquad (15.12)$$

[68] In this case it is supposed that wages show a lag behind activity and not, as was done in other cases, that they have "integral character." Cf. 5.

16. *"Wave Condition" and "Long Wave Conditions."*—As has been shown already in some of the cases treated, the solution of the equations or systems of equations may have different time shapes. The simplest cases often show solutions that are exponentials for some value range of the constants and oscillations (damped or even anti-damped) for other ranges. The conditions determining those latter ranges may be called wave conditions. For Cases A and B, the wave condition has been given.

To explain real business cycles by one of the schemes discussed here, not only must the wave condition be satisfied by the constants but the waves must, in addition, be fairly long in comparison to the lags introduced, i.e., to the time unit. This enables us to prescribe more rigorous conditions for the constants, which we may call "long wave conditions." By this term we mean not only the condition for a long wave but also for a wave not differing too much from an undamped one. Waves which are practically damped down long before one cycle has been described should be excluded.

The first approximation to the long wave conditions is found by putting $r=|\xi|=1$ and $\varphi=\arg\xi=0$ (see 15, sub A.). In this case, however, the mistake must be avoided of applying this method to cases where no oscillation at all is possible. The position $|\xi|=1$, $\varphi=0$, must be a limit of values where φ really differs from zero, and not one of values where only r varies. To make sure, the wave condition must be ascertained first and then it is clear that one of the long wave conditions is nothing other than the wave condition with the sign reduced to an equality sign.

So we find "long wave conditions (first approximation)" for

Case A: $\quad a = \dfrac{2}{\epsilon'} - 1, \qquad k = \dfrac{1}{\epsilon'};$

Case B: $\quad a = k = 1 - \epsilon';$

Case C: $\quad a + b = 1, \qquad 2k = a + 1;$

Case D: $\quad a + 1 - 2k = 0, \qquad na + a' - 2km - 2k' = 0.$

These conditions will be a guide in a statistical test of the different schemes as to their accord with reality. (cf. 17). It may be remarked beforehand that the application of this method to the general equation (15.12) leads to the equation:

$$\sum_i^n c_i = 0$$

as one of the long wave conditions. This means that the sum of the coefficients of the integral terms must be zero; when only one integral term is present, its coefficient should be zero. As we have to do with a first approximation only, it need not be exactly zero but it should be small; schemes of the form (15.12) only then lead to long, not too much damped waves when integral terms are of small importance.

This has a remarkable consequence for those schemes in which wages are supposed to be variable and of the special form discussed in 5 (2), i.e., that they move parallel to the integral of profit margins or of activity. When such schemes have, in addition, no other integral terms of opposite sign and no other dynamic elements such as are included in the general equation (15.12), it will be impossible to explain real business cycles with their help. To state it in another way, the influence of wage changes as they have occurred in reality (I make an exception for the extraordinary wage changes caused by the codes in the U.S.A. in 1933) cannot be one of the chief factors in business cycles.

It is possible to give a *second approximation* by assuming $r=1+\delta$, where δ is a small quantity and φ is also a small value. It is in accordance with reality to suppose more exactly that δ is of the same order of magnitude as φ^2 and φ^3 is negligible. As an example, Case C has been treated in this way. The result is that the long wave conditions (2nd appr.) are (writing—6ψ for φ^2):

$$(1 - bk)(1 + 2\delta + 12\psi) - 2k(1 + \delta + 3\psi) + a + bk + ab\delta = 0, \text{ and}$$
$$(1 - bk)(2 + 4\delta + 8\psi) - 2k(1 + \delta + \psi) + ab = 0.$$

It is possible to solve these equations for a and k:

$$a = \frac{b(1 + 2\delta + 4\psi) - (1 + 3\delta + \psi)}{6\psi b^2 - b(2\delta + \psi) - (1 + \delta + \psi)} \text{ and}$$

$$k = \frac{2 - b}{2} + \frac{b - 1}{2}(\delta(2b - 1) - \psi(6b^2 - 7b + 6)),$$

from which the following table is calculated, showing values for b, a, and k, that lead to given values for r and φ.

17. *Statistical verification of "long wave conditions."* To find out whether these schemes can explain real business cycles and which of them most resembles reality, a statistical determination of the constants used in our formulas becomes necessary.

There are two somewhat different ways to measure these constants, which may be called the *structural* and the *historical* method. The difference may be illustrated by discussing the measurement of a. This constant measures the quotient of corresponding percentage changes in prices and production along the supply curve, other things affecting supply being assumed constant. The cause of this correspondence is

TABLE I(A)

VALUES FOR a, b, AND k, CORRESPONDING WITH $r=0.9$ AND DIFFERENT PERIODS

TABLE I(B)

VALUES FOR a, b, AND k CORRESPONDING WITH A PERIOD OF 26 AND DIFFERENT r'S.

		Periods[a]			Value of r			
		21	26	36	0.90	0.95	1.05	1.10
$b=$ 0.10	$a=$	0.88	0.89	0.90	0.89	0.91	0.94	0.96
	$k=$	0.71	0.71	0.71	0.71	0.81	0.98	1.06
0.20	$a=$	0.85	0.86	0.87	0.86	0.87	0.89	0.90
	$k=$	0.64	0.63	0.63	0.63	0.72	0.87	0.94
0.30	$a=$	0.81	0.82	0.83	0.82	0.82	0.84	0.85
	$k=$	0.56	0.56	0.55	0.56	0.62	0.77	0.82
0.40	$a=$	0.78	0.78	0.79	0.78	0.78	0.79	0.79
	$k=$	0.48	0.47	0.47	0.47	0.55	0.66	0.70
0.50	$a=$	0.74	0.74	0.74	0.74	0.74	0.74	0.74
	$k=$	0.39	0.39	0.38	0.39	0.46	0.55	0.59
0.75	$a=$	0.62	0.63	0.63	0.63	0.62	0.62	0.61
	$k=$	0.17	0.16	0.15	0.16	0.22	0.30	0.33

[a] The time unit is chosen equal to half the production lag.

to be seen chiefly in the fact of unequal cost of production in different enterprises in the same industry. When prices increase, a larger part of enterprise becomes profitable. The velocity with which this part grows, or the increase in this part for a unit change in prices, measures a. This a might, therefore, be determined from a cost distribution function, in any case from a timeless structural datum. It might, however, also be determined in the usual statistical way of determining supply curves from historical data on prices and production figures considered to correspond to these prices. In general, since data at our disposal do not permit following the structural method, the historical method has been chosen.

There are many difficulties to be faced which, to a large extent, are caused by two circumstances, the lack of adequate data and the great simplification of reality adopted by the schemes.[69] In the cases we are

[69] No attention is given, for example, to the special character of the building trades, which largely produce consumers' goods but goods that are not in general sold to the consumers and do not appear in our equation. On the other hand, in some cases, no attention is paid to the fact that a considerable part of consumers' income is spent on housing rent, forming a rather rigid amount. This circumstance can be taken account of by subtracting a constant figure from the

now interested in, the customary way is to reach exact statements: the flight into detailed study, is now, in principle, closed. I am, therefore, convinced of the highly approximate character of the following attempts.

The first verification attempt relates to the pure lag schemes A and B. It attempts to determine constants a and $2k$ of these schemes. In either of these cases, a manifests itself as the quotient of two amplitudes, that of the sales volume of consumers' goods and that of prices. This will be easily seen when it is remembered that the lags appearing in the equations are but small, so small that, in connection with the irregular components in each series, they cannot be estimated with any accuracy. The amplitudes should be measured in units equal or proportional to the equilibrium value of either variable. This equilibrium value has been assumed constant in our schemes but it is easily seen that no changes occur in these schemes by assuming that value itself variable. To reach a larger agreement with reality, we do so and suppose trend values to be equilibrium values.[70] In some cases, irregular movements have first been eliminated by using three year moving averages. Trends are obtained by straight lines. Values of a and $2k$ have been obtained by simply dividing the average absolute value of percentage deviations from trend for the series mentioned in the headings of the table by the corresponding absolute value for the price series. When there are different series that can be estimated to approximate the variable in question, alternative calculations have been made. It should be borne in mind, finally, that prices must be prices of finished products paid by consumers, so retail prices or even cost of living indexes have been used. In a similar way, $2k$ is the amplitude quotient for income spent and prices. The results are given in Table II.

The figures given are very rough. In some cases, standard deviations have been calculated, in other cases the degree of uncertainty is illustrated by the influence of the omission of one or more years or by the deviations between competing series for the same economic category. Standard deviations are multiplied by 3.

Comparing these results with the long wave conditions for Case A, we can state that, as these conditions require both for a and for k values >1, this case cannot explain business cycles in England and the

amount spent by consumers; the influence on our constant k will be that this constant will become larger in about the same proportion. In the same way, other corrections are possible, but it seems wiser to postpone these corrections until the whole subject is treated in a more detailed manner.

[70] The problem that lies behind this hypothesis is fully recognized, but it is considered of secondary importance in the question now discussed.

TABLE II

VALUES FOR a AND k IN SCHEMES A AND B.

The left half of each table relates to a, the right half to k. In the column heading, the series used for the quantity of consumers' goods sold (in the case of a) or the series used for income spent (in the case of k) is indicated; in the rows, the series used for prices is mentioned. The period covered is indicated in the column headings.

ENGLAND

	Production of consumers' goods calculated from London & Cambridge Economic Service Bulletin 1921–1931	Production of consumers' goods (*Vierteljahrsh. z. Konj.*) 1921–1930	National Income (cf. chart) 1921–1930	1921–1931	Actual income of tax payers 1921–1930
Cost of Living	0.6	0.8	0.3	0.5	0.4

GERMANY, 1924–32

	Retail sales: cost of living (without rent)	Production of consumers' goods (old series, *I.f.K.*)	Production of consumers' goods (new series, *I.f.K.*)	National Income (without Reparation Payments)	Labor Income	Income spent for Free[a] Consumption
Cost of living, food only	0.9	1.5	0.9	1.0	1.2	0.8
Cost of living, excl. rent	1.0	1.7	1.0	1.1	1.3	1.0
Cost of living, total	1.0	1.7	1.0	1.1	1.3	0.9

[a] Cf. *Vierteljahrshefte zur Konjunkturforschung*, 1934, (8 Jg, H. 3), p. 161.

UNITED STATES

	Production of consumers' goods, Wagemann index, 1920–1930	Production of consumers' goods League of Nations index, 1921–1932	Production of foods 1919–1933	Department Store Sales: Retail prices (food)	Pay rolls			National Income minus estimated savings (Walsh)	
					1919–1933	1920–1932	1920–1930	1920–1932	1920–1930
Retail prices, food	0.4	1.1 ±0.8	0.4	0.5	0.9	0.8 ± 0.7	0.6	0.5	0.4

United States, but it can, however, explain them in the case of post-war Germany. For that country, values for a and k are near those required in the special case where $\epsilon' = 1$ (the *Zeitschrift* case).

For Case B, on the contrary, a and k must have values below 1, and this is in accordance with English and American figures as well as for a "world" figure obtained by averaging English, American, and German figures.

In general, it may be stated that the order of magnitude of the constants is the one necessary to our schemes but detailed research will be necessary to make final decisions.

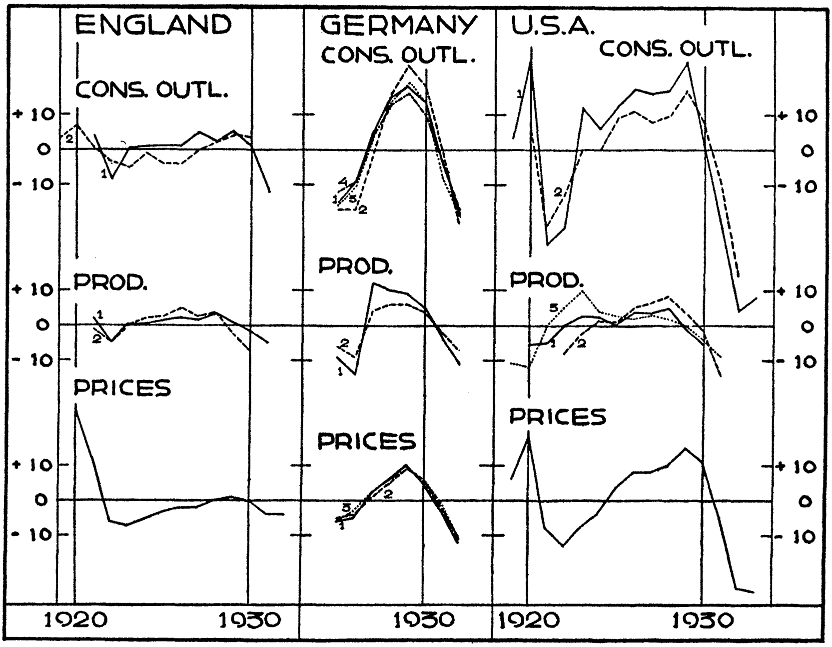

FIGURE 4.—Some data on consumers' outlay, supply of consumers' goods, and retail prices, used in the statistical verification of schemes A and B.

Germany.—"Consumers' Outlay": (1) National Income minus Reparations; (2) Labour Income; (3) National Income; (4) Estimated Consumers' Outlay.

"Production": (1) Production of Consumers" Goods, old series (Konjunkturstat. Handbuch); (2) Production of Consumers' Goods, new series (Wochenbericht, 9–5–1934).

"Prices": (1) Cost of living: food only; (2) Cost of living without rent; (3) Cost of living, total.

England.—"Consumers' Outlay": (1) National Income (1920–23: *The Economist;* 1924–'31: Colin Clark); (2) "Actual Income" of Tax Payers.

"Production": (1) Production of Consumers' Goods, calc. by myself from London and Cambridge Econ. Service indexes; (2) Idem, calc. by Institut f. Konjunkturforschung, Sonderheft 31.

"Prices": Cost of Living.

United States.—"Consumers' Outlay": (1) Total Pay Roll; (2) Income minus Savings (Walsh, *Qu. J. of Econ.* XLVIII (1934), p. 546).

"Production": (1) Production of Consumers' Goods (*Inst. f. Konj.*, Sonderheft 31); (2) Production of Consumers' Goods (League of Nations, *Monthly Bull.* 1934, p. 338); (3) Production of Foods.

Although a rough approximation to reality may be obtained with Scheme B, several objections rise on closer examination of the figures. The correlation between the three variables considered is not satisfactory, as shown in Figure 4. Again, phase differences in scheme B are not the same as in reality, as far as it is possible to tell at all.

This explains the second attempt, an attempt to verify the more complicated schemes C and D. The same remarks on the difficulties resulting from the inadequacy of data and the approximate character of the schemes must be repeated here. In addition, the figures to be given for a' and k' depend on the time unit chosen. I have used one quarter of a year. The figures given are the regression coefficients obtained by a simple least squares calculation.

TABLE III

VALUE OF COEFFICIENTS a, a', $2k$, AND $2k'$, IN THE EQUATIONS:

$$u = ap(t+n) + a'\dot{p}(+tn) + a''t,$$

$$x = 2kp(t+m) + 2k'\dot{p}(t+m) + 2k''t,$$

in which u = supply of consumers' goods as approximated by these formulas,
x = amount spent on these goods,
p = retail price level of finished products,[a]
$\dot{p}$ = average quarterly change of p, and
t = time.

All variables are measured as deviations from their average value over the period considered, with this average value as a unit (time unit = one quarter of a year, $\dot{p}$-unit corresponding to price and time unit). Coefficients are regression coefficients (least square method) divided by correlation coefficients r.

[a] Calculations have been made for wholesale finished products prices; the coefficients found have been multiplied by the proportion between corresponding deviations from trend of wholesale prices and the cost of living index.

A. UNITED STATES

n	Period covered by data for p	a	a'	r	m	Period covered by data for x	$2k$	$2k'$	r
−1	1921–1932	1.62	2.50	0.97					
0	1921–1932	1.73	2.18	0.97	0	1921–1932	1.20	4.37	0.79
		0.60	1.47					5.25	
0	1920–1932	0.72	4.41	0.74					
0	1920–1933	0.76	4.55	0.68					
0	1921–1933	1.67	2.18	0.96					
0	1922–1933	1.57	0.77	0.97					
1	1921–1932	1.83	1.82	0.95	1	1921–1932	1.64	3.12	0.81
2	1921–1932	1.80	0.94	0.92	2	1921–1923	1.94	1.21	0.84
3	1921–1932	1.61	0.04	0.94	3	1921–1932	1.95	−1.11	0.92

B. GERMANY

n	Period covered by data for p	a	a'	r	m	Period covered by data for x	$2k$	$2k'$	r
0	1925–1933	0.96	3.51	0.90	0	1925–1933	1.85	0.84	0.96
1	1926–1933	0.96	2.17	0.92	1	1925–1933	1.87	−0.77	0.96
2	1926–1933	1.23	1.17	0.92					
3	1926–1933	1.41	−0.11	0.94					

A comparison with the long wave conditions for scheme D shows that, although these conditions are seldom exactly fulfilled, the discrepancies are often within the (rather large) limits of uncertainty in the coefficients.

Lag Patterns.—The verifications discussed so far are of a rough nature. It will be found that many alternative forms differing, for example, only in the fact that some small lags have been introduced lead to the same long wave conditions. For a more detailed verification, the lags themselves should be verified statistically. Attempts in this direction have up to the present yielded only meagre results, chiefly because of the influence of irregular movements and the lack of exact data as to different stages of the process of production. To show some

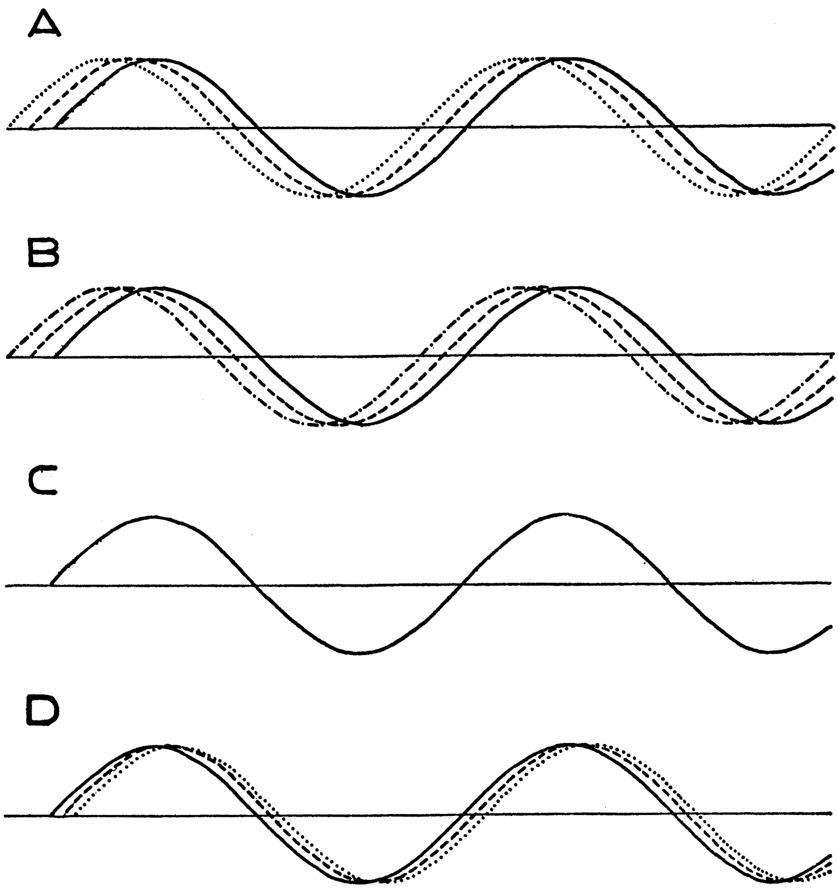

FIGURE 5.—*Lag patterns for some simple business cycle schemes.* (*A* for $\epsilon'=1$, *B* for $\eta=1$, *C* for $b=2$ and *D* for $n>0$) ······ Prices, ----- Income spent on consumables, —— Supply of consumables, -·-·-·-·- Consumables sold. Curves that have not been drawn coincide with the supply curve.

of the details which I have in mind here, lag patterns have been drawn in Figure 5 for Cases *A–D*, indicating the phase differences that follow from each scheme for some of the important variables. The statistical observation of these phase differences is the difficult point; the large number of perturbations occurring in reality entails, in most cases, considerable changes in the order in which different variables pass through a given phase.

Sensibility of period to variation of constants.—One common property of the schemes discussed here must be emphasized. The period is very sensitive to changes in the constants when it is of the length corresponding to the one observed in reality. Although it may be recognized[71] that business cycles have periods that are far from constant, the variability shown by the periods of the short lag schemes seems too large. This circumstance seems to point to the necessity for further amendments of the schemes.

A similar argument against short lag schemes is that an interruption of the upward movement during a relatively short period (e.g., three times the production lag adopted) will not, in general, be followed by a continuance of the original movement. This seems to have occurred sometimes, however, in reality.

Asymmetry of waves.—One of the problems not solved by the simple schemes presented is the asymmetry of waves, i.e., the fact that the upward movement is generally slower than the downward. Apart from special speculative influences that fall outside the schemes hitherto considered, this may be attributed to the fact that the production lag is larger during the expansion than during contraction. In the first period, there will be situations in which stocks of raw materials and of semi-manufactured products are not sufficient to extend production immediately. The production of raw materials must, then, first be extended and the production lag in these stages has to be added to that for the stage under discussion. In a phase of downward moving production this is not true, since the lag will then only be equal to that of the last stage.

18. *Long lag schemes.*—In the schemes meant here, dynamical relations of a lag character play a rôle in which lags occur of the same order of magnitude as the wave lengths themselves. They have chiefly the significance of life times of certain objects such as investment goods. They interpret business cycles in a way more or less like this: suppose there is, for some reason, a high production of investment goods in certain years; then, the long life time of investment goods

[71] As is stated by Hamburger, "Analogie des fluctuations économiques et des oscillations de relaxation," *Indices du mouvement des affaires*, Paris, IX (1931), Supplément.

will entail a sub-normal production of these goods in years to follow. When, in addition, there is a certain tendency among these life times to a grouping in the neighborhood of a given period, after this period there will be a revival of investment goods' production. This interpretation appears in the theories of Marx, Tugan Baranowski, Schönheyder, Cassel, Robertson, and De Wolff, but not as a closed mathematical scheme. I have attempted this for the simplest case, viz., when all investment goods are supposed to have one and the same constant life time.[72] The result is that, without any additional dynamical element, cyclical movements can be generated, the period of which is closely connected with the given life time. It need not, however, equal that life time, and cases are possible where it is only about $\frac{5}{6}$ of that constant.

This simple case corresponds to the simplest possibility for replacement, mentioned in 4. Corresponding to the more complicated cases treated there, generalized schemes are possible. First, it may be supposed that the economic situation has influence on the moment of replacement (the case considered in 23), and, second, the assumption may be made that there are different sorts of machines each having its own life time. Details of this case have been treated by Frisch.[73]

Starting with a discontinuous frequency distribution of the different durations of life over the total number of machines, he shows that, after the lapse of a certain time, the movement of reinvestment as a function of time becomes very irregular when one considers each year separately (Cf. Figure 6). When moving averages are taken, however, a smoother curve is obtained which shows damped oscillations. The cases with a continuous frequency distribution, analyzed in a very elegant way by Frisch, also lead to damped oscillations.

Frisch assumes that the different sorts of machines in no way compete, viz., that neither do they themselves compete in the production of the same finished product nor do their products, if different, compete in consumption. Introduction of such possibilities of substitution have the tendency to restore a purely periodical movement.

As a fourth case, we have individual random differences in life time. A ten-year-old machine, like a five-year-old, may be replaced by a machine whose life time has a given probability distribution, but no given value. This case has been treated by Lotka, Kurz, and Vos,[74] who

[72] *Loc. cit.*, p. 315.

[73] Mimeographed lectures, 1933, 852.

[74] Alfred J. Lotka, "The Progenity of a Population Element," *American Journ. of Hygiene*, Nov. 1928. "The Spread of Generations," *Human Biology*, Sept. 1929. Edwin B. Kurtz, *Life Expectancy of Physical Property*, New York, 1930. Vos, "Conjuncturcyclus en Techniek," de *Social. Gids* 1934, p. 464, and a paper to be published later.

reached the conclusion that, depending on the frequency distribution of life times, a movement with some cycles, or one without any cycle, may result.

The theories showing life times of replaced machinery as the only dynamic element or, more precisely, as the chief dynamic element have often been criticized. This criticism is usually of a negative and incomplete character. With regard to the statistical data in this field, the following information is now at our disposal: (a) rather rough data on depreciation rates used in practice and giving only average values, not

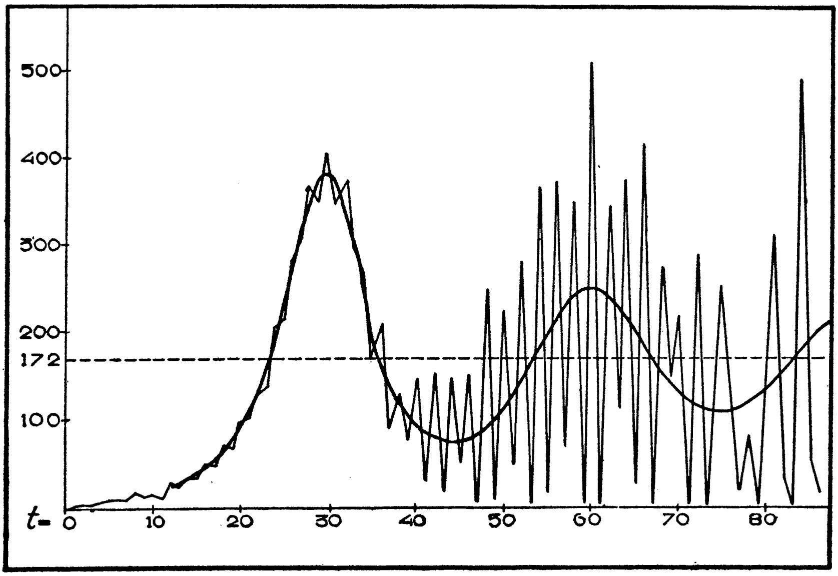

FIGURE 6.

frequency distributions, and (b) data on estimated technical and economical life time by producers of machinery; these are also averages, of course, but are classified according to categories of machinery. Vos gives a selection of data from the *Handbuch der Rationalisierung*, (published by the Reichskuratorium für Wirtschaftlichkeit, 1930) and from *Abschreibungssätze für Anlagen in Maschinenfabriken*, (published by the Verein Deutscher Maschinenbau-Anstalten, 1930). Most of the data relating to prime movers show a technical duration of life of about 20 years, an economic one of 8–15 years. For other machinery (textile, paper, metal industries), data are given only on the economic duration of life, averaging also around 8–15 years. (c) Some actual frequency distributions for automobiles (*Quarterly Journal of Economics*, Feb. 1933) show the following distribution;

Duration of life:	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13 years
Percent of cars:	1	2	3	5	8	11	16	12	12	12	8	6	4	1

Mills (*Statistical Methods*) gives the following figures as to telegraph poles:

Duration of life:	0–2	2–4	4–6	6–8	8–10	10–12	12–14	14–16	16–18	18–20	20–22 years
Percentage of poles:	2	10	14	15	16	15	11	7	6	3	0

A frequency distribution of railway rolling stock in Holland was given in *De Nederlandsche Conjunctuur* (Aug. 1934). Some particulars are given below:

Life time in years:		Median	Quartiles
Locomotives	(437)	49	45;56
Passenger cars	(1030)	51	45;58
Freight cars	(5408)	52	33;59[a]
Luggage cars	(344)	49	43;59

[a] This distribution shows two maxima at 32 and 60 years.

The total number of each category included is given in brackets. It is obvious, however, that these data are highly incomplete.

One serious objection remains against theories that consider replacement waves as the chief factor in business cycles in that they neglect *new* investment too much. The only attempt to make a more or less complete estimate of replacement and new investment is that of the Berlin Institut für Konjunkturforschung,[74a] which gives the following figures (in billions RM):

Value of	1924	1925	1926	1927	1928	1929	1930	1931
Replacement	5.2	5.8	6.2	6.6	7.0	7.0	6.8	6.5
New investment	2.7	4.5	5.2	7.2	7.3	7.0	4.0	

There is one advantage in these theories, the larger stability of period in the face of changes in constants. Relatively large changes in the constants appearing in these schemes have only moderate influence on the period. The influence of life time itself is, of course, about proportional. But that influence is not large in comparison with the very large influence of some constants in the "short lag schemes" on the period (Cf. 15).

The ideal solution seems to be, therefore, the combination of short lag and long lag schemes in generalized types.

[74a] *Konjunkturstatistisches Handbuch,* 1933, p. 48.

SECTION IV. THE CHARACTER OF THE MOVEMENTS

19. *"Active" and "symptomatic" variables.*—In this section I give some general remarks on the quantitative behaviour of solutions, not relating to specific schemes discussed in section III but holding true for a rather large class of schemes. As examples, I shall choose the schemes mentioned, but the reader will realize that the statements are of a more general validity. It makes no difference whether we have a system of dynamic equations (irreducible to one single equation) or only one dynamic equation, after some substitutions, of course.

The first question is the different nature of the variables, which may be called active and passive (or symptomatic). By the latter, I mean variables that appear in only one equation of the original system (before substitution, etc.) and which are, therefore, automatically eliminated by omitting that equation; the latter is simply the definition equation of the variable in question. The economic significance of this mathematical property is that the variable in question is itself determined, or, to express it in a more suggestive way, dragged along, by the cyclic movement but does not reinfluence that movement. Mathematically, there is an interesting difference between the two groups of variables. The degree and, more generally, the functional form in which the active variables occur in their equations has a large influence on the nature of the movements of all variables. The form in which the symptomatic variables occur has influence only on their own movements. Suppose, for example, that in the irreducible system of equations all variables and their derivatives or integrals (relative to time) occur linearly. Then, the movements which these variables can show have the property that they may be multiplied by an arbitrary constant. Small movements owing, say, to small shocks and large movements due to large perturbations are strictly proportional. Moreover, the solutions are sums of very simple mathematical functions, that is, exponentials and damped sines. When the active variables would have appeared in, say, a second or third power or in transcendental forms in the irreducible system, the solutions are far more complicated, as will be shown later. It does not matter, however, whether symptomatic variables appear in such forms or not. Suppose one variable B to be symptomatic in some scheme and linked to one of the active variables A by some equation $B=f(A)$. The function may be such that, when A describes a sine curve, B describes a curve with flat minimum and a very pointed maximum, which seems a closer approach to reality for money rates, for example.

The rôle of active and symptomatic variables is, perhaps, best illustrated by considering the construction of a theoretical scheme in its successive stages. The theorist begins with some equations in which a number of variables appear. The number of variables will at first, generally be larger than the number of equations. He must proceed until these numbers are equal, otherwise the system is not closed. We shall assume that he sets aside such equations as merely introduce one new variable and are thus to be considered only as definition equations for these variables, being themselves "symptomatic." When the system is closed after only a small number of active variables has been introduced, this means that a small group of phenomena is chosen from the economic total and their interrelations are considered to be the causes of the cyclical movements, the other variables being forced to move in dependence on them. This is more or less the case in all simpler schemes. The remarkable thing is that it is possible to choose quite different phenomena in this small circle. However, the more realistic the schemes are, the larger the circle of active variables it is necessary to introduce.

20. *Equilibrium points and small movements.*—An important preliminary step in the solution of the equations is the determination of the equilibrium points, i.e., the *constant* values of the variables that satisfy the equations. Substituting constant values for variables entails a substantial simplification of the equations, as all derivatives with respect to time vanish, all integrals reduce to linear terms, etc. A system of linear equations will, of course, lead to only one equilibrium point (provided its determinant is not zero), but non-linear equations may yield more. Thus, Kalecki's equation shows only one point, as does the simplest case I considered. Some of my more complicated schemes, however, show two equilibria and it is easy to construct schemes that show many. Vinci's system, however, yields only one equilibrium solution.

The equilibria found may, or may not, be *stable*. The criterion is the nature of small movements around the equilibrium. These small movements may be divided into three groups: (1) movements directed, in the long run,[75] away from the equilibrium; (2) movements directed, in the long run, towards the equilibrium, or oscillating movements with positive (or zero) damping; and (3) oscillating movements with negative damping (anti-damping).

The equilibria relating to the first type of small movement may be called unstable, those relating to the second, stable, and those to the third group, quasi-stable.

Whether an equilibrium belongs to a certain group depends, in general, on the special values of the constants. For some values it

[75] This expression means that a finite time can be indicated after which the movements will continue in the direction shown without interruption.

may be stable, for others unstable or quasi-stable. Thus, in Kalecki's case, an unstable equilibrium exists when

$$e^{-m}(m + \theta n) \leqq \frac{1}{e},$$

in which m, n, and θ, are his constants. In the case discussed in 15A (for $\epsilon' = 1$), an unstable equilibrium for $p = 0$ exists for

$$\theta k > \frac{aP}{A} + 1,$$

where, in the case $aP/A = 1$, equilibrium is stable when

$$\theta k \leqq 2.$$

It is interesting that, in this last case, one equilibrium is stable and the other unstable and that the stable one is the one with the highest value for the variable p (price).

Let x, y, etc., be the variables of any dynamic set-up. We choose the origin of our system of variables at the equilibrium point, i.e., in equilibrium we have $x = 0$, $y = 0$, etc. Then x, y, etc., are simply the deviations from equilibrium shown by our variables. It follows that our equations do not contain any constant terms but only terms with one or more factors, x, y, $\dot{x}$, $\dot{y}$, $\int_{t-\epsilon}^{t} x\, d\tau$, or $x(t-\theta)$, etc., where t is time, (τ, when integration variable), ϵ and θ constants. If we also had constant terms, the equilibrium point would not satisfy our equations.

Since we now suppose the movements to be small, all terms containing products of two or more variables (derivatives, integrals, or "lagged variables," included) may be neglected. Thus, we have to study only equations that are linear in these variables, in the wide sense of the word.

This being so, the nature of the components of the movement will be determined by the roots of a characteristic equation. If one of these roots has a positive real part, unstable equilibria will exist. Stable equilibria occur if none of the real parts are positive. If all the roots are actually real and non-positive, then we will have a movement, in the sense mentioned above, directed in the long run towards equilibrium. When there are complex roots, we will have oscillations of decreasing amplitude (positive damping), as soon as all moduli are below unity.

Quasi-equilibria will occur when there are any complex roots having a modulus larger than 1. This entails that the system show at least one oscillation whose amplitude increases beyond any limit as time passes.

This sort of oscillation is impossible in physics as far as endogenous oscillations are concerned, as they are incompatible with conservation of energy. As such a law does not necessarily apply to all economic schemes, they are there possible.

21. *Non-linear equations.*—New problems arise when the irreducible system or original equations is not linear. Then, the theorems concerning small movements around the equilibrium or equilibria are not valid for larger movements. A general discussion becomes, in most cases, very difficult. By examples I shall now show what sorts of new possibilities are then created.

In my paper on oscillatory movements in ECONOMETRICA (1933) p. 44, I discussed a case where it was possible, by introducing a new variable, to transform a non-linear into a linear one. The new variable $\omega(t)$ depends on the old one by

$$\omega(t) = \log \left[bp(t) + 1\right].$$

One of the possible movements of ω is a sine curve of arbitrary amplitude. The corresponding movement of $p(t)$ is similar to the movement of the "symptomatic" variable considered in 19.

A more complicated case is treated in my paper in the Wagemann *Festschrift*, in the sense that some numerical calculations have been made. These calculations show that, for the same set of constants, the period of the movement changes with the initial condition. For instance, when the initial impulse is assumed to be increasing, the period increases, reaching infinity for a finite value of the initial disturbance. Parallel with this growing period, the form of the oscillations is changed. When the movement comes near the second, unstable equilibrium that exists in this case, it tends to move for a long time parallel to that equilibrium and so to prolong the part of the oscillatory movement situated between the two equilibria. When the original disturbance surpasses a certain limit, however, the movement breaks through the unstable equilibrium and the movement does not come back.

It should be stressed that only the first beginnings in this field have been made. On the mathematical side of the problems, Volterra's book, *Sur la théorie mathématique de la lutte pour la vie*, may be mentioned.

I restrict myself to economic aspects. First, it may be stated that higher degree equations are very likely to occur in economics. The simple relation between price, quantity sold, and amount paid, is a quadratic relation. As the number of variables included increases, there are many other reasons for obtaining higher degree equations.[76]

There is one rather general phenomenon leading to non-linear func-

[76] Koopmans, "De mogelijkheid van meervoudig evenwicht," *De Economist*, 1932, pp. 679, 766, and 841, has shown this in an elementary way.

tions in our equations, the phenomenon that might be indicated by *saturation*. Its essential feature is that certain variables are not allowed, by natural, technical, or institutional circumstances, to surpass a given limit. An example is given in the case where a maximum output exists for technical reasons. Assuming, for simplicity, that this output depends only on price, the relation between price and output will be nonlinear; output as a function of price will have a horizontal asymptote. The simplest case presents itself, of course, when the saturable variable is a symptomatic one, whereas the active ones only appear in linear equations. We then have the following situation. When the intensity of the initial perturbancy grows, the active variables show amplitudes proportional to this intensity but the symptomatic ones show amplitudes that, after a certain limit in the intensity has been reached, grow slower and slower, the saturation limit being its limit. In an analogous fashion, the form of the waves in these symptomatic variables is different from that of the active variables, and these waves will be flattened.

When active variables also show saturation, we have a special case of non-linear irreducible equations. The question has not yet been investigated in this way, although the phenomenon as such is often mentioned, even in literary theories. I can only refer to one aspect. It is a source of rather frequent misunderstanding in this field that, when the saturation limit is reached, this would mean at the same time that the turning point, in the sense of the point where the downward movement begins, is reached. It is possible to give a very simple example where this does not hold good. It is the case of the simple "Hanau oscillation," showing inverse waves of prices and deliveries.[77] When, in this scheme, one of the variables has a maximum prescribed by technical conditions, we may have a situation where the maximum is maintained a certain finite time before the downward movement begins.

22. *Some further problems regarding maximum amplitude.*—The following are some further suggestions regarding saturation and maximum amplitude. We know by experience as well as by theoretical considerations that a large number of different activities develop with a certain parallelism during the upward swing of the wave and shrink, in about the same way, during the downward movement. Of course, the variables themselves need not be proportional, but their deviations from equilibrium may be approximately proportional. What will now determine the maximum amplitude of the whole system? That activity whose amplitude limit is reached *first*. The limit may be not so sharply defined; it may be somewhat elastic, but there will be a limit beyond which, for example, in the short run, production of pig iron, or the amount of loans, or the number of workers, cannot be extended. The

[77] Cf. my paper in ECONOMETRICA I (1933), 38.

limit will be a wider one for one phenomenon than for the other and one phenomenon will have the tendency to more expansion than the other, and so forth. How can it be determined statistically which category will first reach its limits? We can say, as a first approximation, the good or service whose price shows a sharp peak, whereas other prices, and especially the output or amount used of the good or service itself, only show flat peaks. As examples, I see pig iron prices before some of the earlier crises (1873, 1890, 1900), discount rates in nearly all crises, and, in some cycles, stock prices (1929) or freight rates (1900). As examples of the opposite, we can mention wages and prices of agricultural products. For this reason, I see more realism in theories that point to a scarcity in capital goods or in short credit than in theories pointing to a scarcity in labor or organic materials.

SECTION V: COORDINATION OF THEORIES

23. "*Alternative dynamizations.*"—The existence of a rather large number of competing theories is not a satisfactory situation. It would be most desirable to have one generally accepted explanation. The way to approach the problem is, first, to confront the theories and, second, to combine their really valuable parts. To carry such a synthesis through is no easy task. In this synthesis there are two different phenomena which I should like to call (a) alternative and (b) different schemes and dynamizations. By alternative schemes I mean schemes or theories pretending to represent the same phenomena but doing so in a different way. By different schemes I mean schemes representing different phenomena, though, of course, they pretend to represent an approach to the same thing. For example, Kalecki's and my schemes both pretend to represent an approach to a society showing business cycles, but, in Kalecki's scheme, elementary phenomena are included other than in mine. He introduces an influence from total capital on amount invested, while I introduce an influence of prices on amount invested. In this respect these schemes are, therefore, different schemes.

First, consider alternative schemes or, as the principal characteristic of the schemes is the "dynamizations" in it, alternative dynamizations. Taken in the true sense of the word, they are few. An example is the two ways in which variation of wages may be connected with variation in employment by a simple lag equation or by a differential relation. Another example is the adaptation of different "time shape functions of production," partly the controversy between Hayek and myself as discussed in 9. A third is the rules governing replacement, the controversy between (a) Kalecki, assuming constant replacement, (b) De Wolff, assuming replacement varying according to age of objects, and (c) others, assuming replacement varying with the economic situation. It is clear that such questions can only be settled by statistical evidence.

Somewhat different questions may be treated in this category, viz., questions of *different degrees of approximation*. Can the supply function (cf. 3, 15), for example, be assumed as linear or should a higher degree function be chosen? Can a constant lag be assumed between price and delivery, or should it be chosen variable? Can a fixed life time be assumed for means of production or should a variable life time be taken? Although, in these questions also, statistical evidence must be decisive as to the necessity of a certain higher approximation, theoretical analysis can inform us on the character and, in some cases, the direction of the "corrections" that such higher approximations entail. Some examples may be given here.

E.[78] *Influence of small lag variations on Kalecki's scheme*

Suppose θ (12) is no longer constant but depends on some variable x (to be chosen later on) by the equation $\theta=\theta_0+\theta_1 x$. This will change the equations given in 12. The first of these equations, (12.1), showing the relation between $L(t)$ and $I(t-\theta)$, will be changed due to the following circumstances. Total investment orders, $I(t)$, at time t correspond to deliveries, $L(t+\theta)$, at time $t+\theta(t)$. Investment orders, $I(t+1)$, at time $t+1$ correspond to deliveries $L(t+1+\theta(t+1))$. Whereas one time unit elapses between the orders, between the deliveries $1+\theta(t+1)-\theta(t)$ $=1+d\theta(t)/dt$ units of time elapse. According to the value of the last term, there will have taken place a "tightening," or an "evacuation," of orders and, therefore, we have, instead of (12.1),

$$L(t+\theta) = \frac{I(t)}{1+\dot{\theta}\cdot(t)},$$

or, as for $\dot{\theta}(t)$ we can write $\theta_1\dot{x}$,

$$(1+\theta_1\dot{x})L(t+\theta) = I(t).$$

No change will occur in equation (12.2), but equation (12.3) may be replaced by

$$W = \int_t^{t+\theta} L(\tau)d\tau,$$

in which θ is now a variable. This entails that

$$\frac{dA}{dt} = \frac{d}{dt}\frac{W}{\theta} = \frac{\dot{W}}{\theta} - \frac{W}{\theta^2}\dot{\theta} = \frac{\dot{W}}{\theta} - \frac{W}{\theta^2}\theta_1\dot{x}$$

$$= \frac{L(t+\theta)-L(t)}{\theta} - \frac{\int_t^{t+\theta}L(\tau)d\tau}{\theta^2}\theta_1\dot{x}.$$

[78] A-D have been given in 15.

Equation (12.4) of 12, written in differentiated form,

$$\dot{I}(t) = m\dot{A}(t) - n\dot{K}(t),$$

can now be transformed with the help of:

$$L(t) = L_0 + L_1(t); \ \dot{I} = (1 + \theta_1\dot{x})\dot{L}(t + \theta) + \theta_1\ddot{x}L(t + \theta)$$

$$\text{and} \int_t^{t+\theta} L(\tau)d\tau = \theta L_0 + \int_t^{t+\theta} L_1(\tau)d\tau,$$

in which L_0 represents the equilibrium value of $L(t)$, and $L_1(t)$ is supposed to be small. When we choose for x,

$$x = \int_t^{t+\theta} L_1(\tau)d\tau,$$

i.e., when we assume that the production lag depends on activity, we find after some computations that Kalecki's equation is replaced by a new one, obtained by putting

$\theta_0(1 - \theta_1L_0)$	instead of θ,
$\dfrac{n}{1 - \theta_1L_0}$	instead of n,
$m(1 - \theta_1L_0) - \theta_1L_0$	instead of m.

F. Influence of small variations of life-duration on the scheme referred to in 18 (Note 72)

As a second example, I treat the influence of small variations in life time of means of production caused by changes in the economic situation, basing myself on a scheme given in my *Zeitschrift* paper. In this scheme, the economic situation is characterized by the price level of capital goods q, and the only thing to be varied is that, instead of supposing the duration of life to be constant, we now assume it to depend on q in the following way:

$$\eta = \eta_0 - \rho q(t), \tag{23.1}$$

where η may now mean the life time of means of production replaced at time t.

Our equations (given in *Zeitschrift f. Nationalokonomie*, 1934, p. 316)

$$\int_{t-\eta}^{t} q(\tau)d\tau = gq(t), \qquad g = \frac{a}{b}\,\frac{e + f - A}{A + aP - e}, \tag{23.2}$$

must now be changed in accordance with (23.1). In the first of equations (23.2), at both sides a constant term has been omitted, viz., the terms $\int_{t-\eta}^{t} Q d\tau = \eta Q$ at the left side, and some constant K at the right side, that is equal to ηQ by virtue of the long run equilibrium condition. As, however, η is now no longer a constant, we have to reconsider this term. It will be replaced by

$$\eta_0 Q - \rho q(t) Q,$$

and our equation becomes

$$\int_{t-\eta}^{t} q(\tau) d\tau = (g + \rho Q) q(t).$$

The effect is simply that g is replaced by a somewhat larger g. The influence of this larger g on the period depends on g's value. For $g=1$, a minimum period is reached, for $g<1$, an increase means decreased period, but for $b>1$, an increase means an increased period; as to r, which measures inversely the degree of damping, this moves contrary to g, and so an increase of g means an *increase* in *damping*.

Mathematical approximation.—Apart from questions of different degrees of approximation to relations that must be found empirically, theoretical relations may also be approximated in order to facilitate calculations. We have already met examples of this sort. Integrals over short intervals—in comparison with periods or quasi-periods—may be replaced by a mean value times interval length, integrals over longer periods by sums, derivatives by differences, etc., and *vice versa.* So it will be possible, for example, to get rational, instead of irrational or transcendental, equations.

Another possibility consists in a reduction of all lagged variables to derivatives, by

$$f(t - \theta) = f(t) - \theta f'(t),$$

which holds, of course, only for small θ's, that is, small as compared with the period of the movements considered. In this way, it is possible to reduce a large number of equations to *pure differential equations.* Of the last possibility, an example is given in the next paragraph.

24. *"Different" dynamizations.*—By far the most important difficulty we encounter when we desire to coordinate theories is that of the confrontation and combination of "different" dynamizations. We have seen in Section II that there is a large number of dynamic relations among the most important economic variables. The number of alternatives is so large that it is virtually impossible to take account of all of them. This has lead several authors to make a choice and the choices have been different. One is lead to ask what choice was best? But the question is not posed well. None is absolutely right or absolutely wrong. Progress is not to be sought through a choice but a combina-

tion. Our aim should always be to reach conclusions of the following kind: "when these coefficients are small and those large, author A gives a better approximation; when those are small and these large, author B does. When all coefficients are of importance, a mixed theory emerges deviating from author A's theory by this and from author B's theory by that peculiarity."

G. A combination of some principles of Frisch's and mine

As a very simple example, I combine Frisch's theory on the interrelation between consumers' goods and capital goods production with my method of introducing prices into the schemes. Let the production of consumers' goods be $x(t)$, and that of capital goods $y(t)$; then, according to Frisch, we have:

$$y(t) = mx(t) + \mu\dot{x}(t). \tag{24.1}$$

The production of consumers' goods itself may depend on the price level of consumers' goods, $p(t)$, and be, for example,

$$x(t) = bp(t) + c\dot{p}(t). \tag{24.2}$$

Apart from a constant factor of proportionality, the right member in (24.2) may be looked upon as an approximation to the price level existing at some time a little before t when $c<0$ (and $b>0$), or a little after t when $c>0$. A *lag* of x after p (for $c<0$) may be caused by the length of the production period; a lag of p behind x may be caused by the fact that producers determine production by paying attention to probable further price development or by the fact that it is not the price level that determines production but production (as dictated by consumption) that determines prices.

The price level itself will be determined at the consumers' goods markets, for which we have

$$(P + p)(X + x) = K + L(x + y). \tag{24.3}$$

Here, according to our usual notation, P, X, and Y, denote equilibrium values, L the average income earned by additional producers for each unit of additional product. Limiting ourselves again to small movements (small p, x, and y) and choosing units such that $P=X=1$, we have,

$$p + x = L(x + y). \tag{24.4}$$

The solution of our system, (24.1), (24.2), and (24.4), gives us

$$p + x = L(x + mx + \mu\dot{x}).$$

Writing λ for $L(1+m)-1$, which will, in all probability, be positive, we have

$$p = \lambda x + \mu L\dot{x},$$

and thus,

$$p = \lambda bp + (\lambda c + \mu Lb)\dot{p} + \mu Lc\ddot{p},$$

or,

$$\gamma p + (\lambda c + \mu Lb)p + \mu Lc\ddot{p} = 0. \qquad (\gamma = \lambda b - 1).$$

This is the equation for a harmonic oscillation, if

$$4\gamma\mu Lc > (\lambda c + \mu Lb)^2.$$

I shall not attempt a full discussion but only state that no oscillations at all are possible when γ and c have opposite signs and that, in this example, another form of equation is obtained into which many combined schemes can be brought by suitable approximations. A fuller discussion and further attempts to generalize this scheme should be postponed until better statistical information is available and some of the controversies discussed in Section II are clarified.

In this connection it is important to remember that "different theories," in the economic sense of the word, may often lead to the same mathematical formulas, the only difference being in the economic sense of some of the constants. I have already pointed out, for instance, that the influence of wages and productivity of labor on profit margin w are both proportional to

$$-\int_0^t p(\tau)d\tau \qquad \text{(cf. 5)}.$$

The same might be said of the influence of the total capital invested on profit calculations, i.e., when entrepreneurs are led by the percentage profits bear to this capital. Then the directive of production will be

$$\frac{W + w(t)}{C + c(t)},$$

W and C meaning equilibrium values of profits and capital and $w(t)$ and $c(t)$ being deviations from equilibrium values of these variables. This expression may be approximated by

$$\frac{1}{C}\left\{W + w(t) - \frac{W}{C}c(t)\right\}.$$

The last term in the brackets will be proportional to $\int_0^t p(\tau)d\tau$, as the *growth* of capital is proportional, in its cyclic component, to $p(t)$. Thus, this last term is of the same structure as the productivity and the wages term.

SECTION VI. PROBLEMS OF BUSINESS CYCLE POLICY

25. *Consequences of a given policy.*—The ultimate practical purpose of all business cycle theory and business cycle research is, of course, to help solve the problems of business cycle policy. The theoretical nature of these problems is that of variation problems; each case of business policy may be looked upon as a variation of some constants of a business cycle scheme. In this respect, they belong in the same class as the problems depending on the influence of other ("natural") perturbances of cyclical movements, such as exceptional crop figures. Some of these problems are analogous to "shock problems" in pendulum physics; another part may be compared to the problem of changing the length of a pendulum. A third category consists in a shifting of the turning point of the pendulum.

In mathematical terms, this is about equivalent to the distinction between three sorts of variation problems, (a) variation of the initial conditions, (b) variation of the coefficients of one or more of the equations, and (c) variation or introduction of an additive term in one or more of the equations of the system.

Apart from this distinction, another can be made, that between "single" and "composed" variations. By a "single" variation, a transition from one given value of a constant to another is meant; by composed variations, a system of consecutive changes in one or different constants.

Some examples will illustrate these notions.[79] Three types of variation problem may be treated: (I) Influence of varying crops on cycles; (II) Influence of "money injections"; and (III) Influence of wage changes, as far as these are not endogenous.

To study the consequences of given economic types of perturbations, one has to choose a suitable scheme.

In our examples, I take as the undisturbed scheme the one discussed in 15, *A* for $\epsilon' = 1$, i.e., the simplest pure lag scheme. Using $p(t)$ as variable, the supply of consumers' goods at time t is $A + ap(t-2)$. The amount spent on the consumers' goods market at time t is

$$K + 2kp(t - 1),$$

and the equation determining the movement of $p(t)$ will be:

$$\{A + ap(t - 2)\}\{P + p(t)\} = K + 2kp(t - 1). \quad (25.1)$$

In this equation we will now consider the three variation problems mentioned.

[79] As I shall point out, problems of type *a* and type *c* are reducible to each other; single variation problems in class *c*, for example, can be transformed into composed variation problems in class *c*. Other transformations also exist.

I. Influence of variations in crops

Variations in crop yields have as a consequence that supply consists of a "regular" part, $A+ap(t-2)$, cf. equation (25.1), and an irregular one, for which we may write $A'(t)$. At moment t we have, therefore, instead of (25.1)

$$\{A + ap(t-2) + A'(t)\}\{P + p(t)\} = K + 2kp(t-1). \qquad (25.2)$$

At each moment, this means that the "new" price, $P+p(t)$, is not determined by the "regular" equation (25.1) leading, with small p's, to a sine curve, but by a somewhat changed equation (25.2) which may be looked upon as the regular equation corresponding to another value of $p(t-2)$, viz.,

$$p'(t-2) = p(t-2) + \frac{A'(t)}{a}. \qquad (25.3)$$

In the case of *small* movements, we may say that the price movement all at once follows another sine curve and such a changing of the curve followed presents itself each time $A'(t) \neq 0$. Considering the problem in this way, we have to do with a variation in initial conditions. Of course, it may also be looked upon as a combination of a variation in additive constants and coefficients. This leads to the creation of changing "harmonics" by erratic shocks, according to Frisch's scheme.

II. Influence of "money injections"

When, at moment t, an amount $K'(t)$ is added to the amount spent at the consumers' goods market (e.g., when public works are paid for with created money), we get

$$\{A + ap(t-2)\}\{P + p(t)\} = K + 2kp(t-1) + K'(t). \qquad (25.4)$$

This can be interpreted by saying that the price, not at moment $t-2$, but at the moment $t-1$, has changed, the amount of variation being

$$\frac{K'(t)}{2k}.$$

In accordance with expectations, a positive increment of $p(t-2)$, as in problem I, has a negative influence on $p(t)$, whereas a positive increment of $p(t-1)$, as in problem II, has a positive influence on $p(t)$.

Restricting ourselves to small movements, in which case we get linear equations, we can interpret the influence of the variation $K'(t)$ also as a changing of the equilibrium level of prices, formerly P. Indeed, we can write for

$$Ap(t) + aPp(t-2) = 2kp(t-1) + K'(t), \qquad (25.5)$$

equivalent to (25.4), the equation

$$A\{p(t) - p_0\} + aP\{p(t-2) - p_0\} = 2k\{p(t-1) - p_0\}, \quad (25.6)$$

when we take

$$(A + aP - 2k)p_0 = K'(t). \quad (25.7)$$

This means that there are two ways in which $p(t)$ can be calculated, first, as the value belonging to $p(t-2)$ and $p(t-1)+K'(t)/2k$ in the old mechanism and related to the old equilibrium level and, second, as p_0 plus the value belonging to $p(t-2)-p_0$ and $p(t-1)-p_0$ in the old mechanism, but related to a new equilibrium level $P+p_0$.

In this case of small movements, we can even give two further interpretations; the additional term may equally well be considered as a variation of $-K'(t)/A$ in $p(t)$ as a variation of $-K'(t)/aP$ in $p(t-2)$.

When, in only one elementary time period (i.e., a period of length 1), an "injection" takes place, we have, for a succession of periods, the equations:

$$\begin{aligned} Ap(2) + aPp(a) &= 2kp(1) \\ Ap(3) + aPp(1) &= 2kp(2) \\ Ap(4) + aPp(2) &= 2kp(3) + K' \\ Ap(5) + aPp(3) &= 2kp(4) \\ Ap(6) + aPp(4) &= 2kp(5), \end{aligned} \quad (25.8)$$

the period of injection being period 4. Considering the variation as a variation in $p(2)$, the variation is a "single" variation in initial conditions, i.e., the movement jumps but once from one curve to another. Considering K' as a variation in equilibrium price level, however, we have to do with two jumps, one after another. First, a jump from equilibrium level P to $P+p_0$ takes place and, second, a jump back to P. We have, then, to do with a "composed" variation of class c, which is evidently identical with a "single" variation of class a.

III. Influence of wage changes

As, in the original scheme adopted here, no wage changes occur, all changes in wages are exogenous. A wage change has two sets of consequences, first, costs of production and, therefore, production plans are changed and, second, the income spent at the consumers' goods markets is changed. The first consequence means that, after a wage change, p in the supply and income functions has to be changed to

$$p - p_0$$

and a and k into $a\alpha$ and $k\beta$; p_0, α, and β, depending on the change in wages adopted. The second consequence entails that labor incomes and, therefore, incomes spent corresponding to a given productive activity have changed, i.e., K becomes $K\gamma$ and $k\beta$ again changes into $k\beta\gamma$. Thus, after the wage change has been worked out fully, equation (25.1) has turned into

$$\{A+a\gamma p(t-2)-a\gamma p_0\}\{P+p(t)\}=K\gamma+2k\beta\gamma\{p(t-1)-p_0)\} \quad (25.9)$$

which, in general, means a change in the mechanism, i.e., in period and degree of dampening, contrary to the other examples given. But this is not the only new feature. The new terms and factors occurring in (25.9) do not come in at the same moment, but at consecutive moments, the order of which may still be different owing to different assumptions. So, it is in every case clear that changes in supply will be later than those in incomes in this scheme. But, in addition, γ may or may not come in at the same time as β and p_0 come in at the right side. And this influences in a very important degree the shape of the disturbance.[80]

It will be seen that, in this third problem, we have to do with a composed variation of initial conditions as well as of coefficients.

Having considered these three variation problems, we can now state that, in general, a perturbation may change four fundamental aspects of the movement, the amplitude, phase, period, and degree of dampening. In common discussions on business cycle policy this is seldom recognized, for often only immediate effects are taken account of. Another statement of importance is that the effect often depends in a very large degree on the phase of the original movement in which the perturbation presents itself.

26. *The problem of the "optimal" policy; some remarks on "neutral" money.*—Having indicated, at least in principle, how the consequences of a given business cycle policy are to be found, we can now enunciate the "last" problem in this field, that is, which policy is the "best" one, the "optimal"? To answer this question, we should dispose of some general ophelimity function and calculate for which policy a course of events would occur giving a maximum to this general ophelimity function.[81] Of course, at this moment any practical calculations of this sort are impossible. I shall not try to give them here.

I shall only make some remarks on the frequent and, I think, justifiable assumption that stabilization of business cycles in employment is the optimum policy.

Stabilization of cycles can be obtained in different ways, viz., by

[80] Cf. my paper: "La politique des salaires, les cycles économiques et les mathématiques," *Revue des Sciences Économiques*, févr. 1935, p. 17.

[81] Cf. in this respect my tentative "Konjunkturforschung und Variationsrechnung," *Archiv fur Sozialwissenschaft und Sozialpolitik* LXI (1929), 533.

(1) reducing the amplitude of the cycles only; or (2) reducing the period of the cycles only; or (3) increasing the dampening of the cycles only; or, finally, by combinations of these types of policy. An example of the first form of stabilization is given by some cases of wage changes, examples of the second categories by some cases of "money injections"[82] or a "raw material monetary standard."[83] Of these three types, the first one is essentially less advisable than the other two, as a new perturbation of equilibrium will immediately change the amplitude and may make it large again. This is unlikely to occur in the second and third types, as a certain degree of continuity will also hold the amplitudes of the movements within limits of these cases. Thus the second and third method are to be preferred.[84]

Another difficulty is also evaded by the second and third types of stabilization policy, viz., the difficulty arising from the fact that the equilibrium value of prices and other variables continuously changes as a result of innumerable variations in data. A practical approximation to this equilibrium level by computation will hardly be possible before a highly organized economic control exists. By adopting, however, a business cycle policy of type 2 or 3, i.e., by changing our economic mechanism so as to show only very short or very much dampened oscillations, we could arrange it so that this equilibrium level would be realized automatically. I hope to add further contributions to this problem within a short time.

It should be borne in mind, however, that the examples of class (2) above relate only to highly simplified schemes and that they do not necessarily behave similarly for other schemes. The remarks made, therefore, are valid for these examples of policy only in so far as the simple schemes are considered. But they relate to many more general examples as well.

I may add a few remarks on a special form of stabilization policy that has been proposed and worked out by several authors during the last decennium, though not by all of them in the same way. This policy, often indicated as a neutral money policy, has perhaps been defined and defended in the most explicit and exact way by Koopmans.[85] To summarize his argument, the state of things which obeys the Walrasian system of equations is an optimal situation. As the equality of supply and demand (in terms of goods) is disturbed by the hoarding and creat-

[82] Cf. the papers mentioned in notes 79 and 56.

[83] Cf. the paper mentioned in 24, Concluding remarks, 2.

[84] Cf. Frisch, mimeographed lectures, 1933. Frisch, in particular, advocates method 3. He calls it the principle of "the oil brake"; a brake is introduced whose strength is proportional to the *velocity* of the movement.

[85] J. G. Koopmans, "Zum problem des 'neutralen' Geldes", published in *Beitrage zur Geldtheorie*, herausgegeben von F. A. Hayek, 1933.

ing of money, the monetary system must be such that each hoarding and creating is counteracted by equal amounts of money created or hoarded. This should be reached by credit policies, i.e., discount and open market policies. Koopmans shows especially that a neutral monetary system is not equivalent to a constant price level monetary system.

This last statement, is, no doubt, valid. I should, however, like to make some remarks on the preceding argument. In the first place, the identification of the optimum situation with a Walras situation is, in my view, very questionable. Since it seems that Koopmans himself recognizes this, it may be left out of the discussion. My main objection is, then, that the realization of the Walrasian situation is impossible when we have a *permanently changing situation* in addition to some elements that make absolutely impossible an immediate reaction of all variables to any change in data. Elements such as the production period and the long duration of means of production make it impossible that a quick adaptation to new data should take place. Therefore, the Walrasian situation can only accidentally be realized. This is the reason why it seems better to discuss the desirability of a given stabilization policy without any connection with the Walrasian system.

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LINKED CITATIONS

[Footnotes]

[6] **The Farmers' Response to Price**
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